Hadron spectroscopy on the lattice

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Outline

I. Introduction
II. Impressive results of hadron spectroscopy
III. Lattice spectroscopy relevant of BESIII
IV. Challenges and oportunities
V Summary

I. What are hadrons

Quark Model

$SU(3)$ multiplets
+ charmonium
+ bottomium
+ D, B meson
+ Heavy flavored baryons

Experiments

- Ground state hadrons
  stable particles in strong interaction
- Resonances

Effective field theory

Hadrons as degrees of freedom

Hadron molecules

Final state interaction

(see talks by F. Guo, Z. Guo J. Wu)

Exotics:

glueball,
hybrids,
XYZ
(multi-quark states)

Lattice QCD:

experiments (and beyond) on computers

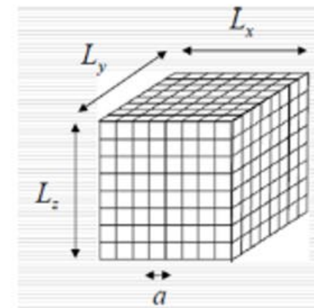
The lattice formulation of QCD---Lattice QCD

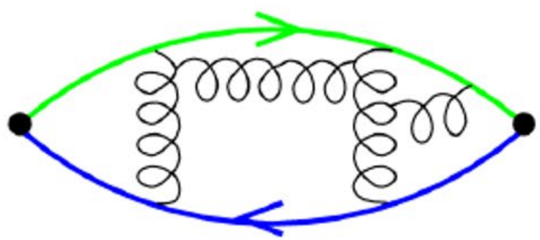
$$Z = \int \mathcal{D}A_\mu \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; e^{-S}$$

$$S = S_{gauge} + S_{quarks} = \int d^4x \left(\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) - \textstyle\sum_i \log(\mathrm{Det} M_i)$$

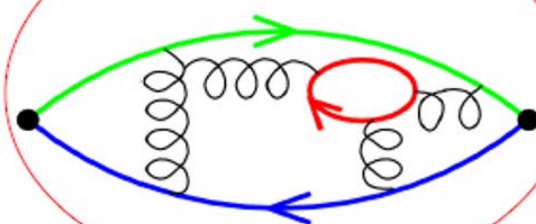
$$Z = \int \mathcal{D}A_\mu \, \det M \; e^{\int d^4x \, (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu})}.$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \, \mathcal{O} \, e^{-S} \; .$$





(A) Quenched QCD: quark loops neglected



(B) Full QCD

Dominated in the present era

The methods for the hadron spectroscoapy in lattice QCD

- Interpolation field operators --- starting point for a meson (-like) system with given J^{PC} and flavor quantrum numbers:

$$\mathcal{O}_i: \quad \bar{q}_1 \Gamma q_2 \quad [\bar{q}_1 \Gamma_1 q][\bar{q} \Gamma_2 q_2] \quad [q_1^T \Gamma_1 q][\bar{q} \Gamma_2 \bar{q}_2^T], \ldots\ldots$$

- Two-point functions --- Observables

$$\mathcal{C}_{ij}(t) = \left\langle 0 \middle| \mathcal{O}_i(t) \mathcal{O}_j^+(0) \middle| 0 \right\rangle$$
$$= \sum_n \langle 0 | \mathcal{O}_i | n \rangle \left\langle n \middle| \mathcal{O}_j^+ \middle| 0 \right\rangle e^{-E_n t}$$

In principle, all the physical states with the same quantum numbers $|n\rangle$ contribute to the two point functions $\mathcal{C}_{ij}(t)$ as the eigenstates of the QCD Hamiltonian with the energy eigenvalue E_n:

- " one-particle state" : $E_n = m_n$
- " two-particle state" : $E_n = \sqrt{m_1^2 + \vec{p}^2} + \sqrt{m_2^2 + \vec{p}^2} + \Delta E, \ \vec{p} = \frac{2\pi}{L} \vec{n}$
- ……

Comparison of the hadron spectra

Euclidean spacetime lattice		Minkowski continuum spacetime
One particle states	⟷	Stable particles
Multiple particle states with discrete relative spatial Momentum (scattering States in a finite volume	⟶	Bound states of hadrons; Resonances
		Continuum scattering states

All the energies are Discretized.

Luescher's Relation:

$$E_n = (m_1^2 + p^2)^{1/2} + (m_2^2 + p^2)^{1/2}$$

$$\tan \delta(p) = \frac{\sqrt{\pi}\ p\ L}{2\ \mathcal{Z}_{00}\left(1; (\frac{pL}{2\pi})^2\right)}$$

Resonances

$$T(p) = \frac{-\sqrt{s}\,\Gamma(p)}{s - m_R^2 + i\sqrt{s}\,\Gamma(p)} = \frac{1}{\cot\delta(p) - i}$$

$$\Gamma(p) = g^2 \frac{p^{2l+1}}{s}\ , \quad \frac{p^{2l+1}}{\sqrt{s}} \cot\delta(p) = \frac{1}{g^2}(m_R^2 - s)$$

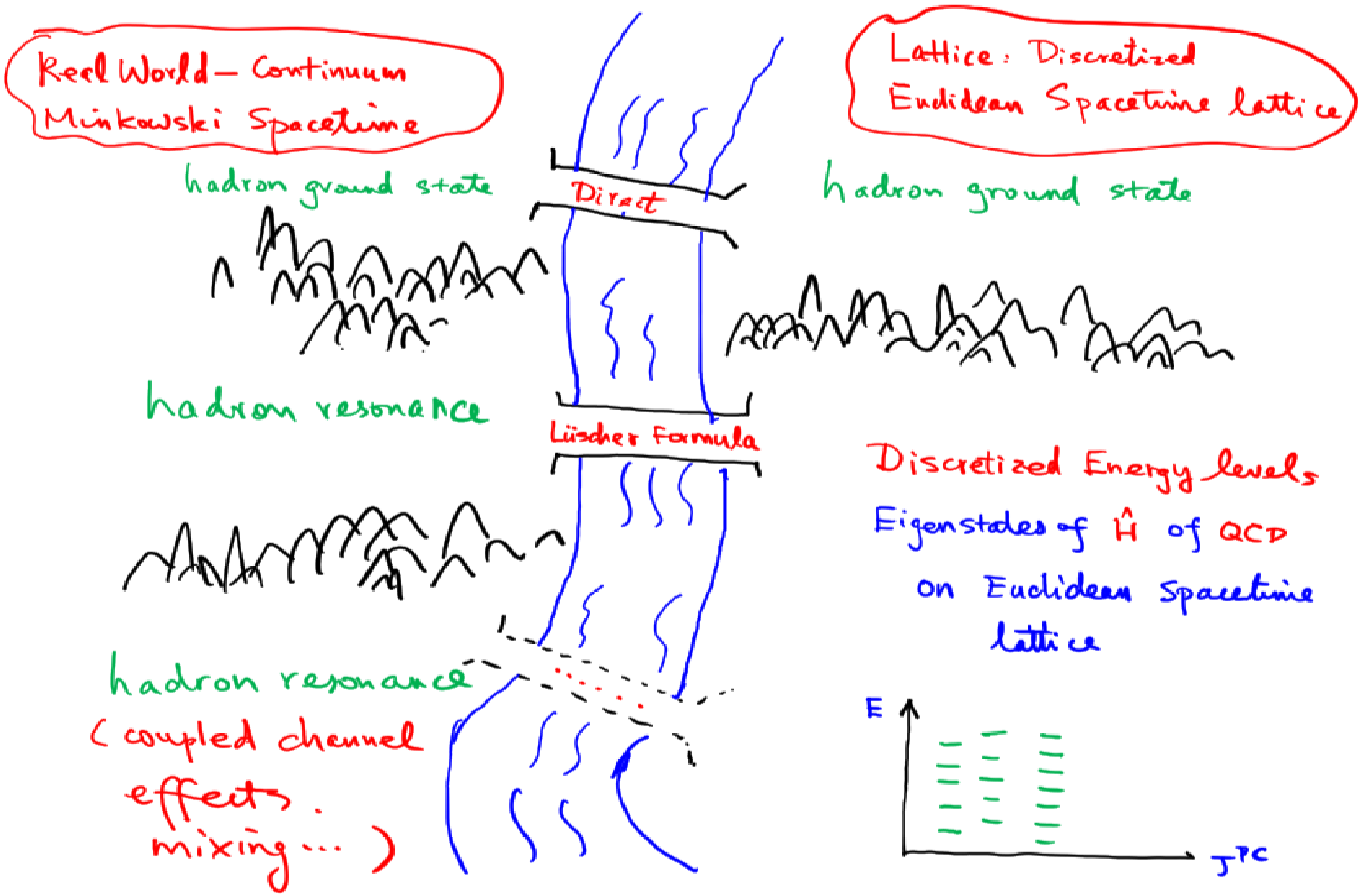
Bound states

$$p\cot(\delta_0(p)) = \frac{1}{a_0} + \frac{1}{2} r_0 p^2\ , \quad -|p_B| = \frac{1}{a_0} - \frac{1}{2} r_0 |p_B|^2$$

$$T = \frac{\ }{\cot(\delta_l(p_B)) - i} = \infty$$

$$m_B = E_{H_1}(p_B) + E_{H_2}(p_B)\ ,\ p_B = i|p_B|$$

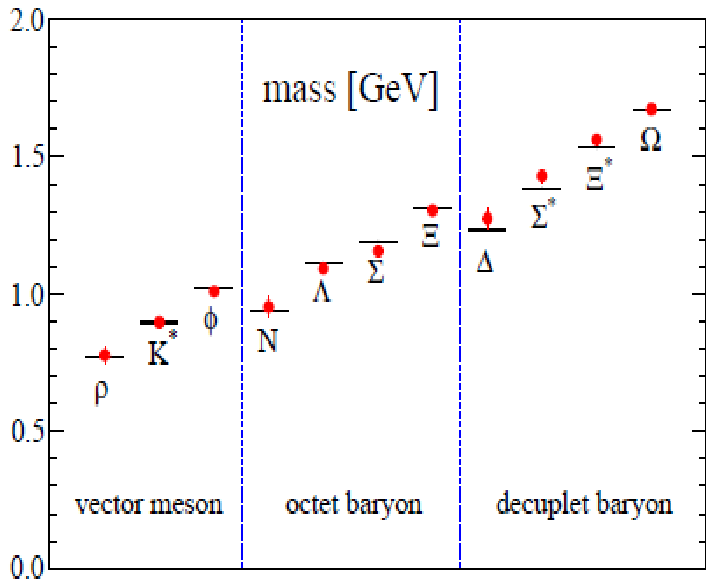
Present status of lattice QCD study on hadron spectroscopy



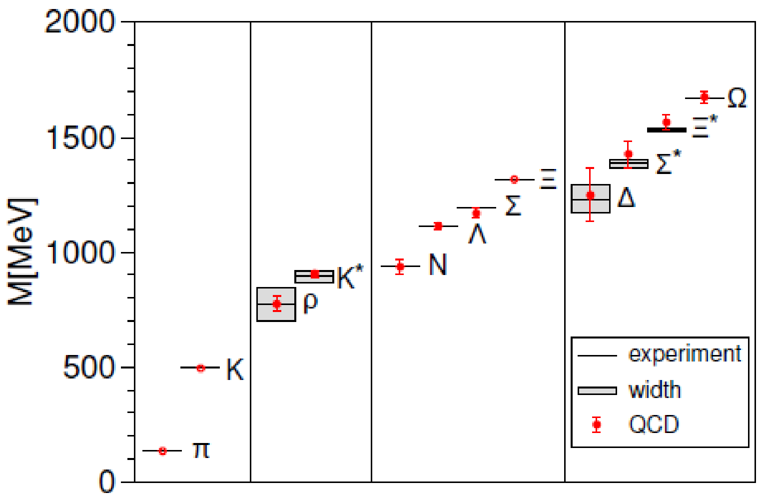
II. Impressive lattice results

- **Ground state hadrons**

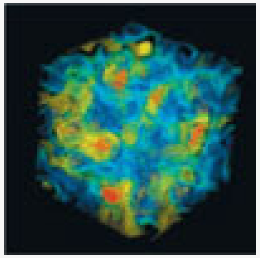
$$C(t) = \langle 0|\sum_{x} \mathcal{O}_f(x,t)\mathcal{O}_i(0)|0\rangle = \sum_n \frac{\langle 0|\mathcal{O}_f|n\rangle\langle n|\mathcal{O}_i|0\rangle}{2E_n} e^{-E_n t}.$$



S. Aoki et al. (PACS-CS Collab.) Phys. Rev. D 79, 034503 (2009).



S.Durr et al.(BMW Collab.), Science 322, 1224 (2008).



Science：2008年十大科学进展

质子质量的"预测"

2. X(3872)

S. Prelovsek & L. Leskovec, PRL111(2013)192001

Operators:

$$O^{\bar{c}c}_{1-8} = \bar{c}\hat{M}_i c(0) \qquad \text{(only } I=0\text{)} \qquad (2)$$

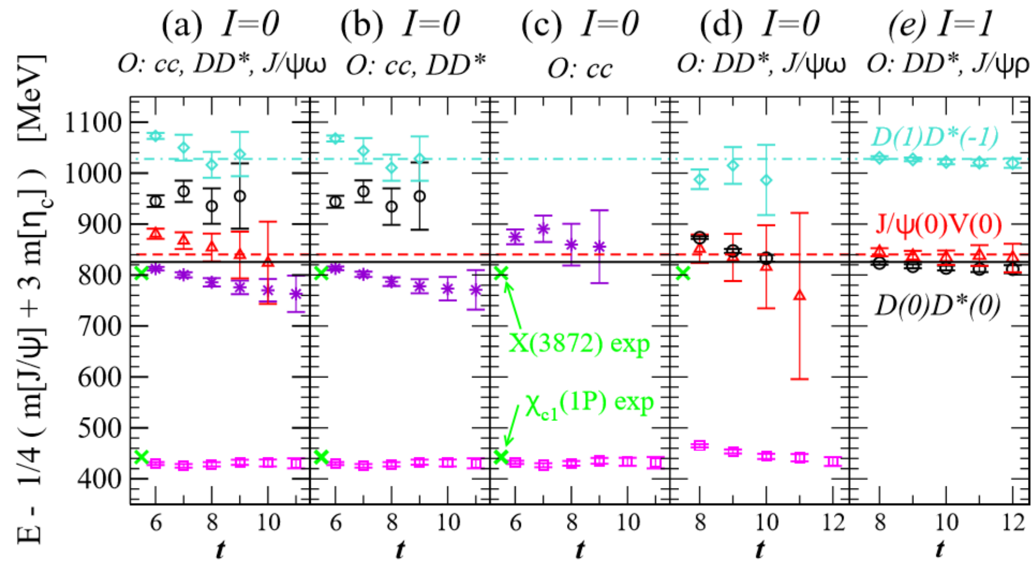
$$O^{DD^*}_1 = [\bar{c}\gamma_5 u(0)\ \bar{u}\gamma_i c(0) - \bar{c}\gamma_i u(0)\ \bar{u}\gamma_5 c(0)] + f_I\{u \to d\}$$

$$O^{DD^*}_2 = [\bar{c}\gamma_5\gamma_t u(0)\ \bar{u}\gamma_i\gamma_t c(0) - \bar{c}\gamma_i\gamma_t u(0)\ \bar{u}\gamma_5\gamma_t c(0)] + f_I\ \{u \to d\}$$

$$O^{DD^*}_3 = \sum_{e_k=\pm e_{x,y,z}} [\bar{c}\gamma_5 u(e_k)\ \bar{u}\gamma_i c(-e_k) - \bar{c}\gamma_i u(e_k)\ \bar{u}\gamma_5 c(-e_k)] + f_I\ \{u \to d\}$$

$$O^{J/\psi V}_1 = \epsilon_{ijk}\ \bar{c}\gamma_j c(0)\ [\bar{u}\gamma_k u(0) + f_I\ \bar{d}\gamma_k d(0)]$$

$$O^{J/\psi V}_2 = \epsilon_{ijk}\ \bar{c}\gamma_j\gamma_t c(0)\ [\bar{u}\gamma_k\gamma_t u(0) + f_I\ \bar{d}\gamma_k\gamma_t d(0)]\ ,$$



$$p \cdot \cot\delta(p) = \frac{2\ Z_{00}(1;q^2)}{\sqrt{\pi}L}\ , \qquad q^2 \equiv \left(\frac{L}{2\pi}\right)^2 p^2$$

$$p\cot\delta(p) = \frac{1}{a_0^{DD*}} + \frac{1}{2} r_0^{DD*} p^2\ .$$

$$a_0^{DD^*} = -1.7 \pm 0.4\ \text{fm}\ , \quad r_0^{DD^*} = 0.5 \pm 0.1\ \text{fm}\ .$$

$$\cot\delta(p_{BS}) = i.$$

$X(3872)$	$m_X - \frac{1}{4}(m_{\eta_c} + 3m_{J/\psi})$	$m_X - (m_{D^0} + m_{D^{0*}})$
$\text{lat}^{L\to\infty}$	815 ± 7 MeV	-11 ± 7 MeV
exp	804 ± 1 MeV	-0.14 ± 0.22 MeV

3. Hadron Spectroscopy Collaboration (HSC)——discourse hegemony

- **Sophisticated lattice techniques**
  - **distillation method——perambulators**
  - **usually large operator basis for a specific quantum number**
  - **complicated data analysis procedure**
  - **S-matrix——hadron-hadron interactions**
    resonances
- **Exhaustive calculations of light meson spectroscopy**
  Exhaustive calculations of light baryon spectroscopy

  The spectrum pattern are compatible with QM expectations

- **Seemingly established a standard for lattice studies on hadron spectroscopy**

Selected work of HSC—— 1^{-+} hybrid π_1 decays

A.J. Woss (HSC Collaboration), arXiv:2009.10034(hep-lat)

- **Luescher quantization condition** for hadron-hadron scatterings on finite lattices

$$\det[1 + i\rho t(1 + iM)] = 0$$

- $\rho(E_{cm})$ and $M(E_{cm}, L)$ are known functions.
- The target of this study is to determine the t matrix based on a proper parameterization.
- Worked on **6 lattices** with different lattice sizes.
- Starting from the **exact $SU_F(3)$ flavor symmetry**
  - $m_u = m_d = m_s \approx m_s^{phys}$

$\pi_1 \to M_1 M_2$

The flavor symmetry decomposition of $M_1 M_2$ system

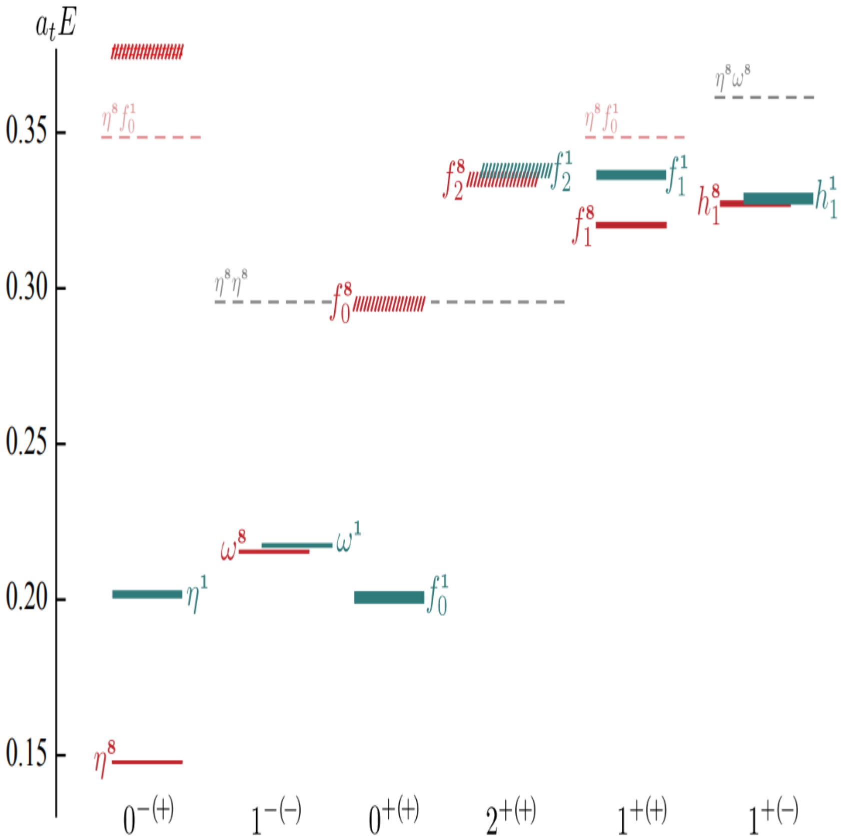
$$\mathbf{8} \otimes \mathbf{1}, \qquad \mathbf{1} \otimes \mathbf{8}$$

$$\mathbf{8} \otimes \mathbf{8} \to \mathbf{1} \oplus \mathbf{8_1} \oplus \mathbf{8_2} \oplus \mathbf{10} \oplus \mathbf{10^*} \oplus \mathbf{27}$$

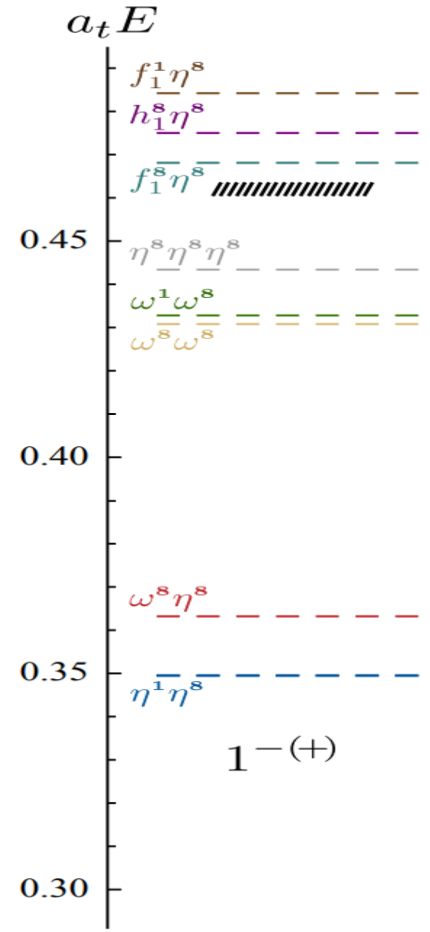
Possible composition of the final states of $M_1 M_2$

$$\eta^1\eta^8, \omega^8\eta^8, \omega^8\omega^8, \omega^1\omega^8, f_1^8\omega^8, h_1^8\eta^8, f_1^1\eta^8$$

$(L/a_s)^3 \times (T/a_t)$	N_{vecs}	N_{cfgs}	N_{tsrcs}
$12^3 \times 96$	48	219	24
$14^3 \times 128$	64	397	16
$16^3 \times 128$	64	529	4
$18^3 \times 128$	96	358	4
$20^3 \times 128$	128	501	4
$24^3 \times 128$	160	607	4



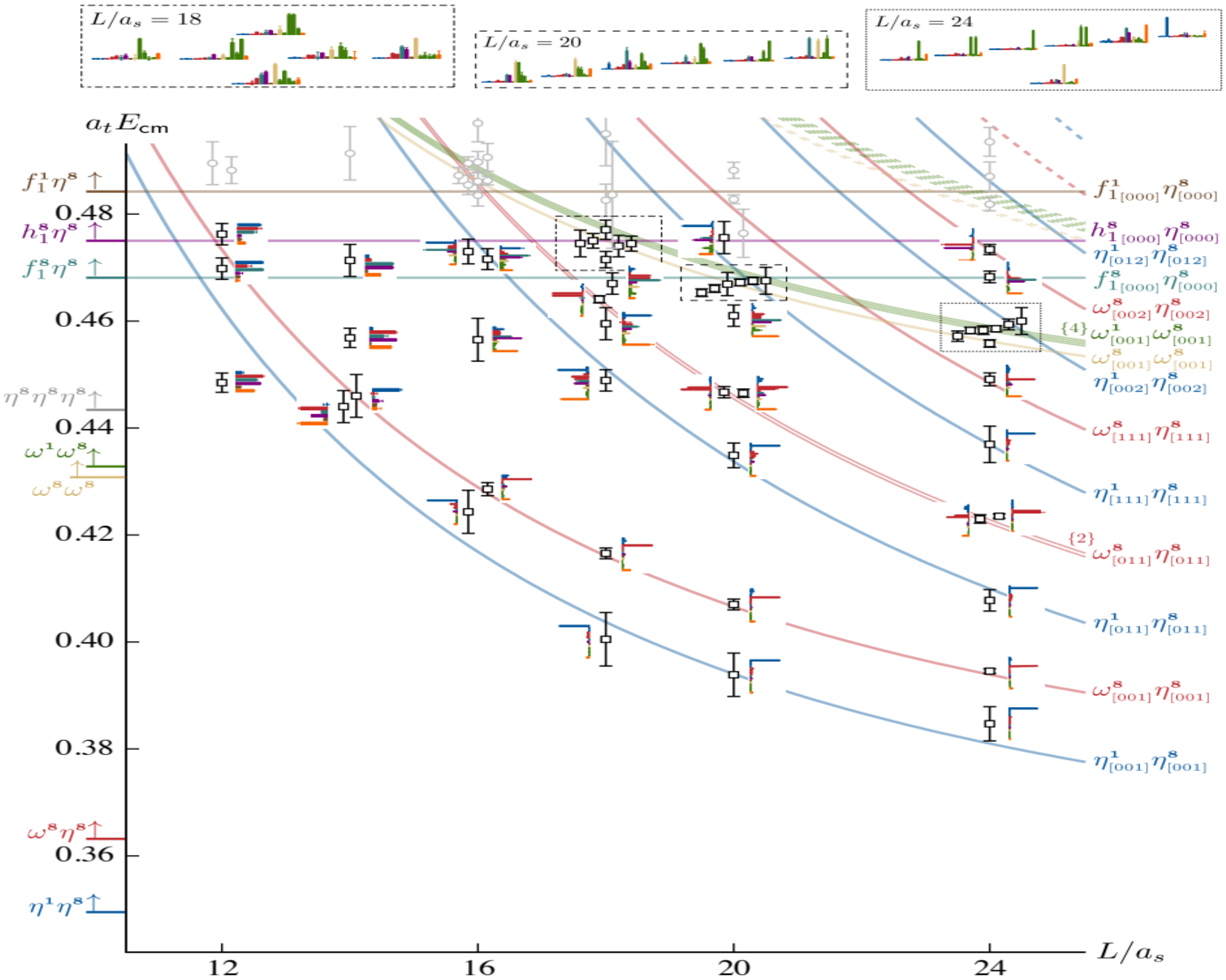
The masses of the light hadrons involved.



$$(\bar{\psi}\Gamma\psi)_i = \underbrace{\epsilon_{ijk}(\bar{\psi}\gamma_j\psi)\, B_k}_{1^{--}\otimes 1^{+-}\to 1^{-+}},$$

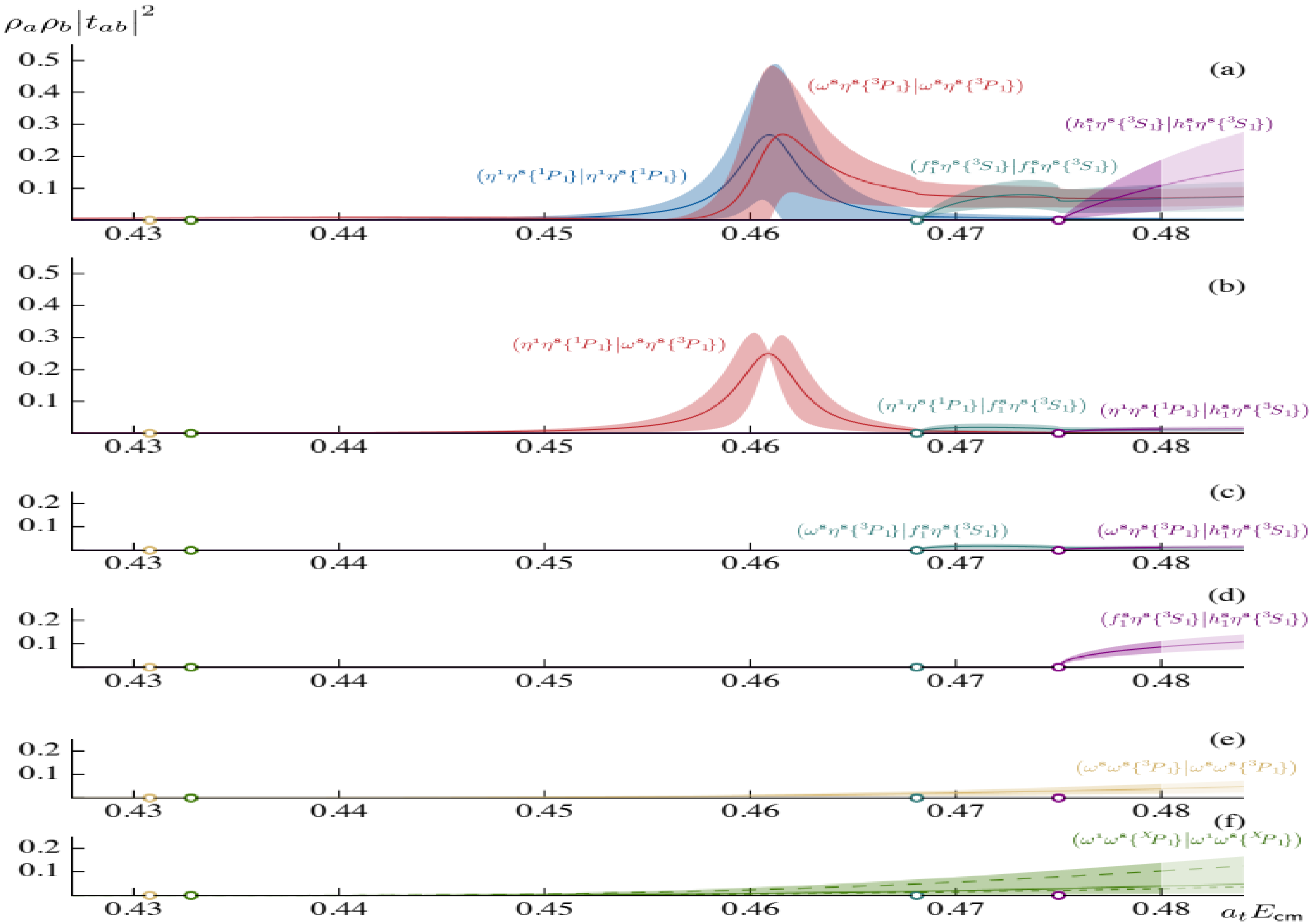
$L/a_s = 12$	$L/a_s = 14$	$L/a_s = 16$	$L/a_s = 18$	$L/a_s = 20$	$L/a_s = 24$
$18 \times \bar{\psi}\mathbf{\Gamma}\psi$	$18 \times \bar{\psi}\mathbf{\Gamma}\psi$	$18 \times \bar{\psi}\mathbf{\Gamma}\psi$	$18 \times \bar{\psi}\mathbf{\Gamma}\psi$	$18 \times \bar{\psi}\mathbf{\Gamma}\psi$	$18 \times \bar{\psi}\mathbf{\Gamma}\psi$
$\eta^{\mathbf{1}}_{[001]}\eta^{\mathbf{8}}_{[001]}$	$\eta^{\mathbf{1}}_{[001]}\eta^{\mathbf{8}}_{[001]}$	$\eta^{\mathbf{1}}_{[001]}\eta^{\mathbf{8}}_{[001]}$	$\eta^{\mathbf{1}}_{[001]}\eta^{\mathbf{8}}_{[001]}$	$\eta^{\mathbf{1}}_{[001]}\eta^{\mathbf{8}}_{[001]}$	$\eta^{\mathbf{1}}_{[001]}\eta^{\mathbf{8}}_{[001]}$
$f^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$\omega^{\mathbf{8}}_{[001]}\eta^{\mathbf{8}}_{[001]}$	$\omega^{\mathbf{8}}_{[001]}\eta^{\mathbf{8}}_{[001]}$	$\omega^{\mathbf{8}}_{[001]}\eta^{\mathbf{8}}_{[001]}$	$\omega^{\mathbf{8}}_{[001]}\eta^{\mathbf{8}}_{[001]}$	$\omega^{\mathbf{8}}_{[001]}\eta^{\mathbf{8}}_{[001]}$
$\omega^{\mathbf{8}}_{[001]}\eta^{\mathbf{8}}_{[001]}$	$f^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$f^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$\eta^{\mathbf{1}}_{[011]}\eta^{\mathbf{8}}_{[011]}$	$\eta^{\mathbf{1}}_{[011]}\eta^{\mathbf{8}}_{[011]}$	$\eta^{\mathbf{1}}_{[011]}\eta^{\mathbf{8}}_{[011]}$
$h^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$h^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$\eta^{\mathbf{1}}_{[011]}\eta^{\mathbf{8}}_{[011]}$	$\{2\}\ \omega^{\mathbf{8}}_{[011]}\eta^{\mathbf{8}}_{[011]}$	$\{2\}\ \omega^{\mathbf{8}}_{[011]}\eta^{\mathbf{8}}_{[011]}$	$\{2\}\ \omega^{\mathbf{8}}_{[011]}\eta^{\mathbf{8}}_{[011]}$
$f^{\mathbf{1}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$f^{\mathbf{1}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$h^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$f^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$\omega^{\mathbf{8}}_{[001]}\omega^{\mathbf{8}}_{[001]}$	$\eta^{\mathbf{1}}_{[111]}\eta^{\mathbf{8}}_{[111]}$
	$\omega^{\mathbf{8}}_{[001]}\omega^{\mathbf{8}}_{[001]}$	$f^{\mathbf{1}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$h^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$f^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$\omega^{\mathbf{8}}_{[111]}\eta^{\mathbf{8}}_{[111]}$
	$\{4\}\ \omega^{\mathbf{1}}_{[001]}\omega^{\mathbf{8}}_{[001]}$	$\{2\}\ \omega^{\mathbf{8}}_{[011]}\eta^{\mathbf{8}}_{[011]}$	$\omega^{\mathbf{8}}_{[001]}\omega^{\mathbf{8}}_{[001]}$	$\{4\}\ \omega^{\mathbf{1}}_{[001]}\omega^{\mathbf{8}}_{[001]}$	$\omega^{\mathbf{8}}_{[001]}\omega^{\mathbf{8}}_{[001]}$
	$\eta^{\mathbf{1}}_{[011]}\eta^{\mathbf{8}}_{[011]}$	$\omega^{\mathbf{8}}_{[001]}\omega^{\mathbf{8}}_{[001]}$	$\{4\}\ \omega^{\mathbf{1}}_{[001]}\omega^{\mathbf{8}}_{[001]}$	$\eta^{\mathbf{1}}_{[111]}\eta^{\mathbf{8}}_{[111]}$	$\{4\}\ \omega^{\mathbf{1}}_{[001]}\omega^{\mathbf{8}}_{[001]}$
	$\{2\}\ \omega^{\mathbf{8}}_{[011]}\eta^{\mathbf{8}}_{[011]}$	$\{4\}\ \omega^{\mathbf{1}}_{[001]}\omega^{\mathbf{8}}_{[001]}$	$f^{\mathbf{1}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$h^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$\eta^{\mathbf{1}}_{[002]}\eta^{\mathbf{8}}_{[002]}$
			$\eta^{\mathbf{1}}_{[111]}\eta^{\mathbf{8}}_{[111]}$	$\omega^{\mathbf{8}}_{[111]}\eta^{\mathbf{8}}_{[111]}$	$f^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$
			$\omega^{\mathbf{8}}_{[111]}\eta^{\mathbf{8}}_{[111]}$	$f^{\mathbf{1}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$	$\omega^{\mathbf{8}}_{[002]}\eta^{\mathbf{8}}_{[002]}$
				$\omega^{\mathbf{8}}_{[002]}\eta^{\mathbf{8}}_{[002]}$	$h^{\mathbf{8}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$
					$f^{\mathbf{1}}_{1\,[000]}\eta^{\mathbf{8}}_{[000]}$
					$\eta^{\mathbf{1}}_{[012]}\eta^{\mathbf{8}}_{[012]}$

Operator sets on the six lattices



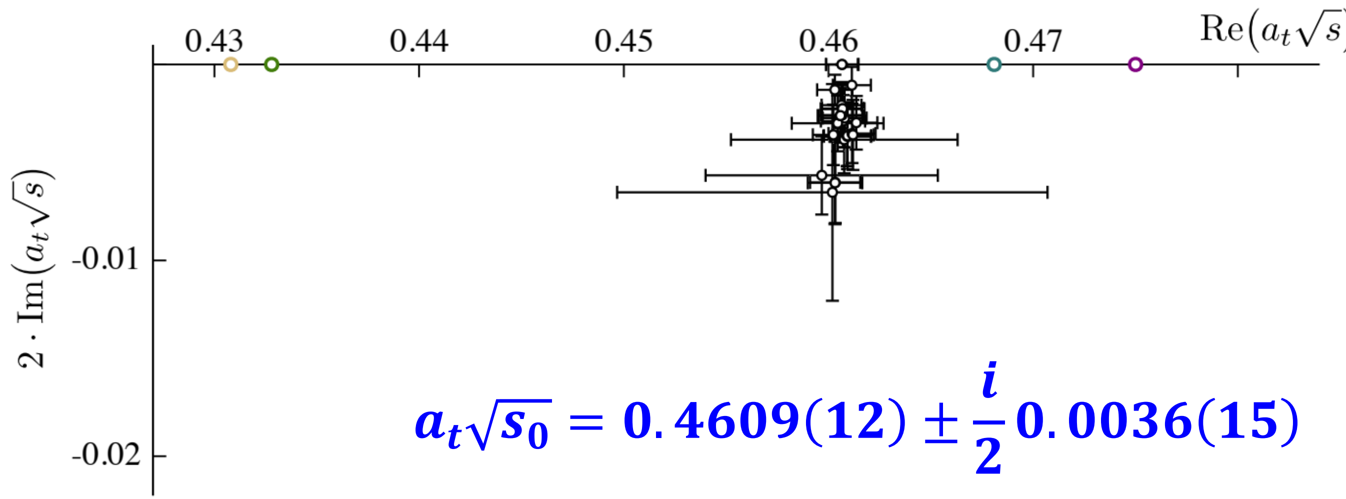
Finite box levels

$$S = 1 + 2i\sqrt{\rho}\,t\sqrt{\rho}$$



resonances, and it is common to interpret the real and imaginary components of the pole position s_0 in terms of the mass m_R and width Γ_R, via $\sqrt{s_0} = m_R \pm \frac{i}{2}\Gamma_R$. Near the pole, the t-matrix takes the form,

$$t_{\ell SJa,\ell' S' Jb} \sim \frac{c_{\ell SJa}\, c_{\ell' S' Jb}}{s_0 - s}$$



Ordinary and extraordinary hadrons

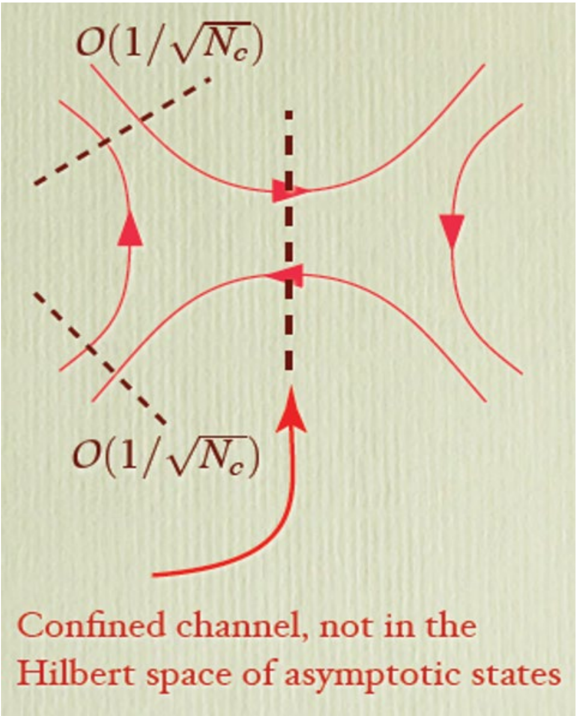
R.L. Jaffe's talk (YKIS-2006, Kyoto), arXiv:hep-ph/0701038.

Ordinary hadrons:

Hadrons that exist in the large N_c limit as confined states.
Namely, quark vacumm polarization is switched off.

They appear in the continuum scattering as resonances
Their width shrink to zero when $N_c \to \infty$

Resonance formation takes place by transition from meson-meson continuum (multiquark) to a confined channel that has no asymptotic states.



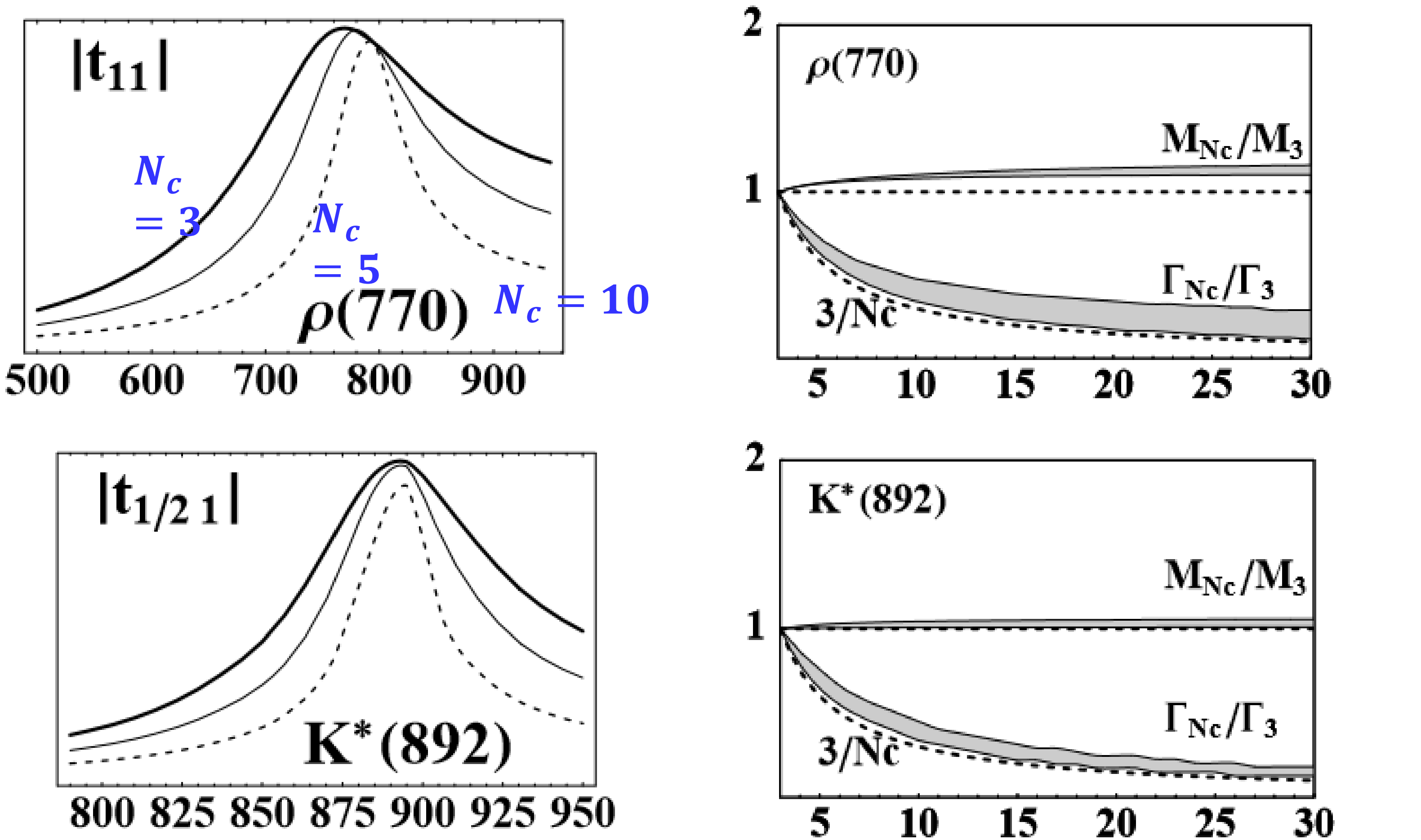
Scattering amplitude: $O\left(\frac{1}{N_c}\right)$

Meson decay width: $O\left(\frac{1}{N_c}\right)$

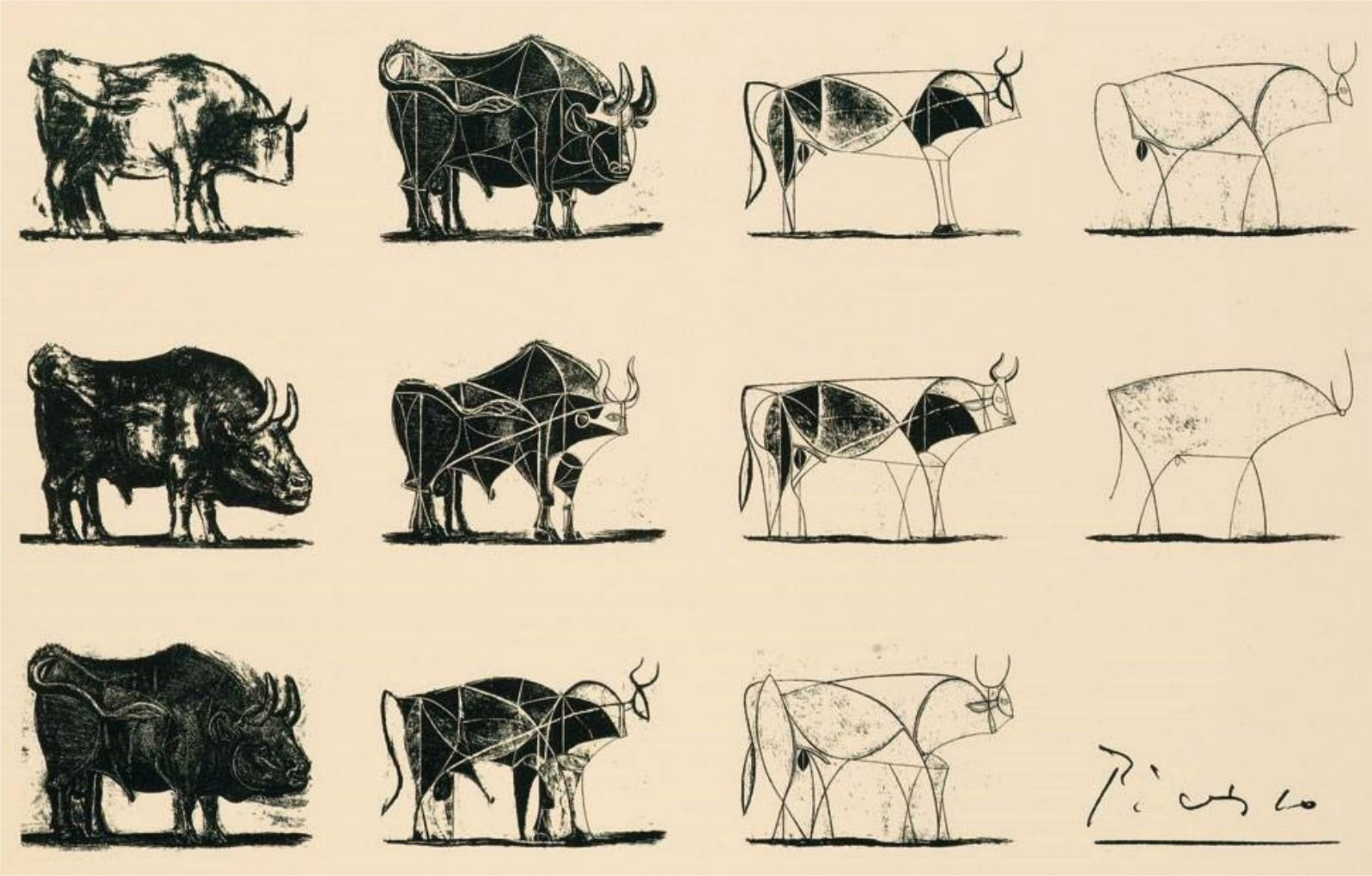
Extraordinary hadrons: vanish when $N_c \to \infty$

Meson-meson scattering from ChPT with different N_c

J.R. Pelaez, Phys. Rev. Lett. 92, 102001 (2004)



$\rho(770)$ and $K^*(892)$ hehave as expected. Their masses are roughly independent of N_c, while their widths go to zero when $N_c \to \infty$



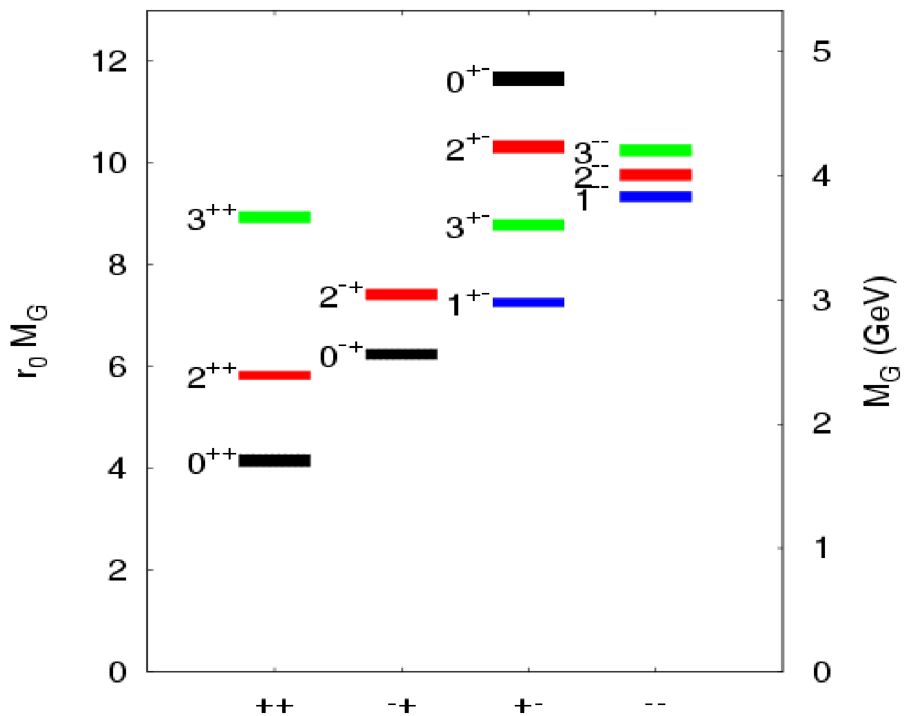
Full-QCD lattice study ← **Quenched Approximation**

III. Hadron spectroscopy relevant of BESIII

I). Glueball

- **Quenched LQCD predicts glueball spectrum**

  Lowest-lying glueballs have masses in the range 1~3GeV



J^{PC}	$r_0 M_G$	M_G (MeV)
0^{++}	4.16(11)(4)	1710(50)(80)
2^{++}	5.83(5)(6)	2390(30)(120)
0^{-+}	6.25(6)(6)	2560(35)(120)
1^{+-}	7.27(4)(7)	2980(30)(140)
2^{-+}	7.42(7)(7)	3040(40)(150)
3^{+-}	8.79(3)(9)	3600(40)(170)
3^{++}	8.94(6)(9)	3670(50)(180)
1^{--}	9.34(4)(9)	3830(40)(190)
2^{--}	9.77(4)(10)	4010(45)(200)
3^{--}	10.25(4))(10)	4200(45)(200)
2^{+-}	10.32(7)(10)	4230(50)(200)
0^{+-}	11.66(7)(12)	4780(60)(230)

Y. Chen et al, Phys. Rev. D 73, 014516 (2006)

- **Preliminary full-QCD results of glueball spectrum**

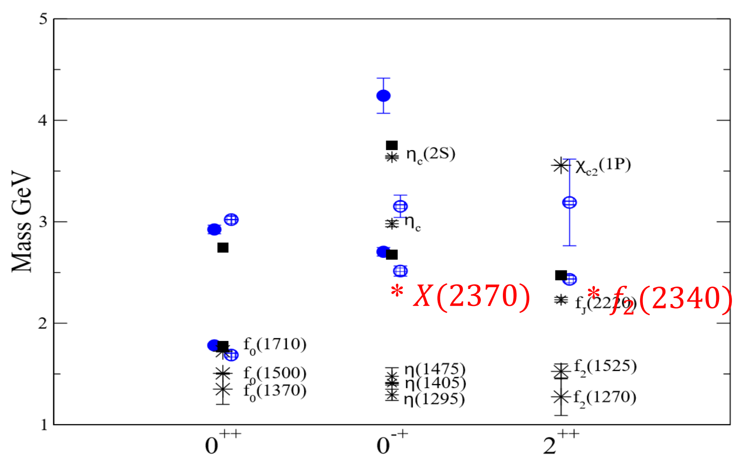
	m_π (MeV)	$m_{0^{++}}$ (MeV)	$m_{2^{++}}$ (MeV)	$m_{0^{-+}}$ (MeV)
$N_f=2$	938	1417(30)	2363(39)	2573(55)
	650	1498(58)	2384(67)	2585(65)
$N_f=2+1$ [22]	360	1795(60)	2620(50)	—
quenched [13]	—	1710(50)(80)	2390(30)(120)	2560(35)(120)
quenched [14]	—	1730(50)(80)	2400(25)(120)	2590(40)(130)

$N_f=2$: **W. Sun et al (CLQCD), Chin. Phys. C 42, 093103 (2018)**

[14] C. Morningstar and M. Peardon, Phys. Rev. D 60, 034509, 1999

[13] Y. Chen et al, Phys. Rev. D 73, 014516, 2006

[22] E. Gregory et al., JHEP 10 (2012) 170, arXiv:1208.1858(hep-lat)



Filled Squares： QQCD
Open circles: full QCD, coarse lattice
Closed circles: full QCD, fine lattice

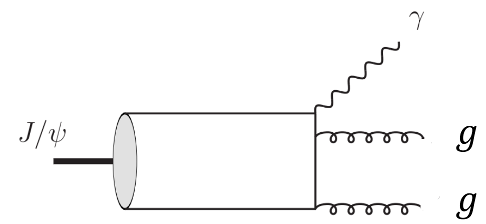
C.M. Richards et al., [UKQCD Collab.], Phys. Rev. D82, 034501 (2010).

No meson or two-meson operators have been involved yet!

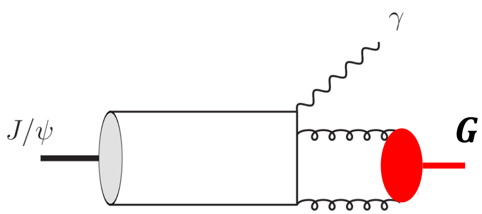
- J/ψ radiative decays —— best hunting ground for glueballs

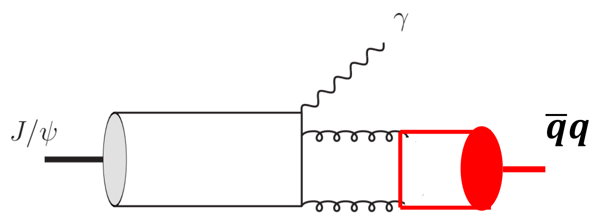
Gluon abundant in J/ψ decays

Gluon is flavor singlet —— isospin filter



- J/ψ radiative decay products—— $\bar{q}q$ meson vs. glb





$O(1)$

Suppressed by $O(\alpha_s^2)$

- Serve as criteria for the experimental identification of glueball.

- **Radiative decay width:**

$$\Gamma(i \to \gamma f) = \int d\Omega_q \frac{1}{32\pi^2} \frac{|\vec{q}|}{M_i^2} \frac{1}{2J_i+1} \times \sum_{r_i, r_j, r_\gamma} \left| M_{r_i, r_j, r_\gamma} \right|^2,$$

- **Transition amplitudes:**

$$M_{r_i, r_f, r_\gamma} = \epsilon_\mu^*(\vec{q}, r_\gamma) \langle f(\vec{p}_f, r_f) | j_{\text{em}}^\mu(0) | i(\vec{p}_i, r_i) \rangle$$

- **Multipole decomposition:**

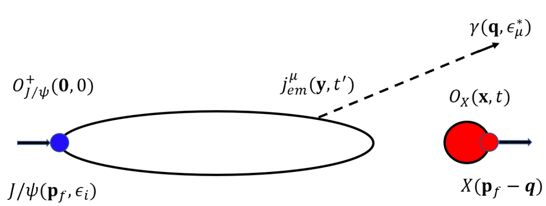
$$\langle f(\vec{p}_f, r_f) | j_{\text{em}}^\mu(0) | i(\vec{p}_i, r_i) \rangle = \sum_k \alpha_k^\mu(p_i, p_f) F_k(Q^2),$$

- **Decay width expressed in terms of the form factors**

$$\Gamma(i \to \gamma f) \propto \sum_k F_k^2(0).$$

- **So the major task is to calculate the matrix elements, which can be derived from the three-point functions**

$$\Gamma^{(3)\mu i}(\vec{p}_f, \vec{q}; t_f, t) = \frac{1}{T} \sum_{\tau=0}^{T-1} \sum_{\vec{y}} e^{+i\vec{q}\cdot\vec{y}} \left\langle O_G(\vec{p}_f, t_f + \tau) j^\mu(\vec{y}, t + \tau) O_{J/\psi}^{i,+}(\tau) \right\rangle$$



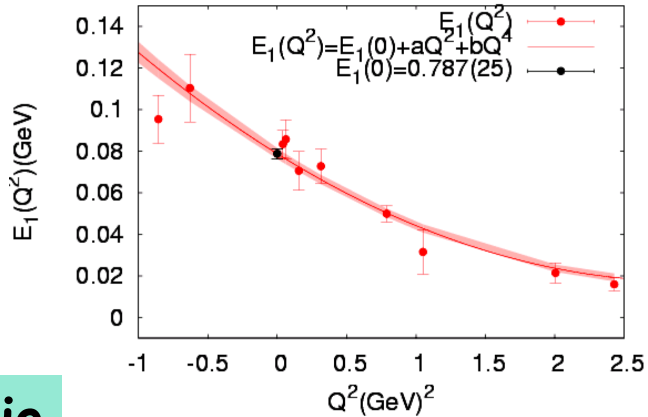
A). J/psi radiatively decaying to the scalar glueball

(L.Gui, et al. (CLQCD Collaboration), Phys. Rev. Lett. 110, 021601 (2013))

$$\Gamma(J/\psi \to \gamma G_{0^+}) = \frac{4}{27}\alpha \frac{|p|}{M_{J/\psi}^2}|E_1(0)|^2$$

Interpolated on-shell form factor E1(0) and its continuum limit

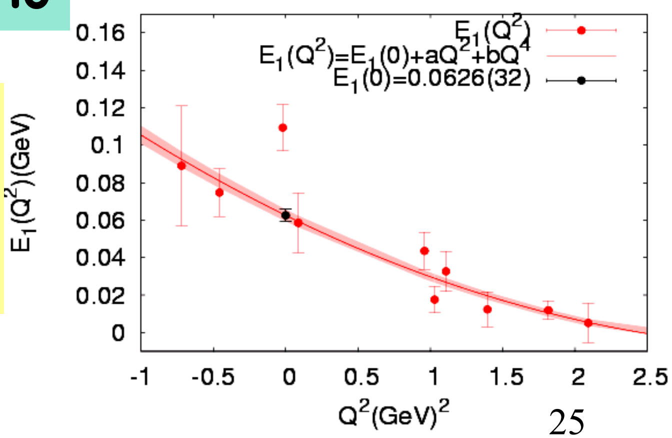
β	M_G(GeV)	$Z_V^{(s)}(a)$	$E_1(0,a)$ (GeV)	Γ(keV)
2.4	1.360(9)	1.39(2)	0.0787(25)	-
2.8	1.537(7)	1.11(1)	0.0626(32)	-
∞	1.710(90) [3]	-	0.0536(57)	0.35(8)



The predicted width and the branch ratio

$$\Gamma(J/\psi \to \gamma G_{0^+}) = \frac{4}{27}\alpha \frac{|p|}{M_{J/\psi}^2}|E_1(0)|^2 = 0.35(8)keV$$

$$\Gamma/\Gamma\_tot = 0.33(7)/93.2 = 3.8(9) \times 10^{-3}$$



Experimental results of $f_0(1500)/f_0(1710)$???

LQCD prediction

$$\Gamma(J/\psi \to \gamma G_{0^+}) = 0.35(8)keV, \quad \Gamma/\Gamma_{tot} = 3.8(9) \times 10^{-3}$$

P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys., 083C01 (2020)

$$J/\psi \to \gamma f_0(1710) \to \gamma K\bar{K} \qquad (9.5^{+1.0}_{-0.5}) \times 10^{-4}$$
$$J/\psi \to \gamma f_0(1710) \to \gamma\pi\pi \qquad (3.8 \pm 0.5) \times 10^{-4}$$
$$J/\psi \to \gamma f_0(1710) \to \gamma\omega\omega \qquad (3.1 \pm 1.0) \times 10^{-4}$$
$$J/\psi \to \gamma f_0(1710) \to \gamma\eta\eta \qquad (2.4^{+1.2}_{-0.7}) \times 10^{-4}$$

$$\Rightarrow Br(J/\psi \to \gamma f_0(1710)) > 1.9 \times 10^{-3}$$

Using $Br(f_0(1710) \to K\underline{K}) = 0.36 \quad \Rightarrow \quad Br(J/\psi \to \gamma f_0(1710)) = 2.4\times10^{-3}$

$Br(f_0(1710) \to \pi\pi) = 0.15 \quad \Rightarrow \quad Br(J/\psi \to \gamma f_0(1710)) = 2.7\times10^{-3}$

In contrast,

$$J/\psi \to \gamma f_0(1500) \to \gamma\pi\pi \qquad (1.01 \pm 0.34) \times 10^{-4}$$
$$J/\psi \to \gamma f_0(1500) \to \gamma K_S^0 K_S^0 \qquad (1.59 \pm 0.16^{+0.18}_{-0.56}) \times 10^{-5}$$

$$Br(f_0(1500) \to \pi\pi) = (34.5 \pm 2.2)\%$$
$$Br(f_0(1500) \to K\bar{K}) = (8.5 \pm 1.0)\%$$

$$\Rightarrow Br(J/\psi \to \gamma f_0(1500)) = 2.9 \times 10^{-4}$$

Recent BESIII results from PWA

$J/\psi \to \gamma X \to \gamma\pi\pi$

BES, PLB642(2006)441

$$Br(J/\psi \to \gamma f_0(1710) \to \gamma\pi\pi) = (4.01 \pm 1.0) \times 10^{-4}$$

$$Br(J/\psi \to \gamma f_0(1500) \to \gamma\pi\pi) = (1.01 \pm 0.34) \times 10^{-4}$$

$$Br(J/\psi \to \gamma f_0(1370) \to \gamma\pi\pi) = ----$$

$J/\psi \to \gamma X \to \gamma\eta\eta$

BESIII, PRD87(2013)092009

$$Br(J/\psi \to \gamma f_0(1710) \to \gamma\eta\eta) = \left(2.35^{+1.27}_{-0.77}\right) \times 10^{-4}$$

$$Br(J/\psi \to \gamma f_0(1500) \to \gamma\eta\eta) = \left(1.65^{+0.57}_{-1.50}\right) \times 10^{-5}$$

$$Br(J/\psi \to \gamma f_0(1370) \to \gamma\eta\eta) = ----$$

$J/\psi \to \gamma X \to \gamma K_s K_s$

BESIII, arXiv:1808.06946 (hep-ex)

$$Br(J/\psi \to \gamma f_0(1710) \to \gamma K_s K_s) = \left(2.00^{+0.03+0.31}_{-0.02-0.10}\right) \times 10^{-4}$$

$$Br(J/\psi \to \gamma f_0(1500) \to \gamma K_s K_s) = \left(1.59^{+0.16+0.18}_{-0.16-0.59}\right) \times 10^{-5}$$

$$Br(J/\psi \to \gamma f_0(1370) \to \gamma K_s K_s) = \left(1.07^{+0.08+0.36}_{-0.07-0.34}\right) \times 10^{-5}$$

Obviously, in each process,
$f_0(1710)$ are produced 10 times more than $f_0(1500)$.

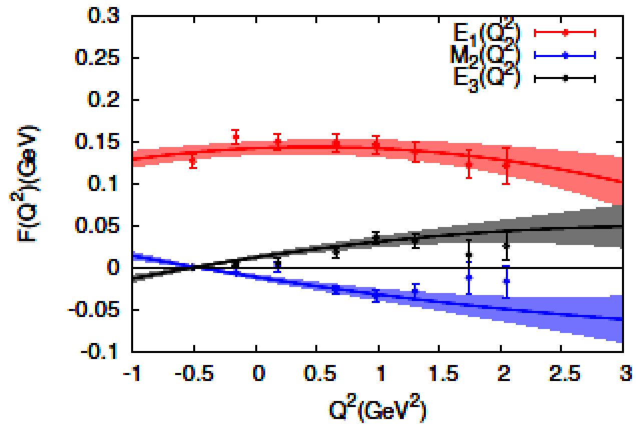
B). J/psi radiatively decaying to the tensor glueball

(Y.B. Yang ,et al .(CLQCD Collaboration), Phys. Rev. Lett. 111, 091601 (2013))

$$\Gamma(J/\psi \to \gamma G_{2^+}) = \frac{4}{27}\alpha \frac{|p|}{M_{J/\psi}^2}\left[|E_1(0)|^2 + |M_2(0)|^2 + |E_3(0)|^2\right]$$

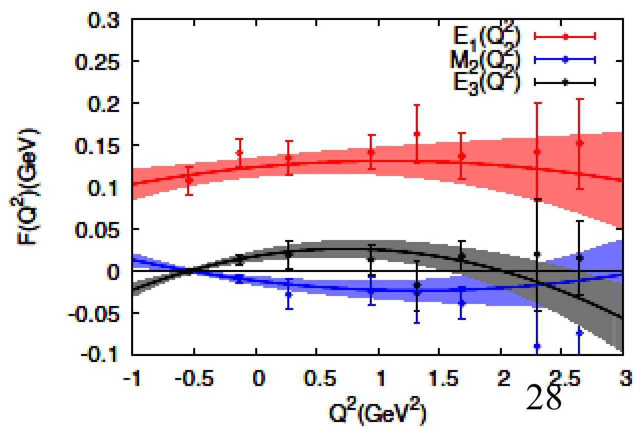
- **The form factors we obtained from the lattice QCD**

β	M_T (GeV)	E_1 (GeV)	M_2 (GeV)	E_3 (GeV)
2.4	2.360(20)	0.142(07)	−0.012(2)	0.012(2)
2.8	2.367(25)	0.125(10)	−0.011(4)	0.019(6)
∞	2.372(28)	0.114(12)	−0.011(5)	0.023(8)



- **We also carry out a similar lattice study on the tensor glueball production rate in J/psi radiative decay.**

$$\Gamma(J/\psi \to \gamma G_{2^+}) = 1.01(22)keV$$
$$\Gamma(J/\psi \to \gamma G_{2^+})/\Gamma_{tot} = 1.1(2) \times 10^{-2}$$



LQCD prediction

$$\Gamma(J/\psi \to \gamma G_{2^+}) = 1.01(22)keV$$
$$\Gamma(J/\psi \to \gamma G_{2^+})/\Gamma_{tot} = 1.1(2) \times 10^{-2}$$

$J/\psi \to \gamma X \to \gamma\eta\eta$ BESIII, PRD87(2013)092009

$$Br(J/\psi \to \gamma f_2(2340) \to \gamma\eta\eta) = \left(5.60^{+0.62+2.37}_{-0.65-2.07}\right) \times 10^{-5}$$

$J/\psi \to \gamma X \to \gamma K_s K_s$ BESIII, arXiv:1808.06946 (hep-ex)

$$Br(J/\psi \to \gamma f_2(2340) \to \gamma K_s K_s) = \left(5.54^{+0.34+3.82}_{-0.40-1.49}\right) \times 10^{-5}$$

$J/\psi \to \gamma X \to \gamma\varphi\varphi$ BESIII, PRD93(2016)112011

$$Br(J/\psi \to \gamma f_2(2340) \to \gamma\phi\phi) = \left(1.91 \pm 0.14^{+0.72}_{-0.73}\right) \times 10^{-4}$$

It is desirable to do a systematic analysis of decay modes $J/\psi \to \gamma VV$

3. Pseudoscalar glueball relevant

- The production rate of the pseudoscalar glueball in J/ψ radiative decays from LQCD (L.-C. Gui et al. , Phys. Rev. D 100, 054511 (2019))

$$\Gamma(J/\psi \to \gamma G_{ps}) = 0.022(7)\ \mathrm{keV}$$

$$Br(J/\psi \to \gamma X(2370)) = 2.3(8) \times 10^{-4}$$

← Not that large!

- If the kenetic factor is subtracted, we have the effective couplings

$$\Gamma(J/\psi \to \gamma X) = \frac{1}{3}\alpha g_X^2 \frac{|\mathbf{q}|^3}{m_{J/\psi}^2},$$

$$g_X = \left[\frac{24\Gamma_{J/\psi}}{\alpha}\frac{Br(J/\psi \to \gamma X)m_{J/\psi}^5}{(m_{J/\psi}^2 - m_X^2)^3}\right]^{\frac{1}{2}}$$

TABLE VI: The g_X of flavor-singlet pseudoscalar mesons.

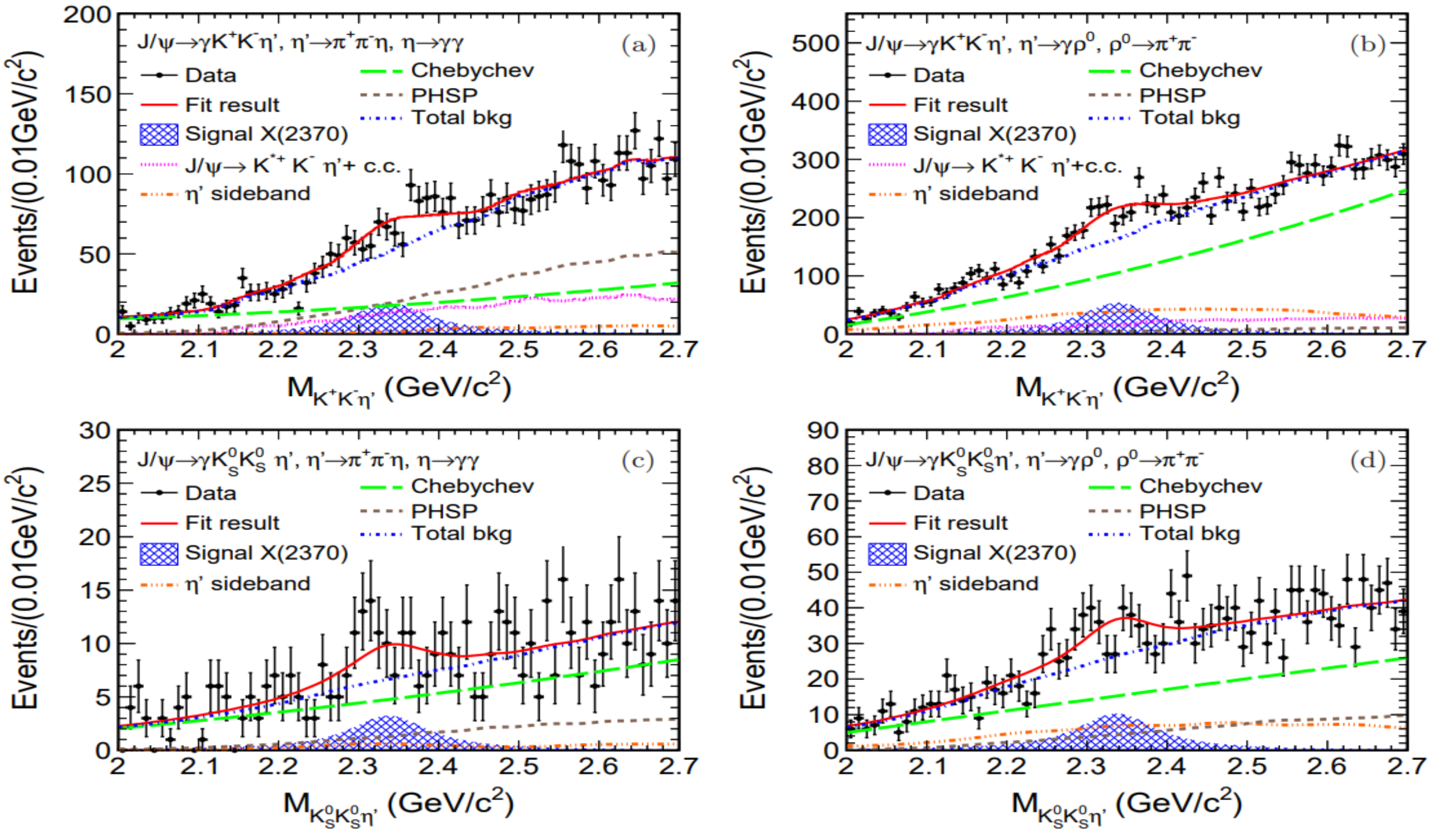
Pseudoscalar (X)	g_X
η	0.0108(2)
η'	0.0259(8)
$\eta(1405/1475)$	0.0313(41)
$\eta(1760)$	0.0255(25)
$X(1835)$	0.0123(12)
$\eta(2225)$	0.0167(17)
$G(0^{-+})$	0.0144(27)

- The effective couplings are comparable for glueball and mesons.
- The $U_A(1)$ anomaly may play an important role here.

Lattice QCD result: $Br(J/\psi \to \gamma X(2370))$=2.3(8)$\times 10^{-4}$

$J/\psi \to \gamma X(2370) \to \gamma K^+K^-\eta'$ $(1.79 \pm 0.23 \pm 0.65) \times 10^{-5}$

$J/\psi \to \gamma X(2370) \to \gamma K_S^0 K_S^0 \eta'$ $(1.18 \pm 0.32 \pm 0.39) \times 10^{-5}$



BESIII, Eur. Phys. J. C 80 (2020) 8, 746.

2) Glueball component of η_c and its implication

(R.Q. Zhang et al. , arXiv: 2107.12749 (hep-lat))

- η_c total width is quite large: $\Gamma_{\eta_c} = 32.0(7)$ MeV
- Lattice QCD predict the mass of the pseudoscalar glueball to be around 2.4-2.6 GeV
- This motivates the possibility of sizable gluonic component in η_c
- If $\eta(1405)$ and $\eta(1475)$ are the same state, there is no need for a pseudoscalar glueball round 1.3-1.5 GeV

  J.-J. Wu et al., Phys. Rev. Lett. 108, 081803 (2012)

- There may be mixing between $\bar{c}c(\,^1S_0\,)$ and the PS glueball

  Y.-D. Tsai, H.-n. Li, and Q. Zhao, Phys. Rev. D 85, 034002 (2012)

  W. Qin, Q. Zhao, and X.-H. Zhong, Phys. Rev. D 97, 096002 (2018)

Gauge configurations: $N_f = 2$ degenerate charm quarks permit the mixing between $c\bar{c}$ and glueball

ensemble	$L^3 \times T$	β	$a_s(\text{fm})$	ξ	$m_{J/\psi}(\text{MeV})$	N_{cfg}
I	$16^3 \times 128$	2.8	0.1026	5	2743	~ 6000
II	$16^3 \times 128$	2.8	0.1026	5	3068	~ 6000

- $c\bar{c}$ operator: $O_{\bar{c}c} = \bar{c}\gamma_5 c$ Glueball operator: O_G
- Mixing model

$$\hat{H} = \begin{pmatrix} m_G & x \\ x & m_{c\bar{c}} \end{pmatrix} \qquad \begin{pmatrix} |X\rangle \\ |\eta_c\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |G\rangle \\ |c\bar{c}\rangle \end{pmatrix}$$

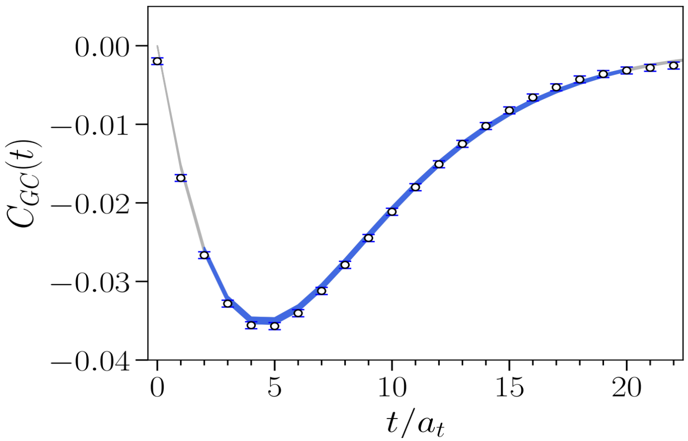
- Under some assumptions, one has

$$C_{GC}(t) \approx -Z \sin\theta \left(e^{-m_X t} - e^{-m_{\eta_c}}\right) + \cdots$$

$$\sin\theta \approx \frac{x}{m_{c\bar{c}} - m_G}$$

- **Very precise data**
- **Well described by the function form**

$$C_{GC}(t) \approx -Z\sin\theta\,(e^{-m_X t} - e^{-m_{\eta_c}}) + \cdots$$



ensemble	Γ	m_{η_1}(MeV)	m_{g_1}(MeV)	θ_1	x_1(MeV)
I	γ_5	2691(2)	2317(51)	7.7(1.1)°	48(7)
	$\gamma_5\gamma_4$	2685(1)	2317(43)	6.8(8)°	43(6)
	avg.	2686(1)	2317(46)	7.1(9)°	46(7)
II	γ_5	2987(9)	2308(63)	4.9(6)°	59(8)
	$\gamma_5\gamma_4$	3013(3)	2385(40)	4.2(3)°	46(6)
	avg.	3010(4)	2363(47)	4.3(4)°	49(7)

Effects of the mixing on the total width of η_c

- Assuming $X(2370)$ is predominantly a glueball, $\Gamma_X \approx 100$ MeV

$J/\psi \to \gamma X(2370) \to \gamma \pi^+ \pi^- \eta'$:

$$M_{X(2370)} = 2341.6 \pm 6.5(\text{stat.}) \pm 5.7(\text{syst.}) \text{ MeV}$$
$$\Gamma_{X(2370)} = 117 \pm 10(\text{stat.}) \pm 8(\text{syst.}) \text{ MeV}$$

$J/\psi \to \gamma X(2370) \to \gamma K \bar{K} \eta'$:

$$M_{X(2370)} = 2376.3 \pm 8.7(\text{stat.})^{+3.2}_{-4.3}(\text{syst.}) \text{ MeV}$$
$$\Gamma_{X(2370)} = 83 \pm 17(\text{stat.})^{+44}_{-6}(\text{syst.}) \text{ MeV},$$

- The mixing angle and the mass shift of PS charmonium

$$\sin\theta = \frac{x}{m_{\eta_c} - m_X} \approx 0.080(10)$$

$$\delta m_{c\bar{c}} = m_{\eta_c} - m_{c\bar{c}} \approx \frac{x^2}{m_{\eta_c} - m_X} \approx +3.9(9) \text{ MeV}$$

- The decays of η_c and $X(2370)$ into light hadrons:
  can be viewed as decaying into two gluons first and then hadronizing

$$\Gamma_{\eta_c} \approx \Gamma(\eta_c \to gg) = \frac{1}{32\pi}\frac{1}{m_{\eta_c}}|\mathcal{M}(c\bar{c} \to gg)|^2$$

$$\Gamma_X \approx \Gamma(X \to gg) = \frac{1}{32\pi}\frac{1}{m_X}|\mathcal{M}(X \to gg)|^2$$

- This also applies to PS charmonium, such that ($\Gamma_X \approx 100$ MeV)

$$\frac{|\mathcal{M}(X \to gg)|}{|\mathcal{M}(c\bar{c} \to gg)|} \approx \left(\frac{m_X\Gamma_X}{m_{\eta_c}\Gamma_{c\bar{c}}}\right)^{1/2}$$

Expt. known

$$\frac{\Gamma_{\eta_c}}{\Gamma_{c\bar{c}}} \approx \left|\cos\theta + \sin\theta\frac{|\mathcal{M}(X \to gg)|}{|\mathcal{M}(c\bar{c} \to gg)|}\right|^2 \approx 1 + 2\sin\theta\left(\frac{m_X\Gamma_X}{m_{\eta_c}\Gamma_{\eta_c}}\right)^{1/2}\left(\frac{\Gamma_{\eta_c}}{\Gamma_{c\bar{c}}}\right)^{1/2}$$

- Thus, even though we cannot calculate the width directly, we have

$$\Gamma_{\eta_c}/\Gamma_{c\bar{c}} \approx 1.27(4), \qquad \delta\Gamma_{\eta_c} \approx +7.2(8)\ \text{MeV}$$

4. Preliminary results at the physical point
(see Feiyu Chen' talk)

- $N_f = 2 + 1$ dynamical confiugrations generated by RBC/UKQCD Collaboration.

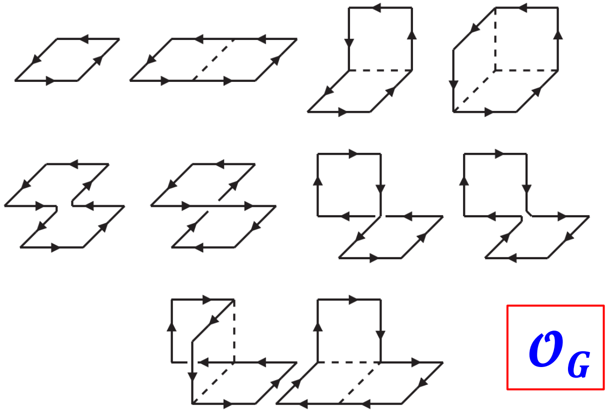
- Accessed through the agreement between χQCD Collaboration (PI: Prof. K.-F. Liu of Univ. Kentucky)

TABLE I. Parameters of 48I and 64I ensemble.

$L^3 \times T$	a (fm)	m_π (MeV)	La (fm)	N_{conf}
$48^3 \times 96$	0.1141(2)	~ 139	~ 5.5	364
$64^3 \times 128$	0.0836(2)	~ 139	~ 5.3	300

- Physical m_π, m_K, large volume, but small size of ensembles—— "physical point"

- **Gauge invariant gluonic operators for glueballs —— build up in terms of Wilson loops**



- **AA-operators for glueballs**

$$\mathcal{O}_{AA}^{(LM;S)}(\vec{r}) = \frac{1}{N_r}\sum_{|\vec{r}|=r}\sum_{\vec{x}} c_{ij}(S)Y_{LM}(\hat{r})A_i(\vec{x}+\vec{r})A_j(\vec{x})$$

S: the total spin of two gauge field;
(LM): the orbital quantum number between two gauge fields;
N_r: the multiplicity of $\vec{r}$ with $|\vec{r}| = r$

- **AA-operators are not gauge invariant** ➡ **Coulomb gauge!**

- **Bethe-Salpeter wave functions** from the $\mathcal{O}_{AA} - \mathcal{O}_G$ correlation functions

Optimized glueball operators: $\left\langle \mathcal{O}_G^{(n)}(t)\mathcal{O}_G^{(n)}(0)\right\rangle \approx e^{-m_n t} + \cdots$

$\mathcal{O}_{AA} - \mathcal{O}_G$ **correlation functions:**

$$\left\langle \mathcal{O}_{AA}(t)\mathcal{O}_G^{(n)}(0)\right\rangle \propto \langle \Omega|\mathcal{O}_{AA}(r)|n\rangle \left\langle n\left|\mathcal{O}_G^{(n)}\right|\Omega\right\rangle e^{-m_n t} \approx \Phi_n(r)e^{-m_n t} + \cdots$$

0^{++} 0^{-+}

2^{++} 2^{++}

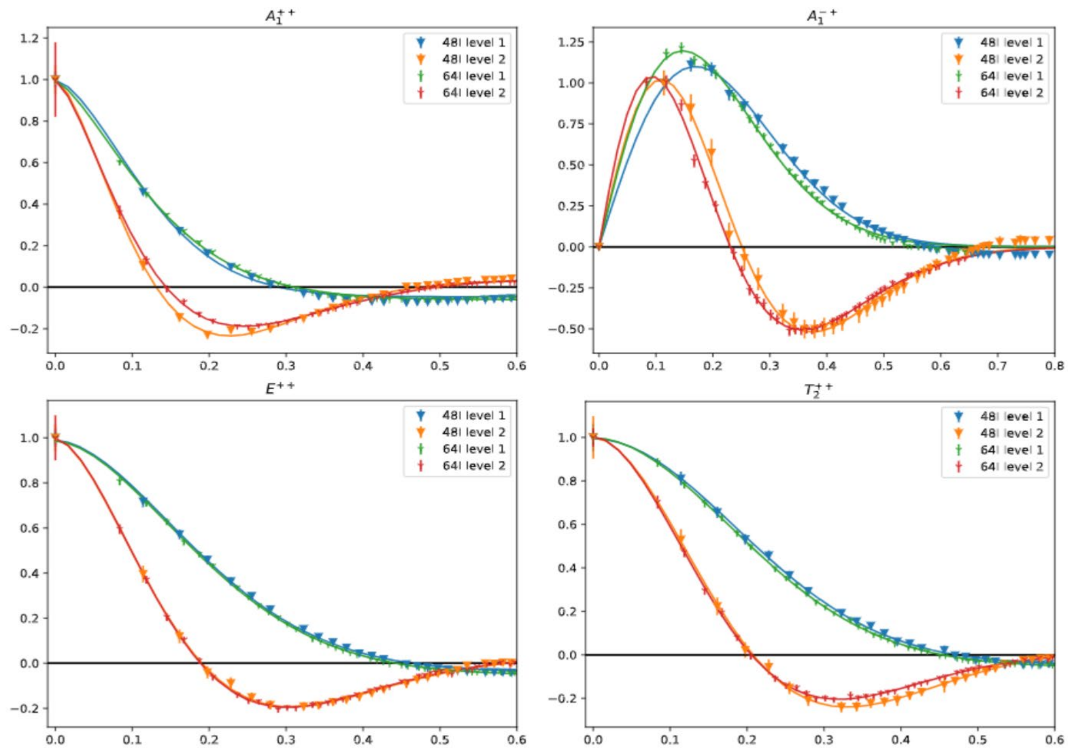
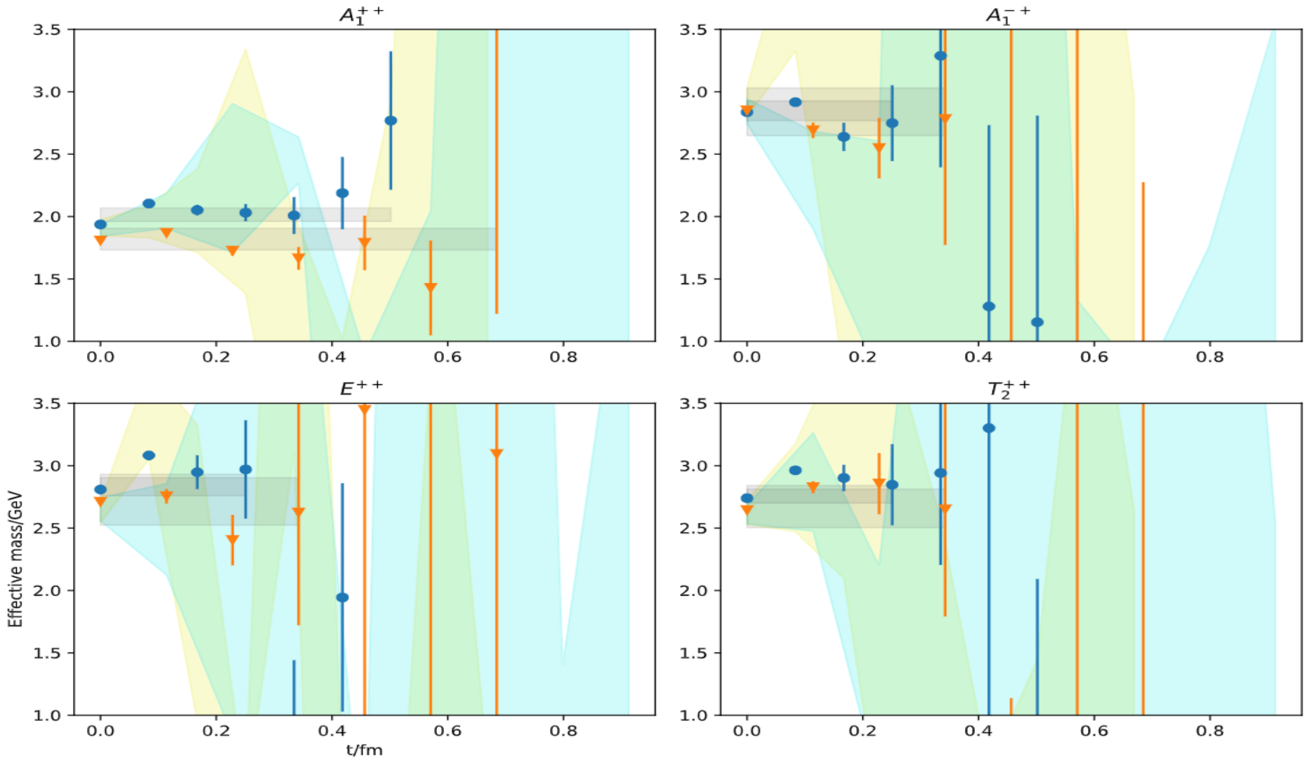


FIG. 2. Normalized BS wave functions of ground and first excited states on 48I and 64I lattice.

- **Effective mass plateaus**
  - $m_{eff}(t) = \ln\frac{C(t)}{C(t+1)} \to const.$ if $C(t)$ is exponential



0	A_1^{++}	E^{++}	T_2^{++}	A_1^{-+}
48I	1.82 ± 0.09	2.6 ± 0.2	2.7 ± 0.02	2.8 ± 0.2
64I	1.96 ± 0.08	2.7 ± 0.1	2.7 ± 0.2	2.8 ± 0.2

II). XYZ particles on the lattice

1. Zc States

(CLQCD Collabl, PRD89(2014)094506, PRD92(2015)054507, CPC43(2019)103103

Zc(3900) : first observed as a structure in $J/\psi\pi^+$ invariant mass spectrum, its "mass" is close to the $D\overline{D^*}$ threshold

Zc(4025) : first observed as a structure in $h_c\pi^+$ invariant mass spectrum, its "mass" is close to the $D^*\overline{D^*}$ threshold.

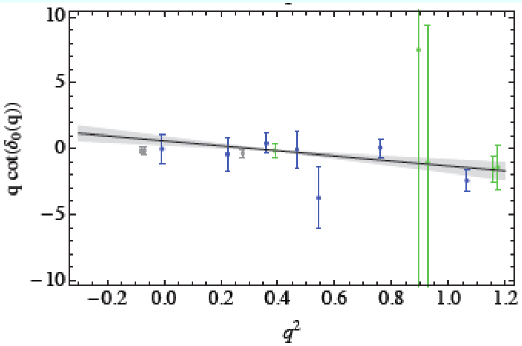
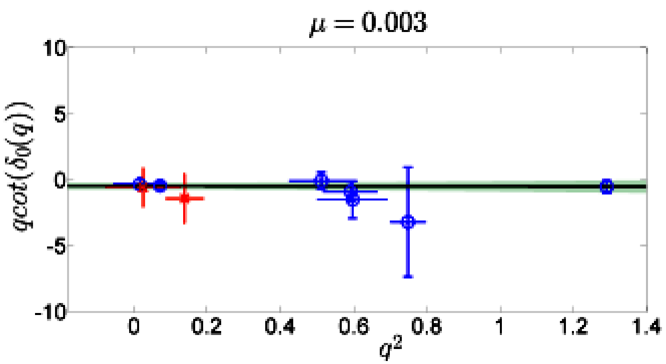


TABLE VI. The values for a_0 and r_0 in physical units obtained from the numbers for the correlated fit in Table IV.

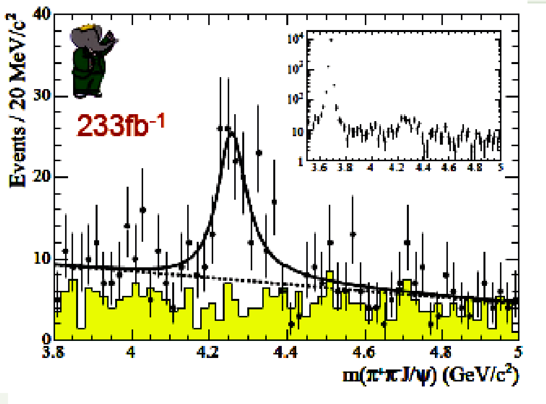
	$\mu = 0.003$	$\mu = 0.006$	$\mu = 0.008$
a_0[fm]	−0.67(1)	−2.1(1)	−0.51(7)
r_0[fm]	−0.78(3)	−0.27(7)	0.82(27)



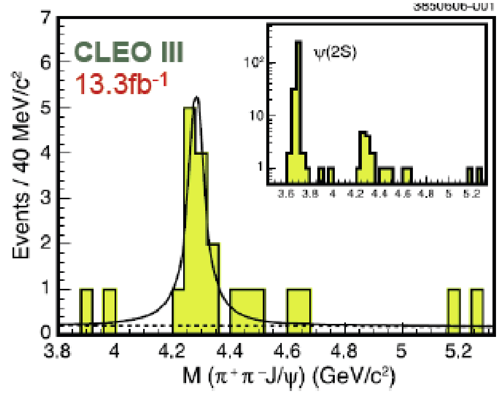
	$\mu = 0.003$	$\mu = 0.006$	$\mu = 0.008$
a_0[fm]	$-0.76^{+0.14}_{-0.21}$	$-0.86^{+0.22}_{-0.22}$	$-0.59^{+0.19}_{-0.25}$
r_0[fm]	$-0.0022^{+0.18}_{-0.19}$	$-0.14^{+0.15}_{-0.18}$	$0.64^{+0.50}_{-0.51}$

In the J^P=1^+ channel, the scattering lengths are negative, indicating a weak repulsive interaction between D(D*) and D*bar. These results does not support a bound state in this channel. However, since the pion mass is still much higher than the physical pion mass, we cannot rule out the possible appearance of a bound state. A more systematic lattice study is demanding.

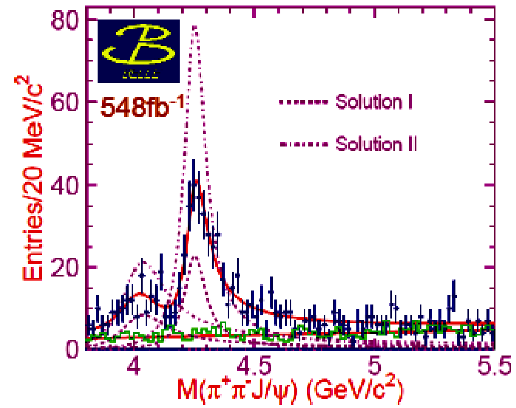
2. Y(4260) relevant study from quenched lattice QCD



BaBar,PRL95,142001(05)



CLEO-III,PRD74, 091104(R)(2006)



Belle,PRL99,182004(07)

1. Observed in the initial state radiation process

$$e^+e^- \to \gamma_{ISR}X$$
$$X \to J/\psi\pi^+\pi^-$$
$$J/\psi \to l^+l^-$$

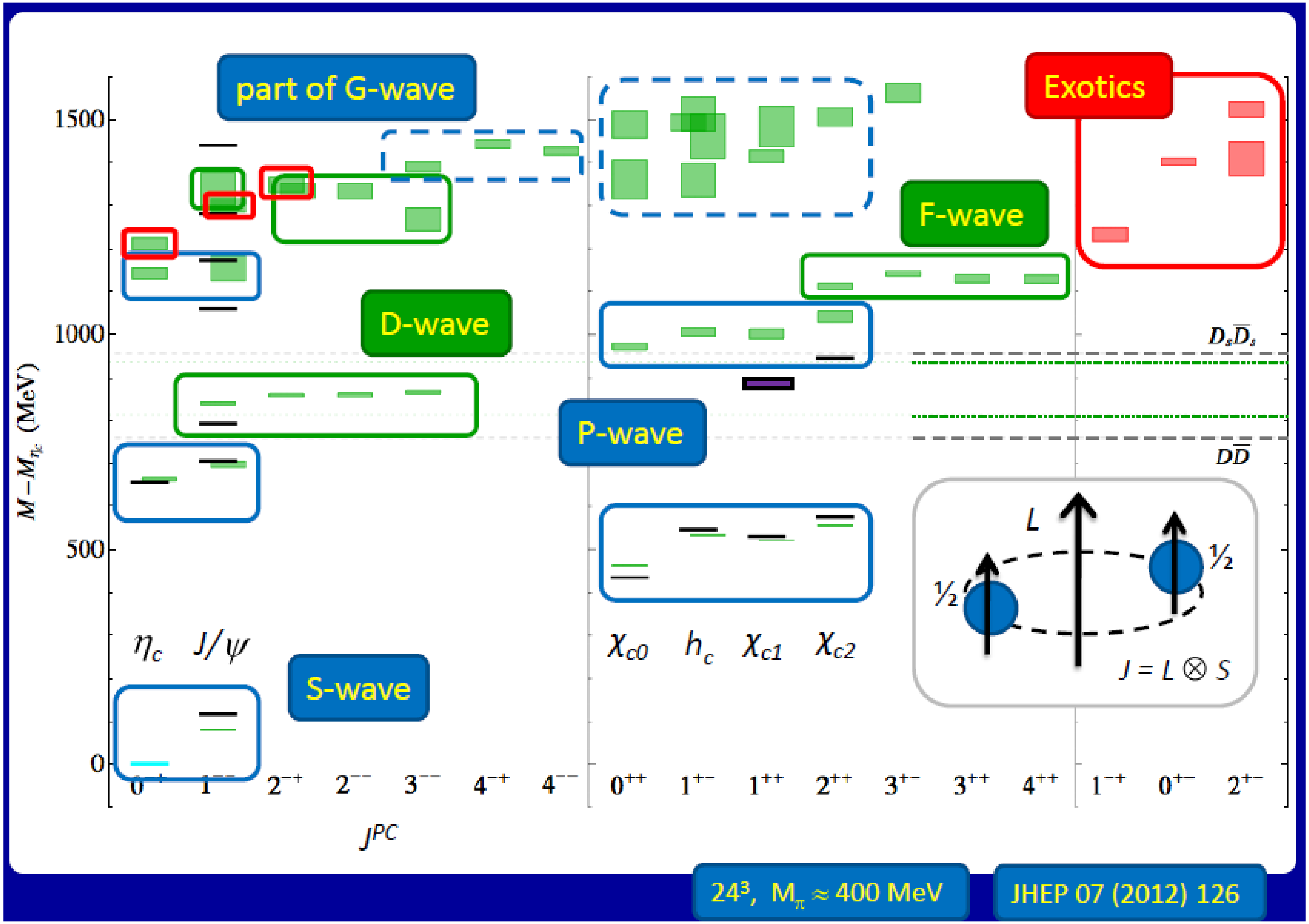
2. The resonance parameter (PDG2012)

$$M_X = 4263(8)MeV$$
$$\Gamma_X = 95(14)MeV$$

3. The leptonic decay width

$$\Gamma(Y(4260) \to e^+e^-)\Gamma(Y(4260) \to J/\psi\pi^+\pi^-)/\Gamma_{tot} = 5.8eV$$

Latest charmonium spectrum from lattice QCD



A. Leptonic decay width of vector hybrid charmonium

Y. Chen, W. Chiu et al., Chin. Phys. C40, 081002 (2016)

The leptonic decay constant of the exotic state can be calculated directly in lattice QCD.

The decay constant of a vector meson is defined as

$$\langle 0|\bar{q}\gamma_\mu q|V(\vec{p},r)\rangle = m_V f_V \varepsilon_\mu(\vec{p},r)$$

can be derived by calculating the two point function

$$\sum_{\vec{x}} \langle 0|\bar{q}\gamma_\mu q(\vec{x},t) O^{(w)}(0)|0\rangle = \sum_{i,r} \frac{1}{2M_i} \langle 0|\, \bar{q}\gamma_\mu q|V_i,r\rangle\langle V_i,r|O^{(w)}|0\rangle e^{-M_i t}$$

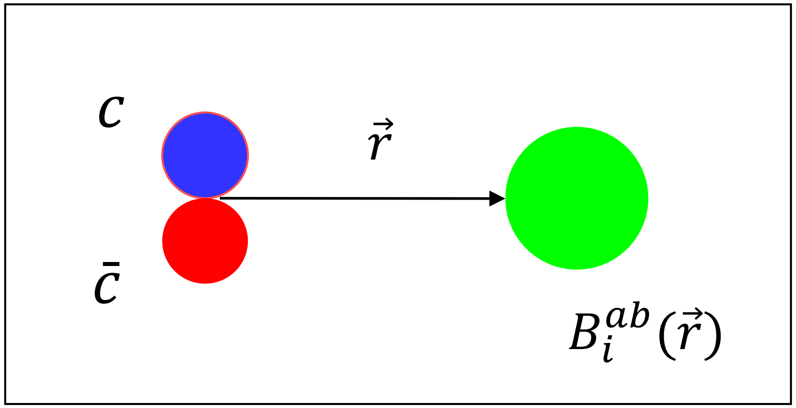
Using the formula $\Gamma(V_{c\bar{c}} \to e^+e^-) = \frac{16\pi}{27}\alpha_{\mathrm{QED}}^2 \frac{f_V^2}{M_V}$

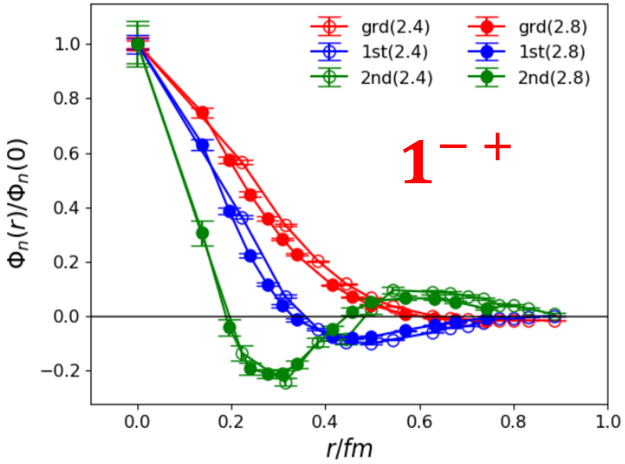
One can predict $\Gamma(X \to e^+e^-) < 40\,\mathrm{eV}.$

This result is confirmed (G. Ray and C. McNeile, arXiv:2110.14101(hep-lat))

B. "Color-Halo" picture for hybrid charmonium

Y. Ma et al., Chin. Phys. C 45, 093111 (2021)





"Halo charmonium":

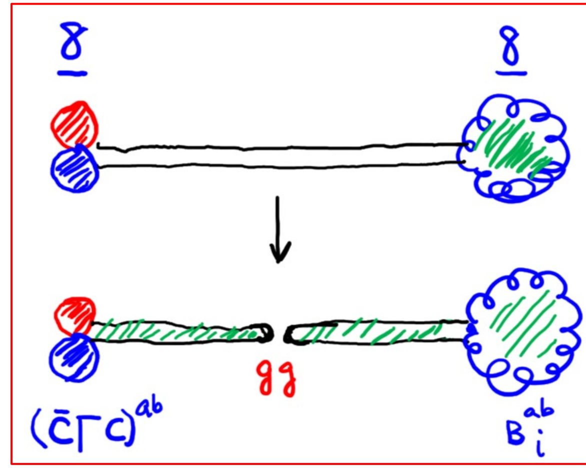
A relatively localized kernal of color octet ccbar surrounded by a gluonic cloud.

The glounic cloud can be easily hadronized into light hadrons by emitting or absorbing a soft gluon.

Hybrids decay into a conventional charmoium by emitting light hadrons.

#node	$m(1^{--})$ (GeV)	$m(0^{-+})$ (GeV)	$m(1^{-+})$ (GeV)	$m(2^{-+})$ (GeV)
0	3.109(5)	3.010(4)	-	-
0	3.703(82)	3.672(76)	-	-
0	4.591(69)	4.551(63)	4.309(2)	4.419(3)
1	5.460(31)	5.393(28)	5.693(12)	5.779(12)
2	8.226(99)	8.286(109)	7.661(31)	7.708(29)

Discussion in the “halo-charmonium” picture



$J/\psi\pi^+\pi^-$ **mode:**
relative S-wave between J/ψ **and** $\pi^+\pi^-$

$\chi_{c0}\omega$ **mode:**
relative S-wave between χ_{c0} **and** ω

$h_c\pi^+\pi^-$ **mode: relative P-wave between** h_c **and** $\pi^+\pi^-$

The $c\bar{c}$ in the halo charmonium is spin singlet (S=0),

$J/\psi\pi^+\pi^-$	**mode:**	J/ψ **(S=1), spin flipping, m_c suppressed, no refugal barrier**
$\chi_{c0}\omega$	**mode:**	χ_{c0} **(S=1), spin flipping, m_c suppressed, no refugal barrier**
$h_c\pi^+\pi^-$	**mode:**	h_c **(S=0), no spin flipping, but suppressed by the refugal barrier .**

In this picture, it is understandable that the above three modes have similar cross section at $\sqrt{s} \sim 4.22\ GeV$

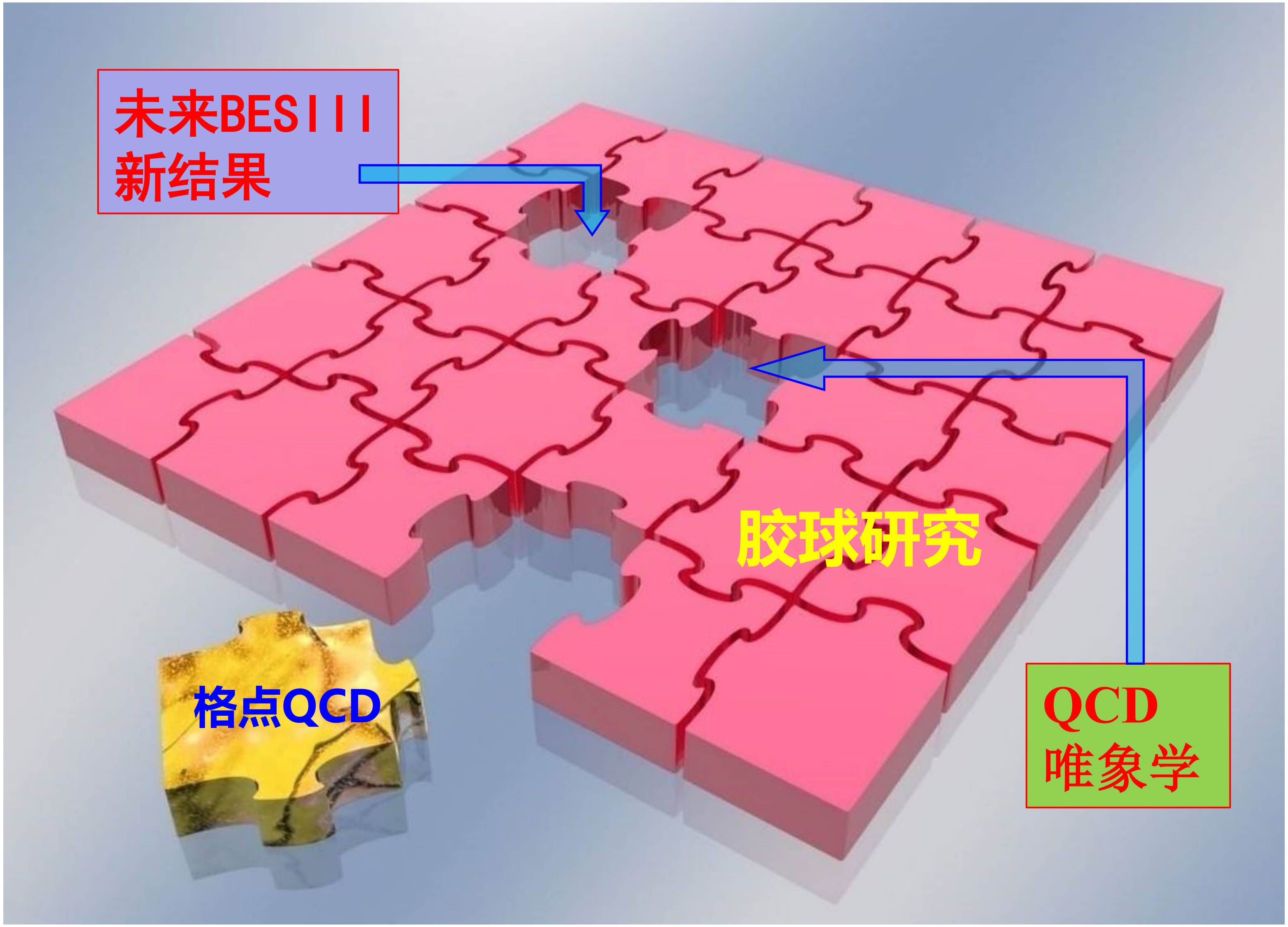
IV. Challenges and Opportunities

- **Light hadron spectroscopy is still challenging for the full-QCD lattice study.**

- 国产计算资源逐渐赶上了：

  高能所和华师大的 GPU 机群
  中科院的“先导一号”（比Summit 快）
  还有若干E级超算投入使用

- 希望为BESIII 提供更多的、更可靠的理论输入。


未来BESIII
新结果
胶球研究
格点QCD
QCD
唯象学

Prospects

- A large $N_f = 2$ gauge ensemble ($N_{conf} \sim 7000$) at $m_\pi \approx 350$ MeV (by Ming Gong)
- Perambulators of light quarks have been generated (by Wei Sun)
- Partial width $J/\psi \to \gamma\eta,\ \gamma\eta_1(1^{-+})$ under going (hopefully by the end of 2021)
- We are generating gauge ensembles with $N_f = 1$ dynamical strange quarks (by Ming Gong)
- Scalar glueball-$s\bar{s}$ meson mixing (2022)
- Partial width $J/\psi \to \gamma s\bar{s}(0^{++}), \gamma s\bar{s}(2^{++})$ (2022)

IV. Summary

- Glueball spectrum has been investigated both from quenched lattice QCD and full-QCD lattice study. There are not large unquenched effects observed. Apart from the spectrum, lattice QCD can also provide useful theoretical information to the production properties of glueballs.

- XYZ particles are also hot topics in lattice QCD study, but the results are preliminary.

- There are still many difficulties for lattice QCD to study exotic hadrons, from both the theoretical tools and numerical calculations.

Thanks!