Demystifying the two-pole structure

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Based on: Arxiv:2008.02684,Arxiv:2008.08002,ZYZ&ZX

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Outline

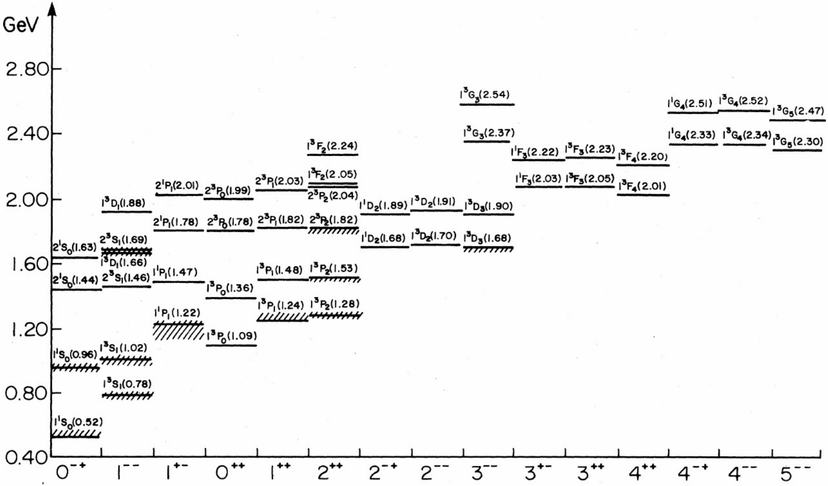
Quark model

Quark Potential Model

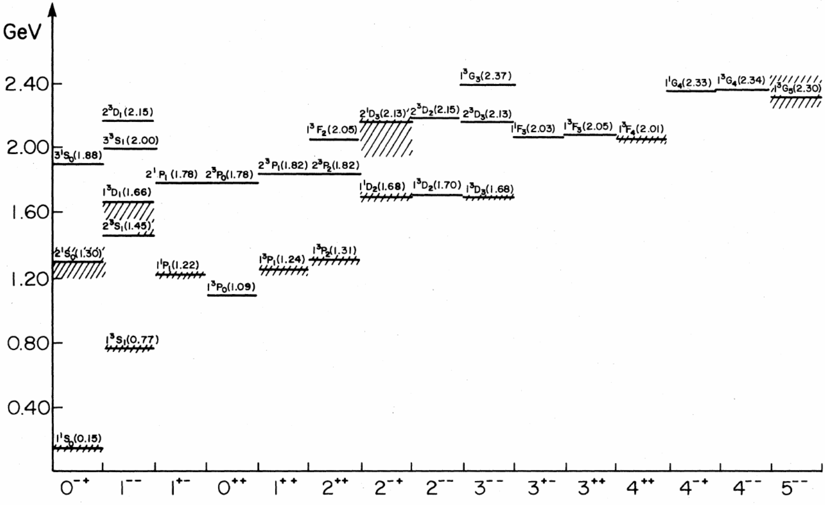
One Gluon exchange potential plus confinement (*Godfrey-Isgur, PRD 32,189(1985)*):

$$H = H_0 + V, \quad V = H_{ij}^{conf} + H_{ij}^{hyp} + H_{ij}^{so} + H_A$$

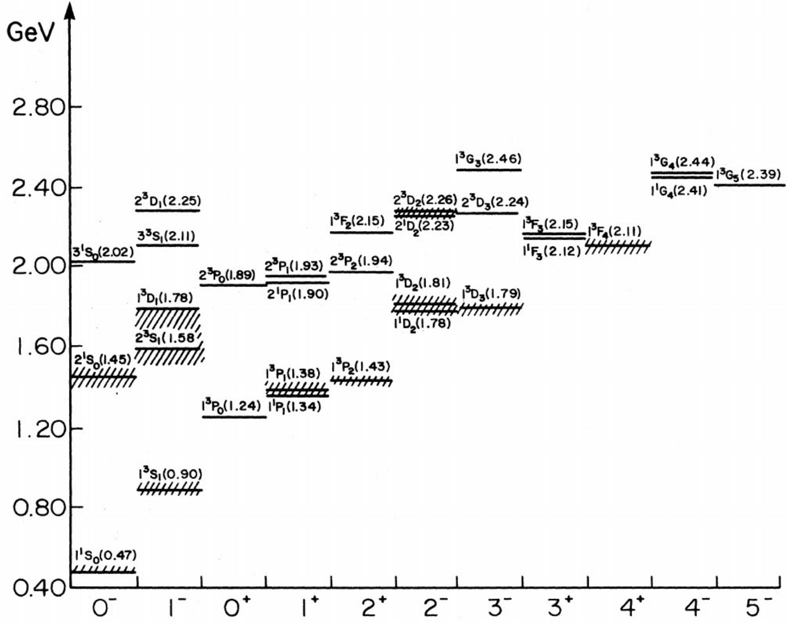
- Pros: Including the relativistic correction, it can describe the light meson spectra as well as heavy ones.
- Cons: The low lying 0^+ spectrum, σ, κ, $f_0(980)$, $a_0(980)$, and states above the open flavor threshold.
- Reason: it does not include the effects of the interaction between hadrons in the spectrum



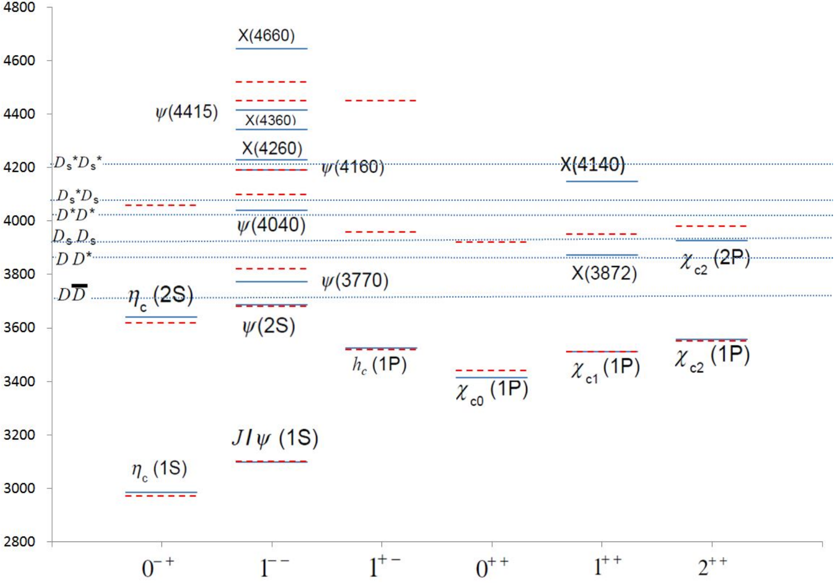
- The lowest 0^{++} isoscalar is 1.09GeV.
- No $\sigma(500)$.



- The lowest 0^{++} isovector is $1.09 GeV$.
- No $a_0(980)$.


GeV
2.80
2.40
2.00
1.60
1.20
0.80
0.40
0⁻ 1⁻ 0⁺ 1⁺ 2⁺ 2⁻ 3⁻ 3⁺ 4⁺ 4⁻ 5⁻
$1^1S_0(0.47)$
$2^1S_0(1.45)$
$3^1S_0(2.02)$
$1^3S_1(0.90)$
$2^3S_1(1.58)$
$1^3D_1(1.78)$
$3^3S_1(2.11)$
$2^3D_1(2.25)$
$1^3P_0(1.24)$
$2^3P_0(1.89)$
$1^3P_1(1.38)$
$1^1P_1(1.34)$
$2^3P_1(1.93)$
$2^1P_1(1.90)$
$1^3P_2(1.43)$
$2^3P_2(1.94)$
$1^3F_2(2.15)$
$1^3D_2(1.81)$
$1^1D_2(1.78)$
$2^3D_2(2.26)$
$2^1D_2(2.23)$
$1^3D_3(1.79)$
$2^3D_3(2.24)$
$1^3G_3(2.46)$
$1^3F_3(2.15)$
$1^1F_3(2.12)$
$1^3F_4(2.11)$
$1^3G_4(2.44)$
$1^1G_4(2.41)$
$1^3G_5(2.39)$

- No κ .



red ones are GI's prediction.

Unitarized quark model

Ann.Phys.123(1979),1; NPB203(1982)268; PRL,49(1982),624; Z.Phys,C61(1994),525, Törnqvist;Z.Phys.C30, 615(1986),Beveren et al. ;PRD83(2011),014010,Zhou&ZX

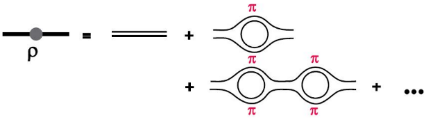
- Assume that $q\bar{q}$ bare bound states (seeds) generated by QCD are coupled to the pseudoscalar mesons.
- Take into account the hadron loop effect in the propagators of the bare states.

$$P = \frac{1}{m_0 - s + \Pi(s)},$$

$$\mathrm{Re}\Pi(s) = \frac{1}{\pi}\mathcal{P}\int_{s_{th}}^{\infty} dz\, \frac{\mathrm{Im}\Pi(z)}{z-s}$$

$$\mathrm{Im}\Pi(s) = -\sum_i G(s)^2 = -\sum_i g_i^2 \frac{k_i(s)}{\sqrt{s}} F_i(s)^2 \theta(s - s_{th,i})$$

$F_i(s) = \exp[-k_i^2(s)/2k_0^2]$ form factor.



Unitarized quark model

- Coupled channel effect: α, β the bare states, i, j diffferent channels

$$
\begin{aligned}
T_{ij} &= \sum_{\alpha,\beta} G_{i\alpha} P_{\alpha\beta} G^*_{j\beta}, \\
\{P^{-1}\}_{\alpha\beta}(s) &= (m^2_{0,\alpha} - s)\delta_{\alpha\beta} + \Pi_{\alpha\beta}(s), \\
\mathrm{Im}\Pi_{\alpha\beta}(s) &= -\sum_i G_{i\alpha}(s) G^*_{\beta i}(s) = -\sum_i g_{\alpha i} g_{\beta i} \frac{k_i(s)}{\sqrt{s}} F^2_i(s)\theta(s - s_{th,i}), \\
\mathrm{Re}\Pi_{\alpha\beta}(s) &= \frac{1}{\pi}\mathcal{P}\int_{s_{th}}^{\infty} dz\, \frac{\mathrm{Im}\Pi_{\alpha\beta}(z)}{z - s}
\end{aligned}
$$

- There can be mixing between α and β states by coupling to the same channel.
- This T matrix satisfies the Unitarity automatically.

SPP INTERACTION

- A simple model describing the Scalar-Pseudoscalar-Pseudoscalar interaction including the OZI violation interaction:

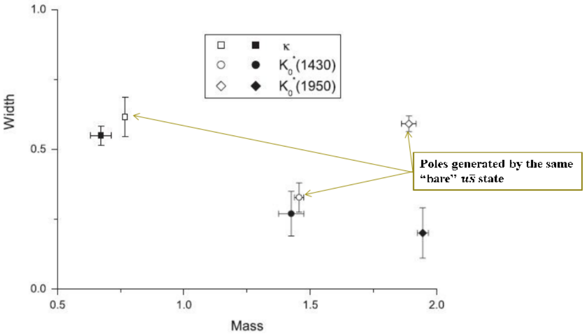
$$\mathcal{L}_{SPP} = \alpha\, Tr[SPP] + \beta\, Tr[S]\, Tr[PP] + \gamma\, Tr[S]\, Tr[P]\, Tr[P].$$

$$S = \begin{pmatrix} \frac{a^0+f_n}{\sqrt{2}} & a^+ & \kappa^+ \\ a^- & \frac{(-a^0+f_n)}{\sqrt{2}} & \kappa^0 \\ \kappa^- & \bar{\kappa}^0 & f_s \end{pmatrix}, P = \begin{pmatrix} \frac{\sqrt{3}\pi^0+\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{-\sqrt{3}\pi^0+\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta_8 \end{pmatrix} + \frac{\eta_1}{\sqrt{3}},$$

$f_n = n\bar{n} \equiv (u\bar{u} + d\bar{d})/\sqrt{2}$ and $f_s = s\bar{s}$.

- Also consider the isospin breaking effect and $\eta\eta'$ mixing.
- Combined fit of three cases: $I = \frac{1}{2}$, $K\pi$; $I = 0$, $\pi\pi$; $I = 1$, $\pi\eta$ S-wave scattering.

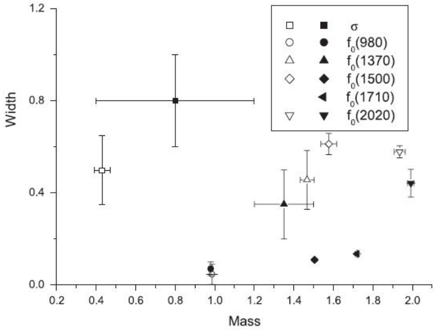
Numerical results



- $I = 1/2$, $K\pi$ S-wave: three poles generated by $u\bar{s}$ state

$$\sqrt{s^{II}} = 0.767_{\pm 0.009} - i0.308_{\pm 0.035} \quad (\kappa, \text{Dynamical}),$$
$$\sqrt{s^{III}} = 1.456_{\pm 0.018} - i0.164_{\pm 0.026}, \quad (K_0^*(1430)\text{from seed, shadow})$$
$$\sqrt{s^{IV}} = 1.890_{\pm 0.029} - i0.296_{\pm 0.014}, \quad (K_0^*(1950)\text{from seed, shadow})$$

Numerical results

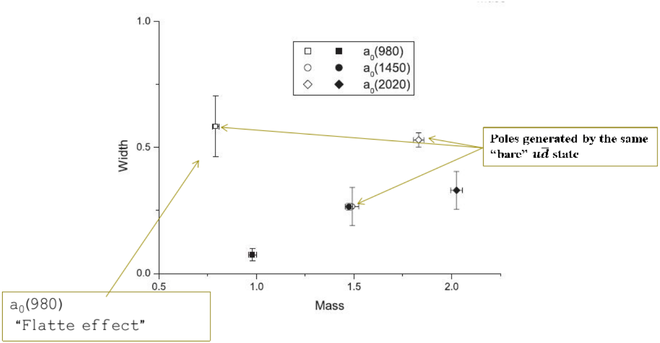


- $I = 0$, all poles generated by $n\bar{n}$ and $s\bar{s}$ have the corresponance in PDG

$$\sqrt{s^{II}} = 0.430_{\pm 0.040} - i0.249_{\pm 0.075}, \quad (\sigma(500)), \quad \sqrt{s^{II}} = 0.986_{\pm 0.015} - i0.023_{\pm 0.022}, \quad (f_0(980)),$$
$$\sqrt{s^{IV}} = 1.467_{\pm 0.035} - i0.228_{\pm 0.064}, \quad (f_0(1370)), \quad \sqrt{s^{V}} = 1.577_{\pm 0.040} - i0.306_{\pm 0.023}, \quad (f_0(1500)$$
$$\sqrt{s^{VI}} = 1.935_{\pm 0.028} - i0.289_{\pm 0.013}, \quad (f_0(2020)),$$

- $f_0(1370)$: PDG , $1200 - 1500$ MeV, Belle 1.47GeV (PRD78,052004).
- Not including $\rho\rho$, 4π effects.

Numerical results

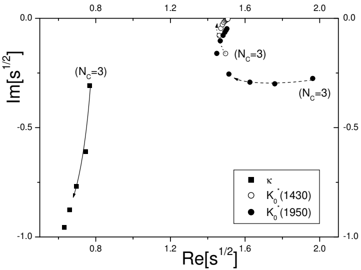


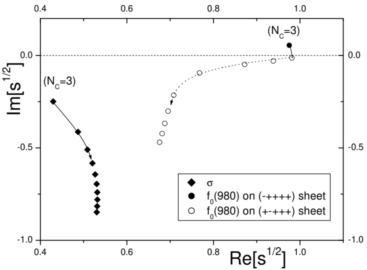
- Poles:

$$\sqrt{s^{II}} = 0.792_{\pm 0.015} - i0.292_{\pm 0.060}, \quad (a_0(980))$$
$$\sqrt{s^{III}} = 1.491_{\pm 0.034} - i0.133_{\pm 0.038}, \quad (a_0(1450))$$
$$\sqrt{s^{IV}} = 1.831_{\pm 0.027} - i0.265_{\pm 0.014}, \quad .$$

- $a_0(1830)$ predicted, maybe related to $a_0(1950)(a_0(2020))$ seen by Brystal Barrel Collab. (PLB452,173)

Large N_c trajectories of resonances





- σ,κ, $a_0(980)$,$f_0(980)$ move farther away from the real axis.
- σ,κ, $a_0(980)$, $f_0(980)$: Dynamically generated by the interaction between seeds and pseudoscalars.

What we learned

- A whole picture: $q\bar{q}$ interaction with two particle continuum $\rightarrow$ dynamically generated state.
- Pole from Seed + Dynamically generated pole: Two-Pole structures.
- σ,κ, $a_0(980)$, $f_0(980)$ nonet are dynamically generated different from the interaction of seeds and continua.
- Shadow poles in different sheets from the same seed.

Problems

The model is too simple

- Seed positions are fitted: should be predicted by quark model – GI model.
- The form factor is put by hand, couplings are fitted: should come from dynamics.
- How to understand the dynamically generated state: the mechanism.
- Dynamically generated states: relation with the bare states and continuum states?

Quark pair creation model & Form factor

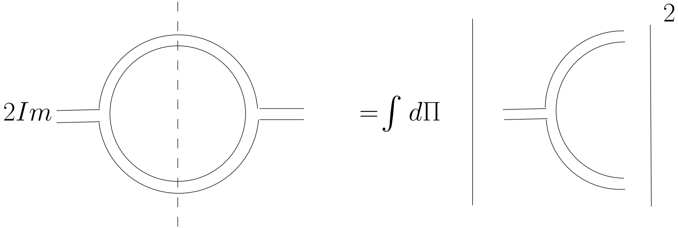
Bare from GI & Dynamical Form-factor, QPC

PRD84,(2011)034023, EPJA,50(2013),125 ZYZ, ZX

Consider the inverse propagator: for $q_1\bar{q}_2$ meson, m_0 bare mass

$$\mathbb{P}^{-1}(s) = m_0^2 - s + \Pi(s) = m_0^2 - s + \sum_n \Pi_n(s),$$

$$\mathrm{Re}\Pi_n(s) = \frac{1}{\pi}\mathcal{P}\int_{s_{th,n}}^{\infty} \mathrm{d}z \frac{\mathrm{Im}\Pi_n(z)}{(z-s)},$$



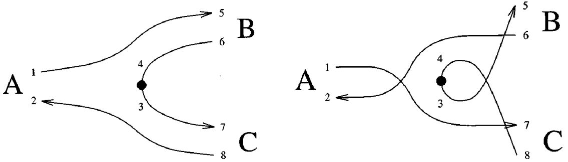
- Bare mass from GI: Corrections to GI spectrum
- Dynamically form factor: QPC(3P0) model, nonrelativistic, only for heavy meson. (PRD29(1984),110,Törnqvist)

Dynamical Form factor: QPC

Three-vertex: Quark Pair Creation model (3P0), creation of a quark-antiquark pair ($q_3\bar{q}_4$) from the vacuum, (PRD53(1996),3700,Blundell,Godfrey)

$$T = -3\gamma \sum_m \langle 1m1-m|00\rangle \int d^3\vec{p_3}\, d^3\vec{p_4}\, \delta^3(\vec{p_3}+\vec{p_4}) \mathcal{Y}_1^m\left(\frac{\vec{p_3}-\vec{p_4}}{2}\right) \chi^{34}_{1-m}\phi^{34}_0\omega^{34}_0 b_3^\dagger(\vec{p_3}) d_4^\dagger(\vec{p_4}),$$

$$\langle BC|T|A\rangle = \delta^3(\vec{P}_f-\vec{P}_i)\mathcal{M}^{M_{J_A} M_{J_B} M_{J_C}}.$$



Once subtracted dispersion relation:

$$\mathbb{P}^{-1}(s) = m^2_{pot} - s + \sum_n \frac{s-s_0}{\pi} \int_{s_{th,n}}^{\infty} \mathrm{d}z \frac{\mathrm{Im}\Pi_n(z)}{(z-s_0)(z-s)},$$

- Wave function: SHO
- We discussed the D, D_s (PRD84,(2011)034023), and $c\bar{c}$ spectra (EPJA,50(2013),125 ZYZ, ZX).

Nonrelativistic Friedrichs Model

Understanding the dynamical pole:Friedrichs model[Commun. Pure Appl. Math.,1(1948),361, Friedrichs]

$$H = H_0 + V$$

- One bare state $|1\rangle$ and a continuum state $|\omega\rangle$: free Hamiltonian

$$H_0 = \omega_0 |1\rangle\langle 1| + \int_0^\infty \omega |\omega\rangle\langle\omega| \mathrm{d}\omega$$

- Interaction:

$$V = \lambda \int_0^\infty [f(\omega)|\omega\rangle\langle 1| + f^*(\omega)|1\rangle\langle\omega|]\mathrm{d}\omega$$

This model is exactly solvable.

Solutions:

Eigenvalue equation:

$$H|\Psi(E)\rangle = (H_0 + V)|\Psi\rangle = E|\Psi(E)\rangle.$$

- Continuum: Eigenvalue $E > 0$, real Solution: define

$$\eta^{\pm}(E) = E - \omega_0 - \lambda^2 \int_0^{\infty} \frac{f(\omega) f^*(\omega)}{E - \omega \pm i\epsilon} \mathrm{d}\omega$$

$$|\Psi_{\pm}(E)\rangle \quad = \quad |E\rangle + \lambda \frac{f^*(E)}{\eta^{\pm}(E)} \left[|1\rangle + \lambda \int_0^{\infty} \frac{f(\omega)}{E - \omega \pm i\epsilon} |\omega\rangle \mathrm{d}\omega \right]$$

- S-matrix:

$$S(E, E') = \delta(E - E') \left(1 - 2\pi i \frac{\lambda f(E) f^*(E)}{\eta^+(E)} \right).$$

- Discrete states:The zero point of $\eta(E)$ corresponds to eigenvalues of the full Hamiltonian — discrete states.

Discrete state solutions:Bound states

$$\eta^I(E) = E - \omega_0 - \lambda^2 \int_0^\infty \frac{f(\omega)f^*(\omega)}{E-\omega} \mathrm{d}\omega = 0$$
$$\eta^{II}(E) = \eta^I(z) - 2i\pi G(z), \quad G \equiv \lambda^2 f(E) f^*(E)$$

- Bound states: solutions on the first sheet real axis below the threshold.
$$|z_B\rangle = N_B\Big(|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{z_B - \omega}|\omega\rangle \mathrm{d}\omega\Big)$$
where $N_B = (\eta'(z_B))^{-1/2} = (1 + \lambda^2 \int d\omega \frac{|f(\omega)|^2}{(z_B-\omega)^2})^{-1/2}$, such that $\langle z_B|z_B\rangle = 1$.
- Elementariness: $Z = N_B^2$;
Compositeness: $X = N_B^2 \lambda^2 \int d\omega \frac{|f(\omega)|^2}{(z_B-\omega)^2}$.
- Eg. If $\omega_0 < 0$, there could be a bound state. In the weak coupling limit, it $\to |1\rangle$,
- Eg. there could also be dynamically generated bound state in the strong coupling.

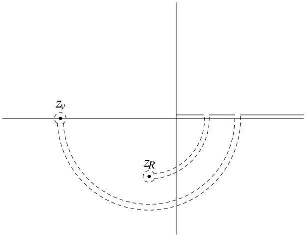
Discrete state solutions: Virtual states

- Virtual states: Solutions on the second sheet real axis below the threshold.

$$|z_v^{\pm}\rangle = N_v^{\pm}\Big(|1\rangle + \lambda \int_0^{\infty} \frac{f(\omega)}{[z_v - \omega]_{\pm}} |\omega\rangle \mathrm{d}\omega\Big), \quad \langle \tilde{z}_v^{\pm}| = \langle z_v^{\mp}|,$$

  where
  $N_v^{-} = N_v^{+*} = (\eta'^{+}(z_v))^{-1/2} = (1 + \lambda^2 \int d\omega \frac{|f(\omega)|^2}{[(z_v - \omega)_+]^2})^{-1/2}$,
  such that $\langle \tilde{z}_v^{\pm}|z_v^{\pm}\rangle = 1$.

- When $\omega_0 < 0$, a bound state generated from $|1\rangle$ is always accompanied with a virtual state in weak coupling.

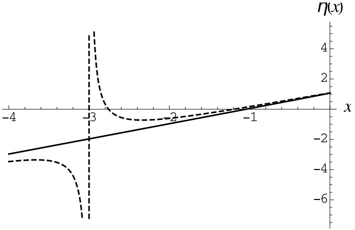


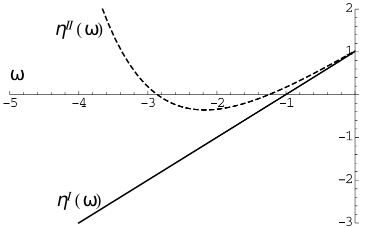
Discrete state solutions: Virtual states

PRD94(2016),076006, ZYZ&ZX

- Dynamical virtual state comes from the singularity of the form factor, analytically continued $G(\omega) = |f(\omega)|^2$:

$$\eta^I = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega} \mathrm{d}\omega$$
$$\eta^{II}(\omega) = \eta^I(\omega) + 2\pi i \lambda^2 \, G^{II}(\omega) = \eta^I(\omega) - 2\lambda^2 \pi i \, G(\omega),$$



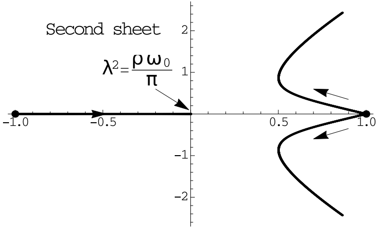


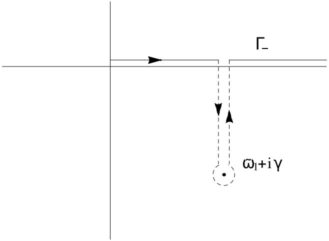
- Virtual state generated from the bare states: $\omega_0 < 0$

Discrete state solutions: Resonance

- Resonant states: $\omega_0 >$ threshold, the discrete state becomes a pair of solutions z_R, z_R^*, on the second sheet of the complex plane. $\hat{H}|z_R\rangle = z_R|z_R\rangle$

$$|z_R\rangle = N_R\Big(|1\rangle + \lambda\int_0^\infty \mathrm{d}\omega\,\frac{f(\omega)}{[z_R-\omega]_+}|\omega\rangle\Big),$$
$$|z_R^*\rangle = N_R^*\Big(|1\rangle + \lambda\int_0^\infty \mathrm{d}\omega\,\frac{f(\omega)}{[z_R^*-\omega]_-}|\omega\rangle\Big),$$





GENERALIZATION: *JMP.58(2017),062110;JMP58(2017), 072102; ZYZ&ZX*

To use this model in the real world

- Include more discrete states and more continua: coupled channel interaction.
- Partial wave decomposition: The angular momentum space states can be expressed in the Friedrichs model.
- Include interaction among continuum: in general not solvable.

$$
\begin{aligned}
H = & \sum_{i=1}^{D} M_i |i\rangle\langle i| + \sum_{i=1}^{C} \int_{M_{i,th}}^{\infty} d\omega_i \omega_i |\omega_i; i\rangle\langle\omega_i; i| \\
& + \sum_{i_2,i_1} \int_{M_{i_1,th}} \mathrm{d}\omega' \int_{M_{i_2,th}} \mathrm{d}\omega\, f_{i_2,i_1}(\omega',\omega)|\omega'; i_2\rangle\langle\omega; i_1| \\
& + \sum_{i=1}^{D}\sum_{j=1}^{C} \int_{M_{j,th}} d\omega\, g_{i,j}(\omega)|i\rangle\langle\omega; j| + h.c.
\end{aligned}
$$

Application: $X(3872)$, *PRD96(2017),054031; ZYZ& ZX*

Application:

- Using the GI's bare mass and wave function as input.
- Using the QPC model to provide the form factor.
- We can calculate the η function and solve $\eta(z) = 0$
- Coupling bare $\chi_{c1}(2P)$ (3953MeV) with DD^*, D^*D^*.
- $X(3872)$ & $X(3940)$ may be two-pole structure:
  $X(3872)$, natrually dynamically generated;
  $\chi_{c1}(2P)$ seed $\rightarrow 3917 - 45i$ MeV $\sim$ may be related to $X(3940)$.
- If $X(3872)$ is a bound state: it has a large portion of DD^*.
- This information helps us in understanding its decay.
  PRD100(2019),094025; PRD97(2018), 034011, ZYZ&ZX
- Also application in BB^* system: the $X(3872)$ counterpart
  PRD99 (2019),034005; ZYZ& ZX

Problems

- Nonrelativistic Friedrichs model: the dispersion integrel is in E not in s as in the relativistic dispersion relation.

$$\eta^+(E) = E - \omega_0 - \lambda^2 \int_0^\infty \frac{f(\omega)f^*(\omega)}{E-\omega+i\epsilon}\mathrm{d}\omega$$

  Possible solution:
  introducing the creation & annilation operators.

- The QPC model is non-relativistic: can not discribe the interaction between low lying mesons with light quarks.

  Possible solution: the relativistic kinetics and Lorentz boosts

Relativistic Generalization: Friedrichs & QPC

Relativistic free two particle states:

JMP4(1963),490, Macfarlane; Nuo.Cim,34,1289,McKerrell

We need the state in the angular momentum representation.

- In c.m. frame, choose $w^2 = (q_1 + q_2)^2$, boost the two particle system to have total 3-momentum $\vec{p}$, $E = (\vec{p}^2 + w^2)^{1/2}$.
- In c.m. frame, choose a z direction, we can define orbital angular momentum l and total spin s, and total angular momentum j, m. Then, we boost it to have 3-momantum $\vec{p}$.

$$|\vec{p}m[wj]ls\rangle = \Big(\frac{w}{2q_1^0 q_2^0}\Big)^{1/2}\Big(\frac{2l+1}{4\pi}\Big)^{1/2}\sum_{l_z s_z} C(lsjl_z s_z m)$$
$$\times \int d\varphi \sin\theta d\theta D^{l*}_{l_z 0}(\varphi,\theta,0)\sum_{\mu_1\mu_2}\sum_{\nu_1,\nu_2} C(s_1 s_2 s \nu_1 \nu_2 s_z) D^{s_1}_{\mu_1\nu_1}(\bar{R}(p_1,\ell(p)))$$
$$\times D^{s_2}_{\mu_2\nu_2}(\bar{R}(p_2,\ell(p)))|p_1\mu_1 p_2\mu_2\rangle$$

- We treat the two particle states together, using an annihilation and creation operator: $B^\dagger_{\vec{p}m[wj]ls}|0\rangle = |\vec{p}m[wj]ls\rangle$

$$[B_{\vec{p}'m'[w'j']l's'}, B^\dagger_{\vec{p}m[wj]ls}] = \delta^{(3)}(\vec{p}-\vec{p}')\frac{\delta(q-q')}{q^2}\delta_{mm'}\delta_{ss'}\delta_{ll'}\delta_{jj'}$$
$$= \beta^{-1}\delta^{(3)}(\vec{p}-\vec{p}')\delta(E-E')\delta_{mm'}\delta_{ss'}\delta_{ll'}\delta_{jj'}, \quad \beta(E) = \frac{q q_1^0 q_2^0 E}{w^2}$$

Relativisitc Friedrichs-Lee model

JMP39(1998),2995,Antoniou; Arxiv:2008.02684,ZYZ&ZX

Hamiltonian:

$$P_0 = \int d^3\mathbf{k}\beta(E)\,dE\,E\,B^\dagger(E,\mathbf{k})B(E,\mathbf{k}) + \int d^3\mathbf{k}\,\omega(\mathbf{k})\,a^\dagger(\mathbf{k})a(\mathbf{k})$$

$$+ \int d^3\mathbf{k}\,\beta(E)\,dE\,\alpha_0(E,\mathbf{k})\left(a(\mathbf{k}) + a^\dagger(-\mathbf{k})\right)\left(B^\dagger(E,\mathbf{k}) + B(E,-\mathbf{k})\right)$$

$$\omega(\mathbf{k}) = \sqrt{m^2+\mathbf{k}^2}, \quad \alpha(E,\mathbf{k}) = \alpha^*(E,-\mathbf{k})$$

α: interaction form factor between the discret state and the continuum.

Eigenvalue problem: find $b^\dagger$ s.t.

$$[H, b^\dagger(E)] = Eb^\dagger(E)$$

Relativisitc Friedrichs-Lee model

Solution:

- Continuum: $E > E_{th}$

$$b^\dagger_{in}(E,\mathbf{p}) = B^\dagger(E,\mathbf{p}) - \frac{2\omega(\mathbf{p})\alpha(k(E,\mathbf{p}))}{\eta_+(E,\mathbf{p})}\Big[\int_{M_{th}} dE'\beta(E')\alpha(k(E',\mathbf{p}))\Big[\frac{B^\dagger(E',\mathbf{p})}{(E'-E-i0)} - \frac{B(E',-\mathbf{p})}{(E'+E+i0)}\Big] - \frac{1}{2\omega(\mathbf{p})}\Big((\omega(\mathbf{p})+E)a^\dagger(\mathbf{p}) - (\omega(\mathbf{p})-E)a(-\mathbf{p})\Big)\Big],$$

$$\eta_\pm(s) = s - \omega_0^2 - \int_{s_{th}} ds' \frac{\rho(s')}{s-s'\pm i0}, \quad \rho = 2\omega_0 \frac{k\varepsilon_1\varepsilon_2}{W}|\alpha(k)|^2$$

- S-matrix

$$S(E,\mathbf{p};E',\mathbf{p}') = \delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta(E-E')\Big(1 - 2\pi i\frac{\rho(s)}{\eta_+(s)}\Big).$$

- Discrete state: at the solution of $\eta(z) = 0$

$$b^\dagger(E_0,\mathbf{p}) = N\Big[\frac{(\omega(\mathbf{p})+E_0)}{\sqrt{2\omega(\mathbf{p})}}a^\dagger(\mathbf{p}) - \frac{(\omega(\mathbf{p})-E_0)}{\sqrt{2\omega(\mathbf{p})}}a(-\mathbf{p}) - \sqrt{2\omega(\mathbf{p})}\int_{M_{th}} dE'\beta(E')\Big[\frac{\alpha(k(E',\mathbf{p}))}{E'-E_0}B^\dagger(E',\mathbf{p}) - \frac{\alpha(k(E',\mathbf{p}))}{E'+E_0}B(E',-\mathbf{p})\Big]\Big],$$

For bound state $N = \frac{1}{\sqrt{2E_0}}\Big[1 + 2\omega(\mathbf{p})\int_{M_{th}} dE'\beta(E')\frac{2E'|\alpha(k(E',\mathbf{p}))|^2}{(E'+E_0)^2(E'-E_0)^2}\Big]^{-1/2}$

Relativisitc QPC

PRC86(2012),055205,Fuda; Arxiv:2008.02684,ZYZ&ZX

- A $q\bar{q}$ bound state can be expressed using a Mock state:

$$|A(\tilde{W},{}^{2s_A+1}l_{A\,j_A m_{j_A}})(\mathbf{p})\rangle = \sum_{m_l m_s}\sum_{\substack{m_1 m_2\\ m_1' m_2'}}\int d^3\mathbf{k}\psi^A_{l_A m_{l_A}}(\mathbf{k})|\mathbf{p}_1, s_1 m_1'\rangle\otimes|\mathbf{p}_2, s_2 m_2'\rangle\phi^{12}_A\omega^{12}_A$$
$$\times D^{s_1}_{m_1' m_1}[r_c(l_c(p),k_1)]D^{s_2}_{m_2' m_2}[r_c(l_c(p),k_2)]\langle s_1 s_2 m_1 m_2|s_A m_{s_A}\rangle\langle l_A s_A m_{l_A} m_{s_A}|j_A m_{j_A}\rangle$$
$$\times\Big(\frac{\varepsilon_1(\mathbf{p}_1)}{\varepsilon_1(\mathbf{k})}\frac{\varepsilon_2(\mathbf{p}_2)}{\varepsilon_2(-\mathbf{k})}\frac{W_{12}(\mathbf{k})}{E_{12}(\mathbf{p},\mathbf{k})}\Big)^{1/2}.$$

- Quark pair creation: $H_I = \gamma\int \mathrm{d}^3x\bar{\psi}(x)\psi(x), \quad t=0,$

$$T = -\sqrt{8\pi}\gamma\int\frac{d^3\mathbf{p}_3\, d^3\mathbf{p}_4}{\sqrt{\varepsilon_3(\mathbf{p}_3)\varepsilon_4(\mathbf{p}_4)}}\delta^{(3)}(\mathbf{p}_3+\mathbf{p}_4)\sum_m\sum_{m_3 m_4}\langle 1,m,1,-m|0,0\rangle$$
$$\times\mathcal{Y}^m_1\big(\frac{\mathbf{p}_3-\mathbf{p}_4}{2}\big)\langle 1/2,m_3,1/2,m_4|1,-m\rangle\phi^{34}_0\omega^{34}_0 b^\dagger_{m_3}(\mathbf{p}_3)d^\dagger_{m_4}(\mathbf{p}_4),$$

- From the matrix element $\langle BC|T|A\rangle$, we obtain form factor α for the Friedrichs model.

Relativistic Friedrichs-QPC scheme

- Relativistic Friedrichs model: Inverse resolvent, $\eta(z)$

$$\eta(s) = s - \omega_0^2 - \int_{s_{th}} ds' \frac{\rho(s')}{s - s'}, \quad \rho = 2\omega_0 \frac{k\varepsilon_1\varepsilon_2}{W}|\alpha(k)|^2$$

  Solve $\eta(z) = 0$, find poles of S-matrix: resonance, bound state, virtual state.
- Relativized quark model: GI, bare mass(a little tuned), wave function
- Relativistic QPC: only one parameter γ
- Spectrum: Broader range, inluding the light meson, and heavy meson together.

Two-pole structures

Two pole structures

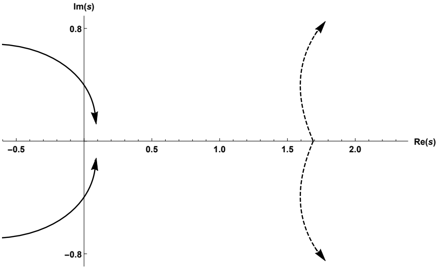
When $\gamma = 4.3$ GeV, Single channel approximation: general appearance of two-pole structures

"discrete"	"continuum"	GI mass	Input	poles	experiment states	PDG values [15]
$\frac{u\bar{u}+d\bar{d}}{\sqrt{2}}(1^3P_0)$	$(\pi\pi)_{I=0}$	1.09	1.3	$\sqrt{s_{r1}} = 1.34 - 0.29i$	$f_0(1370)$	$1.35^{\pm 0.15} - 0.2^{\pm 0.05}i$
				$\sqrt{s_{r2}} = 0.39 - 0.26i$	$f_0(500)$	$0.475^{\pm 0.075} - 0.275^{\pm 0.075}i$
$u\bar{s}(1^3P_0)$	$(\pi K)_{I=\frac{1}{2}}$	1.23	1.42	$\sqrt{s_{r1}} = 1.41 - 0.17i$	$K_0^*(1430)$	$1.425^{\pm 0.05} - 0.135^{\pm 0.04}i$
				$\sqrt{s_{r2}} = 0.66 - 0.34i$	$K_0^*(700)$	$0.68^{\pm 0.05} - 0.30^{\pm 0.04}i$
$s\bar{s}(1^3P_0)$	$K\bar{K}$	1.35	1.68	$\sqrt{s_{r1}} = 1.71 - 0.16i$	$f_0(1710)$	$1.704^{\pm 0.012} - 0.062^{\pm 0.009}i$
				$\sqrt{s_b} = 0.98, \sqrt{s_v} = 0.19$	$f_0(980)$	$0.99^{\pm 0.02} - 0.028^{\pm 0.023}i$
$\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}(1^3P_0)$	$\pi\eta$	1.09	1.3	$\sqrt{s_{r1}} = 1.26 - 0.14i$	$a_0(1450)$	$1.474^{\pm 0.019} - 0.133^{\pm 0.007}i$
				$\sqrt{s_{r2}} = 0.70 - 0.42i$	$a_0(980)$	$0.98^{\pm 0.02} - 0.038^{\pm 0.012}i$
$c\bar{u}(1^3P_0)$	$D\pi$	2.4	2.4	$\sqrt{s_{r1}} = 2.58 - 0.24i$	$D_0^*(2300)$	$2.30^{\pm 0.019} - 0.137^{\pm 0.02}i$
				$\sqrt{s_{r2}} = 2.08 - 0.10i$		
$c\bar{s}(1^3P_0)$	DK	2.48	2.48	$\sqrt{s_{r1}} = 2.80 - 0.23i$		
				$\sqrt{s_b} = 2.24, \sqrt{s_v} = 1.8$	$D_{s0}^*(2317)$	$2.317^{\pm 0.0005} - 0.0038^{\pm 0.0038}i$
$b\bar{u}(1^3P_0)$	$\bar{B}\pi$	5.76	5.76	$\sqrt{s_{r1}} = 6.01 - 0.21i$		
				$\sqrt{s_{r2}} = 5.56 - 0.07i$		
$b\bar{s}(1^3P_0)$	$\bar{B}K$	5.83	5.83	$\sqrt{s_{r1}} = 6.23 - 0.17i$		
				$\sqrt{s_b} = 5.66, \sqrt{s_v} = 5.3$		
$c\bar{c}(2^3P_1)$	$D\bar{D}^*$	3.95	3.95	$\sqrt{s_{r1}} = 4.01 - 0.049i$	$X(3940)$	
				$\sqrt{s_b} = 3.785$	$X(3872)$	$3.87169^{\pm 0.00017}$

Two-pole structures

Two pole structure, a general phenomenon:
when the coupling γ is turned on

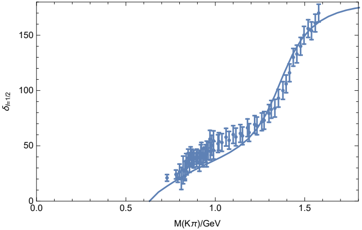
- Coupling a seed $q\bar{q}$ state with the nearest open flavor state in S-wave — another new dynamical state ("dynamical pole").
- The seed will move into the second sheet — a pair of resonance poles ("bare pole").
- The dynamical pole come from far away on the second sheet towards the real axis: Resonance or virtual state or /and bound state poles.

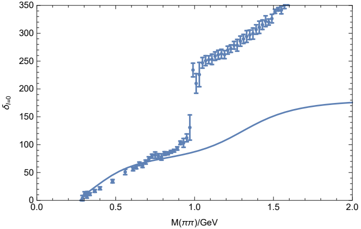


Two pole structures: light scalars, phase shift sum rule

Seeds $\frac{u\bar{u}+d\bar{d}}{\sqrt{2}}$, $u\bar{s}$, $s\bar{s}$, $\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$

- $f_0(500)/\sigma$, $f_0(1370)$; $K_0^*(700)/\kappa$, $K_0^*(1430)$; $f_0(980)$,$f_0(1710)$; $a_0(980)$, $a_0(1450)$ are two pole structures.
- Phase: one single channel approximation is good, two poles contribute a 180° phase shift.





As $s \to \infty$, $T \propto \sin\delta\, e^{i\delta} \sim \frac{\rho(s)}{\eta_+(s)} \to 0$, $\delta \to n \times 180^\circ$,
Here $n = 1$.

$c\bar{u}$ SEED, $b\bar{u}$ SEED

- $c\bar{u}$ seed couples to $D\pi$: $D_0^*(2300)$, two broad poles

$$\gamma = 4.3: \quad \sqrt{s_1} = 2.08 - i0.10; \quad \sqrt{s_2} = 2.58 - i0.24$$

$$\gamma = 3: \quad \sqrt{s_1} = 2.21 - i0.28; \quad \sqrt{s_2} = 2.39 - i0.18$$

- Two-poles From Unitarized χPT: $D_0^*(2300)$, two poles

  PLB582(2004),39,EEK et.al; PLB641(2006),278, FK.Guo, et. al.; PLB,767(2017),465, MA,et.al.:

$$\sqrt{s_1} = 2.105 - i0.102; \quad \sqrt{s_2} = 2.451 - i0.134$$

  PRD92(2015),094008,ZH.Guo et.al.:

$$\sqrt{s_1} = 2.114 - i0.111; \quad \sqrt{s_2} = 2.473 - i0.140$$

- $b\bar{u}$ couples to $\bar{B}\pi$:

$$\gamma = 4.3: \sqrt{s_1} = 5.556 - i0.07; \quad \sqrt{s_2} = 6.01 - i0.21$$

$$\gamma = 3.0: \quad \sqrt{s_1} = 5.62 - i0.13; \quad \sqrt{s_2} = 5.85 - i0.26$$

  Unitarized χPT:

$$\sqrt{s_1} = 5.537 - i0.116; \quad \sqrt{s_2} = 5.840 - i0.025$$

$c\bar{s}$, $b\bar{s}$ SEEDS

- $c\bar{s}$ couples to DK: $D^*_{s0}(2317)$, dynamically generated;

$$\gamma = 4.3 : \sqrt{s_b} = 2.24, \quad \sqrt{s_v} = 1.8, \quad \sqrt{s_{r1}} = 2.80 - 0.23i$$

$$\gamma = 3.0 : \sqrt{s_b} = 2.32, \quad \sqrt{s_v} = 1.9, \quad \sqrt{s_{r1}} = 2.68 - 0.26i$$

- $b\bar{s}$ couples to $\bar{B}K$:

$$\gamma = 4.3 : \quad \sqrt{s_b} = 5.66, \quad \sqrt{s_v} = 5.3, \quad \sqrt{s_{r1}} = 6.23 - 0.17i$$

$$\gamma = 3.0 : \quad \sqrt{s_b} = 5.72, \quad \sqrt{s_v} = 5.4, \quad \sqrt{s_{r1}} = 6.11 - 0.22i$$

$X(3872)$

- $c\bar{c}(2^3P_1)$ couples to $D\bar{D}^*$: $X(3872)$ dynamically generated

$$\gamma = 4.3: \quad \sqrt{s_b} = 3.785, \quad \sqrt{s_{r1}} = 4.01 - 0.049i$$

$$\gamma = 3.0: \quad \sqrt{s_b} = 3.84, \quad \sqrt{s_r} = 3.99 - 0.045i$$

General features of the two pole structure:

Coupling of a seed with a continuum: dynamically generate a new state

- Nontrivial form factor: Scattering of mesons, composite of $q\bar{q}$. Non-local interaction.
- The dynamically generated state may come from far away from the seed: in general from the singularity of the form factor
- If single channel approximation is applicable, the two poles together may roughly contribute a phase shift of 180°.
- Whether the dynamical state is a bound state, virtual state or resonance depend on the specific wave function of the particles in the interation.
- Conjecture: for S-wave coupling of $q\bar{q}$ with the nearest open-flavor continuum, the two-pole structure may be near the physical region and it is highly possible to have observable effect in the experiments.

Conclusion & Outlook

- Coupling a seed with the continuum when all particles are composite may in general generate a dynamically new state.
- We combine the relativistic Friedrichs model and the relativistic QPC model : this scheme can be applied to both light mesons and heavy mesons.
- $f_0(500)/f_0(1370)$, $f_0(980)/f_0(1710)$, $K_0^*(700)/K_0^*(1430)$, $a_0(980)/a_0(1450)$, $X(3872)/X(3940)$, all result from the two-pole mechanism.
- Prediction: $D_0^*(2210)/D_0^*(2390)$, $D_{s0}^*(2317)/D_{s0}^*(2680)$, $B_0^*(5620)/B_0^*(5850)$, $B_{s0}^*(5720)/B_{s0}^*(6110)$. $B_{s0}^*(5720)$ should be very narrow.
- There could be other two-pole structures to be discovered.
- Coupled channel generalization.
- This mechanism may be much more general beyond hadron physics.

Thanks !

Appendix

S-Matrix Unitarity

- S-Matrix: $S_{\beta\alpha} = \langle \beta_{out} | \alpha_{in} \rangle$
- Unitarity: $SS^\dagger = 1$, $\quad S = 1 + iT$

$$1 = (1 - iT^\dagger)(1 + iT) = 1 + iT - iT^\dagger + T^\dagger T \quad \Rightarrow \quad -i(T - T^\dagger) = T^\dagger T$$

- For initial k_1, k_2 particle,

$$\langle \{\vec{q}_i\} | T | \vec{k}_1 \vec{k}_2 \rangle = (2\pi)^4 \delta^{(4)}\big(k_1 + k_2 - \sum q_i\big) \mathcal{M}(k_1, k_2 \to \{q_i\})$$

- Unitarity: forward scattering

$$2\mathrm{Im}\mathcal{M}(a \to a) = \sum_f \int d\Pi_f \mathcal{M}^*(a \to f) \mathcal{M}(a \to f)$$

- Optical theorem:

$$\mathrm{Im}\mathcal{M}(k_1, k_2 \to k_1, k_2) = 2E_{cm} p_{cm} \sigma_{\text{tot}}$$

Partial wave amplitude and unitarity

- Partial wave decomposition for spinless particles amplitude: $2 \to 2$ amplitude

$$A_l(s) = \frac{1}{32\pi}\int_{-1}^{1} d\cos\theta\, P_l(\cos\theta) A(s, t(s,\cos\theta))$$

$$A(s,t,u) = 16\pi \sum_l (2l+1) P_l(\cos\theta) A_l(s)$$

- Partial wave S-matrix: $S_l = 1 + 2i\rho(s)A_l$, $S_l(s)S_l^*(s) = 1$, $\rho(s) = 2k/E$.
- Partial-wave Unitarity:

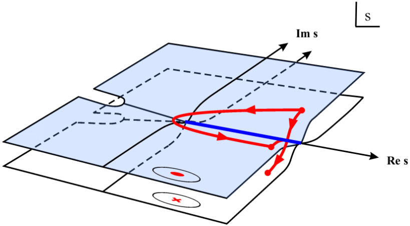
$$\mathrm{Im}A(s) = \frac{1}{2i}(A(s+i\epsilon) - A(s-i\epsilon)) = \rho(s)|A(s)|^2$$

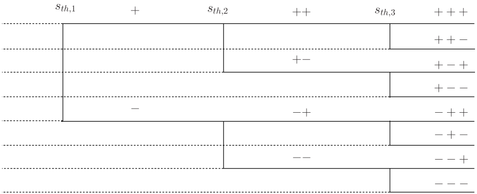
- Coupled channel: $S_{ij} = \delta_{ij} + 2i\sqrt{\rho_i}A_{ij}\sqrt{\rho_j}$

$$\mathrm{Im}A_{ij} = \sum_k A_{ik}\rho_k(s)A_{kj}^*, \quad \text{for on-shell internal } k \text{ states}$$

Pole, zero of S-matrix and states

The partial wave S-matrix can be analytically continued to the complex s-plane:





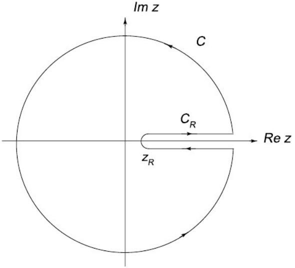
- Unitary cut: $s > s_{th,j}$
- Single channle: $S^I = \frac{1}{S^{II}(s)}$
- First sheet zero $\leftrightarrow$ second sheet pole $\rightarrow$ virtual state, or resonance $s = (M - i\Gamma/2)^2$.
- First sheet pole: on the real axis below threshold, — bound state.
- Coupled channel: Riemann sheets doubled

Dispersion relation

The amplitude $A(s)$ is a real analytic function after analytic continuation:

- Real analytic function: $A(z^*) = A^*(z)$
- Unitarity: $\mathrm{Im}A(s) \neq 0$, for $s > z_R \Rightarrow$ right hand cut
- Cauchy theorem: if $A(s)$ converges at infinity, no poles on the first sheet

$$A(z) = \frac{1}{2\pi i}\int_C dz' \frac{A(z')}{z'-z}$$
$$= \frac{1}{\pi}\int_{z_R} dz' \frac{\mathrm{Im}\,A(z')}{z'-z}$$



This is called dispersion relation.

- Once subtracted dispersion relation:

$$A(z) = A(z_0) + \frac{z-z_0}{\pi}\int_{Z_R}^{\infty} dz' \frac{\mathrm{Im}\,A(z')}{(z'-z_0)(z'-z)}$$