Small Scale Anisotropies in Slab Turbulence

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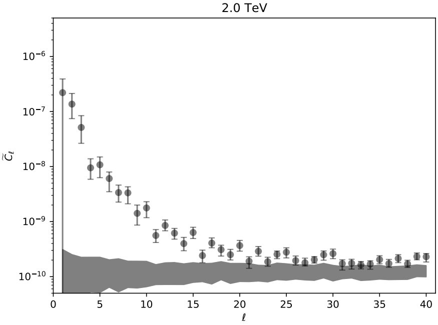
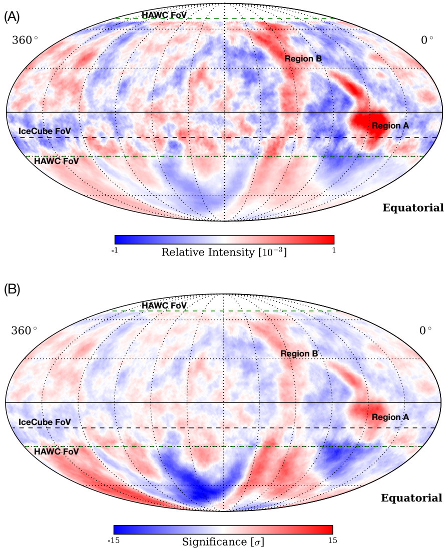
Motivation

- High statistics observatories have observed anisotropies down to small angular scales.
- The standard theory of cosmic ray transport can not account for these anisotropies.





Abeysekara *et al.*, ApJ 871 (2019) 96

Abeysekara *et al.*,ApJ 865 (2018) 57

Small-scale turbulence and ensemble averaging

- In standard diffusion, compute C_ℓ from $\langle f \rangle$:

$$C_\ell^{\text{std}} = \frac{1}{4\pi} \int \mathrm{d}\hat{\mathbf{p}}_1 \int \mathrm{d}\hat{\mathbf{p}}_2 \, P_\ell(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \langle f(\hat{\mathbf{p}}_1) \rangle \langle f(\hat{\mathbf{p}}_2) \rangle$$

- However, in an individual realisation of δB, $\delta f = f - \langle f \rangle \neq 0$

$$\langle C_\ell \rangle = \frac{1}{4\pi} \int \mathrm{d}\hat{\mathbf{p}}_1 \int \mathrm{d}\hat{\mathbf{p}}_2 \, P_\ell(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle$$

- If $f(\hat{\mathbf{p}}_1)$ and $f(\hat{\mathbf{p}}_2)$ are correlated,

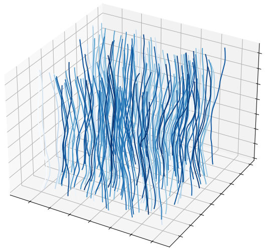
$$\langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle \geq \langle f(\hat{\mathbf{p}}_1) \rangle \langle f(\hat{\mathbf{p}}_2) \rangle \quad \Rightarrow \quad \langle C_\ell \rangle \geq C_\ell^{\text{std}}$$

Source of the small scale anisotropies?

Giacinti & Sigl, PRL **109** (2012) 071101

Ahlers, PRL **112** (2014) 021101, Ahlers & Mertsch, ApJL 815 (2015) L2, Pohl & Rettig, *Proc. 36th ICRC* (2016) 451, López-Barquero *et al.*, ApJ 830 (2016) 19, López-Barquero *et al.* ApJ 842 (2017) 54

Numerical Simulations



- Track particles back through synthetic magnetic field turbulence by solving the Newton Lorentz equation

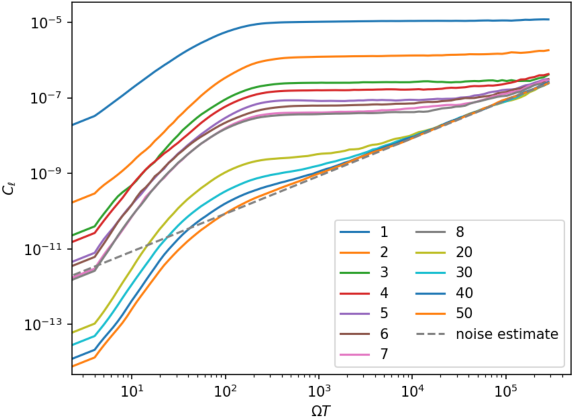
$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

- Exploit Liouville's theorem (phase-space density is conserved along trajectories)

$$\mathrm{d}f = 0 \quad \Rightarrow \quad f(\mathbf{x} = \mathbf{x}_\oplus, \mathbf{p}_i, t) = f_{\text{ini}}(\mathbf{x}_i(t_0), \mathbf{p}_i(t_0)).$$

- Assume isotropic but inhomogeneous initial state $f_{\text{ini}}(\mathbf{x}, \mathbf{p})$.

Convergence of the Angular Power Spectrum



- For large backtracking times a steady state is reached.
- Noise becomes important for large ℓ. $\mathcal{N} \propto T/N_{\text{part}} \Rightarrow$ Large number of particles needed.

Gradient ansatz

Mertsch & Ahlers (2019)

- Vlasov equation:

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \nabla_{\mathbf{r}} f + q\mathbf{v} \times (\langle \mathbf{B} \rangle + \delta \mathbf{B}) \cdot \nabla_{\mathbf{p}} f \\ &\simeq \frac{\partial f}{\partial t} + \underbrace{(\hat{\mathbf{p}} \cdot \nabla_{\mathbf{r}} + q(\hat{\mathbf{p}} \times \langle \mathbf{B} \rangle) \cdot \nabla_{\mathbf{p}})}_{\mathcal{L}} f + \underbrace{(q(\hat{\mathbf{p}} \times \delta \mathbf{B}) \cdot \nabla_{\mathbf{p}})}_{\delta \mathcal{L}} f = 0\end{aligned}$$

- Gradient ansatz:

$$f(\mathbf{r}, \hat{\mathbf{p}}) = f_{\oplus}(\hat{\mathbf{p}}) + (\mathbf{r}_{\oplus} - \mathbf{r}) \cdot \mathbf{G}\,,$$

→ Dipolar source term in the Vlasov equation:

$$\frac{\partial f_{\oplus}}{\partial t} + \underbrace{(q(\hat{\mathbf{p}} \times \langle \mathbf{B} \rangle) \cdot \nabla_{\mathbf{p}})}_{\mathcal{L}'} f_{\oplus} + \underbrace{(q(\hat{\mathbf{p}} \times \delta \mathbf{B}) \cdot \nabla_{\mathbf{p}})}_{\delta \mathcal{L}} f_{\oplus} = \hat{\mathbf{p}} \cdot \mathbf{G}$$

Mixing matrices

Mertsch & Ahlers (2019)

- Define propagator:

$$U_{t,t_0} = \mathcal{T} \exp\left[-\int_{t_0}^{t} \mathrm{d}t' \left(\mathcal{L}' + \delta\mathcal{L}(t')\right)\right]$$

- Formal solution of Vlasov equation:

$$f_\oplus(\mathbf{p}, t) = U_{t,t_0} f_\oplus(\mathbf{p}, t_0) + \int_{t_0}^{t} \mathrm{d}t' \, U_{t,t'} \, c \, \hat{\mathbf{p}} \cdot \mathbf{G}$$

$\rightarrow$ Differential equation for $\langle C_\ell \rangle$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle C_\ell \rangle(t) + \left(\frac{\delta_{\ell\ell_0} - M_{\ell\ell_0}(\Delta T)}{\Delta T}\right)\langle C_{\ell_0}\rangle(t) = \frac{8\pi}{9} K |\mathbf{G}|^2 \delta_{\ell 1}$$

mixing $\ell_0 \rightarrow \ell$ sourcing ℓ

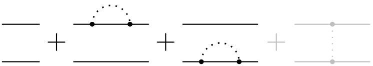
where

$$M_{\ell\ell_0}(t, t_0) = \frac{1}{4\pi} \int \mathrm{d}\hat{\mathbf{p}}_A \int \mathrm{d}\hat{\mathbf{p}}_B \mathrm{P}_\ell(\hat{\mathbf{p}}_A \cdot \hat{\mathbf{p}}_B) \langle U^A_{t,t_0} U^{B*}_{t,t_0} \rangle \frac{2\ell_0 + 1}{4\pi} \mathrm{P}_{\ell_0}(\hat{\mathbf{p}}_A \cdot \hat{\mathbf{p}}_B)$$

Ignoring correlations

- Without "interactions":

$$\langle U^{A}_{t,t_0} U^{B*}_{t,t_0} \rangle \simeq$$



- Corresponds to "usual" pitchangle scattering as in QLT

$$\langle U^{A}_{t,t_0} U^{B*}_{t,t_0} \rangle^{(1a)} = -\Delta T \, D_{\mu\mu} L^2$$

- Assuming that late times don't contribute (as in QLT) we recover $\delta(k\mu - \Omega)$

$$M^{(0)}_{\ell\ell_0} = \delta_{\ell\ell_0}$$

$$M^{(1a)}_{\ell\ell_0} = -4\pi\ell(\ell+1)\left(\frac{2}{3}\Lambda_0(\Delta T) - \frac{1}{3}\Lambda_2(\Delta T)\right)\delta_{\ell\ell_0}$$

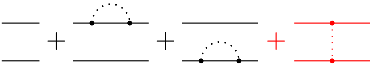
$$\frac{\delta_{\ell\ell_0} - M_{\ell\ell_0}(\Delta T)}{\Delta T} \langle C_{\ell_0} \rangle(t) = \frac{8\pi}{9} K |\mathbf{G}|^2 \delta_{\ell 1} \, ,$$

$\rightarrow$ $M_{\ell\ell_0} \propto \delta_{\ell\ell_0} \Rightarrow$ No mixing!

Including correlations

- With "interactions"

$$\langle U^{A}_{t,t_0} U^{B*}_{t,t_0} \rangle \simeq$$



$$M^{(0)}_{\ell\ell_0} = \delta_{\ell\ell_0}$$

$$M^{(1a)}_{\ell\ell_0} = -4\pi\ell(\ell+1)\left(\frac{2}{3}\Lambda_0(\Delta T) - \frac{1}{3}\Lambda_2(\Delta T)\right)\delta_{\ell\ell_0}$$

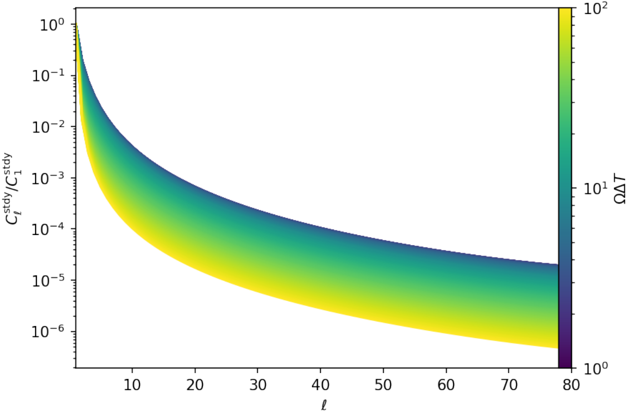
$$M^{(1c)}_{\ell\ell_0} = \pi \sum_{\ell_A,\ell_B} i^{\ell_B-\ell_A}(2\ell_0+1)(2l_A+1)(2l_B+1)$$

$$\times \begin{pmatrix} \ell_A & \ell & \ell_0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_B & \ell & \ell_0 \\ 0 & 0 & 0 \end{pmatrix} \left(1+(-1)^{\ell_A+\ell_B}\right)$$

$$\times \sum_{m_0,m} \left((2\ell_0(\ell_0+1) - 2m_0^2) \begin{pmatrix} \ell_A & \ell & \ell_0 \\ 0 & m & m_0 \end{pmatrix} \begin{pmatrix} \ell_B & \ell & \ell_0 \\ 0 & m & m_0 \end{pmatrix} \kappa_{\ell_A,\ell_B}(\Delta T) \right)$$

→ Gradient source term is mixing into higher harmonics!

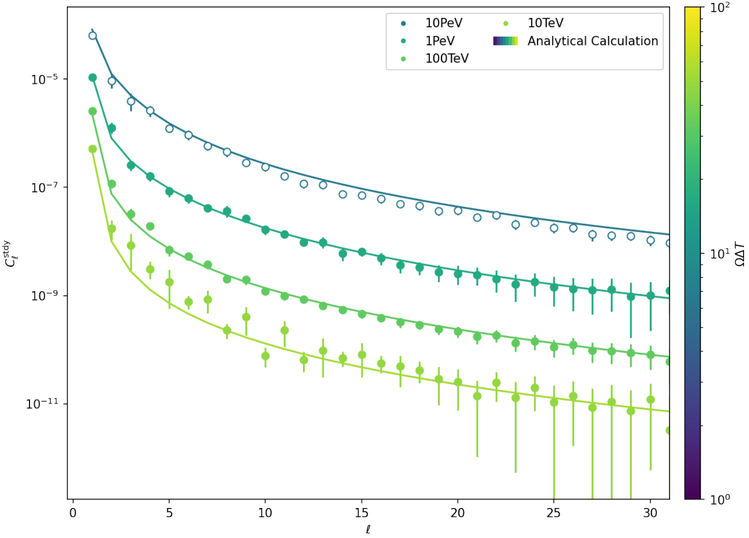
Resulting Angular Power Spectrum



- Result still depends on $\Omega\Delta T$
- $\Omega\Delta T$ should depend on scattering time $\Omega\tau_s$

$$\frac{\delta_{\ell\ell_0} - M_{\ell\ell_0}(\Delta T)}{\Delta T} \langle C_{\ell_0} \rangle (t) = \frac{8\pi}{9} K |\mathbf{G}|^2 \delta_{\ell 1} \,,$$

Results



- Numerical results agree very well with the analytical angular power spectra!

Results

- Expect scaling $\Omega \Delta T \propto (\Omega \tau_s)^{1/3}$
  $\rightarrow$ Confirmed by numerical simulations
- Observations of small scale anisotropies can constrain $\Omega \tau_s$
  $\rightarrow$ Constraints on the turbulent magnetic field!

- Numerical and analytic angular power spectra become steeper for smaller energies
- Observed angular power spectra become flatter for smaller energies

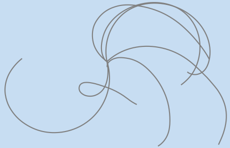
Possible reasons for different scaling

- A feature in the power spectrum
- A very small outer scale
- Slab turbulence does not describe data well
- Finite energy resolution

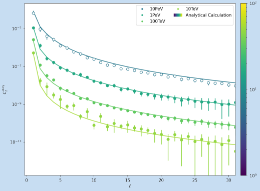
Conclusion

- Small-scale anisotropies are not expected in standard QLT
- May come from the local realization of the turbulent magnetic field
- Could be used to constrain magnetic field turbulence parameters such as outer scale or turbulence level

Numerical Simulations



Analytical Calculation



Constrain Turbulence

Relate $\Omega\Delta T$ to Scattering Time

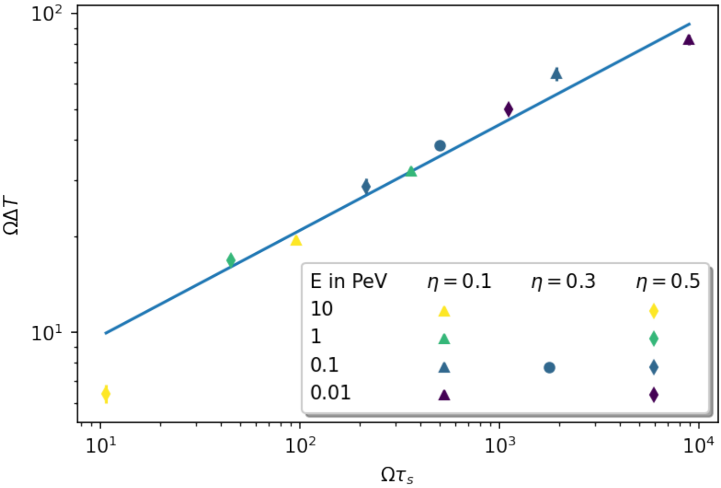
$$\langle U^{A}_{t,t_0} U^{B*}_{t,t_0} \rangle^{(1c)} \propto \left(\frac{\Delta\mu}{\bar{\mu}} \right)^{-1} \sin\left[\frac{\Delta\mu}{\bar{\mu}} \Omega \Delta T \right] \tag{1}$$

with $\Delta\mu = \mu_A - \mu_B$ and $\bar{\mu} = 1/2(\mu_A + \mu_B)$.

- Sinc function $\Rightarrow$ resonance with finite support $\frac{\Delta\mu}{\bar{\mu}} \Omega T \sim 1$
- Assume $\langle (\Delta\mu)^2 \rangle \propto D_{\mu\mu} \Delta T$
- $\frac{\Delta\mu}{\bar{\mu}} \propto \sqrt{D_{\mu\mu} \Delta T} \propto \sqrt{\frac{\Delta T}{\tau_s}}$

$\Rightarrow$ $\Delta T \sim \tau_s^{1/3}$

Relate $\Omega\Delta T$ to Scattering Time



- Numerical simulations confirm $\Omega\Delta T \propto (\Omega\tau_s)^{1/3}$

Energy Scaling