# Analytical and Numerical Evaluation of 2- and 3-Loop Integrals with Massive Lines


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G.A. 754496

# Why Loop Integrals ?

Precision computation of the cross-section in perturbation theory requires the computation of multi-leg / multi loop Feynman Integrals.

Reduction of scalar integrals to Master integrals using IBP

Computation of the MIs

The main bottleneck

# Integration-By-Parts (IBP) identity

Chetyrkin, Tkachov

$$\int \prod_{\alpha=1}^{l} d^d k_\alpha \frac{\partial}{\partial k_{j,\mu}} \left( \frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) = 0$$

Loop momenta

Loop and external momenta

$$\int_{\alpha=1}^{l} \prod d^d k_\alpha \frac{\partial}{\partial k_{j,\mu}} \left( \frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) = \int_{\alpha=1}^{l} \prod d^d k_\alpha \left[ \frac{\partial v^\mu}{\partial k_{j,\mu}} \left( \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \right) - \sum_{j=1}^{N} \frac{a_j}{D_j} \frac{\partial D_j}{\partial k_{j,\mu}} \left( \frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) \right]$$

$$C_1\, I(a_1, \cdots a_N - 1) + \cdots + C_r\, I(a_1 + 1, \cdots a_N) = 0$$

- Gives relations between different scalar integrals with different exponents
- l(l+E) number of equations
- Solve the system symbolically : Recursion relations — LiteRed
- Solve for specific integer value of the exponents : Laporta Algorithm — Fire, Reduze, Kira,..

# Different New Ideas ?

- Kira - A Feynman Integral Reduction Program
  Maierhoefer, Usovitsch, Uwer (2018)
- FIRE6: Feynman Integral REduction with Modular Arithmetic
  Smirnov, A. V. and Chuharev (2019)
- Two-loop five-point massless QCD amplitudes within the integration-by-parts approach
  Chawdhry,Lim,Mitov (2018)
- Integration-by-parts reductions of Feynman integrals using Singular and GPI-Space
  Bendle, Boehm, Decker,Georgoudis, Pfreundt, Rahn, Wasser, Zhang (2019)
- FiniteFlow: multivariate functional reconstruction using finite fields and dataflow graphs
  Peraro (2019)

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$$I = \sum_{i=1}^{\nu} c_i J_i$$


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$$I = \sum_{i=1}^{\nu} c_i J_i$$

$$I \cdot J_i$$

$$J_i \cdot J_j = \delta_{ij}$$

$$I \cdot J_j \left(C^{-1}\right)_{ji}$$

$$J_i \cdot J_j = C_{ij} \neq \delta_{ij}$$

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**Goal : To define an Vector space with inner product for the FIs**

What is the Vector Space $V$ ?

What is the dual vector space $V^*$

What is the scalar product $V \times V^* \to \mathbb{C}$

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**Intersection Theory**

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**Intersection Theory**

# Intersection Theory

Aomoto, Gelfand, Kita, Cho, Matsumoto, Mimachi, Mizera, Yoshida

$$\hat{\varphi}(\mathbf{z})d^m\mathbf{z}$$

Single-valued differential Form

$$I = \underbrace{\int_{\mathcal{C}} u(\mathbf{z})}_{\text{Twisted Cycle}} \overbrace{\varphi(\mathbf{z})}^{\text{Twisted Co-cycle}}$$

$$\langle\varphi|\mathcal{C}]$$

Pairing

$u(\mathbf{z})$ is a multi-valued function

$u(\mathbf{z})$ *vanishes on the boundaries* of $\mathcal{C}$, $u(\partial\mathcal{C}) = 0$

# Basics of Intersection Theory

$$\omega \equiv d \log u$$

$$0 = \int_{\mathcal{C}} d\,(u\,\xi) = \int_{\mathcal{C}} (du \wedge \xi + u\, d\xi) = \int_{\mathcal{C}} u \left( \frac{du}{u} \wedge + d \right) \xi \equiv \int_{\mathcal{C}} u\, \nabla_\omega \xi.$$

$$\nabla_\omega \equiv d + \omega \wedge$$

**Equivalence Class** $\quad {}_\omega\langle\varphi| : \; \varphi \sim \varphi + \nabla_\omega \xi \qquad \int_{\mathcal{C}} u\,\varphi = \int_{\mathcal{C}} u\,(\varphi + \nabla_\omega \xi)$

**Vector Space of n-forms** $\quad H^n_\omega \equiv \{n\text{-forms } \varphi_n \,|\, \nabla_\omega \varphi_n = 0\} / \{\nabla_\omega \varphi_{n-1}\}$ **Twisted Cohomology Group**

**Dual space** $\quad H^n_{-\omega} \qquad \nabla_{-\omega} = d - \omega \wedge$

$$\chi(X) = \sum_{k=0}^{2n} (-1)^k \dim H^k_\omega. \qquad H^{k\neq n}_\omega \text{ vanish.}$$

Aomoto (1975)

$$\nu = (-1)^n \chi(X)$$

$$= (-1)^n \, (n{+}1 - \chi(\mathcal{P}_\omega))$$

$$= \{\text{number of solutions of } \omega{=}0\}$$

*Integral*

$$\langle \varphi_L | \mathcal{C}_L] = \int_{\mathcal{C}_L} u(\mathbf{z}) \varphi_L(\mathbf{z})$$

*Intersection Number*

$$\langle \varphi_L | \varphi_R \rangle$$

*Master Decomposition Formula*

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \left( \mathbf{C}^{-1} \right)_{ji} \langle e_i |$$

**Univariate Intersection Number**

$$\langle \varphi_L | \varphi_R \rangle_\omega = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p} \big( \psi_p \, \varphi_R \big)$$

Matsumoto, Mizera

$$\nabla_{\omega_p} \psi_p = \varphi_{L,p}$$

First Order Differential Equation

# Intersection Theory: Algorithm

Frellesvig, Gasparotto, MKM, Mastrolia, Mattiazzi, Mizera (2019)

### Integrals

$$I = \int_{\mathcal{C}} u\, \varphi = \langle \varphi | \mathcal{C}]$$

### Number of MIs

$$\omega \equiv d\log u(\mathbf{z}) = \sum_{i=1}^{n} \hat{\omega}_i \, dz_i$$

$$\nu = \text{Number of solutions of the system of equations}$$

$$\hat{\omega}_i \equiv \partial_{z_i} \log u(\mathbf{z}) = 0\,, \qquad i = 1, \ldots, n$$

$$I = \sum_{i=1}^{\nu} c_i J_i \qquad\qquad J_i = \langle e_i | \mathcal{C}]$$

### Choice of Bases

$$e_i(\mathbf{z}) \qquad h_i(\mathbf{z})$$

$$\mathbf{C}_{ij} = \langle e_i | h_j \rangle$$

***Metric Matrix***

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i |$$

***Master Decomposition Formula***

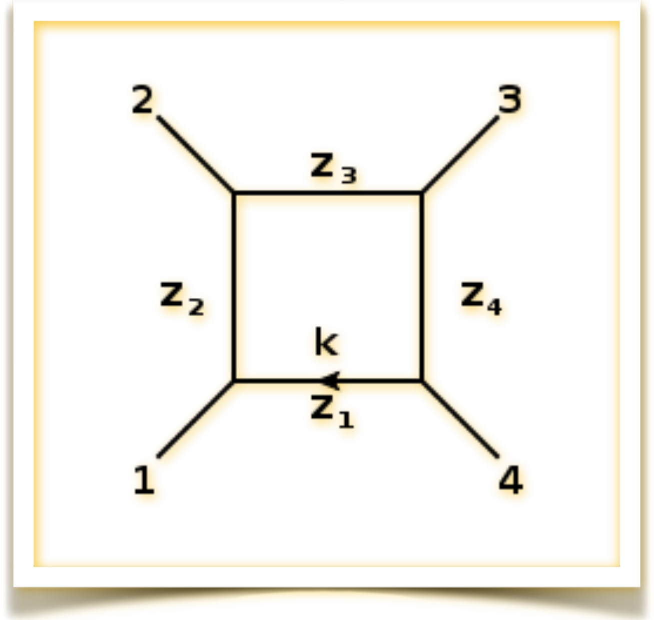
### Computation of Multivariate Intersection Number

Mizera

Recursive Formula :

$$_{\mathbf{n}}\langle \varphi_L^{(\mathbf{n})} | \varphi_R^{(\mathbf{n})} \rangle = -\sum_{p \in \mathcal{P}_n} \operatorname*{Res}_{z_n = p} \left( _{\mathbf{n-1}}\langle \varphi_L^{(\mathbf{n})} | h_i^{(\mathbf{n-1})} \rangle \, \psi_i^{(n)} \right)$$

# Intersection Theory: Example

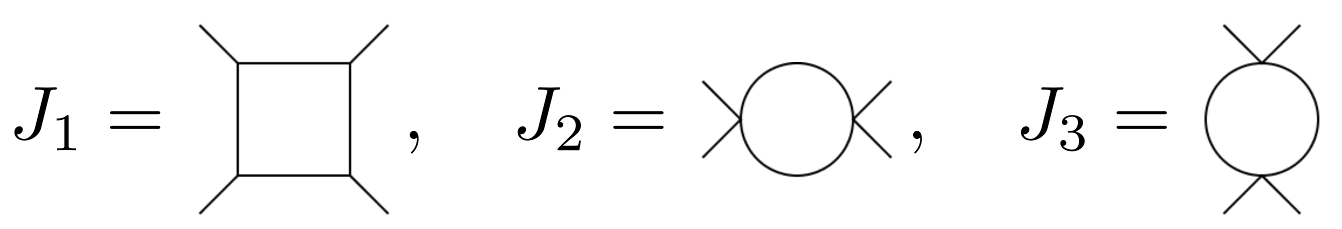



$$u(\mathbf{z}) = \left((st - sz_4 - tz_3)^2 - 2tz_1(s(t+2z_3-z_2-z_4)+tz_3) + s^2z_2^2 + t^2z_1^2 - 2sz_2(t(s-z_3)+z_4(s+2t))\right)^{\frac{d-5}{2}}$$

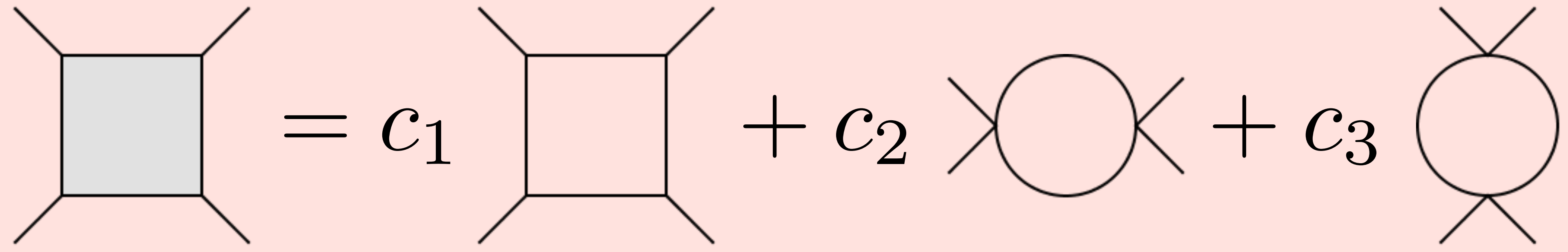
**Baikov Polynomial**

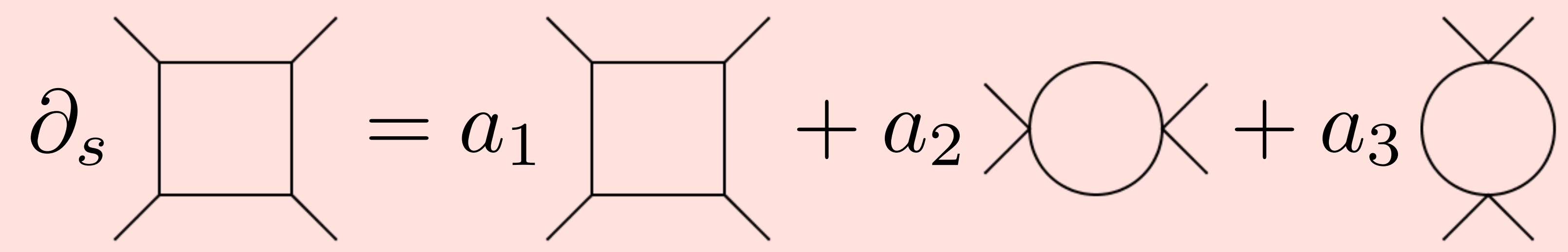
The sectors containing the MIs are

**3 MIs** $\begin{cases} N_{\{1,2,3,4\}} = 1 \\ N_{\{1,3\}} = 1 \\ N_{\{2,3\}} = 1 \end{cases}$

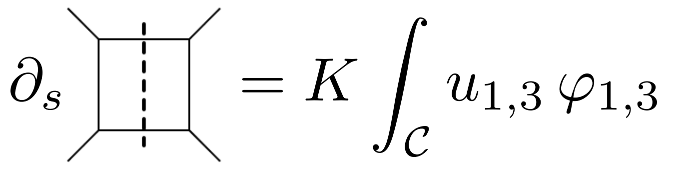



**Integral Decomposition**




**Cut$_{\{1,3\}}$**

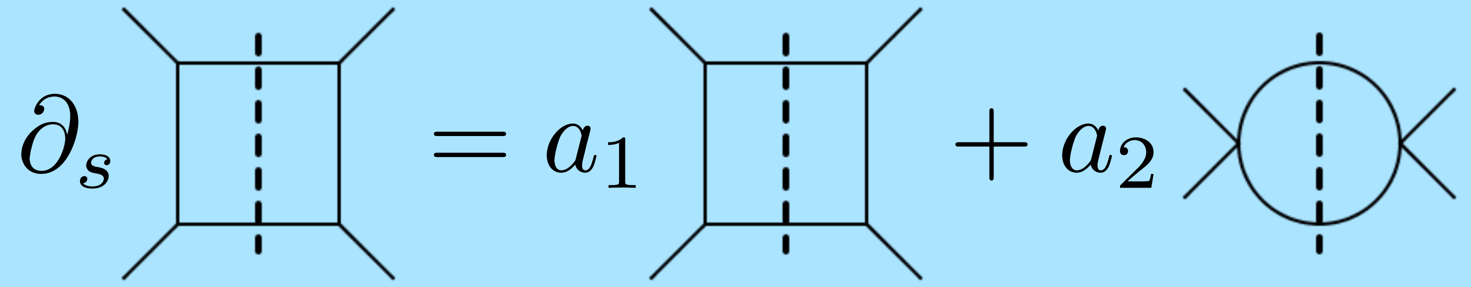



$$\varphi_{1,3} = \hat{\varphi}_{1,3}\, dz_4 \wedge dz_2$$

$$u_{1,3} = z_2^{\rho_2} z_4^{\rho_4}\, u(0, z_2, 0, z_4)$$

$$\hat{\varphi}_{1,3} = \frac{f}{z_2 z_4} \qquad f = \frac{1}{Ku}\frac{\partial(Ku)}{\partial s}$$

Differential Equation

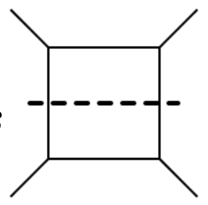



$$a_1 = \sum_{j=1}^{2} \langle \varphi_{1,3} | e_j^{(42)} \rangle \left(\mathbf{C}_{(42)}^{-1}\right)_{j1} = \frac{(d-6)t - 2s}{2s(s+t)}$$

$$a_2 = \sum_{j=1}^{2} \langle \varphi_{1,3} | e_j^{(42)} \rangle \left(\mathbf{C}_{(42)}^{-1}\right)_{j2} = \frac{2(d-3)}{s^2(s+t)}$$

# One Loop Box : DE

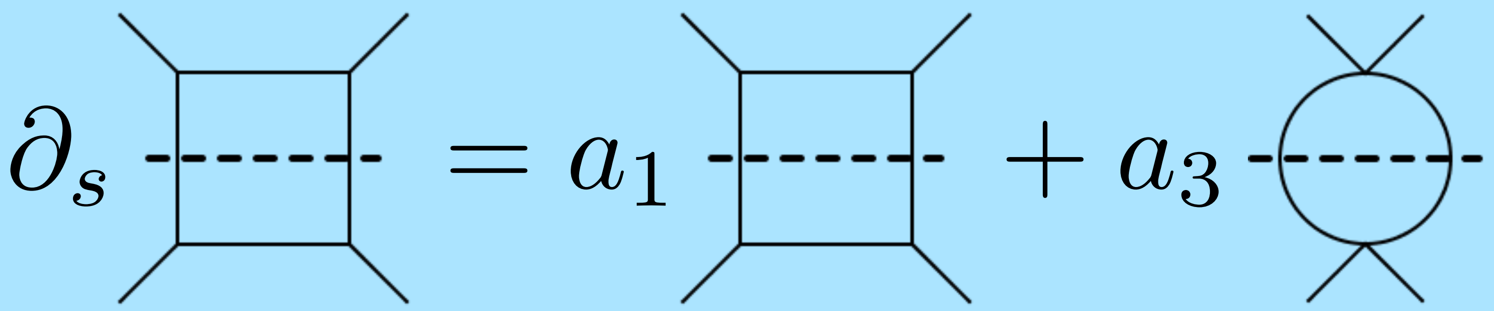
**Cut$_{\{2,4\}}$**

$\partial_s$  $= K \int_{\mathcal{C}} u_{2,4}\, \varphi_{2,4} \qquad \varphi_{2,4} = \hat{\varphi}_{2,4}\, dz_3 \wedge dz_1$

$$u_{2,4} = z_1^{\rho_1} z_3^{\rho_3}\, u(z_1, 0, z_3, 0)$$

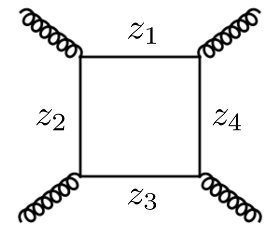
$$\hat{\varphi}_{2,4} = \frac{f}{z_1 z_3} \qquad f = \frac{1}{Ku}\frac{\partial(Ku)}{\partial s}$$

Differential Equation



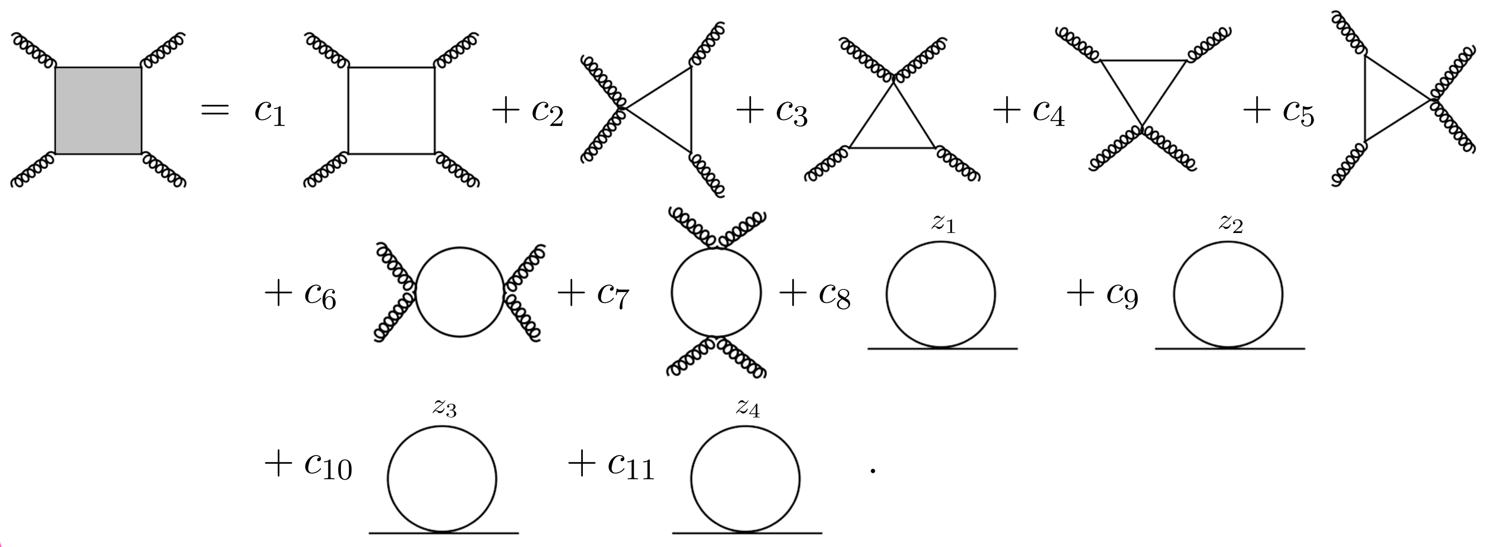

$$a_1 = \sum_{j=1}^{2} \langle \varphi_{1,3} | e_j^{(42)} \rangle \left(\mathbf{C}_{(42)}^{-1}\right)_{j1} = \frac{(d-6)t - 2s}{2s(s+t)}$$

$$a_3 = \sum_{j=1}^{2} \langle \varphi_{2,4} | e_j^{(31)} \rangle \left(\mathbf{C}_{(31)}^{-1}\right)_{j2} = -\frac{2(d-3)}{st(s+t)}$$

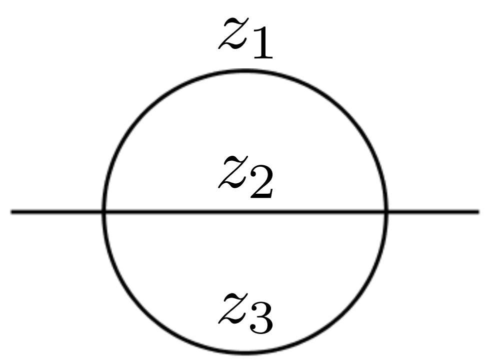
| Integral family | Denominators |
|---|---|
|  $s = (p_1+p_2)^2, \quad t = (p_2+p_3)^2$ | $z_1 = k^2 - m_1^2$<br>$z_2 = (k+p_1)^2 - m_2^2$<br>$z_3 = (k+p_1+p_2)^2 - m_3^2$<br>$z_4 = (k+p_1+p_2+p_3)^2 - m_4^2$ |

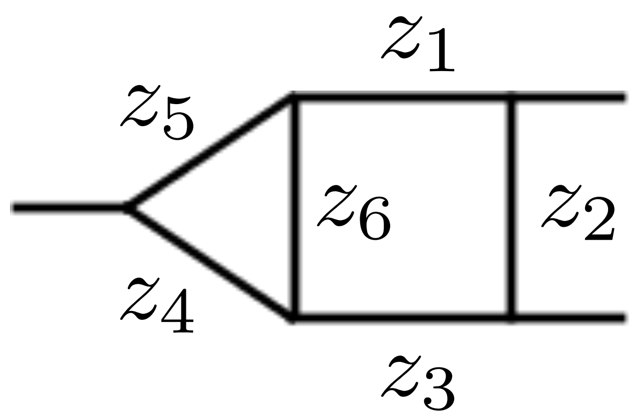


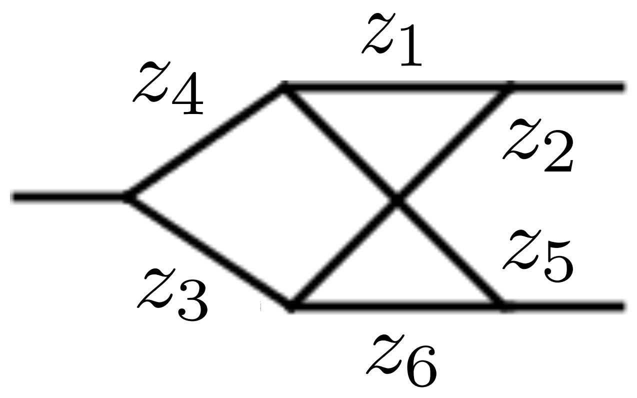
| $\tau$ | $\nu$ | $e$ |
|---|---|---|
| $z_4 = 0$ | $\nu_{\{3\}} = 2$ | $e^{(3)} = \left\{1, \frac{1}{z_3}\right\}$ |
| | $\nu_{\{32\}} = 3$ | $e^{(32)} = \left\{\frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_2 z_3}\right\}$ |
| | $\nu_{\{321\}} = 6$ | $e^{(321)} = \left\{1, \frac{1}{z_2}, \frac{1}{z_1 z_2}, \frac{1}{z_1 z_3}, \frac{1}{z_2 z_3}, \frac{1}{z_1 z_2 z_3}\right\}$ |
| $z_3 = 0$ | $\nu_{\{4\}} = 2$ | $e^{(4)} = \left\{1, \frac{1}{z_4}\right\}$ |
| | $\nu_{\{41\}} = 3$ | $e^{(41)} = \left\{\frac{1}{z_1}, \frac{1}{z_4}, \frac{1}{z_1 z_4}\right\}$ |
| | $\nu_{\{412\}} = 6$ | $e^{(412)} = \left\{1, \frac{1}{z_1}, \frac{1}{z_1 z_2}, \frac{1}{z_1 z_4}, \frac{1}{z_2 z_4}, \frac{1}{z_1 z_2 z_4}\right\}$ |
| $z_2 = 0$ | $\nu_{\{4\}} = 2$ | $e^{(4)} = \left\{1, \frac{1}{z_4}\right\}$ |
| | $\nu_{\{43\}} = 3$ | $e^{(43)} = \left\{\frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_3 z_4}\right\}$ |
| | $\nu_{\{431\}} = 6$ | $e^{(431)} = \left\{1, \frac{1}{z_4}, \frac{1}{z_1 z_3}, \frac{1}{z_1 z_4}, \frac{1}{z_3 z_4}, \frac{1}{z_1 z_3 z_4}\right\}$ |
| $z_1 = 0$ | $\nu_{\{4\}} = 2$ | $e^{(4)} = \left\{1, \frac{1}{z_4}\right\}$ |
| | $\nu_{\{43\}} = 3$ | $e^{(43)} = \left\{\frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_3 z_4}\right\}$ |
| | $\nu_{\{432\}} = 6$ | $e^{(432)} = \left\{1, \frac{1}{z_3}, \frac{1}{z_2 z_3}, \frac{1}{z_2 z_4}, \frac{1}{z_3 z_4}, \frac{1}{z_2 z_3 z_4}\right\}$ |

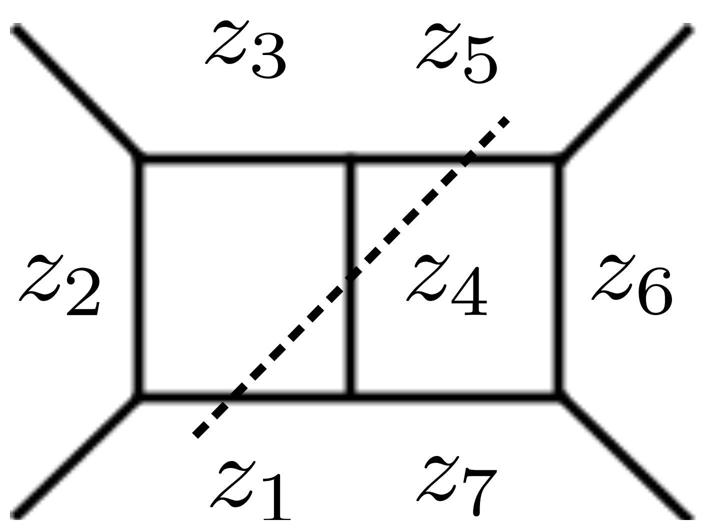


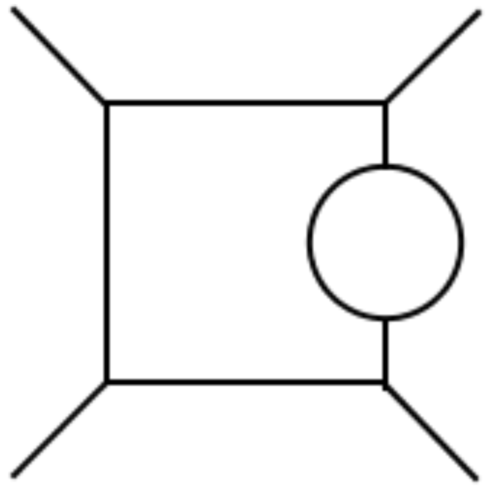
# Work in progress



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...

# Evaluation of MIs

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## Analytical approach

🙂 Stable and fast numerical evaluation

☹️ Not Flexible

☑ Direct Integration (Feynman Parameter, Mellin Barnes Method)

Smirnov (1999); Tausk (1999); Czakon (2005);
Czakon, Gluza, Reimann (2005); Brown (2009); Panzer (2015);

☑ Differential Equation

Kotikov (1991); Remiddi (1997);
Gehrmann, Remiddi (2000); Argeri, Mastrolia (2007)
Henn(2013)

## Numerical approach

🙂 Flexible

☹️ Numerical instabilities, not fast

☑ Sector Decomposition

Hepp (1966); Roth, Denner (1996); Binoth, Heinrich (2000); Carter, Heinrich (2010); Borowka, Carter, Heinrich (2012);
Smirnov, Smirnov, Tentyukov (2011), Bogner, Weinzierl (2008); Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke (2017)

☑ Loop-Tree Duality

Catani, Gleisberg, Krauss, Rodrigo, Winter (2008); Bierenbaum, Catani, Draggiotis, Rodrigo (2010); Runkel, Szr, Vesga, Weinzierl (2019);
Capatti, Hirschi, Kermanschah, Ruijl (2019); Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, Bobadilla (2020);
Ramrez-Uribe, Hernndez-Pinto, Rodrigo, Sborlini, Bobadilla (2020)

☑ Differential Equation

Boughezal, Czakon, Schutzmeier (2007); Czakon (2008); Liu, Ma, Wang (2017); MKM, Zhao (2018)

# MIs Evaluation : DE Method

$I = (I_1, I_2, \cdots, I_M)^{\mathrm{T}}$

$$\frac{\partial I(x;\epsilon)}{\partial x_i} = J_i(x;\epsilon) I(x;\epsilon), \quad i = 1, \cdots, K$$

Matrix, with rational functions entries of x and d

$x = (x_1, x_2, \cdots, x_K)$

- The singularities form the kinematics are governed by the Landau equation

$$\frac{\partial F(x;\epsilon)}{\partial x_i} = \epsilon A_i(x) F(x;\epsilon)$$

Henn(2013)

**Cannonical form**

Dlapa, Henn, Yan (2020)

- Novel algorithm to construct the cannonical system, starting from one Uniformly Transcendental integral.

- There has been a recent proposal to construct *d* log-form integrals of the hypergeometric type, treat them as a representation of Feynman integrals, and project them into master integrals using intersection theory.

Chen, Xu, Yang (2020)

# Differential Equation : Cannonical basis

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- ✦ 3-Loop Higgs gluon form factors Harlander, Prausa, Usovitsch (2019)
- ✦ 2-Loop QCD correction to Higgs + jet production with internal top quark Bonciani, Del Duca, Frellesvig, Henn, Hidding, Maestri, Moriello, Salvatori, Smirnov (2019); Frellesvig,Hidding,Maestri,FM, Salvatori (2019)
- ✦ 3-Loop mixed QCD EW correction (light quarks) to Higgs production in gluon fusion Bonetti, Melnikov, Tancredi
- ✦ 2-Loop mixed QCD EW correction to DY production Bonciani, Di Vita, Mastrolia, Schubert (2016); Heller, Manteuffel, Schabinger (2019); Bonciani, Buccioni, Rana, Triscari, Vicini (2019)
- ✦ 2-Loop mixed QCD EW correction to Higgs+jet production Bonetti, Panzer, Smirnov, Tancredi (2020); Becchetti, Bonciani, Del Duca, Hirschi, Moriello, Schweitzer (2020)
- ✦ 2-Loop QCD correction to top pair production Adams, Choubey, Weinzierl (2018); Di Vita, Gehrmann, Laporta, Mastrolia, Primo, Schubert (2019)
- ✦ 2-Loop QED correction to muon-electron scattering Mastrolia, Passera, Primo, Schubert (2017); Di Vita, Laporta, Mastrolia, Primo, Schubert (2018)

# Differential Equation : Series Expansion

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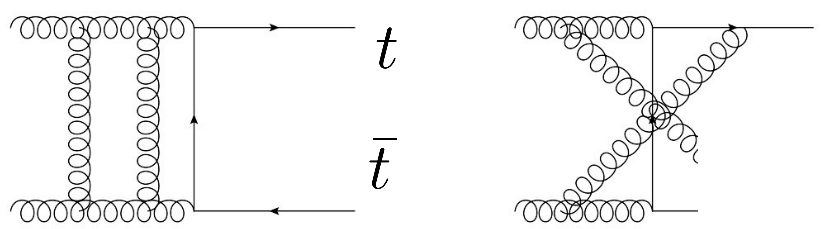
Moriello (2019) Hidding (2020)

Abreu, Ita, Moriello, Page, Tschernow, Zeng (2020)

$$\boxed{\frac{\partial \mathbf{F}(\epsilon,\vec{s})}{\partial s_n} = \mathbf{A}(\epsilon,\vec{s})\mathbf{F}(\epsilon,\vec{s}), \quad n \in \{1,\ldots,N\}}$$

- Fix two points and define a contour between them $\vec{s}(t) = \vec{s}_a + (\vec{s}_b - \vec{s}_a)t, \quad t \in [0,1]$
- DE along the contour becomes : $\frac{d}{dt}\mathbf{F}(\epsilon,t) = \mathbf{A}(\epsilon,t)\mathbf{F}(\epsilon,t)$ $\quad \mathbf{A}(\epsilon,t) = \sum_{n=1}^{N} \mathbf{A}_n(\epsilon,\vec{s}(t))\frac{ds_n(t)}{dt}$
- Becomes one dimensional problem: Transport the solution by series expansion along the contour
- The series solution is found by the Frobenius method.
- Very efficient and allow for high precision in all kinematic regions, which is very suitable for MC integration.
- Has been very successful recently in obtaining the two loop amplitude for Higgs + jet production keeping the top mass, two loop planar five point one mass integrals.

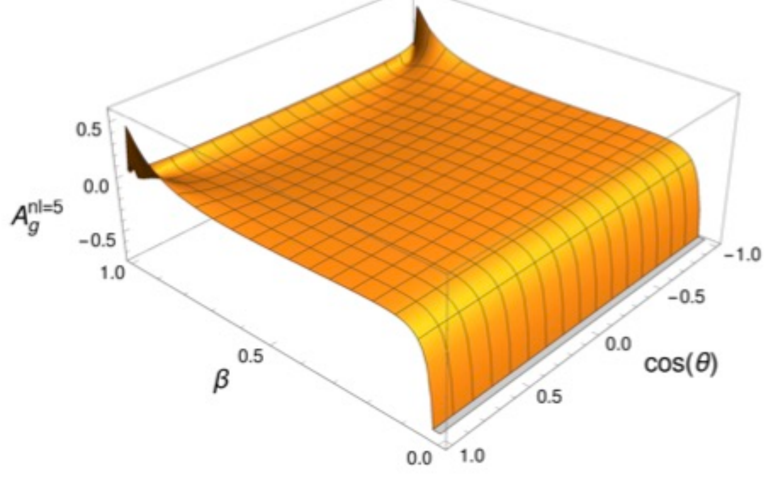
# Numerical Approach




$$m_s = \frac{m_t^2}{s}, \quad x = \frac{t}{s}$$

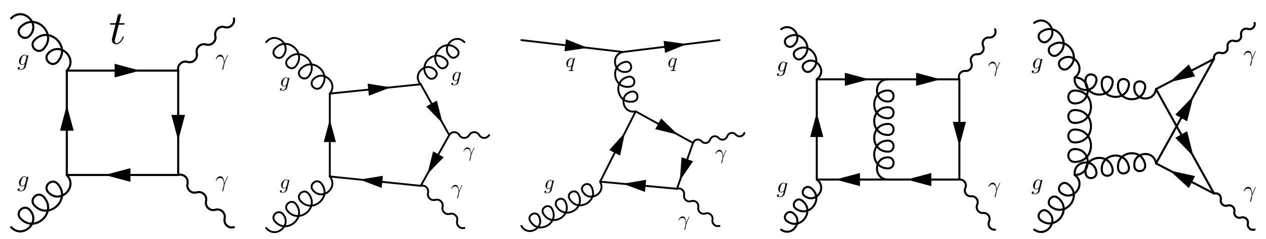
$$m_s \frac{\partial}{\partial m_s} \vec{I} = \hat{A}_m \left(m_s, x, \epsilon\right) \vec{I}$$

$$x \frac{\partial}{\partial x} \vec{I} = \hat{A}_x \left(m_s, x, \epsilon\right) \vec{I}$$

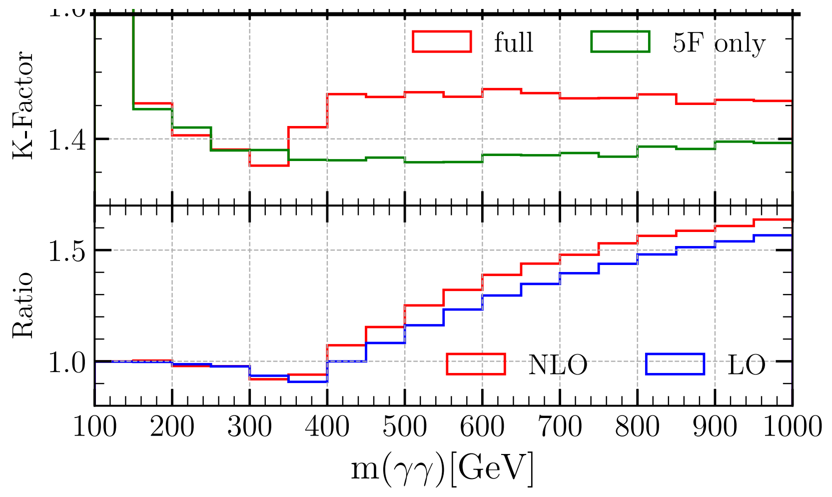



Chen, Czakon, Poncelet

$gg \to \gamma\gamma$

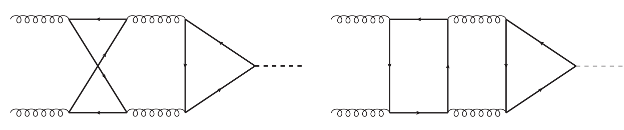


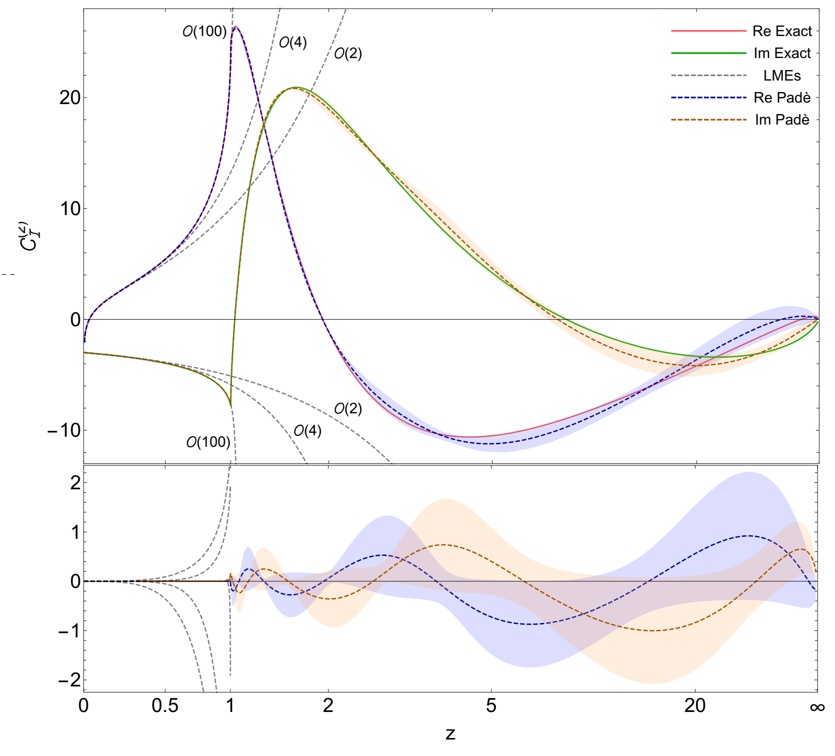
Uses publicly-available software to solve systems of diff. equations !




Maltoni, MKM, Zhao
MKM, Zhao

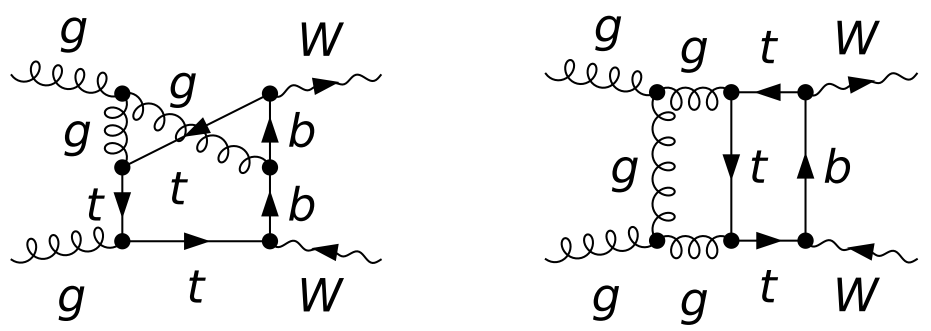
$gg \to H$

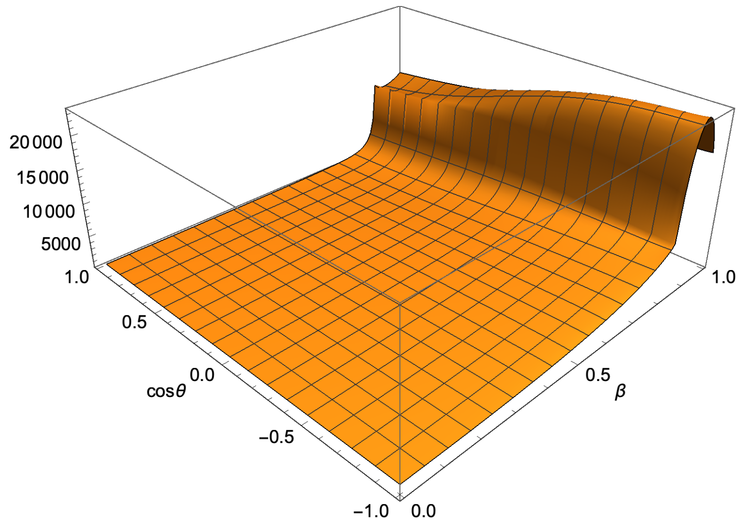





Czakon, Niggetiedt

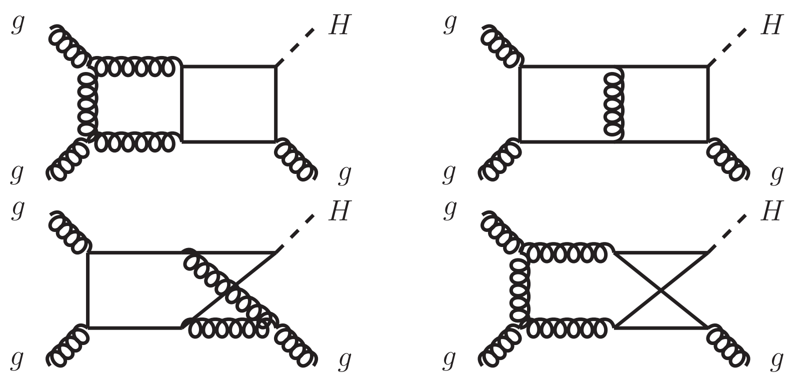
$gg \to WW$

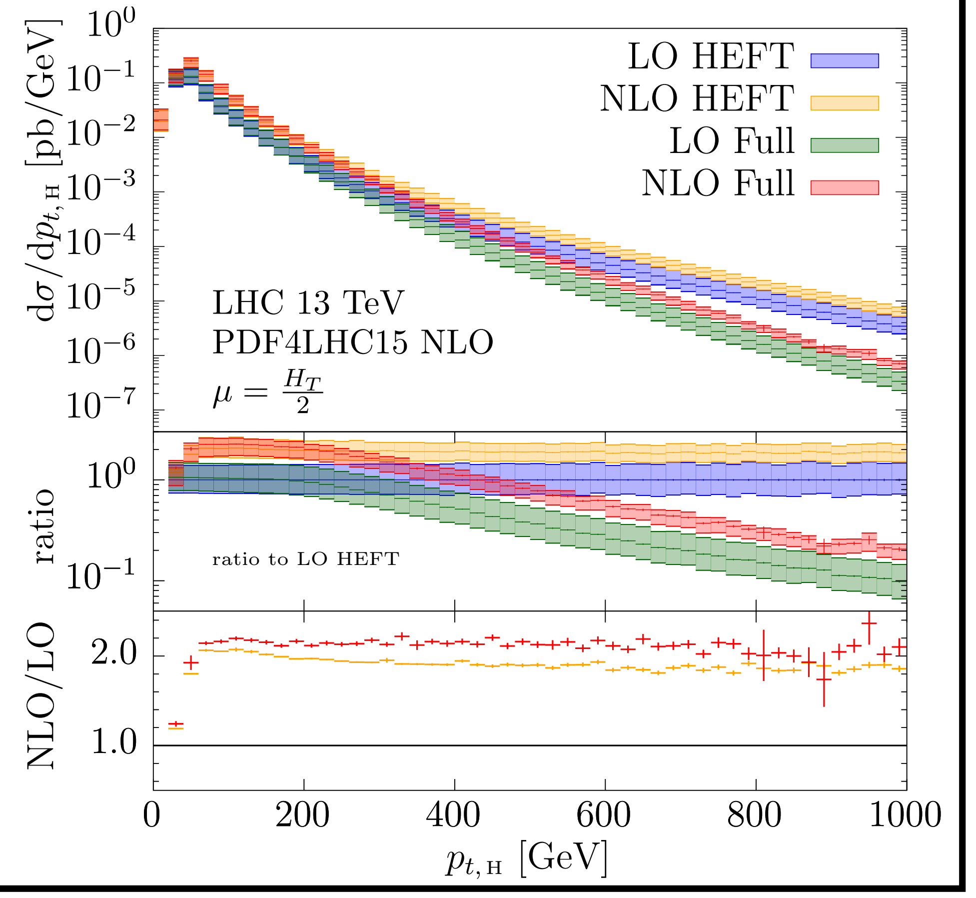


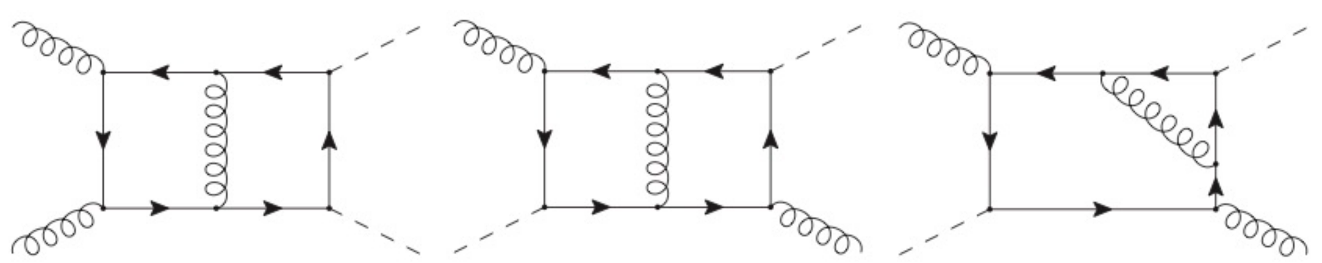

Liu, Ma, Wang
Bronnum-Hanson, Wang

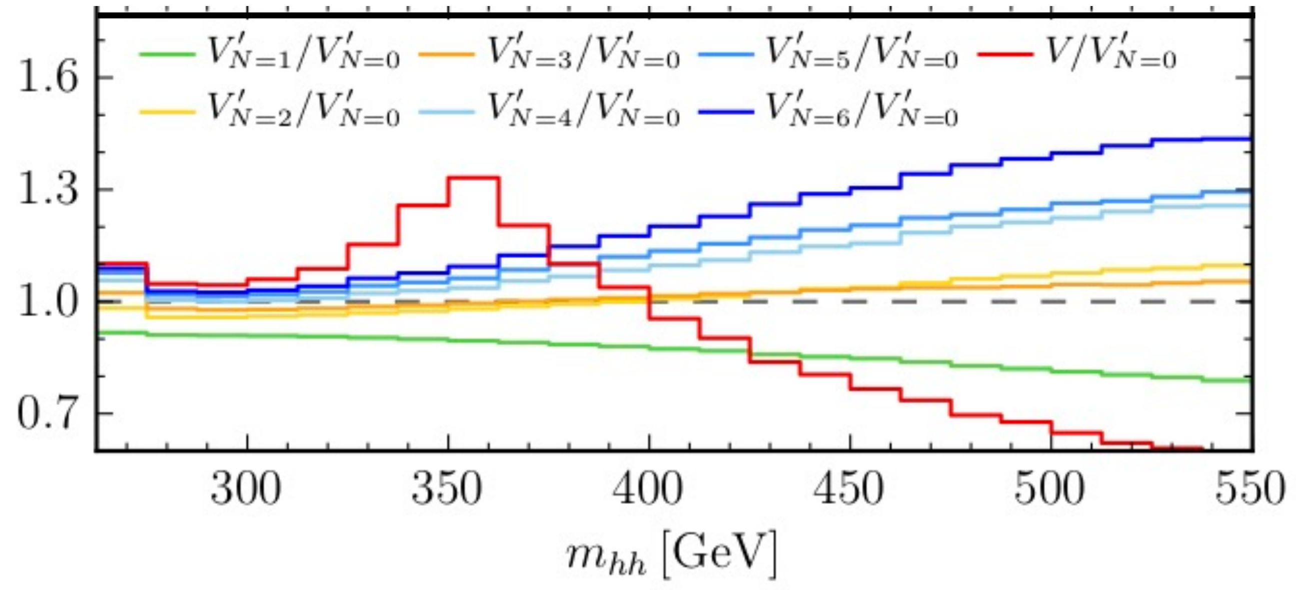
$pp \to H + j$






Kerner, Jones, Luisoni






Borowka, Greiner, Heinrich, Kerner, Jones, Schlenk, Zirke

Also: Baglio, Campanario, Glaus, Mühlleitner, Spira, Streicher

H2020 MSCA COFUND G.A. 754496

## Our Proposal

MKM, X Zhao (2018)

Numerically Solve the DE of the MIs

Use SD

Un-physical point

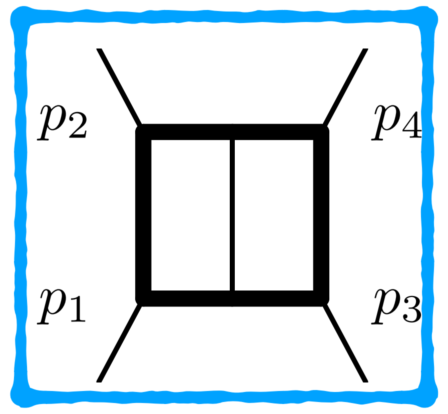
Use DE

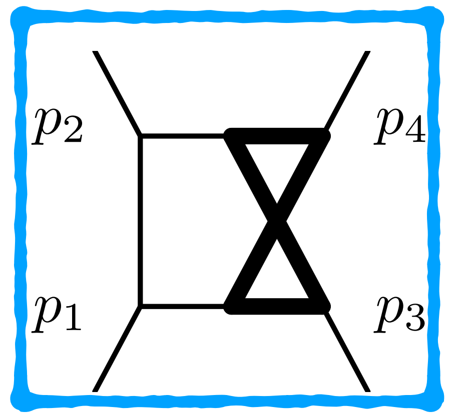
Choose a Contour

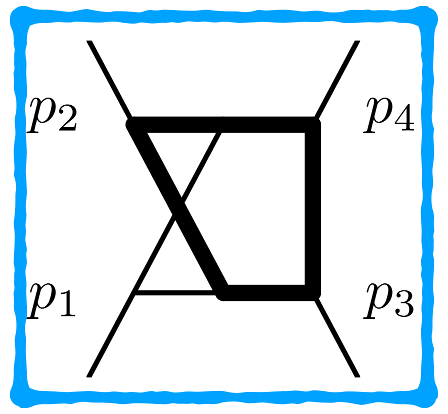
Target Physical point

The contour should not cross any branch cut

| | F1 | F2 | F3 |
|---|---|---|---|
| $s=0$ | N | Y | N |
| $s=4$ | Y | Y | Y |
| $s=-16$ | - | N | - |
| $t=0$ | N | N | N |
| $t=4$ | Y | Y | Y |
| $u=0$ | N | N | N |
| $u=4$ | - | Y | Y |
| $t=u$ | - | N | N |
| $st+4u=0$ | Y/N | Y/N | Y/N |
| $tu+4s=0$ | - | Y/N | Y/N |
| $su+4t=0$ | - | Y/N | Y/N |
| $4t^2-s(t-1)^2=0$ | N | N | N |
| $4u^2-s(u-1)^2=0$ | - | - | N |

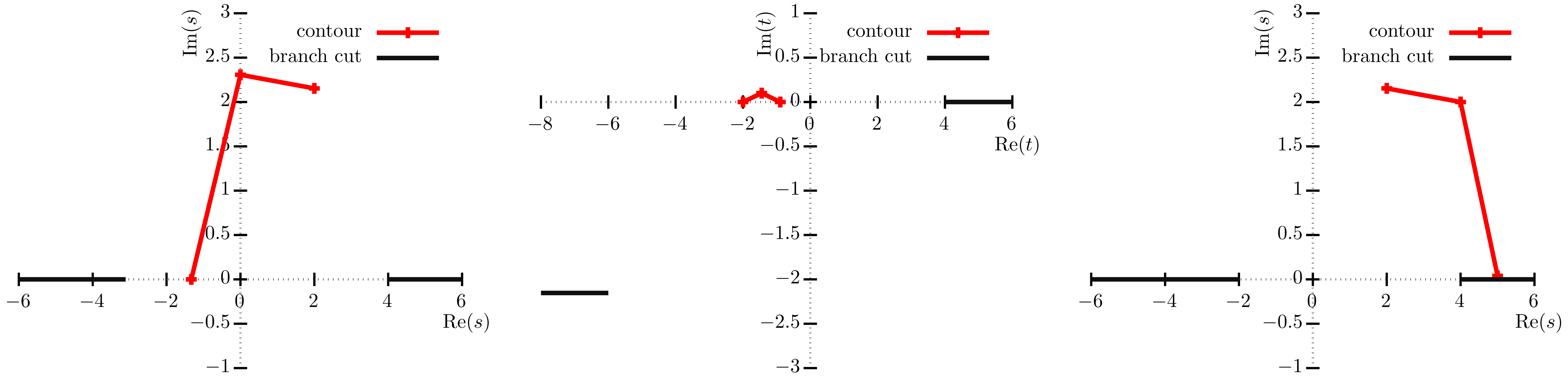



Only Branch point for $s>0, t>0$

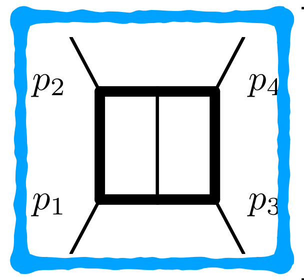
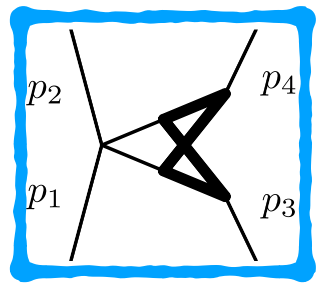
# Differential Equation: Numerical Integration




$$(s_a, t_a) = (-1.33, -0.891)$$
$$(s_b, t_b) = (5, -2)$$

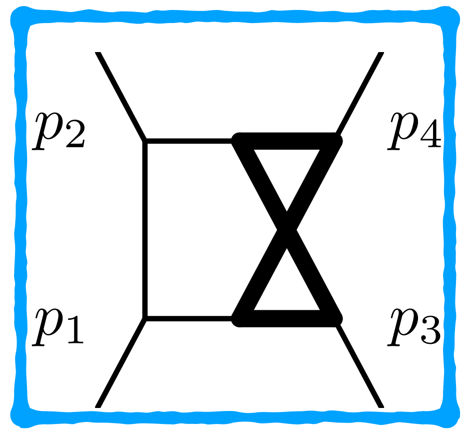
$$
\begin{aligned}
&(s_a, t_a)\\
\to&(i\sqrt{-4s_a}, t_a)\\
\to&(2 + i\sqrt{-s_a} + i\sqrt{s_b - 4}, t_a)\\
\to&(2 + i\sqrt{-s_a} + i\sqrt{s_b - 4}, (t_a + t_b)/2 + 0.1i)\\
\to&(2 + i\sqrt{-s_a} + i\sqrt{s_b - 4}, t_b)\\
\to&(4 + i\sqrt{4(s_b - 4)}, t_b)\\
\to&(s_b, t_b).
\end{aligned}
$$

$$\mathrm{IC1}(s = -1.33, t = -0.891)$$
$$\mathrm{IC2}(s = -1.63, t = -0.632)$$

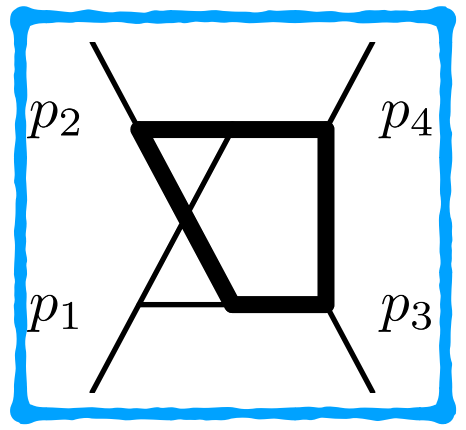



| | $(s = 5, t = -2)$ | $c_0$ | time(s) |
|---|---|---|---|
| $I_1$ | IC1 | $0.573661717 - i0.45602298$ | 0.11 |
| | IC2 | $0.573662051 - i0.45602316$ | 0.10 |
| | Ref. [26] | $0.573661756 - i0.45602309$ | – |
| $I_2^{\mathrm{sub}}$ | IC1 | $-0.077764616 + i0.34306744$ | 0.26 |
| | IC2 | $-0.077764595 + i0.34306737$ | 0.23 |
| | Ref. [54] | $-0.077764620 + i0.34306741$ | – |

$I_2$



$I_3$



| | $(s = 5, t = -2)$ | $c_0$ | $c_1$ | $c_2$ | time(s) |
|---|---|---|---|---|---|
| $I_2$ | IC1 | $0.02188084 - i0.00000002$ | $-0.0870259 + i0.05170117$ | $-0.246416 - i0.17602070$ | 0.26 |
| | IC2 | $0.02188080 + i0.00000001$ | $-0.0870262 + i0.05170118$ | $-0.246417 - i0.17602072$ | 0.23 |
| | pySecDec | $0.02187(3) + i0.00003(3)$ | $-0.0869(3) + i0.0518(4)$ | $-0.248(2) - i0.175(2)$ | $\mathcal{O}(10^4)$ |
| $I_3$ | IC1 | $-0.0599222 + i0.4204527$ | $-1.2093294 + i1.1271787$ | $-3.737851 + i0.435880$ | 0.74 |
| | IC2 | $-0.0599219 + i0.4204528$ | $-1.2093298 + i1.1271798$ | $-3.737851 + i0.435879$ | 0.78 |
| | pySecDec | $-0.05998(7) + i0.42048(8)$ | $-1.2100(7) + i1.1262(7)$ | $-3.737(3) + i0.430(3)$ | $\mathcal{O}(10^4)$ |

# Conclusion

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- Novel Algebraic Property Unveiled
  - The algebra of Feynman Integrals is controlled by intersection numbers
  - Intersection Numbers : Scalar Product/Projection between Feynman Integrals
  - Useful for both Physics and Mathematics
- Novel decomposition method
  - Direct decomposition in a Integral Basis
  - No Intermediate relation required
- Evaluation of MIs
  - DE has been the most successful in case of analytical or numerical
  - Series expansion of the DE looks very promising
  - Numerical techniques are becoming stable and efficient, thanks to the ideas of integrating the DE.

# Thank you

# Back-up

# Basics of Intersection Theory

$$\omega \equiv d\log u$$

$$0 = \int_{\mathcal{C}} d\,(u\,\xi) = \int_{\mathcal{C}} (du \wedge \xi + u\, d\xi) = \int_{\mathcal{C}} u \left( \frac{du}{u} \wedge + d \right) \xi \equiv \int_{\mathcal{C}} u\, \nabla_\omega \xi.$$

$$\nabla_\omega \equiv d + \omega \wedge$$

**Equivalence Class** $\quad {}_\omega\langle\varphi| :\ \varphi \sim \varphi + \nabla_\omega \xi \qquad \int_{\mathcal{C}} u\,\varphi = \int_{\mathcal{C}} u\,(\varphi + \nabla_\omega \xi)$

**Vector Space of n-forms** $\quad H^n_\omega \equiv \{n\text{-forms } \varphi_n \,|\, \nabla_\omega \varphi_n = 0\}/\{\nabla_\omega \varphi_{n-1}\}$ **Twisted Cohomology Group**

**Dual space** $\quad H^n_{-\omega}. \qquad \nabla_{-\omega} = d - \omega\wedge$

$$\chi(X) = \sum_{k=0}^{2n} (-1)^k \dim H^k_\omega. \qquad H^{k\neq n}_\omega \text{ vanish.}$$

Aomoto (1975)

$$\nu = (-1)^n \chi(X)$$

$$= (-1)^n\,(n{+}1 - \chi(\mathcal{P}_\omega))$$

$$= \{\text{number of solutions of } \omega{=}0\}$$

***Integral***

$$\langle \varphi_L | \mathcal{C}_L] = \int_{\mathcal{C}_L} u(\mathbf{z})\varphi_L(\mathbf{z})$$

***Intersection Number***

$$\langle \varphi_L | \varphi_R \rangle$$

***Master Decomposition Formula***

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i |$$

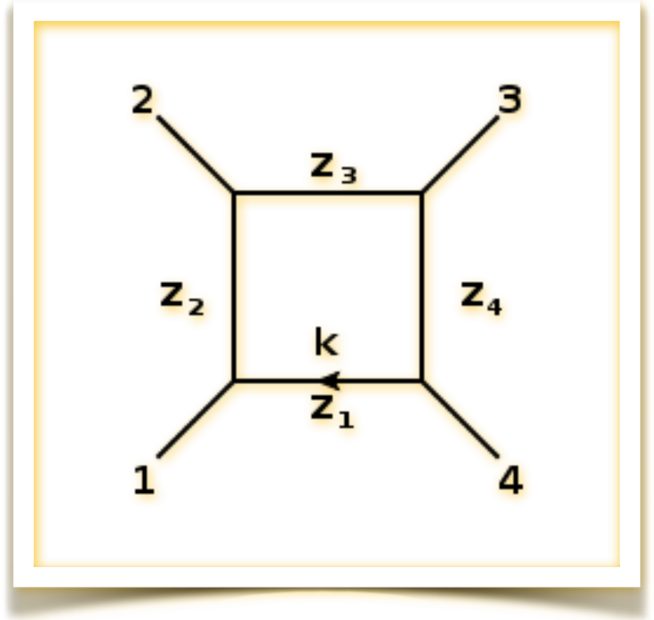
**Univariate Intersection Number**

$$\langle \varphi_L | \varphi_R \rangle_\omega = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p}\big(\psi_p\, \varphi_R\big)$$

Matsumoto, Mizera

$$\nabla_{\omega_p} \psi_p = \varphi_{L,p}$$

First Order Differential Equation

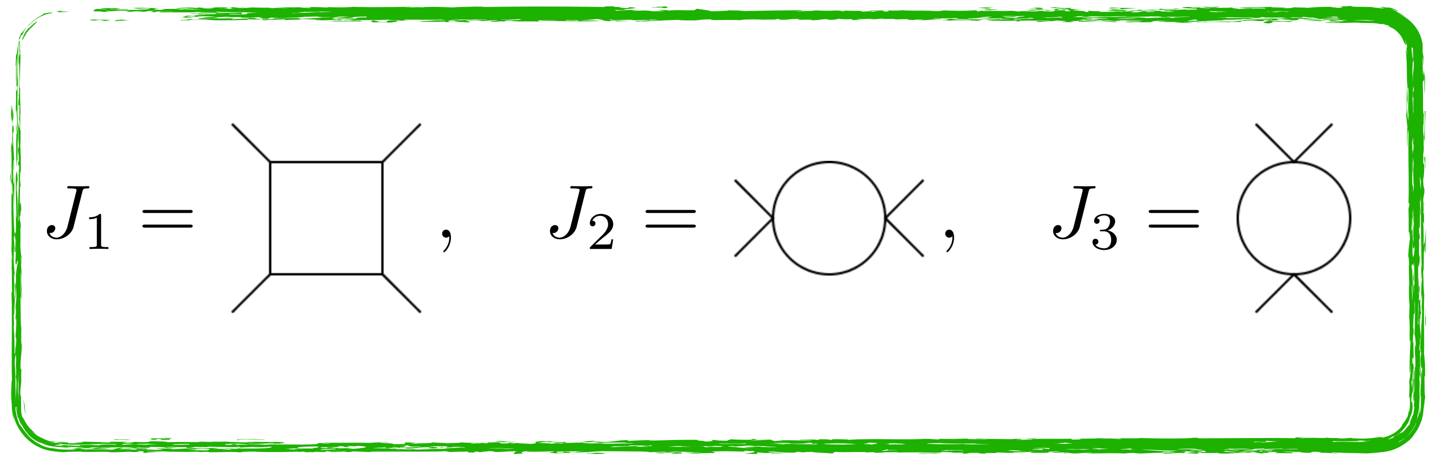



$$u(\mathbf{z}) = \left((st - sz_4 - tz_3)^2 - 2tz_1(s(t+2z_3-z_2-z_4)+tz_3) + s^2z_2^2 + t^2z_1^2 - 2sz_2(t(s-z_3)+z_4(s+2t))\right)^{\frac{d-5}{2}}$$

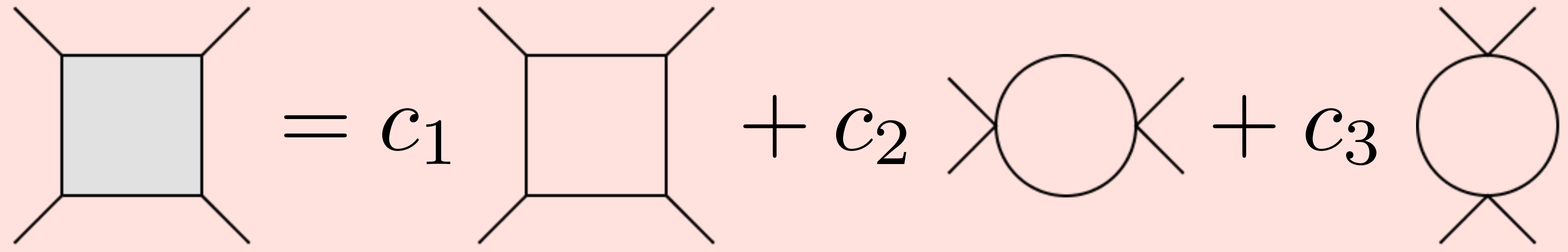
**Baikov Polynomial**

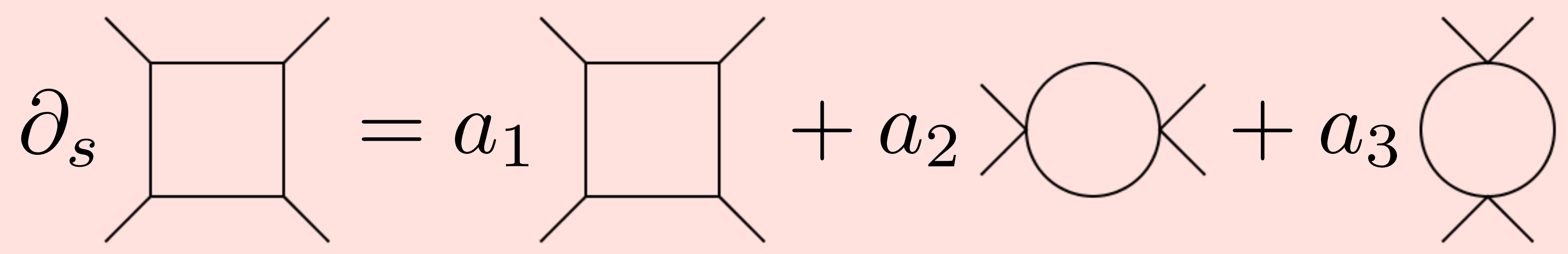
The sectors containing the MIs are

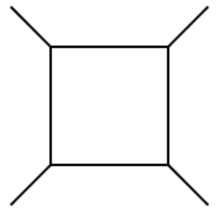
**3 MIs**
$$\begin{cases} N_{\{1,2,3,4\}} = 1 \\ N_{\{1,3\}} = 1 \\ N_{\{2,3\}} = 1 \end{cases}$$




**Integral Decomposition**

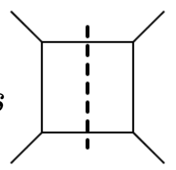


$$= K \int_{\mathcal{C}} \frac{u\, d^4\mathbf{z}}{z_1 z_2 z_3 z_4} \qquad K = (-st(s+t))^{2-\frac{d}{2}}$$

**Cut$_{\{1,3\}}$**

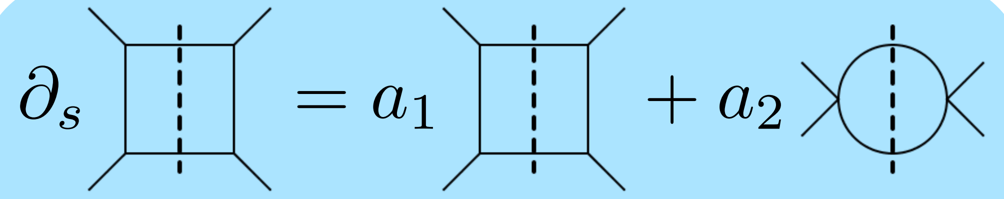
$\partial_s$  $= K \int_{\mathcal{C}} u_{1,3}\, \varphi_{1,3}$ $\qquad \varphi_{1,3} = \hat{\varphi}_{1,3}\, dz_4 \wedge dz_2$

$$u_{1,3} = z_2^{\rho_2} z_4^{\rho_4}\, u(0, z_2, 0, z_4)$$

$$\hat{\varphi}_{1,3} = \frac{f}{z_2 z_4} \qquad f = \frac{1}{Ku}\frac{\partial(Ku)}{\partial s}$$

| Number of MIs | The choice of Bases |
|---|---|
| $\nu_{(42)} = 2$ | $\hat{e}_1^{(42)} = \dfrac{1}{z_2 z_4}, \quad \hat{e}_2^{(42)} = 1$ |
| $\nu_{(4)} = 2$ | $\hat{e}_1^{(4)} = \dfrac{1}{z_4}, \quad \hat{e}_2^{(4)} = 1$ |

Differential Equation

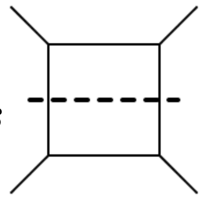



$$a_1 = \sum_{j=1}^{2} \langle \varphi_{1,3} | e_j^{(42)} \rangle \left(\mathbf{C}_{(42)}^{-1}\right)_{j1} = \frac{(d-6)t - 2s}{2s(s+t)}$$

$$a_2 = \sum_{j=1}^{2} \langle \varphi_{1,3} | e_j^{(42)} \rangle \left(\mathbf{C}_{(42)}^{-1}\right)_{j2} = \frac{2(d-3)}{s^2(s+t)}$$

# One Loop Box : DE

**Cut**$_{\{2,4\}}$

$\partial_s$  $= K \int_{\mathcal{C}} u_{2,4}\, \varphi_{2,4}$ $\qquad \varphi_{2,4} = \hat{\varphi}_{2,4}\, dz_3 \wedge dz_1$

$$u_{2,4} = z_1^{\rho_1} z_3^{\rho_3}\, u(z_1, 0, z_3, 0)$$

$$\hat{\varphi}_{2,4} = \frac{f}{z_1 z_3} \qquad f = \frac{1}{Ku}\frac{\partial(Ku)}{\partial s}$$

Number of MIs

$\nu_{(31)} = 2$

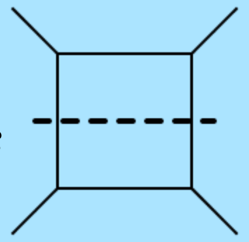
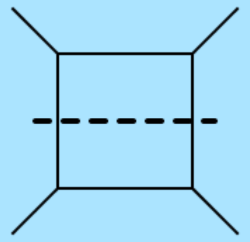
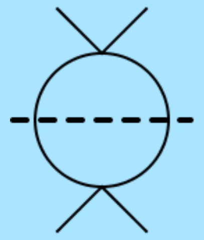
$\nu_{(3)} = 2$

The choice of Bases

$\hat{e}_1^{(31)} = \dfrac{1}{z_1 z_3} \qquad \hat{e}_2^{(31)} = 1$

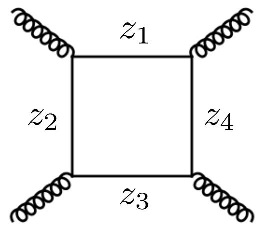
$\hat{e}_1^{(3)} = \dfrac{1}{z_3} \qquad \hat{e}_2^{(3)} = 1$

Differential Equation

$\partial_s$  $= a_1$  $+ a_3$ 

$$a_1 = \sum_{j=1}^{2} \langle \varphi_{1,3} | e_j^{(42)} \rangle \left(\mathbf{C}_{(42)}^{-1}\right)_{j1} = \frac{(d-6)t - 2s}{2s(s+t)}$$

$$a_3 = \sum_{j=1}^{2} \langle \varphi_{2,4} | e_j^{(31)} \rangle \left(\mathbf{C}_{(31)}^{-1}\right)_{j2} = -\frac{2(d-3)}{st(s+t)}$$

| Integral family | Denominators |
|---|---|
|   $s = (p_1+p_2)^2, \quad t = (p_2+p_3)^2$ | $z_1 = k^2 - m_1^2$<br>$z_2 = (k+p_1)^2 - m_2^2$<br>$z_3 = (k+p_1+p_2)^2 - m_3^2$<br>$z_4 = (k+p_1+p_2+p_3)^2 - m_4^2$ |

| $\tau$ | $\nu$ | $e$ |
|---|---|---|
| $z_4 = 0$ | $\nu_{\{3\}} = 2$ | $e^{(3)} = \left\{1, \frac{1}{z_3}\right\}$ |
| | $\nu_{\{32\}} = 3$ | $e^{(32)} = \left\{\frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_2 z_3}\right\}$ |
| | $\nu_{\{321\}} = 6$ | $e^{(321)} = \left\{1, \frac{1}{z_2}, \frac{1}{z_1 z_2}, \frac{1}{z_1 z_3}, \frac{1}{z_2 z_3}, \frac{1}{z_1 z_2 z_3}\right\}$ |
| $z_3 = 0$ | $\nu_{\{4\}} = 2$ | $e^{(4)} = \left\{1, \frac{1}{z_4}\right\}$ |
| | $\nu_{\{41\}} = 3$ | $e^{(41)} = \left\{\frac{1}{z_1}, \frac{1}{z_4}, \frac{1}{z_1 z_4}\right\}$ |
| | $\nu_{\{412\}} = 6$ | $e^{(412)} = \left\{1, \frac{1}{z_1}, \frac{1}{z_1 z_2}, \frac{1}{z_1 z_4}, \frac{1}{z_2 z_4}, \frac{1}{z_1 z_2 z_4}\right\}$ |
| $z_2 = 0$ | $\nu_{\{4\}} = 2$ | $e^{(4)} = \left\{1, \frac{1}{z_4}\right\}$ |
| | $\nu_{\{43\}} = 3$ | $e^{(43)} = \left\{\frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_3 z_4}\right\}$ |
| | $\nu_{\{431\}} = 6$ | $e^{(431)} = \left\{1, \frac{1}{z_4}, \frac{1}{z_1 z_3}, \frac{1}{z_1 z_4}, \frac{1}{z_3 z_4}, \frac{1}{z_1 z_3 z_4}\right\}$ |
| $z_1 = 0$ | $\nu_{\{4\}} = 2$ | $e^{(4)} = \left\{1, \frac{1}{z_4}\right\}$ |
| | $\nu_{\{43\}} = 3$ | $e^{(43)} = \left\{\frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_3 z_4}\right\}$ |
| | $\nu_{\{432\}} = 6$ | $e^{(432)} = \left\{1, \frac{1}{z_3}, \frac{1}{z_2 z_3}, \frac{1}{z_2 z_4}, \frac{1}{z_3 z_4}, \frac{1}{z_2 z_3 z_4}\right\}$ |