# QED factorization for the leptonic B meson decays

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in collaboration with Y. K. Huang, Y. L. Shen, Y. M. Wang, X. C. Zhao

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# Outline

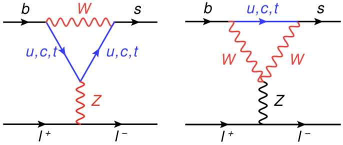
1. Motivation / power enhancement in $B_{d/s} \to \mu^+ \mu^-$
2. QED corrections to $B_{d/s} \to \tau^+ \tau^-$ in SCET
3. Numerical results and discussion
4. Summary and outlook

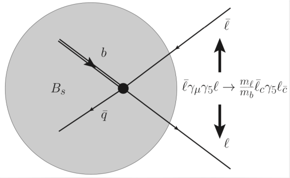
# Power enhancement in $B_{d/s} \to \mu^+ \mu^-$

**In the SM the process is**

- Loop suppressed (FCNC)
- Helicity suppresses (scalar meson decaying into energetic leptons through vector interactions)
- QCD contained in the meson decay constant $f_B$ for purely leptonic final state, in the absence of QED.

$$\langle 0 \,|\bar{q}(0)\gamma^\mu\gamma_5 b(0)|\, \bar{B}_q(p)\rangle = i f_{B_q} p^\mu$$



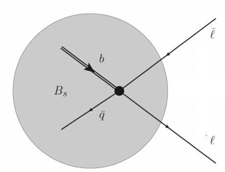




$$Q_{10} = \frac{\alpha_{\rm em}}{4\pi}\,(\bar{q}\gamma^\mu P_L b)\,\left(\bar{\ell}\gamma_\mu\gamma_5\ell\right)$$

$\rightarrow$ Highly suppressed in the SM and can be computed with a good precision!

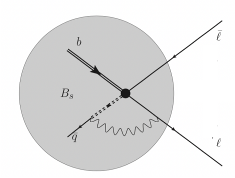
# Power enhancement in $B_{d/s} \to \mu^+ \mu^-$

**Without QED**




Local Annihilation $r \lesssim \frac{1}{m_b}$

**With QED** [M.Beneke,C.Bobeth,R.S.,arXiv:1708.09152]




Non-local Annihilation separated by $r \sim 1/\sqrt{m_b \Lambda_{\rm QCD}}$ (smaller than $r \sim 1/\Lambda_{\rm QCD}$)

$\Rightarrow$ a dynamical enhancement by a power of $m_b/\Lambda_{\rm QCD}$ and by large logarithms $\ln m_b \Lambda_{\rm QCD}/m_\mu^2 \to$ 1 % of Br[$B_{d,s} \to \mu^+\mu^-$]

four times the size of previous estimates of NLO QED effects $\alpha_{\rm em}/\pi \sim 0.3\%$

the order of non-parametric theoretical uncertainty ( $\mathcal{O}(\alpha_s^3), \mathcal{O}(\alpha_{\rm em}^2), \mathcal{O}(\alpha_s \alpha_{\rm em}), \mathcal{O}(m_b^2/m_W^2)$)

# Motivation for QED corrections to $B_{d/s} \to \tau^+\tau^-$

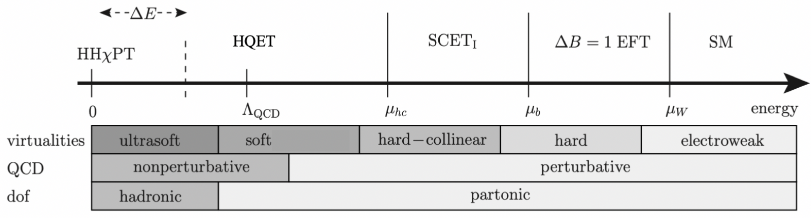
$$m_\tau \sim \sqrt{m_b \Lambda_{\rm QCD}} \sim \text{hard collinear}, \quad m_\mu \sim \Lambda_{\rm QCD} \sim \text{soft}$$

- Does the power enhancement QED effect exist in $B_{d/s} \to \tau^+\tau^-$?
- Analysis of the complete NLO QED correction (besides the dominant power-enhanced contribution).
- Relevant for lepton universality test.

# Scales and EFTs

Hierarchy of energy scales:

$M_W \sim 80\,\text{GeV} \gg m_b \sim 4.2\,\text{GeV} \gg \text{few times } \Lambda_{\text{QCD}} \gg \Delta E \sim 60\,\text{MeV}$




(figure from [Beneke, Bobeth, Szafron '19])

✓ short-distance QED at $\mu \gtrsim m_b \to$ Wilson coefficients of weak eff. Lagrangian

→ NNLO QCD [T. Hermann, M. Misiak, M. Steinhauser, 2013] NLO EW [C. Bobeth, M. Gorbahn, E. Stamou, 2014]

✓ ultrasoft photons with $\mu_{\text{us}} \ll \Lambda_{\text{QCD}}$ see pointlike mesons ("universal") → based on eikonal approximation. e.g. [A. von Manteuffel, R. M. Schabinger and H. X. Zhu '14]

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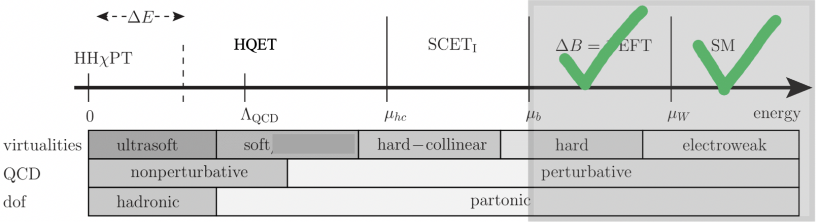


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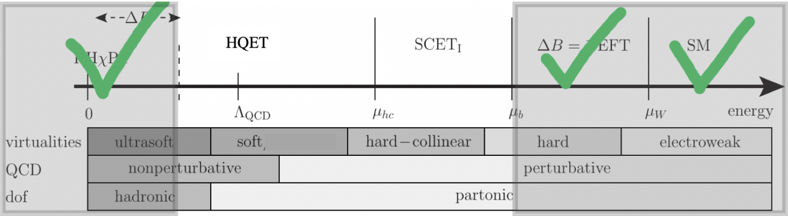


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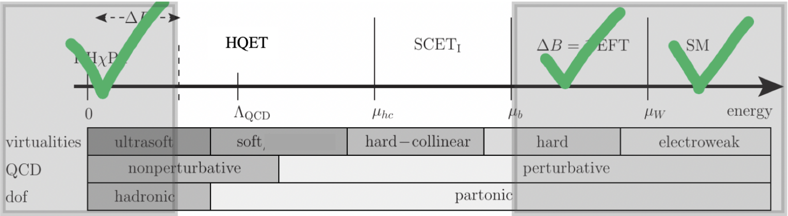


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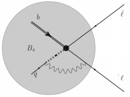
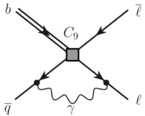
**Goal:** QED corrections between $m_b$ and $\Lambda_{\text{QCD}}$ can resolve partonic structure!
Non-universal, "structure dependent "



# Power counting for $B_q \to \tau^+\tau^-$ and SCET

**Kinematics of $B_q \to \tau^+\tau^-$ and power counting**




The momentums of initial and final state for

$b\,(p_b) + q\,(\ell_q) \to \ell\,(p_\ell) + \bar{\ell}\,(p_{\bar{\ell}})$ with $p^\mu = (n \cdot p, \bar{n} \cdot p,\, p_\perp)$ are

$$\left.\begin{aligned} p_b^\mu &= m_b v^\mu + \ell_b\,, \quad \ell_{b,q} \sim \Lambda_{\rm QCD} \\ p_\ell^\mu &\sim (m_b,\, \Lambda_{\rm QCD}, 0) \sim m_b(1,\, \Lambda_{\rm QCD}/m_b, 0) \\ p_{\bar{\ell}}^\mu &\sim (\Lambda_{\rm QCD}, m_b,\, 0) \sim m_b(\Lambda_{\rm QCD}/m_b, 1, 0) \end{aligned}\right\} \text{kinematic invarants}$$

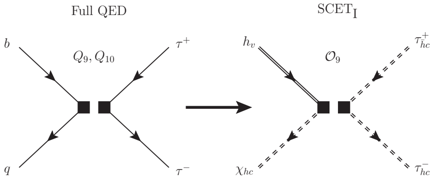
Different from $\ell = \mu$, $p_\ell^2 = m_\tau^2 = m_b\Lambda_{\rm QCD}$

Modes for virtual photon with momentum k: with scaling parameter $\lambda^2 = \frac{\Lambda_{\rm QCD}}{m_b}$

| mode | relative scaling | absolute scaling | virtuality $k^2$ |
|---|---|---|---|
| hard | (1,1,1) | $m_b(1,1,1)$ | $m_b^2$ |
| hard-collinear | $(1, \lambda^2, \lambda)$ | $(m_b, \Lambda_{\rm QCD}, \sqrt{m_b\Lambda_{\rm QCD}})$ | $m_b\Lambda_{\rm QCD}$ |
| hard anti-collinear | $(\lambda^2, 1, \lambda)$ | $(\Lambda_{\rm QCD}, m_b, \sqrt{m_b\Lambda_{\rm QCD}})$ | $m_b\Lambda_{\rm QCD}$ |
| soft | $(\lambda^2, \lambda^2, \lambda^2)$ | $(\Lambda_{\rm QCD}, \Lambda_{\rm QCD}, \Lambda_{\rm QCD})$ | $\Lambda_{\rm QCD}^2$ |

QED $\to$ $\mathrm{SCET_I}$ $\to$ HQET $\otimes$ $\mathrm{SCET_I}$

**1. Integrate out hard scale $m_b$ and match QED $\to$ $\mathrm{SCET_I}$**




- weak EFT operators

$$Q_9 = \frac{\alpha_{\rm em}}{4\pi}\,(\bar{q}\gamma^\mu P_L b)\sum_\ell \bar{\ell}\gamma_\mu \ell$$

$$Q_{10} = \frac{\alpha_{\rm em}}{4\pi}\,(\bar{q}\gamma^\mu P_L b)\sum_\ell \bar{\ell}\gamma_\mu \gamma_5 \ell$$

$$Q_7 = \frac{e}{(4\pi)^2}\bar{m}_b\,[\bar{q}\sigma^{\mu\nu}P_R b]\,F_{\mu\nu}$$

- Operators for a hard-collinear light quark in $\mathrm{SCET_I}$ in position space,

$$\widetilde{\mathcal{O}}_9(s,t) = g^\perp_{\mu\nu}\,\left[\bar{\chi}_C\,(sn_+)\gamma^\mu_\perp P_L h_v(0)\right]\,\left[\bar{\ell}_C\,(tn_+)\,\gamma^\nu_\perp \ell_{\bar{C}}(0)\right]$$

$$\widetilde{\mathcal{O}}_{10}(s,t) = i\varepsilon^\perp_{\mu\nu}\,\left[\bar{\chi}_C\,(sn_+)\gamma^\mu_\perp P_L h_v(0)\right]\,\left[\bar{\ell}_C\,(tn_+)\,\gamma^\nu_\perp \ell_{\bar{C}}(0)\right]$$

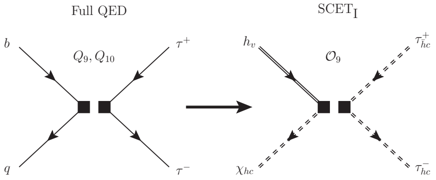
$n_+ \to n_-$ for anti-hard-collinear quark, $\widetilde{\mathcal{O}}_{\overline{9},\overline{10}}$

$Q_7$ can match to $\widetilde{\mathcal{O}}_9$: $Q_7 = \frac{2Q_\ell}{u}\,\mathcal{O}_9$

$$\widetilde{\mathcal{O}}_9(s,t) = \widetilde{\mathcal{O}}_{10}(s,t) \Leftarrow \frac{\not{\bar{n}}}{2}\gamma^\perp_\mu\gamma_5 = i\epsilon^\perp_{\mu\nu}\gamma^\perp_\nu \text{ in d=4} \quad (1)$$

# $\mathcal{O}(\alpha_{\text{em}})$ Matching Calculation

**1. Integrate out hard scale $m_b$ and match QED $\to \text{SCET}_\text{I}$**

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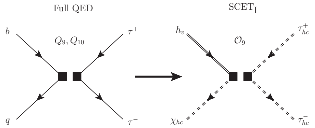
The Fourier-transformed SCET operators are defined as

$$\mathcal{O}_i(u) = n_+p_C \int \frac{dr}{2\pi} e^{-iur(n_+p_C)} \widetilde{\mathcal{O}}_i(0, r) \,,$$

where $u = n \cdot p_\ell / n \cdot p_C$ carried by the lepton field.



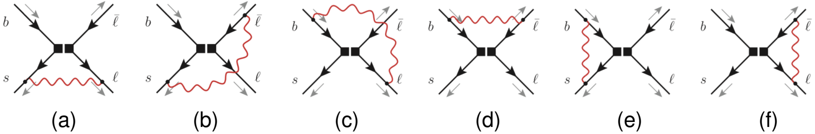

- Matching equation in momentum space,

$$\sum_k C_k\,(\mu_b)\; Q_k = \int du\, H_9\,(u, \mu_b)\; \mathcal{O}_9(u) \qquad (2)$$

→ At Leading Order,

$$H_9^{(0)}\,(u, \mu_b) \;=\; \mathcal{N}\left[C^{(0)}_{9\,\mathrm{eff}}\,(u, \mu_b) + C^{(0)}_{10}\,(u, \mu_b) - \frac{2Q_\ell}{u} C_7^{\mathrm{eff}}\,(u, \mu_b)\right] \qquad (3)$$

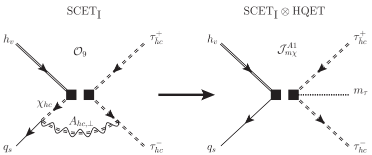
→ To Next to Leading Order (one loop hard functions $H_9^{(1)}$),




$$H_9^{(1)}\,(u, \mu_b) \;=\; \mathcal{N}\left[C^{(1)}_{7,\,9}\,(\mu_b) + C^{(1)}_{10}\,(\mu_b)\right] - H_9^{(0)}\,(u, \mu_b)\; Z^{(1)}_{99} \,, \qquad (4)$$

where the second terms in r.h.s of both equations above are the IR subtraction.

## 2. Integrate out intermediate hard-coll. scale $\sqrt{m_b \Lambda_{\rm QCD}}$ and match onto HQET⊗SCET




- To convert hard-collinear quark into a soft quark to get a non-vanishing overlap the B-meson state, we need power suppressed interaction

$$\mathcal{L}^{(1)}_{\xi q}(x) = \bar{q}_s(x_-)\left[W_{\xi C} W_C\right]^\dagger (x) i \not{D}_{C\perp} \xi_C(x) + \text{h.c.} \quad (5)$$

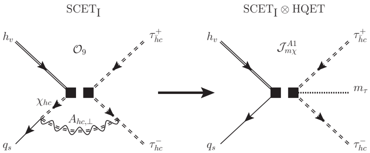
Small component of hard-collinear the same as soft momentum → Soft fields become delocalised along the light- cone: $\bar{u}_s(\ell_q)\ \not{\epsilon}_\perp (k) \frac{\not{n}}{2} \frac{1}{\Lambda_{\rm QCD}}$, it is power enhaced factor.

- the (hard-) collinear photon, $A_{C\perp}$ from $D_{C\perp}$, would be followed by the fusion

$$\bar{\ell}_C + A_{\perp C} \to m_\tau\, \bar{\ell}_C \quad (6)$$

through the leading power Lagrangian,

$$\mathcal{L}^{(0)}_m(y) = m_\tau\, \bar{\ell}_C \left[ i \not{D}_{C\perp}, \frac{1}{i n_+ D_C} \right] \frac{\not{n}_+}{2} \ell_C \quad (7)$$




$$\mathcal{O}_9(u) \to \int d\omega\, J_m(u;\omega)\, \mathcal{J}^{A1}_{m\chi}(\omega)$$

- Therefore, we will match the time-ordered product of the $\mathrm{SECT_I}$ operators $\mathcal{O}_9(u)$ with $\mathcal{L}^{(1)}_{\xi q}(x)$ and $\mathcal{L}^{(0)}_m(y)$,

$$\left\langle \ell(p_\ell)\,\bar{\ell}\,(p_{\bar{\ell}}) \left| \int d^4x \int d^4y\, T\left\{\mathcal{O}^{\mathrm{I}}_9(u),\, \mathcal{L}^{(1)}_{\xi q}(x),\, \mathcal{L}^{(0)}_m(y)\right\}\right| b\,(p_b)\,q\,(\ell_q)\right\rangle\,, \tag{8}$$

to matrix element of operator in HQET $\otimes$ $\mathrm{SECT_I}$,

$$\widetilde{\mathcal{J}}^{A1}_{m\chi}(v) \quad = \left[\bar{q}_s\,(vn_-)\,Y\,(vn_-,0)\,\frac{\not{n}_-}{2}\,P_L\,h_v(0)\right]\,\left[Y^\dagger_+\,Y_-\right]\,\left[\bar{\ell}_C(0)\,(4m_\ell P_R)\,\ell_{\overline{C}}(0)\right] \tag{9}$$

$[M.Beneke, T.Feldmann'03]$

- At tree level, the jet function (hard collinear function) $J_m(u,\omega)$

$$J^{(0)}_m(u;\omega;\mu=\mu_{hc}) = \frac{\alpha}{4\pi}\,Q_\ell Q_s\,\frac{\bar{u}}{\omega}\,\ln\left(1+\frac{u}{\bar{u}}\,\frac{\omega\, m_b}{m^2_\ell}\right)\,\theta(u)\theta(\bar{u})\,, \tag{10}$$

where $\omega = \bar{n}\cdot\ell_q \sim \Lambda_{\mathrm{QCD}}$, power enhancement factor and large logarithms

$$\mathcal{A}_9 \sim \int_0^1 du\, 2\, H_9(u) \int_0^\infty d\omega\, J_m(u;\omega)\, \langle \ell^+\ell^- | \widetilde{\mathcal{J}}^{A1}_{m\chi} | \bar{B}_q \rangle \tag{11}$$

**3. Matrix elements of operators $\widetilde{\mathcal{J}}^{A1}_{m\chi,\bar{\chi}}$ in HQET $\otimes$ $\mathrm{SECT_I}$**

$$\widetilde{\mathcal{J}}^{A1}_{m\chi}(v) = \left[\bar{q}_s\,(vn_-)\; Y\,(vn_-,0)\; \frac{\not{n}_-}{2}\; P_L\, h_v(0)\right] \left[Y_+^\dagger Y_-\right]\,(0) \left[\bar{\ell}_C(0)\,(4m_\ell P_R)\;\ell_{\overline{C}}(0)\right]$$

$$\langle \ell^+\ell^- | \widetilde{\mathcal{J}}^{A1}_{m\chi} | \bar{B}_q \rangle = \langle 0 | \widetilde{\mathcal{J}}_s | \bar{B}_q \rangle\; \langle \ell^- | \widetilde{\mathcal{J}}_C | 0 \rangle\; \langle \ell^+ | \widetilde{\mathcal{J}}_{\bar{C}} | 0 \rangle\,, \tag{12}$$

where

$$(-4)\; \langle 0 | \widetilde{\mathcal{J}}_s | \bar{B}_q \rangle = \frac{\left\langle 0 \left| \bar{q}_s\,(vn_-)\, Y\,(vn_-,0)\;\; \not{n}_-\gamma_5 h_v(0)\, \left[Y_+^\dagger Y_-\right]\,(0) \right| \bar{B}_q(p) \right\rangle}{\left\langle 0 \left| \left[Y_+^\dagger Y_-\right]\,(0) \right| 0 \right\rangle} \tag{13}$$

$$\langle \ell^- | \widetilde{\mathcal{J}}_C | 0 \rangle\; \langle \ell^+ | \widetilde{\mathcal{J}}_{\bar{C}} | 0 \rangle = \langle 0 | [Y_+^\dagger Y_-](0) | 0 \rangle\; \bar{\ell}_C(0)\,(4m_\ell P_R)\;\ell_{\overline{C}}(0) \tag{14}$$

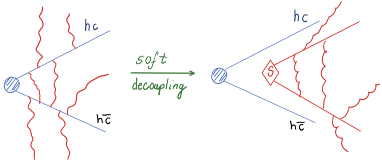
# Soft function/$B$-meson LCDA

Soft matrix element

$$(-4)\,\langle 0|\widetilde{\mathcal{J}}_s|\bar{B}_q\rangle = \frac{\left\langle 0 \left| \bar{q}_s\,(vn_-)\,Y\,(vn_-,0)\;\not{n}_-\gamma_5 h_v(0) \left[Y_+^\dagger Y_-\right](0) \right| \bar{B}_q(p) \right\rangle}{\left\langle 0 \left| \left[Y_+^\dagger Y_-\right](0) \right| 0 \right\rangle}$$

$$\equiv -i\mathscr{F}_{B_q} m_{B_q} \int_0^\infty d\omega e^{-i\omega v}\Phi_+(\omega) \quad [Beneke, Bobeth, Szafron'19]$$

- Wilson lines $\left[Y_+^\dagger Y_-\right](0)$ are process dependent, consequence of soft photon decoupling




$[Fig\ from\ Szafron'\ talk]$

- $\Phi_+(\omega)$ is the B-meson LCDA generalized to QED for $B_s \to \ell^+\ell^-$ and process dependent decay constant $\mathscr{F}_{B_q}$ can be defined similarly

$\rightarrow$ soft function becomes process dependent!

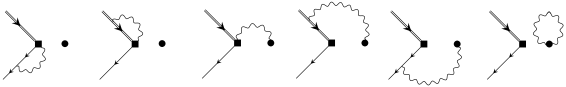
# Soft function/$B$-meson LCDA

QED expansion at soft scale

$$\mathcal{F}_{B_q}(\mu_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_{\rm em}(\mu_s)}{4\pi}\right)^n F_{B_q}^{(n)}(\mu_s)$$

$$\mathcal{F}_{B_q}(\mu_s)\,\Phi_+(\omega;\mu_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_{\rm em}(\mu_s)}{4\pi}\right)^n F_{B_q}^{(n)}(\mu_s)\,\phi_+^{(n)}(\omega;\mu_s)$$

- The leading term is the standard B meson decay constant and LCDA.
- Higher-order terms define non-universal, non-local QCD (more precisely, HQET) matrix elements that have to be evaluated nonperturbatively. For example at one loop,



- At **LL or NLL** accuracy, only the universal objects $F_{B_q}^{(0)}(\mu_s)$ and $\phi_+^{(0)}(\omega;\mu_s)$ are needed.

# Factorization of the amplitude

including the renormalized hard (anti)coll. on-shell matrix elements,

$$\langle \ell^- (p_\ell) \,|[\text{ hard coll. }]|\, 0 \rangle = Z_\ell \bar{u}_C (p_\ell)\,, \quad \langle \ell^+ (p_{\bar\ell}) \,|[\text{hard anticoll. }]|\, 0 \rangle = Z_{\bar\ell} v_{\bar C} (p_{\bar\ell}) \tag{15}$$

we can now derive the factorized expression for the matrix elements of HQET $\otimes$ $\mathrm{SECT_I}$ in the form,

$$\left\langle \ell^+ (p_{\bar\ell})\, \ell^- (p_\ell) \left| \mathcal{J}^{A1}_{m\chi}(\omega, w) \right| \bar{B}_q(p) \right\rangle = T_+ m_{B_q} \mathcal{F}_{B_q} \Phi_+(\omega) \tag{16}$$

where

$$T_+(\mu) \equiv (-i) m_\ell(\mu) Z_\ell(\mu) Z_{\bar\ell}(\mu) \left[ \bar{u}_C (p_\ell)\, P_R v_{\overline{C}} (p_{\bar\ell}) \right] \tag{17}$$

The complete expression for $B \to \tau\tau$ amplitude is now obtained by adding the hard function and hard-(anti)collinear matching coefficients,

$$i\mathcal{A}_9 = T_+ \int_0^1 du\, 2\, H_9(u) \int_0^\infty d\omega\, J_m(u;\omega)\, m_{B_q}\, \mathcal{F}_{B_q}\, \Phi_+(\omega) \tag{18}$$

# Factorization of the amplitude

including the renormalized hard (anti)coll. on-shell matrix elements,

$$\left\langle \ell^- (p_\ell) \left| [\text{ hard coll. }] \right| 0 \right\rangle = Z_\ell \bar{u}_C (p_\ell)\,, \quad \left\langle \ell^+ (p_{\bar{\ell}}) \left| [\text{hard anticoll. }] \right| 0 \right\rangle = Z_{\bar{\ell}} v_{\bar{C}} (p_{\bar{\ell}}) \tag{15}$$

we can now derive the factorized expression for the matrix elements of HQET $\otimes$ $\mathrm{SECT_I}$ in the form,

$$\left\langle \ell^+ (p_{\bar{\ell}})\, \ell^- (p_\ell) \left| \mathcal{J}^{A1}_{m\chi}(\omega, w) \right| \bar{B}_q(p) \right\rangle = T_+ m_{B_q} \mathcal{F}_{B_q} \Phi_+(\omega) \tag{16}$$

where

$$T_+(\mu) \equiv (-i) m_\ell(\mu) Z_\ell(\mu) Z_{\bar{\ell}}(\mu) \left[\bar{u}_C (p_\ell)\, P_R v_{\overline{C}} (p_{\bar{\ell}})\right] \tag{17}$$

The complete expression for $B \to \tau\tau$ amplitude is now obtained by adding the hard function and hard-(anti)collinear matching coefficients,

$$i\mathcal{A}_9 = T_+ \int_0^1 du\, 2\, H_9(u) \int_0^\infty d\omega\, J_m(u;\omega)\, m_{B_q}\, \mathcal{F}_{B_q}\, \Phi_+(\omega) \tag{18}$$

# Resummed amplitude

- The Hard function and Soft function are evoluted to $\mu_{hc}$
- Anomalous dimension is (almost) known
  - → SCET I: B-type current with fermion number 2 [M. Beneke, M. Garny, R. Szafron, J. Wang JHEP 1803, 001 2018]
  - → soft part [B.Lange, M. Neubert, Phys.Rev.Lett. 91, 102001, 2003] + additional contribution from the soft Wilson lines [Beneke, Bobeth, Szafron '19]
- The resummed result to LL

$$i\mathcal{A}_9 = T_+(\mu_{hc})\, m_{B_q} \int_0^1 du \int_0^\infty d\omega \Big[ U_h(\mu_b, \mu_{hc})\, U_s^{\text{QED}}(\mu_{hc}, \mu_s; \omega) H_9(u; \mu_b) + U_{\bar{h}}(\mu_b, \mu_{hc})\, U_{\bar{s}}^{\text{QED}}(\mu_{hc}, \mu_s; \omega) H_{\bar{9}}(u; \mu_b) \Big] J_m(u; \omega; \mu_{hc})\, F_{B_q}(\mu_{hc})\, \phi_+(\omega; \mu_{hc}) \tag{19}$$

# Numerical prediction: complete QED virtual correction

- The non-radiative branching fraction of $B_q \to \tau^+ \tau^-$ for central values of the parameters are

$$\mathrm{Br}^{(0)}(B_d \to \tau^+\tau^-) = \left(2.07959_{\text{(LO)}} - 0.00094_{\text{(NLO)}}\right) \times 10^{-8}$$
$$\mathrm{Br}^{(0)}(B_s \to \tau^+\tau^-) = \left(6.83576_{\text{(LO)}} - 0.00311_{\text{(NLO)}}\right) \times 10^{-7}$$

$\to$ Complete NLO+LL QED virtual correction (hard and hard collinear functions) changes the branching fraction by: $\sim 0.04\%$

$$J_m^{(0)}(u;\omega;\mu=\mu_{hc}) = \frac{\alpha}{4\pi}\, Q_\ell Q_s\, \frac{\bar{u}}{\omega} \ln\left(1 + \frac{u}{\bar{u}}\, \frac{\omega\, m_b}{m_\ell^2}\right) \theta(u)\theta(\bar{u})\,, \qquad (20)$$

where $\omega = \bar{n} \cdot \ell_q \sim \Lambda_{QCD}$, power enhancement factor and large logarithms

- compared with $B_{d,s} \to \mu^+\mu^-$, power-enhanced correction ((hard) collinear functions) $\sim 0.4\%$, [Beneke, Bobeth, Szafron '17, '19]

$$J_9(u;\omega;\mu=\mu_{hc}) \sim \frac{\alpha}{4\pi}\, Q_\ell Q_s\, \frac{\bar{u}}{\omega} \ln\left(\frac{\omega\, m_b}{m_\ell^2}\right) \theta(u)\theta(\bar{u})\,, \qquad (21)$$

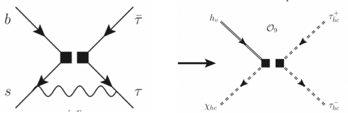
# Summary and outlook

- structure depended QED corrections can be calculated in **SCET** and **HQET**
  - $\rightarrow$ convolution of hard function (have included NLO), jet function and QED specific B-meson LCDAs at NLO completely for $B_q \rightarrow \tau^+\tau^-$
  - $\rightarrow$ interesting effect – power suppressed interaction $\mathcal{L}^{(1)}_{\xi q}(x)$ lead to power enhanced correction $1/\Lambda_{QCD}$
- QED factorization more complicated than QCD-alone due to charged external states
  - $\rightarrow$ one cannot naively generalise QCD to QCD+QED
- Outlook
  - $\rightarrow$ QED factorization and electromagnetic effects in other processes ($B \rightarrow K^{(*)}\ell^+\ell^-$)

*Thank you!*

Backup-Slides

- Expanding the matching equation at NLO




$$\sum_k C_k^{(1)}(\mu_b)\langle Q_k\rangle^{(0)} = \sum_i \int du H_i^{(0)}(u,\mu_b)\langle \mathcal{O}_i(u)\rangle^{(1)} + H_i^{(1)}(u,\mu_b)\langle \mathcal{O}_i(u)\rangle^{(0)} \tag{22}$$

$$\text{with} \quad \langle \mathcal{O}_i(u)\rangle^{(1)} = Z_{ij}^{(1)}\langle \mathcal{O}_j(u)\rangle^{(0)}, \quad \text{in Dimension regulation}$$

$$H_9^{(1)}(u,\mu_b) = \mathcal{N}\left[C_{7,9}^{(1)}(\mu_b) + C_{10}^{(1)}(\mu_b)\right] - H_9^{(0)}(u,\mu_b)\, Z_{99}^{(1)}, \tag{23}$$

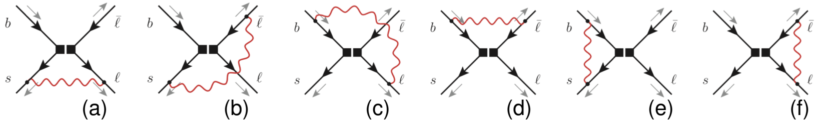
For the case of an anti-hard-collinear quark $\mathcal{O}_{\bar{9}}$, the hard function is

$$H_{\bar{9}}^{(1)}(u,\mu_b) = \mathcal{N}\left[C_{\bar{7},\bar{9}}^{(1)}(\mu_b) - C_{\overline{10}}^{(1)}(\mu_b)\right] - H_{\bar{9}}^{(0)}(u,\mu_b)\, Z_{\bar{9}\bar{9}}^{(1)}, \tag{24}$$

where the second terms in in r.h.s of both equations above are the IR subtraction.

$$C_{\bar{7},\bar{9},\overline{10}}^{(1)} = C_{7,9,10}^{(1)} \; ? \Leftarrow \text{exchange two leptons} \tag{25}$$

- $C_k^{\text{(a-d)}}$ are antisymmetric under the exchange of the hc and anti-hc (exchange two leptons)




$$C_k^{\text{(a)}} = -C_{\bar{k}}^{\text{(b)}}, \quad C_{\bar{k}}^{\text{(a)}} = -C_k^{\text{(b)}}, \quad C_k^{\text{(c)}} = -C_{\bar{k}}^{\text{(d)}}, \quad C_k^{\text{(d)}} = -C_{\bar{k}}^{\text{(c)}},$$
$$C_k^{\text{(e)}} = C_{\bar{k}}^{\text{(e)}}, \quad C_k^{\text{(f)}} = C_{\bar{k}}^{\text{(f)}}, \quad \text{k=7, 9, 10} \tag{26}$$

- performing the charge conjugation (C) just for final leptons

$$\langle \gamma\,|Q_i, \mathcal{L}_{QED}|\,\ell^+\ell^-\rangle \xrightarrow{\text{C}} -\langle 0\,|\mathcal{O}_i|\,\ell^+\ell^-\rangle \tag{27}$$

More general form ($\text{N}^\text{m}\text{LO}$)

$$\langle m\,\gamma\,|Q_i, \mathcal{L}_{QED}|\,\ell^+\ell^-\rangle \xrightarrow{\text{C}} (-1)^m\,\langle 0\,|\mathcal{O}_i|\,\ell^+\ell^-\rangle\,, \tag{28}$$

where $m$ stands for the number of photon attached to lepton sector

$$\begin{aligned} H_9^{(1)}(u,\mu_b) + H_{\bar{9}}^{(1)}(u,\mu_b) &= 2\,\mathcal{N}\left[C_{7,9}^{(e-f)}(\mu_b) + C_{10}^{(a-d)}(\mu_b)\right] \\ &= 2\,\mathcal{N}\left[C_{\bar{7},\bar{9}}^{(e-f)}(\mu_b) + C_{\overline{10}}^{(a-d)}(\mu_b)\right] \end{aligned} \tag{29}$$

Then $H_9^{(1)}$ equals to $H_{\bar{9}}^{(1)}$ written as

$$H_{9/\bar{9}}^{(1)}(u,\mu_b) = \mathcal{N}\left[C_{7,9}^{(e-f)}(\mu_b) + C_{10}^{(a-d)}(\mu_b)\right] \tag{30}$$

$$
\begin{aligned}
i\mathcal{A}_9 = &-\frac{i}{2}\,\frac{\alpha_{\rm em}\left(\mu_{hc}\right)}{4\pi}\,Q_\ell\,Q_q\,m_\ell\,m_{B_q}\,F_{B_q}\mathcal{N}\,\left[\bar{\ell}_C\left(1+\gamma_5\right)\ell_{\bar{C}}\right] \\
&\int_0^1 du\,\bar{u}\int_0^\infty \frac{d\omega}{\omega}\left\{\left[U_h(\mu_b,\mu_{hc})\,U_s^{\rm QED}\left(\mu_{hc},\mu_s;\omega\right)-\,U_{\bar{h}}(\mu_b,\mu_{hc})\,U_{\bar{s}}^{\rm QED}\left(\mu_{hc},\mu_s;\omega\right)\right]\,H^{(0)}_{9\leftarrow 10}(\mu_b)\right. \\
&\left.+\left[U_h(\mu_b,\mu_{hc})\,U_s^{\rm QED}\left(\mu_{hc},\mu_s;\omega\right)+\,U_{\bar{h}}(\mu_b,\mu_{hc})\,U_{\bar{s}}^{\rm QED}\left(\mu_{hc},\mu_s;\omega\right)\right]\,\left[H^{(0)}_{9\leftarrow 7,9}(\mu_b)+H^{(1)}_{9/\bar{9}}(\mu_b)\right]\right\} \\
&\times\;\phi_+\left(\omega;\mu_{hc}\right)\ln\left(1+\frac{u}{\bar{u}}\,\frac{m_b\,\omega}{m_\ell^2}\right). \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
H^{(0)}_{9\leftarrow 10} &= -\,H^{(0)}_{\bar{9}\leftarrow \overline{10}} = C_{10} \\
H^{(0)}_{9\leftarrow 7,9} &= H^{(0)}_{\bar{9}\leftarrow \bar{7},\bar{9}} = C_9^{\rm eff} - \frac{2\,Q_\ell}{u}\,C_7^{\rm eff} \\
H^{(1)}_{9/\bar{9}} &= C^{(e)}_{7,\,9} + C^{(f)}_{7,\,9} + C^{(a-d)}_{10} - \text{IR subtractions} \\
&= (C_9^{\rm eff} - \frac{2\,Q_\ell}{u}\,C_7^{\rm eff})\,\frac{\alpha_{\rm em}}{4\pi}\,Q_bQ_s\,\left[-\ln^2\frac{\tilde{r}}{\bar{u}} - 2\ln\frac{\tilde{r}}{\bar{u}} + \frac{1}{2}\ln^2\tilde{r} + 2\,{\rm Li}_2(-\frac{u}{\bar{u}}) - 4 - \frac{\pi^2}{12}\right] \\
&+ (C_9^{\rm eff} - \frac{2\,Q_\ell}{u}\,C_7^{\rm eff})\,\frac{\alpha_{\rm em}}{4\pi}\,Q_\ell^2\,\left[-\ln^2\frac{-u-i0}{\tilde{r}} + 3\ln\frac{-u-i0}{\tilde{r}} - 8 + \frac{\pi^2}{6}\right] \\
&+ C_9^{\rm eff}\,\frac{\alpha_{\rm em}}{4\pi}\,Q_bQ_s\,\left[\ln\tilde{r} - \frac{\bar{u}}{u}\ln\bar{u}\right] \\
&+ C_{10}\,\frac{\alpha_{\rm em}}{4\pi}\,Q_\ell Q_s\,\left[-\ln^2\frac{u}{r} - \ln^2\frac{-\bar{u}-i0}{r} + \frac{2\ln u}{\bar{u}} + \ln^2 r + 3\ln r + 2\,{\rm Li}_2\left(-\frac{\bar{u}}{u}\right) + 10 + \frac{\pi^2}{6}\right]
\end{aligned}
\tag{32}
$$

where $Q_\ell = -1, Q_q = -1/3$.

# fields B LCDA and uncertainties

| Field | heavy quark | light quark | | | leptons | | photon (gluon) | | |
|---|---|---|---|---|---|---|---|---|---|
| | $h_v$ | $\chi_C$ | $\chi_c$ | $q_s$ | $\ell_C$ | $\ell_c$ | $A_C\,(G_C)$ | $A_c\,(G_c)$ | $A_s\,(G_s)$ |
| Scaling | $\lambda^3$ | $\lambda$ | $\lambda^2$ | $\lambda^3$ | $\lambda$ | $\lambda^2$ | $(1,\lambda,\lambda^2)$ | $(1,\lambda^2,\lambda^4)$ | $\lambda^2(1,1,1)$ |

we will employ the following three-parameter model for leading twist B-meson LCDA
$\phi^+(\omega) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\frac{\omega}{\omega_0^2}e^{-\frac{\omega}{\omega_0}}U\left(\beta-\alpha, 3-\alpha, \frac{\omega}{\omega_0}\right)$

parametric origin:

$B_s$ meson decay con- stant $f_{B_s}$, quark-mixing element $V_{cb}$, top-quark mass

Non-parametric uncertainties:

are due to the omission of higher-order corrections $\alpha_s$,$\alpha_{em}$,$\alpha_s\alpha_{em}$ in the QCD and QED couplings $\alpha_s$ and $\alpha_{em}$, respectively,