Evolution functions at 1 loop

I. CONVENTIONS AND NOTATIONS

The evolution of a low energy function (LCDA, PDF) is equivalent to the renormalization in the low energy EFT. When one tries to establish the factorization theorem, this renormalization scale will become the factorization scale.

The covariant derivative is defined as

$$D_\mu = \partial_\mu - igT^a A^a_\mu. \tag{1}$$

(maybe it is more convenient to use $n^+ = (n^0 + n^3)/\sqrt{2}$?) We use the light-cone coordinates:

$$V^+ = \frac{V^0 + V^3}{\sqrt{2}}, \quad V^- = \frac{V^0 - V^3}{\sqrt{2}}, \tag{2}$$

and it is convenient to introduce two light-cone vectors $n_\pm$ with the expression:

$$n_{+\mu} = \frac{1}{\sqrt{2}}(1,0,0,1), \quad n_{-\mu} = \frac{1}{\sqrt{2}}(1,0,0,-1). \tag{3}$$

$$n_+ \cdot n_- = 1, \quad a_\mu = n_- \cdot a n_{+\mu} + n_+ \cdot a n_{-\mu} + a_{\perp\mu}, \tag{4}$$

The P is moving on n_- direction and the z is at n_+

$$P_\mu = n_+ P n_{-\mu}\,, \qquad z_\mu = n_- z n_{+\mu}\,. \tag{5}$$

For the light-like momentum, we have

$$P \cdot z = n_+ \cdot P n_- \cdot z. \tag{6}$$

$$d^4k = dk^+ dk^- d^2 k_\perp, \quad \mu^{4-d} d^d k = \mu^{2\epsilon} dk^+ dk^- d^{d-2} k_\perp, \tag{7}$$

II. BRODSKY-LEPAGE EVOLUTION

The LCDA for a light meson is formally defined as:

$$\langle \pi(P) | (\bar{u} W_c)(z) \gamma_\mu \gamma_5 (W_c^\dagger d)(0) | 0 \rangle = -i f_\pi P_\mu \int_0^1 dx e^{ixP\cdot z} \Phi(x,\mu), \tag{8}$$

and the LCDA is normalized as

$$\int_0^1 dx \Phi(x,\mu) = 1. \tag{9}$$

The Wilson line is defined as

$$W_c(z) = \mathrm{Pexp}\left(ig_s \int_{-\infty}^0 ds n_+ A(z + s n_+) \right) \tag{10}$$

The Brodsky-Lepage evolution for a LCDA is given as:

$$\mu \frac{d}{d\mu} \Phi(x,\mu) \;=\; \frac{\alpha_s(\mu)}{\pi} C_F \int_0^1 dy V_0(x,y) \Phi(y,\mu). \tag{11}$$

$$V_0(x,y) = \left[\frac{1-x}{1-y} \left(1 + \frac{1}{x-y} \right) \theta(x-y) + \frac{x}{y} \left(1 + \frac{1}{y-x} \right) \theta(y-x) \right]_+ . \tag{12}$$

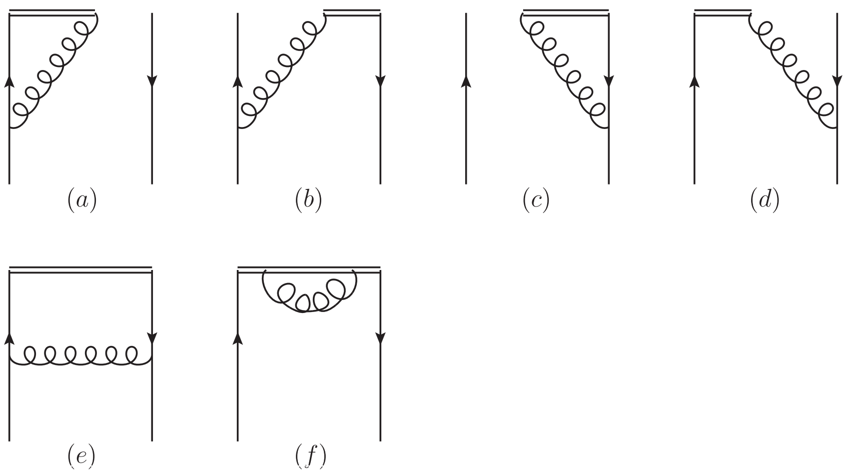

FIG. 1: Brodsky-Lepage evolution kernel at 1 loop. These diagrams are the same for the evolution of PDF, DGLAP.

Here we should understand the LCDA as a renormalized quantity.

In order to derive the evolution kernel, one needs to perform the matching and technically calculate the diagrams in Fig. 1. Notice that in these diagrams the external lines are free quarks. So what one is going to compute is:

$$\langle u(x_0P)\bar{d}(\bar{x}_0P)|(\bar{u}W_c)(z)\gamma_\mu\gamma_5(W_c^\dagger d)(0)|0\rangle = -i\hat{f}_\pi P_\mu \int_0^1 dx e^{ixP\cdot z}\hat{\Phi}(x,\mu), \tag{13}$$

We will assume the P_μ is moving on the n_- direction, and thus it is convenient to calculate the Frourier transformed matrix element:

$$\int \frac{d(n_-\cdot z)}{2\pi} e^{-ixP\cdot z}\langle u(x_0P)\bar{d}(\bar{x}_0P)|(\bar{u}W_c)(z)\not{n}_+\gamma_5(W_c^\dagger d)(0)|0\rangle = -i\hat{f}_\pi\hat{\Phi}(x,\mu)\,. \tag{14}$$

notice that the factor 1/2 arises from the Frourier transformation.

From the viewpoint of composite operator renormalization, one should also add $(Z_2/Z_O - 1)$ times the tree-level result, in order to make the full results finite. The tree-level result of the above matrix element is

$$\langle\Phi\rangle^{(0)} = \frac{1}{n_+\cdot P}\bar{u}(xP)\not{n}_+\gamma_5 d(\bar{x}P)\delta(x-x_0). \tag{15}$$

Thus one can formally define the following result for decay constant and LCDA:

$$-i\hat{f}_\pi = \frac{1}{n_+\cdot P}\bar{u}(xP)\not{n}_+\gamma_5 d(\bar{x}P), \quad \hat{\Phi}^{(0)}(x,\mu) = \delta(x-x_0). \tag{16}$$

Let us calculate the digram (a) in Fig. 1.

$$\begin{aligned}
\langle\Phi\rangle^{(a)} &= \int \frac{d(n_-\cdot z)}{2\pi}e^{-ixP\cdot z}\langle u(x_0P)\bar{d}(\bar{x}_0P)|\bar{u}(z)\not{n}_+\gamma_5\left(-ig_s\int_{-\infty}^0 ds n_+\cdot A(sn_+)d\right)(0)\int d^4z_1 ig_s\big(\bar{d}T^b\not{A}^b d\big)(z_1)|0\rangle \\
&= g_s^2 C_F \int \frac{d(n_-\cdot z)}{2(2\pi)}e^{-ixP\cdot z}\int_{-\infty}^0 ds\int d^4z_1\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}e^{-iq\cdot(z_1-sn_+)}e^{-ik\cdot(0-z_1)}\frac{-i}{q^2}e^{ix_0P\cdot z}e^{i\bar{x}_0P\cdot z_1} \\
&\quad\times\bar{u}(x_0P)\not{n}_+\gamma_5\frac{i\not{k}}{k^2}\not{n}_+ d(\bar{x}_0P) \\
&= g_s^2 C_F\frac{\delta(x-x_0)}{n_+\cdot P}\int d^4z_1\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\frac{1}{in_+\cdot q}\frac{i}{k^2}\frac{-i}{q^2}e^{-iq\cdot(z_1-0)}e^{-ik\cdot(0-z_1)}e^{i\bar{x}P\cdot z_1} \\
&\quad\times\bar{u}(xP)\not{n}_+\gamma_5\not{k}\not{n}_+ d(\bar{x}P).
\end{aligned} \tag{17}$$

The momentum and coordinate integration gives $k = q - \bar{x}P$. One can also use the following simplification method:

$$\bar{u}(xP)\not{n}_+\gamma_5\not{k}\not{n}_+ d(\bar{x}P) = n_+\cdot k\bar{u}(xP)\not{n}_+\gamma_5\not{n}_-\not{n}_+ d(\bar{x}P) = 2n_+\cdot k\bar{u}(xP)\not{n}_+\gamma_5 d(\bar{x}P). \tag{18}$$

Thus we have

$$
\begin{aligned}
\langle\Phi\rangle^{(a)} &= \langle\Phi\rangle^{(0)} \times (-i) g_s^2 C_F \int \frac{d^4q}{(2\pi)^4} 2[n_+\cdot q - \bar{x} n_+\cdot P] \frac{1}{n_+\cdot q} \frac{1}{(q-\bar{x}P)^2} \frac{1}{q^2} \\
&= \langle\Phi\rangle^{(0)} \times i g_s^2 C_F \int \frac{d^4q}{(2\pi)^4} 2[\bar{x} n_+\cdot P - n_+\cdot q] \frac{1}{n_+\cdot q} \frac{1}{(q-\bar{x}P)^2} \frac{1}{q^2} \\
&= \langle\Phi\rangle^{(0)} \times i g_s^2 C_F \int \frac{dq^+ dq^- d^2q_\perp}{(2\pi)^4} 2[\bar{x} n_+\cdot P - n_+\cdot q] \frac{1}{n_+\cdot q} \frac{1}{2[q^+ - \bar{x}P^+]q^- + q_\perp^2} \frac{1}{2q^+q^- + q_\perp^2}.
\end{aligned} \tag{19}
$$

The integration over q^- can be performed using contour integration. The integral is nonzero when $0 < q^+ < \bar{x}P^+$:

$$
\begin{aligned}
\langle\Phi\rangle^{(a)} &= \langle\Phi\rangle^{(0)} \times i g_s^2 C_F \int_0^{\bar{x}P^+} dq^+ \int \frac{d^2q_\perp}{(2\pi)^4} [\bar{x} n_+\cdot P - n_+\cdot q] \frac{1}{n_+\cdot q} (-2\pi i) \frac{1}{-[q^+ - \bar{x}P^+]\frac{q_\perp^2}{q^+} + q_\perp^2} \frac{1}{q^+} \\
&= \langle\Phi\rangle^{(0)} \times (-1) \frac{g_s^2}{8\pi^2} C_F \int_0^{\bar{x}} dy \frac{\bar{x}-y}{\bar{x}y} \frac{1}{\hat{\epsilon}} \\
&= \langle\Phi\rangle^{(0)} \times (-1) \frac{\alpha_s}{2\pi} C_F \int_0^{\bar{x}} dy \frac{\bar{x}-y}{\bar{x}y} \frac{1}{\hat{\epsilon}} \\
&= \delta(x-x_0) \frac{\bar{u}(x_0P) \not{n}_+ \gamma_5 d(\bar{x}_0 P)}{n_+\cdot P} \times (-1) \frac{\alpha_s}{2\pi} C_F \int_0^{\bar{x}} dy \frac{\bar{x}-y}{\bar{x}y} \frac{1}{\hat{\epsilon}}
\end{aligned} \tag{20}
$$

where it should be noted that the integration about $q_\perp^2$ is

$$
\int \frac{d^2q_\perp}{q_\perp^2} = -\frac{\pi}{\hat{\epsilon}}. \tag{21}
$$

The second diagram gives

$$
\begin{aligned}
\langle\Phi\rangle^{(b)} &= \int \frac{d(n_-\cdot z)}{2\pi} e^{-ixP\cdot z} \langle u(x_0P)\bar{d}(\bar{x}_0P)|\bar{u}(z) \left(i g_s \int_{-\infty}^0 ds\, n_+\cdot A(z+sn_+) \right) \not{n}_+ \gamma_5 d(0) \int d^4z_1\, i g_s \big(\bar{d} T^b \not{A}^b d\big)(z_1)|0\rangle \\
&= -g_s^2 C_F \int \frac{d(n_-\cdot z)}{2\pi} e^{-ixP\cdot z} \int_{-\infty}^0 ds \int d^4z_1 \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} e^{-iq\cdot(z+sn_+-z_1)} e^{-ik\cdot(0-z_1)} \frac{-i}{q^2} e^{ix_0P\cdot z} e^{i\bar{x}_0P\cdot z_1} \\
&\quad \times \bar{u}(x_0P) \not{n}_+ \gamma_5 \frac{i\not{k}}{k^2} \not{n}_+ d(\bar{x}_0P) \\
&= -g_s^2 C_F \int d^4z_1 \int \frac{d^4q}{(2\pi)^4} \frac{\delta(x-x_0+q^+/P^+)}{n_+\cdot P} \int \frac{d^4k}{(2\pi)^4} \frac{1}{-in_+\cdot q} \frac{i}{k^2} \frac{-i}{q^2} e^{-iq\cdot(0-z_1)} e^{-ik\cdot(0-z_1)} e^{i\bar{x}_0P\cdot z_1} \\
&\quad \times \bar{u}(xP) \not{n}_+ \gamma_5 \not{k} \not{n}_+ d(\bar{x}P) \\
&= \frac{\bar{u}(x_0P) \not{n}_+ \gamma_5 d(\bar{x}_0P)}{n_+\cdot P} \times (-i g_s^2) C_F \int \frac{d^4q}{(2\pi)^4} \delta(x-x_0-q^+/P^+) 2[\bar{x}_0 n_+\cdot P - n_+\cdot q] \frac{1}{n_+\cdot q} \frac{1}{(q-\bar{x}_0P)^2} \frac{1}{q^2} \\
&= \frac{\bar{u}(x_0P) \not{n}_+ \gamma_5 d(\bar{x}_0P)}{n_+\cdot P} \times (-i g_s^2) C_F \int_0^{\bar{x}_0P^+} dq^+ \delta(x-x_0-q^+/P^+) [\bar{x}_0P^+ - q^+] \frac{-i}{2\pi\times 4\pi} \frac{1}{q^+} \frac{1}{-\bar{x}_0P^+} \frac{1}{\hat{\epsilon}} \\
&= \frac{\bar{u}(x_0P) \not{n}_+ \gamma_5 d(\bar{x}_0P)}{n_+\cdot P} \times (-i g_s^2) C_F \theta(x-x_0) P^+ [\bar{x}_0P^+ - xP^+ + x_0P^+] \frac{-i}{2\pi\times 4\pi} \frac{1}{x-x_0} \frac{1}{P^+} \frac{1}{-\bar{x}_0P^+} \frac{1}{\hat{\epsilon}} \\
&= \frac{\bar{u}(x_0P) \not{n}_+ \gamma_5 d(\bar{x}_0P)}{n_+\cdot P} \times \frac{\alpha_s}{2\pi} C_F \theta(x-x_0) \frac{1-x}{x-x_0} \frac{1}{\bar{x}_0} \frac{1}{\hat{\epsilon}}
\end{aligned} \tag{22}
$$

I will use the Feynman rule method to calculate this diagram again. The momentum q is choosing flowing into the quark line. We have the second diagram:

$$
\begin{aligned}
\langle\Phi\rangle^{(b)} &= C_F\int\frac{d^dq}{(2\pi)^d}\bar{u}(x_0P)\not{h}_+\gamma_5\frac{-i(\bar{x}_0\not{P}-\not{q})}{(\bar{x}_0P-q)^2}ig_s\gamma_\mu d(\bar{x}_0P)\times ig_sn_+^\mu\frac{1}{in_+\cdot q}\frac{-i}{q^2}\delta(xn_+\cdot P-x_0n_+\cdot P-n_+\cdot q)\\
&= -ig_s^2C_F\int\frac{d^dq}{(2\pi)^d}\bar{u}(x_0P)\not{h}_+\gamma_5(\bar{x}_0\not{P}-\not{q})\not{n}_+d(\bar{x}_0P)\frac{1}{(\bar{x}_0P-q)^2}\frac{1}{q^2}\frac{1}{q^+}\delta(xP^+-x_0P^+-q^+)\\
&= -ig_s^2C_F\bar{u}(x_0P)\not{h}_+\gamma_5d(\bar{x}_0P)(1-x)P^+\int\frac{dq^-d^2q_\perp}{(2\pi)^4}\frac{1}{xP^+-x_0P^+}\frac{1}{q^2}\frac{1}{(q-\bar{x}_0P)^2}\\
&= -ig_s^2C_F\bar{u}(x_0P)\not{h}_+\gamma_5d(\bar{x}_0P)\frac{1-x}{x-x_0}\frac{2\pi i}{\bar{x}_0P^+}\frac{\pi}{\hat{\epsilon}}\frac{1}{(2\pi)^4}\theta(x-x_0)\\
&= \frac{1}{P^+}\bar{u}(x_0P)\not{h}_+\gamma_5d(\bar{x}_0P)\frac{\alpha_s}{2\pi}C_F\theta(x-x_0)\frac{1-x}{x-x_0}\frac{1}{\bar{x}_0}\frac{1}{\hat{\epsilon}}.
\end{aligned}
\tag{23}
$$

The third diagram is given as

$$
\begin{aligned}
\langle\Phi\rangle^{(c)} &= \int\frac{d(n_-z)}{2\pi}e^{-ixPz}\langle u(x_0P)\bar{d}(\bar{x}_0P)|\bar{u}(z)\Big(ig_s\int_{-\infty}^0 dsn_+A(z+sn_+)\Big)\not{n}_+\gamma_5d(0)\int d^4z_1ig_s\Big[\bar{u}(z_1)T^b\not{A}^b(z_1)u(z_1)\Big]|0\rangle\\
&= -g_s^2C_F\int\frac{d(n_-z)}{2(2\pi)}e^{-ixPz}\int_{-\infty}^0 ds\int d^4z_1\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}e^{-ik(z_1-z)}\frac{-i}{q^2}e^{-iq(z_1-z-sn_+)}e^{ix_0Pz_1}\\
&\quad\times\bar{u}(x_0P)\not{n}_+\frac{i\not{k}}{k^2}\not{n}_+\gamma_5d(\bar{x}_0P)\\
&= ig_s^2C_F\frac{1}{n_+P}\delta(x-\frac{n_+q}{n_+P}-\frac{x_0n_+P-n_+q}{n_+P})\int\frac{d^4q}{(2\pi)^4}\frac{1}{n_+q}\frac{1}{q^2}\frac{1}{(x_0P-q)^2}2n_+k\Big(\bar{u}(x_0P)\not{n}_+\gamma_5d(\bar{x}_0P)\Big)\\
&= ig_s^2C_F\frac{1}{n_+P}\delta(x-x_0)\int\frac{d^4q}{(2\pi)^4}\frac{1}{n_+q}\frac{1}{q^2}\frac{1}{(x_0P-q)^2}2(x_0n_+P-n_+q)\Big(\bar{u}(x_0P)\not{n}_+\gamma_5d(\bar{x}_0P)\Big)\\
&= ig_s^2C_F\frac{1}{n_+P}\delta(x-x_0)i\frac{1}{8\pi^2}\frac{1}{\hat{\epsilon}}\int_0^{x_0}dy\frac{x_0-y}{x_0y}\Big(\bar{u}(x_0P)\not{n}_+\gamma_5d(\bar{x}_0P)\Big)\\
&= \frac{1}{n_+P}\delta(x-x_0)\Big(\bar{u}(x_0P)\not{n}_+\gamma_5d(\bar{x}_0P)\Big)(-1)\frac{\alpha_s}{2\pi}C_F\frac{1}{\hat{\epsilon}}\int_0^{x_0}dy\frac{x_0-y}{x_0y}\,.
\end{aligned}
\tag{24}
$$

I will use the Feynman rule method to calculate this diagram again.

$$
\begin{aligned}
\langle\Phi\rangle^{(c)} &= C_F\int\frac{d^dq}{(2\pi)^d}\bar{u}(x_0P)ig_s\gamma_\mu\frac{i(x_0\not{P}-\not{q})}{(x_0P-q)^2}\not{n}_+\gamma_5d(\bar{x}_0P)\times ign_+^\mu\frac{1}{n_+\cdot q}\frac{-i}{q^2}\delta(xP^+-x_0P^+)\\
&= ig_s^2C_F\delta(xP^+-x_0P^+)\int\frac{d^dq}{(2\pi)^d}\frac{\bar{u}(x_0P)\not{n}_+(x_0\not{P}-\not{q})\not{n}_+\gamma_5d(\bar{x}_0P)}{(x_0P-q)^2q^+q^2}\\
&= ig_s^2C_F\delta(xP^+-x_0P^+)\bar{u}(x_0P)\not{n}_+\gamma_5d(\bar{x}_0P)\frac{1}{2}\int\frac{dq^+dq^-d^2q_\perp}{(2\pi)^d}\frac{2(x_0P^+-q^+)}{(x_0P-q)^2q^+q^2}\\
&= ig_s^2C_F\delta(xP^+-x_0P^+)\bar{u}(x_0P)\not{n}_+\gamma_5d(\bar{x}_0P)\int\frac{dq^+dq^-d^2q_\perp}{(2\pi)^d}(-2\pi i)\frac{x_0P^+-q^+}{q^+}\frac{1}{q^+\Big[(q^+-x_0P^+)\frac{q_\perp^2}{q^+}-q_\perp^2\Big]}\\
&= \delta(x-x_0)\frac{\bar{u}(x_0P)\not{n}_+\gamma_5d(\bar{x}_0P)}{n_+\cdot P}\times(-1)\frac{\alpha_s}{2\pi}C_F\int_0^{x_0}dy\frac{x_0-y}{yx_0}\frac{1}{\hat{\epsilon}}.
\end{aligned}
\tag{25}
$$

The fourth diagram is given as

$$\begin{aligned}
\langle\Phi\rangle^{(d)} &= \int \frac{d(n_-z)}{2\pi} e^{-ixPz} \langle u(x_0P)\bar{d}(\bar{x}_0P)|\bar{u}(z)\not{n}_+\gamma_5\Big(-ig_s\int_{-\infty}^0 dsn_+A(sn_+)\Big)d(0)\int d^4z_1 ig_s\Big[\bar{u}(z_1)T^b\not{A}^b(z_1)u(z_1)\Big]|0\rangle \\
&= g_s^2C_F\int\frac{d(n_-z)}{2\pi}e^{-ixPz}\int_{-\infty}^0 ds\int d^4z_1\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}e^{-ik(z_1-z)}\frac{-i}{q^2}e^{-iq(z_1-sn_+)}e^{ix_0Pz_1} \\
&\quad\times\bar{u}(x_0P)\not{n}_+\frac{i\not{k}}{k^2}\not{n}_+\gamma_5 d(\bar{x}_0P) \\
&= g_s^2C_F\frac{1}{n_+P}\delta(x-\frac{n_+k}{n_+P})\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\frac{1}{in_+q}\frac{1}{q^2}\frac{1}{k^2}\delta(k+q-x_0P)(2\pi)^4 2n_+k\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big) \\
&= -ig_s^2C_F\frac{1}{n_+P}\delta(x-x_0+\frac{n_+q}{n_+P})\int\frac{d(n_+q)d^2q_\perp d(n_-q)}{(2\pi)^4}\frac{1}{n_+q}\frac{1}{n_+qn_-q+q_\perp^2}\frac{1}{n_-q(n_+q-x_0n_+P)+q_\perp^2} \\
&\quad\times[x_0n_+P-n_+q]\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big) \\
&= \frac{g_s^2}{8\pi}C_F\frac{1}{n_+P}\delta(x-x_0+y)\frac{1}{\hat{\epsilon}}\int_0^{x_0}dy(n_+P)\frac{1}{y^2(n+P)^2}[x_0n_+P-n_+q]\frac{y}{x_0}\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big) \\
&= \frac{\alpha_s}{2\pi}C_F\frac{1}{n_+P}\frac{1}{\hat{\epsilon}}\int_0^{x_0}dy\frac{(x_0-y)y}{y^2x_0}\delta(x-x_0+y)\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big) \\
&= \frac{\alpha_s}{2\pi}C_F\frac{1}{n_+P}\frac{1}{\hat{\epsilon}}\frac{x}{(x_0-x)x_0}\theta(x_0-x)\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big)\,.
\end{aligned} \tag{26}$$

I will use the Feynman rule method to calculate this diagram again.

$$\begin{aligned}
\langle\Phi\rangle^{(d)} &= C_F\int\frac{d^dq}{(2\pi)^d}\bar{u}(x_0P)ig_s\gamma_\mu\frac{i(x_0\not{P}-\not{q})}{(x_0P-q)^2}\not{n}_+\gamma_5 d(\bar{x}_0P)\times(-ig_sn_+^\mu)\frac{1}{iq^+}\frac{-i}{q^2}\delta(xP^+-x_0P^++q^+) \\
&= -ig_s^2C_F\int\frac{d^dq}{(2\pi)^d}\bar{u}(x_0P)\not{n}_+(x_0\not{P}-\not{q})\not{n}_+\gamma_5 d(\bar{x}_0P)\frac{1}{(x_0P-q)^2}\frac{1}{q^+}\frac{1}{q^2}\delta(xP^+-x_0P^++q^+) \\
&= -ig_s^2C_F\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\int\frac{dq^+dq^-d^2q_\perp}{(2\pi)^d}\frac{x_0P^+-q^+}{(x_0P-q)^2}\frac{1}{q^+}\frac{1}{q^2}\delta(xP^+-x_0P^++q^+) \\
&= -ig_s^2C_F\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)xP^+\int_0^{x_0P^+}dq^+\int\frac{d^2q_\perp}{(2\pi)^d}\frac{1}{x_0-x}\frac{1}{P^+}(-2\pi i)\frac{1}{-x_0P^+}\frac{1}{q_\perp^2}\delta(xP^+-x_0P^++q^+) \\
&= \frac{1}{n_+P}\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\frac{\alpha_s}{2\pi}C_F\theta(x_0-x)\frac{x}{(x_0-x)x_0}\frac{1}{\hat{\epsilon}}.
\end{aligned} \tag{27}$$

The fifth diagram is given as

$$\begin{aligned}
\langle\Phi\rangle^{(e)} &= \int\frac{d(n_-z)}{2\pi}e^{-ixPz}\langle u(x_0P)\bar{d}(\bar{x}_0P)|\bar{u}(z)\not{n}_+\gamma_5 d(0)\int d^4z_1 ig_s\Big[\bar{d}(z_1)T^a\not{A}^a(z_1)d(z_1)\Big]\int d^4z_2 ig_s\Big[\bar{u}(z_2)T^b\not{A}^b(z_2)u(z_2)\Big]|0\rangle \\
&= -g_s^2C_F\int\frac{d(n_-z)}{2\pi}e^{-ixPz}\int d^4z_1\int d^4z_2\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4} \\
&\quad\times e^{-ik_2(z_2-z)}\frac{-i}{q^2}e^{-iq(z_2-z_1)}e^{-ik_1(0-z_1)}e^{ix_0Pz_2}e^{i\bar{x}_0Pz_1}\Big(\bar{u}(x_0P)\gamma^\mu\frac{i\not{k}_2}{k_2^2}\not{n}_+\gamma_5\frac{i\not{k}_1}{k_1^2}\gamma_\mu d(\bar{x}_0P)\Big) \\
&= -ig_s^2C_F\frac{1}{n_+P}\delta(x-x_0+\frac{n_+q}{n_+P})\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\frac{1}{(\bar{x}_0P+q)^2}\frac{2q_\perp^2}{(x_0P-q)^2}\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big) \\
&= -ig_s^2C_F\frac{1}{n_+P}\delta(x-x_0+\frac{n_+q}{n_+P})\int\frac{d(n_+q)d^2q_\perp d(n_-q)}{(2\pi)^4}\frac{q_\perp^2}{n_+qn_-q+q_\perp^2}\frac{1}{n_-q(n_+q-x_0n_+P)+q_\perp^2} \\
&\quad\times\frac{1}{n_-q(n_+q+\bar{x}_0n_+P)+q_\perp^2}\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big)\,.
\end{aligned} \tag{28}$$

There are 3 poles for the n_-q integration:

$$n_-q=\frac{-q_\perp^2-i\epsilon}{2n_+q},\qquad n_-q=\frac{-q_\perp^2-i\epsilon}{2(n_+q-x_0n_+P)},\qquad n_-q=\frac{-q_\perp^2-i\epsilon}{2(n_+q+\bar{x}_0n_+P)}\,, \tag{29}$$

when $0 < n_+q < x_0 n_+P$, one should choose the second pole in the upper half plane; when $-\bar{x}_0 n_+P < q^+ < 0$, one should choose the last pole in the lower half plane.

$$
\begin{aligned}
\langle\Phi\rangle^{(e)} &= -ig_s^2 C_F \frac{1}{n_+P}\delta(x - x_0 + \frac{n_+q}{n_+P})\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big)\Bigg\{ - \frac{2i\pi}{(2\pi)^4}\int_{-\bar{x}_0 n_+P}^{0} d(n_+q)\int d^2q_\perp\ q_\perp^2 \\
&\quad\times \frac{1}{\frac{-n_+q}{n_+q+\bar{x}_0n_+P}+1}\frac{1}{(q_\perp^2)^2}\frac{1}{\frac{-n_+q+x_0n_+P}{n_+q+\bar{x}_0n_+P}+1}\frac{1}{n_+q+\bar{x}_0n_+P} \\
&\quad+\frac{2i\pi}{(2\pi)^4}\int_0^{x_0n_+P} d(n_+q)\int d^2q_\perp\ q_\perp^2\frac{1}{\frac{-n_+q}{n_+q-x_0n_+P}+1}\frac{1}{(q_\perp^2)^2}\frac{1}{\frac{-n_+q-\bar{x}_0n_+P}{n_+q-x_0n_+P}+1}\frac{1}{n_+q-x_0n_+P}\Bigg\} \\
&= -ig_s^2 C_F\frac{1}{n_+P}\delta(x-x_0+y)\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big)\Bigg\{\frac{i}{8\pi^2}\frac{1}{\hat{\epsilon}}\int_{-\bar{x}_0}^{0} dy(n_+P)\frac{y+\bar{x}_0}{\bar{x}_0}(y+\bar{x}_0)\frac{1}{y+\bar{x}_0}\frac{1}{n_+P} \\
&\quad-\frac{i}{8\pi^2}\frac{1}{\hat{\epsilon}}\int_0^{x_0} dy(n_+P)\frac{y-x_0}{-x_0}(x_0-y)\frac{1}{y-x_0}\frac{1}{n_+P}\Bigg\} \\
&= \frac{\alpha_s}{2\pi}C_F\frac{1}{n_+P}\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big)\frac{1}{\hat{\epsilon}}\Big[\frac{1-x}{\bar{x}_0}\theta(x-x_0)+\frac{x}{x_0}\theta(x_0-x)\Big]\,. \qquad (30)
\end{aligned}
$$

I will use the Feynman rule method to calculate this diagram again.

$$
\begin{aligned}
\langle\Phi\rangle^{(e)} &= C_F\int\frac{d^dq}{(2\pi)^d}\bar{u}(x_0P)ig_s\gamma_\mu\frac{i(x_0\not{P}+\not{q})}{(x_0P+q)^2}\not{n}_+\gamma_5\frac{-i(\bar{x}_0\not{P}-\not{q})}{(\bar{x}_0P-q)^2}ig_s\gamma_\nu d(\bar{x}_0P)\times\frac{-ig^{\mu\nu}}{q^2}\delta(xP^+-x_0P^+-q^+) \\
&= (-1???)ig_s^2C_F\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\int\frac{d^dq}{(2\pi)^d}\frac{2q_\perp^2}{(x_0P+q)^2}\frac{1}{(\bar{x}_0P-q)^2}\frac{1}{q^2}\delta(xP^+-x_0P^+-q^+) \qquad (31)
\end{aligned}
$$

There are 3 poles for the q^- integration:

$$
q^- = \frac{-q_\perp^2-i\epsilon}{2q^+},\quad q^- = \frac{-q_\perp^2-i\epsilon}{2(q^+-\bar{x}_0P^+)},\quad q^- = \frac{-q_\perp^2-i\epsilon}{2(q^++x_0P^+)}\,, \qquad (32)
$$

when $0 < q^+ < \bar{x}_0P^+$, one should choose the second pole in the upper half plane; when $-x_0P^+ < q^+ < 0$, one should choose the last pole in the lower half plane.

$$
\begin{aligned}
\langle\Phi\rangle^{(e)} &= (-1???)ig_s^2C_F\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\int\frac{dq^+dq^-d^2q_\perp}{(2\pi)^d}\frac{q_\perp^2}{(x_0P+q)^2}\frac{1}{(\bar{x}_0P-q)^2}\frac{1}{q^2}\delta(xP^+-x_0P^+-q^+) \\
&= (-1???)ig_s^2C_F\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\int\frac{d^2q_\perp}{(2\pi)^d} \\
&\quad\times\Bigg\{\int_0^{\bar{x}_0P^+}dq^+2\pi i\frac{q_\perp^2}{q^+-\bar{x}_0P^+}\frac{\delta(xP^+-x_0P^+-q^+)}{q^+\frac{q_\perp^2}{q^+-\bar{x}_0P^+}-q_\perp^2}\frac{1}{(q^++x_0P^+)\frac{q_\perp^2}{q^+-\bar{x}_0P^+}-q_\perp^2} \\
&\quad+\int_{-x_0P^+}^{0}dq^+(-2\pi i)\frac{q_\perp^2}{q^++x_0P^+}\frac{\delta(xP^+-x_0P^+-q^+)}{q^+\frac{q_\perp^2}{q^++x_0P^+}-q_\perp^2}\frac{1}{(q^+-\bar{x}_0P^+)\frac{q_\perp^2}{q^++x_0P^+}-q_\perp^2}\Bigg\} \\
&= -\frac{\alpha_s}{2\pi}C_F\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\frac{1}{\hat{\epsilon}} \\
&\quad\times\Bigg\{\int_0^{\bar{x}_0P^+}dq^+\frac{1}{\bar{x}_0P^+}\frac{q^+-\bar{x}_0P^+}{P^+}-\int_{-x_0P^+}^{0}dq^+\frac{q^++x_0P^+}{-x_0P^+}\frac{1}{-\bar{x}_0P^+-x_0P^+}\Bigg\}\delta(xP^+-x_0P^+-q^+) \\
&= -\frac{\alpha_s}{2\pi}C_F\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\frac{1}{P^+}\frac{1}{\hat{\epsilon}}\left(\theta(x-x_0)\frac{x-1}{1-x_0}-\theta(x_0-x)\frac{x}{x_0}\right) \qquad (33)
\end{aligned}
$$

The QCD self-energy correction gives

$$
\begin{aligned}
\langle\Phi\rangle^{(s.e)} &= (Z_2-1)\frac{1}{n_+\cdot P}\bar{u}(xP)\not{n}_+\gamma_5 d(\bar{x}P)\delta(x-x_0) \\
&= -\frac{\alpha_s}{4\pi}C_F\frac{1}{\hat{\epsilon}}\frac{1}{n_+\cdot P}\bar{u}(xP)\not{n}_+\gamma_5 d(\bar{x}P)\delta(x-x_0) \qquad (34)
\end{aligned}
$$

Add the divergences from these five diagrams up

$$\langle\Phi\rangle^{(all)} = \frac{\alpha_s}{2\pi}C_F\frac{1}{n_+P}\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big)\frac{1}{\hat{\epsilon}}\Bigg\{\underbrace{-\delta(x-x_0)\int_0^{\bar{x}_0}dy\frac{\bar{x}_0-y}{\bar{x}_0 y}}_{T_1}+\underbrace{\theta(x-x_0)\frac{1-x}{x-x_0}\frac{1}{\bar{x}_0}}_{T_2}$$
$$\underbrace{-\delta(x-x_0)\int_0^{x_0}dy\frac{x_0-y}{yx_0}}_{T_3}+\underbrace{\theta(x_0-x)\frac{x}{(x_0-x)x_0}}_{T_4}+\Big(\underbrace{\frac{1-x}{\bar{x}_0}\theta(x-x_0)}_{T_{51}}+\underbrace{\frac{x}{x_0}\theta(x_0-x)}_{T_{52}}\Big)\Bigg\} \quad (35)$$

Next, we define plus function,

$$\int_0^1 dx\Big[f(x)\Big]_+ g(x) \equiv \int_0^1 dx f(x)(g(x)-g(x_0)), \quad (36)$$

so, we have

$$f(x) = \Big[f(x)\Big]_+ + \Big[\int_0^1 dy f(y)\Big]\delta(x-x_0). \quad (37)$$

We will use the T_2 and demonstrate the plus function:

$$\begin{aligned} T_2 &= \Big[\theta(x-x_0)\frac{1-x}{x-x_0}\frac{1}{\bar{x}_0}\Big]_+ + \delta(x-x_0)\int_0^1 dy\theta(y-x_0)\frac{1-y}{y-x_0}\frac{1}{\bar{x}_0} \\ &= \Big[\theta(x-x_0)\frac{1-x}{x-x_0}\frac{1}{\bar{x}_0}\Big]_+ + \delta(x-x_0)\int_0^{1-x_0} dy\frac{\bar{x}_0-y}{y}\frac{1}{\bar{x}_0}, \end{aligned} \quad (38)$$

where the second term exactly cancels the T_1.

Nest we will do some modifications of (35),

$$\begin{aligned} T_2+T_{51} &= \frac{1-x}{\bar{x}_0}\Big[1+\frac{1}{x-x_0}\Big]\theta(x-x_0) \quad \longrightarrow \quad [\mathbf{a}] \\ T_4+T_{52} &= \frac{x}{x_0}\Big[1+\frac{1}{x_0-x}\Big]\theta(x_0-x) \quad \longrightarrow \quad [\mathbf{b}] \end{aligned}$$

So,

$$\begin{aligned} [\mathbf{a}] &= [\mathbf{a}]_+ + \int_0^1 dy\ \mathbf{a}(y)\delta(x-x_0) \\ &= [\mathbf{a}]_+ + \int_{x_0}^1 dy\frac{1-y}{\bar{x}_0}[1+\frac{1}{y-x_0}]\delta(x-x_0) \\ &= [\mathbf{a}]_+ + \int_0^{\bar{x}_0} dy\frac{\bar{x}_0-y}{\bar{x}_0}[1+\frac{1}{y}]\delta(x-x_0) \quad \longrightarrow \quad \mathbf{(1)} \end{aligned}$$

The $1/y$ will cancel the T_1 term. The b term is given as

$$\begin{aligned} [\mathbf{b}] &= [\mathbf{b}]_+ + \int_0^1 dy\ \mathbf{b}(y)\delta(x-x_0) \\ &= [\mathbf{b}]_+ + \int_0^{x_0} dy\frac{y}{x_0}[1+\frac{1}{x_0-y}]\delta(x-x_0) \\ &= [\mathbf{b}]_+ + \int_0^{x_0} dy\frac{x_0-y}{x_0}[1+\frac{1}{y}]\delta(x-x_0) \quad \longrightarrow \quad \mathbf{(2)} \end{aligned}$$

while the $1/y$ integration will cancel T_3. Finally, (35) can be written as

$$\begin{aligned} \langle\Phi\rangle^{(all)} &= \frac{\alpha_s}{4\pi}C_F\frac{1}{n_+P}\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big)\frac{1}{\hat{\epsilon}}2\Big\{T_1+T_3+\mathbf{(1)}+\mathbf{(2)}\Big\} \\ &= \frac{\alpha_s}{4\pi}C_F\frac{1}{n_+P}\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big)\frac{1}{\hat{\epsilon}}\Big\{2[\mathbf{a}]_+ + 2[\mathbf{b}]_+ + \delta(x-x_0)\Big\}. \end{aligned} \quad (39)$$

Consider the QCD self-energy correction,

$$\langle\Phi\rangle^{(all)} + \langle\Phi\rangle^{(s.e)} = \frac{1}{n_+P}\Big(\bar{u}(x_0P)\not{n}_+\gamma_5 d(\bar{x}_0P)\Big)\frac{\alpha_s}{4\pi}C_F\frac{2}{\hat{\epsilon}}\int_0^1 dy\ V_0(x,y)\Phi^{(0)}(y,\mu)\,. \tag{40}$$

where $\Phi^{(0)}(y,\mu) = \delta(x_0 - y)$. So,

$$V_0(x,y) = \Big[\frac{1-x}{1-y}[1+\frac{1}{x-y}]\theta(x-y) + \frac{x}{y}[1+\frac{1}{y-x}]\theta(y-x)\Big]_+\,. \tag{41}$$

One should notice that, after the composite operator renormalization, the divergences will be removed and what will be left are the $\ln(\mu^2)$. So the evolution is given as

$$\mu\frac{d\Phi(x,\mu)}{d\mu} = \frac{\alpha_s}{\pi}C_F\int_0^1 dy\ V_0(x,y)\Phi(y,\mu), \tag{42}$$

this is the Brodsky-Lepage evolution.

For a vector meson, there is a second type of LCDA at leading twist:

$$\langle\rho(\epsilon,P)|(\bar{u}W_c)(z)\frac{\not{n}_+}{2}\gamma_\alpha^\perp(W_c^\dagger d)(0)|0\rangle = -if_\rho\epsilon_\alpha^\perp\int_0^1 dx e^{ixP\cdot z}\Phi^T(x,\mu), \tag{43}$$

it should be noted that the decay constant f_ρ^T also receives renormalization. This can be calculated using the tensor current, where the only contribution is from self-energy diagrams. We have

$$\mu\frac{df_\rho^T(\mu)}{d\mu} = -\frac{\alpha_s}{2\pi}f_\rho^T(\mu). \tag{44}$$

For the LCDA, only diagrams (a,b,c,d) contribute and thus the evolution kernel is given by

$$V_0^T(x,y) = \Big[\frac{1-x}{1-y}\frac{1}{x-y}\theta(x-y) + \frac{x}{y}\frac{1}{y-x}\theta(y-x)\Big]_+\,. \tag{45}$$

for the evolution equation:

$$\mu\frac{d\Phi^T(x,\mu)}{d\mu} = \frac{\alpha_s}{\pi}C_F\int_0^1 dy\ V_0(x,y)\Phi^T(y,\mu). \tag{46}$$

III. DGLAP EVOLUTION

The standard parton distribution function is defined as

$$f_q(x) = \int\frac{d(n_-\cdot z)}{2\pi}e^{-ixp\cdot z}\langle P(p)|(\bar{q}W_c)(z)\frac{\not{n}_+}{2}(W_c^\dagger q)(0)|P(p)\rangle, \tag{47}$$

$$f_g(x) = \sum_j\int\frac{d\omega^-}{2\pi xp^+}e^{-ixP^+\omega^-}\langle P(p)|(F^{+j}W_A)(\omega^-)(W_A^\dagger F^{+j})(0)|P(p)\rangle, \tag{48}$$

where one should notice that the subscript A on W_A denotes the Wilson line is in the adjoint representation.

The renormalized PDF is defined as

$$f_{j/H}(x,\mu) = \int\frac{dz}{z}[Z_{j'}(g,\epsilon)Z_{jj'}(z,g,\epsilon)][Z_{j'}^{-1}f_{(0)j'/H}(x/z,\mu)], \tag{49}$$

where j and j' can be quarks or gluons. The $Z_j^{-1}f_{(0)j'/H}$ is the parton density with renormalized rather than bare fields in its definition. Thus it is calculated using the standard Feynman rules for the theory and for the parton density, counterterms from the Lagrangian are used as needed. In the compensation for the $Z_{j'}^{-1}$ factor, the $Z_{jj'}$ factor is combined with a factor of $Z_{j'}$.

The RG can be generically written as

$$\frac{d}{d\ln\mu} f_{j/H}(\xi,\mu) = \sum_{j'} \int \frac{dz}{z} P_{jj'}(z,g) f_{j'/H}(\xi/z,\mu), \tag{50}$$

where the DGLAP evolution kernel $P_{jj'}$ obeys

$$\frac{d}{d\ln\mu} Z_{jk}(z,g,\epsilon) = \sum_{j'} \int \frac{dz'}{z'} P_{jj'}(z',g,\epsilon) Z_{j'k}(z/z',\mu), \tag{51}$$

namely,

$$P = \frac{d}{d\ln\mu} Z. \tag{52}$$

More explicitly, the DGLAP evolution equation at one-loop is given as

$$\begin{aligned}
\mu\frac{d}{d\mu} f_q(x,\mu) &= \frac{\alpha_s}{\pi}\int_x^1 \frac{dz}{z}\Big\{P_{q\to q}(z) f_q\Big(\frac{x}{z},\mu\Big) + P_{g\to q}(z) f_g\Big(\frac{x}{z},\mu\Big)\Big\}, && (53)\\
\mu\frac{d}{d\mu} f_{\bar q}(x,\mu) &= \frac{\alpha_s}{\pi}\int_x^1 \frac{dz}{z}\Big\{P_{q\to q}(z) f_{\bar q}\Big(\frac{x}{z},\mu\Big) + P_{g\to q}(z) f_g\Big(\frac{x}{z},\mu\Big)\Big\}, && (54)\\
\mu\frac{d}{d\mu} f_g(x,\mu) &= \frac{\alpha_s}{\pi}\int_x^1 \frac{dz}{z}\Big\{P_{q\to g}(z) \Big[f_q\Big(\frac{x}{z},\mu\Big) + f_{\bar q}\Big(\frac{x}{z},\mu\Big)\Big] + P_{g\to g}(z) f_g\Big(\frac{x}{z},\mu\Big)\Big\}, && (55)
\end{aligned}$$

with the evolution kernel

$$\begin{aligned}
P_{q\to q}(z) &= C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right], && (56)\\
P_{q\to g}(z) &= C_F\left[\frac{1+(1-z)^2}{z}\right], && (57)\\
P_{g\to q}(z) &= \frac{1}{2}\left[z^2+(1-z)^2\right], && (58)\\
P_{g\to g}(z) &= 2C_2(G)\left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z) + \left(\frac{11}{12} - \frac{n_f}{18}\right)\delta(1-z)\right], && (59)\\
P_{g\to g}(z) &= 2C_2(G)\left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z)\right] + \frac{1}{2}\left(11 - \frac{2n_f}{3}\right)\delta(1-z) \\
&= 2C_2(G)\left[\frac{1}{z} - 2 + \frac{1}{(1-z)_+} + z(1-z)\right] + \frac{1}{2}\left(11 - \frac{2n_f}{3}\right)\delta(1-z) && (60)
\end{aligned}$$

Feynman diagrams for quark PDFs are exactly the same as the LCDA ones.
At tree level, we may choose the free quarks as the incoming or outgoing states. Then we will obtain

$$f_q^{(0)}(x) = \delta(xp^+ - p^+)\bar u(p)\frac{\not n_+}{2}u(p) \to \delta(xp^+ - p^+)\bar u(p)\frac{1}{2}\mathrm{tr}\left[\frac{\not n_+}{2}\not p\right] = \delta(x-1). \tag{61}$$

The first diagram in Fig. 1 is given as

$$\begin{aligned}
f_q^{(1,a)}(x) &= \int \frac{d^4q}{(2\pi)^4}\bar u(p)(-ig_s t^a n_+^\mu)\frac{\not n_+}{2}\frac{i(\not p - \not q)}{(p-q)^2} igt^a\gamma_\mu u(p) \times \frac{-i}{q^2}\frac{1}{-in_+\cdot q}\delta(p^+ - xp^+)\\
&= ig_s^2 C_F \int \frac{dq^+dq^-d^2q_\perp}{(2\pi)^d} 2(p^+ - q^+)p^+\delta(p^+ - xp^+)\frac{1}{q^2}\frac{1}{q^+}\frac{1}{(q-p)^2}\\
&= ig_s^2 C_F \int \frac{dq^+d^2q_\perp}{(2\pi)^d} 2(p^+ - q^+)\delta(x-1) \times (-2\pi i)\frac{1}{q^+}\frac{1}{2q^+}\frac{1}{(q^+ - p^+)\frac{-q_\perp^2}{q^+} + q_\perp^2}\\
&= -\frac{g_s^2}{8\pi^2}C_F\frac{1}{\epsilon}\delta(x-1)\int dq^+(p^+ - q^+)\frac{1}{q^+p^+}\\
&= -\frac{g_s^2}{8\pi^2}C_F\frac{1}{\epsilon}\delta(x-1)\int_0^1 dy\frac{y}{1-y}. && (62)
\end{aligned}$$

The third diagram gives identically the same contribution.

The second diagram gives the contribution:

$$
\begin{aligned}
f_q^{(1,b)}(x) &= \int \frac{d^4q}{(2\pi)^4}\bar{u}(p)(ig_s t^a n_+^\mu)\frac{\not{n}_+}{2}\frac{i(\not{p}-\not{q})}{(p-q)^2}igt^a\gamma_\mu u(p)\times\frac{-i}{q^2}\frac{1}{-in_+\cdot q}\delta(p^+-xp^+-q^+) \\
&= -ig_s^2C_F\int\frac{dq^+dq^-d^2q_\perp}{(2\pi)^4}2(p^+-q^+)p^+\delta(p^+-xp^+-q^+)\frac{1}{q^+}\frac{1}{q^2}\frac{1}{(q-p)^2} \\
&= -ig_s^2C_F\int\frac{dq^+d^2q_\perp}{(2\pi)^4}2(p^+-q^+)p^+\delta(p^+-xp^+-q^+)\frac{1}{q^+}\times(-2\pi i)\frac{1}{2q^+}\frac{1}{(q^+-p^+)\frac{-q_\perp^2}{q^+}+q_\perp^2} \\
&= -\frac{g_s^2}{8\pi^3}C_F\int dq^+d^2q_\perp(p^+-q^+)p^+\delta(p^+-xp^+-q^+)\frac{1}{q^+}\times\frac{1}{p^+q_\perp^2} \\
&= \frac{g_s^2}{8\pi^2}C_F\frac{1}{\epsilon}\int dq^+(p^+-q^+)\delta(p^+-xp^+-q^+)\frac{1}{q^+} \\
&= \frac{g_s^2}{8\pi^2}C_F\frac{1}{\epsilon}\frac{x}{1-x},
\end{aligned}
\tag{63}
$$

and the fourth diagram gives the same contribution.

The fifth diagram gives

$$
\begin{aligned}
f_q^{(1,f)}(x) &= \int\frac{d^4q}{(2\pi)^4}\bar{u}(p)igt^a\gamma^\mu\frac{i(\not{p}-\not{q})}{(p-q)^2}\frac{\not{n}_+}{2}\frac{i(\not{p}-\not{q})}{(p-q)^2}igt^a\gamma_\mu u(p)\frac{-i}{q^2}\delta(xp^+-p^++q^+) \\
&= -ig_s^2C_F\int\frac{dq^+dq^-d^2q_\perp}{(2\pi)^4}\bar{u}(p)\gamma^\mu(\not{p}-\not{q})\frac{\not{n}_+}{2}(\not{p}-\not{q})\gamma_\mu u(p)\frac{1}{[(q-p)^2]^2}\frac{1}{q^2}\delta(xp^+-p^++q^+) \\
&= -ig_s^2C_F(-2)\int\frac{dq^+dq^-d^2q_\perp}{(2\pi)^4}\bar{u}(p)\not{q}\frac{\not{n}_+}{2}\not{q}u(p)\frac{1}{[(q-p)^2]^2}\frac{1}{q^2}\delta(xp^+-p^++q^+) \\
&= -2ig_s^2C_F\int\frac{dq^+dq^-d^2q_\perp}{(2\pi)^4}\bar{u}(p)\frac{\not{n}_+}{2}u(p)\frac{q_\perp^2}{[(q-p)^2]^2}\frac{1}{q^2}\delta(xp^+-p^++q^+) \\
&= -2ig_s^2C_F\int\frac{dq^+d^2q_\perp}{(2\pi)^4}p^+\times(-2\pi i)\frac{q_\perp^2(q^+)^2}{[p^+q_\perp^2]^2}\frac{1}{2q^+} \\
&= \frac{g_s^2}{8\pi^2}C_F\frac{1}{\epsilon}(1-x).
\end{aligned}
\tag{64}
$$

Adding all contributions and including the self-energy diagrams, we have

$$
\begin{aligned}
f_q^{(1)}(x) &= \frac{g_s^2}{4\pi^2}C_F\frac{1}{\epsilon}\left\{\frac{x}{1-x}-\delta(x-1)\int_0^1 dy\frac{y}{1-y}+\frac{1-x}{2}-\frac{1}{4}\delta(1-x)\right\} \\
&= \frac{g_s^2}{8\pi^2}C_F\frac{1}{\epsilon}\left\{\left(\frac{2}{1-x}\right)_+-(1+x)+\frac{3}{2}\delta(1-x)\right\}.
\end{aligned}
\tag{65}
$$

Thus the Z function is given by the negative of the divergence:

$$
Z_{q/q}(g,z,\epsilon) = -\frac{g_s^2}{8\pi^2}C_F\frac{1}{\epsilon}\left\{\left(\frac{2}{1-z}\right)_+-(1+z)+\frac{3}{2}\delta(1-z)\right\},
\tag{66}
$$

which leads to

$$
P_{q/q}(z)=\frac{dZ}{d\ln\mu}=\frac{g_s^2}{4\pi^2}C_F\left\{\left(\frac{2}{1-z}\right)_+-(1+z)+\frac{3}{2}\delta(1-z)\right\}=\frac{\alpha_s}{\pi}P_{q\to q}(z).
\tag{67}
$$

Now we calculate the quark in gluon contribution with the following matrix element:

$$
f_{q/g}(x) = \int\frac{d(n_-\cdot z)}{2\pi}e^{-ixP\cdot z}\langle g(p)|(\bar{q}W_c)(z)\frac{\not{n}_+}{2}(W_c^\dagger q)(0)|g(p)\rangle,
\tag{68}
$$

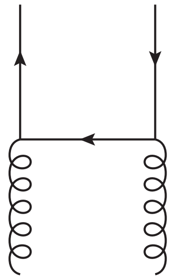

FIG. 2: quark in gluon

where the incoming and outgoing states are gluons with the same transverse polarizations. The Feynman diagram is given in Fig. 2, with the amplitude

$$f^{(1)}_{q/g}(x) = \int \frac{d^4q}{(2\pi)^4}\delta(p^+ - xp^+ - q^+)(-\mathrm{tr})\left[\frac{i(\not{p}-\not{q})}{(p-q)^2} igt^a\gamma^\mu \frac{-i\not{q}}{q^2} igt^b\gamma^\nu \frac{i(\not{p}-\not{q})}{(p-q)^2}\frac{\not{n}_+}{2}\right] \times \epsilon_\mu(T)\epsilon^*_\nu(T), \tag{69}$$

the contraction of polarization can be used as

$$\epsilon_\mu(T)\epsilon^*_\nu(T) \to \frac{1}{2}(-g_{\perp\mu\nu}) = -\frac{1}{2}(g_{\mu\nu} - n_{+\mu}n_{-\nu} - n_{-\mu}n_{+\nu}). \tag{70}$$

The trace part is simplified as

$$\begin{aligned}
N &= (-\mathrm{tr})\left[(\not{p}-\not{q})\gamma^\mu\not{q}\gamma^\nu(\not{p}-\not{q})\frac{\not{n}_+}{2}\right]\frac{1}{2}(-g_{\perp\mu\nu}) \\
&= \frac{1}{2}\mathrm{tr}\left\{\left[(\not{p}-\not{q})\gamma^\mu\not{q}\gamma_\mu(\not{p}-\not{q})\frac{\not{n}_+}{2}\right] - \left[(\not{p}-\not{q})\not{n}_+\not{q}\not{n}_-(\not{p}-\not{q})\frac{\not{n}_+}{2}\right] - \left[(\not{p}-\not{q})\not{n}_-\not{q}\not{n}_+(\not{p}-\not{q})\frac{\not{n}_+}{2}\right]\right\} \\
&= \frac{1}{2}\mathrm{tr}\left\{-2\left[\not{p}\not{q}\not{p}\frac{\not{n}_+}{2}\right] - (p^+ - q^+)\left[\not{n}_-\not{n}_+\not{q}\not{n}_-(\not{p}-\not{q})\frac{\not{n}_+}{2}\right] - (p^+ - q^+)\left[(\not{p}-\not{q})\not{n}_-\not{q}\not{n}_+\not{n}_-\frac{\not{n}_+}{2}\right]\right\} \\
&= \frac{1}{2}\left\{-8(p^+)^2q^- - 4(p^+ - q^+)q_\perp^2 - 4(p^+ - q^+)q_\perp^2\right\} \\
&= \frac{1}{2}\left\{-8(p^+)^2\frac{-q_\perp^2}{2q^+} - 4(p^+ - q^+)q_\perp^2 - 4(p^+ - q^+)q_\perp^2\right\} \\
&= \frac{2q_\perp^2}{q^+}\left\{(p^+)^2 - 2q^+(p^+ - q^+)\right\}
\end{aligned} \tag{71}$$

Thus we have the PDG at one-loop

$$\begin{aligned}
f^{(1)}_{q/g}(x) &= -ig_s^2T_F\delta^{ab}\int \frac{d^4q}{(2\pi)^4}\delta(p^+ - xp^+ - q^+)\frac{1}{q^2}\frac{1}{[(q-p)^2]^2}\frac{2q_\perp^2}{q^+}\left\{(p^+)^2 - 2q^+(p^+ - q^+)\right\} \\
&= -ig_s^2T_F\delta^{ab}\int \frac{dq^+d^2q_\perp}{(2\pi)^4}\delta(p^+ - xp^+ - q^+)(-2\pi i)\frac{1}{2q^+}\frac{(q^+)^2}{(p^+)^2(q_\perp^2)^2}\frac{2q_\perp^2}{q^+}\left\{(p^+)^2 - 2q^+(p^+ - q^+)\right\} \\
&= \frac{g_s^2}{8\pi^2}T_F\delta^{ab}\frac{1}{\epsilon}\left\{1 - 2x(1-x)\right\},
\end{aligned} \tag{72}$$

and thus the counter-term is given as

$$Z_{q/g} = -\frac{g_s^2}{8\pi^2}T_F\delta^{ab}\frac{1}{\epsilon}\left\{1 - 2x(1-x)\right\}. \tag{73}$$

The evolution kernel is

$$P_{q/g} = \frac{d}{d\ln\mu}Z_{q/g} = \frac{\alpha_s}{\pi}T_F[1 - 2x(1-x)] \equiv \frac{\alpha_s}{\pi}P_{g\to q}(x). \tag{74}$$

For the gluon PDF, the tree-level result is given as

$$
\begin{aligned}
f_g^{(0)}(x) &= \sum_j \int \frac{d\omega^-}{2\pi x p^+} e^{-ixp^+\omega^-} \langle g(p)|(F^{+j}W_A)(\omega^-)(W_A^\dagger F^{+j})(0)|g(p)\rangle \\
&= \sum_j \int \frac{d\omega^-}{2\pi x p^+} e^{-ixp^+\omega^-} (ip^+)(-ip^+) e^{ixp^+\omega^-} \epsilon^j(T)\epsilon^{*j}(T) \\
&= \delta(x-1).
\end{aligned} \tag{75}
$$

For the quark to gluon, the Feynman diagram is given in Fig. 3. The amplitude is given as

$$
\begin{aligned}
f_{g/q}^{(1)}(x) &= \sum_j \int \frac{d\omega^-}{2\pi x p^+} e^{-ixp^+\omega^-} \int \frac{d^4q}{(2\pi)^4} e^{i(p^+-q^+)\omega^-} (-i)(xp^+ g_\mu^j - (p-q)^j g_\mu^+) i(xp^+ g_\nu^j - (p-q)^j g_\nu^+) \\
&\quad \times \bar{u}(p) i g t^a \gamma^\nu \frac{i\not{q}}{q^2} i g t^a \gamma^\mu u(p) \times \frac{-i}{(q-p)^2} \frac{-i}{(q-p)^2} \\
&= i g_s^2 C_F \sum_j \frac{1}{xp^+} \int \frac{d^4q}{(2\pi)^4} \delta(p^+ - xp^+ - q^+)(xp^+ g_\mu^j + q^j g_\mu^+)(xp^+ g_\nu^j + q^j g_\nu^+) \bar{u}(p) \gamma^\nu \frac{\not{q}}{q^2} \gamma^\mu u(p) \frac{1}{q^2[(q-p)^2]^2} \\
&= i g_s^2 C_F \frac{1}{xp^+} \int \frac{d^4q}{(2\pi)^4} \delta(p^+ - xp^+ - q^+) \bar{u}(p) \left[(xp^+)^2 \gamma^j \not{q} \gamma^j + q^j q^j \not{n}_+ \not{q} \not{n}_+ + xp^+ \gamma^j \not{q} q^j \not{n}_+ + q^j \not{n}_+ \not{q} x p^+ \gamma^j \right] u(p) \\
&\quad \times \frac{1}{q^2[(q-p)^2]^2} \\
&= i g_s^2 C_F \frac{1}{xp^+} \int \frac{d^4q}{(2\pi)^4} \delta(p^+ - xp^+ - q^+) \bar{u}(p) \left[(xp^+)^2 2 q^- \not{n}_+ - 2q^+ q_\perp^2 \not{n}_+ - 2xp^+ q_\perp^2 \not{n}_+ \right] u(p) \frac{1}{q^2[(q-p)^2]^2} \\
&= i g_s^2 C_F \frac{1}{xp^+} \int \frac{dq^+ d^2 q_\perp}{(2\pi)^4} \delta(p^+ - xp^+ - q^+) \bar{u}(p) \left[(xp^+)^2 2 \frac{-q_\perp^2}{2q^+} \not{n}_+ - 2q^+ q_\perp^2 \not{n}_+ - 2xp^+ q_\perp^2 \not{n}_+ \right] u(p) \\
&\quad \times (-2\pi i) \frac{1}{2q^+} \frac{(q^+)^2}{(p^+)^2 (q_\perp^2)^2} \\
&= \frac{g_s^2}{16\pi^2} C_F \frac{1}{\epsilon} \frac{1}{xp^+} \left[x^2 + 2(1-x)^2 + 2x(1-x) \right] \bar{u}(p) \not{n}_+ u(p) \\
&= \frac{g_s^2}{8\pi^2} C_F \frac{1}{\epsilon} \frac{1 + (1-x)^2}{x}.
\end{aligned} \tag{76}
$$

The counter-term is then derived as

$$
Z_{g/q}(x) = -\frac{g_s^2}{8\pi^2} C_F \frac{1}{\epsilon} \frac{1+(1-x)^2}{x}, \tag{77}
$$

and thus

$$
P_{g/q}(x) = \frac{d}{d\ln\mu} Z_{g/q}(x) = \frac{g_s^2}{4\pi^2} C_F \frac{1+(1-x)^2}{x} = \frac{\alpha_s}{\pi} C_F \frac{1+(1-x)^2}{x} \equiv \frac{\alpha_s}{\pi} P_{q\to g}(x). \tag{78}
$$

Now we come to the gluon PDF at one-loop, whose Feynman diagrams are given in Fig. 4. We quote the gluon self-energy wave function renormalization constant as follows:

$$
Z_3 = 1 - \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \left[\frac{2}{3} n_f - 5 \right] \tag{79}
$$

For the calculation of gluon PDF, I will use the Feynman rules derived in Collins's book, but notice the different

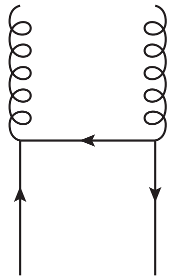

FIG. 3: gluon in quark

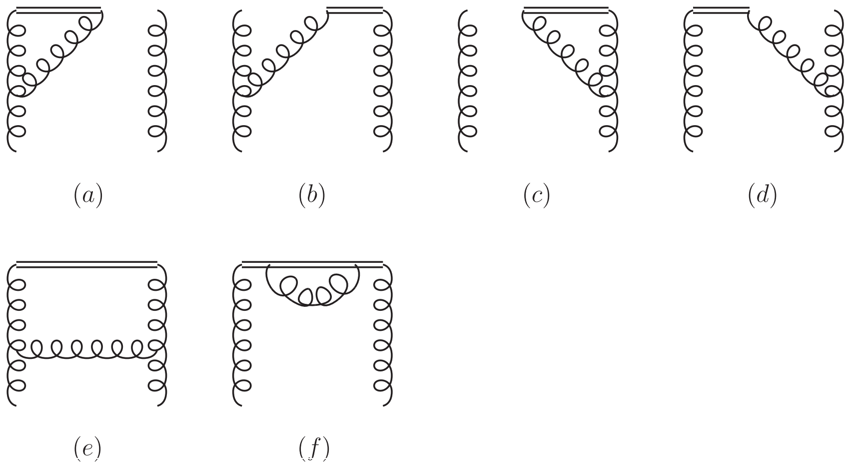

FIG. 4: gluon PDF at one-loop

convention on the coupling constant g_s. The first diagram in Fig. 4 is given as follows:

$$
\begin{aligned}
f_{g/g}^{(1,a)}(x) &= \sum_j \int \frac{d\omega^-}{2\pi x p^+} e^{-ixp^+\omega^-} \int \frac{d^4q}{(2\pi)^4} e^{ip^+\omega^-} ixp^+ g^j_\mu \epsilon^{*\mu} (-i)(xp^+ g^j_\nu - (p-q)^j n_{+\nu}) f^{\alpha\gamma\beta'} \\
&\quad \times \frac{i}{-q^+}(-g_s n_+^\rho) g_s f^{\beta\gamma\beta'} [(p+q)^\nu g^{\rho\sigma} + (-q+p-q)^\sigma g^{\nu\rho} + (-p+q-p)^\rho g^{\nu\sigma}] \epsilon_\sigma \frac{-i}{(q-p)^2} \frac{-i}{q^2} \\
&= -ig_s^2 C_2(G) \delta^{\alpha\beta} \sum_j \int \frac{d\omega^-}{2\pi} e^{-ixp^+\omega^-} \int \frac{d^4q}{(2\pi)^4} e^{ip^+\omega^-} \epsilon^{*j} (xp^+ g^j_\nu + q^j n_{+\nu}) \\
&\quad \times \frac{1}{q^+} [(p+q)^\nu n_+^\sigma + (-q+p-q)^\sigma n_+^\nu + n_+ \cdot (-p+q-p) g^{\nu\sigma}] \epsilon_\sigma \frac{1}{(q-p)^2} \frac{1}{q^2} \\
&= -ig_s^2 C_2(G) \delta^{\alpha\beta} \sum_j \delta(xp^+ - p^+) p^+ \int \frac{d^4q}{(2\pi)^4} \epsilon^{*j} \epsilon^j (q-2p)^+ \frac{1}{q^+} \frac{1}{(q-p)^2} \frac{1}{q^2} \\
&= -ig_s^2 C_2(G) \delta^{\alpha\beta} \delta(xp^+ - p^+) \int \frac{dq^+ d^2 q_\perp}{(2\pi)^4} p^+ (q-2p)^+ (-2\pi i) \frac{1}{q^+} \frac{1}{2p^+ q_\perp^2} \\
&= \frac{g_s^2}{16\pi^2} \frac{1}{\epsilon} C_2(G) \delta^{\alpha\beta} \delta(x-1) \int_0^1 dy \left(1 - \frac{2}{1-y}\right). \qquad (80)
\end{aligned}
$$

Now let us consider the second diagram,

$$\begin{aligned}
f^{(1,b)}_{g/g}(x) &= \sum_j \int \frac{d\omega^-}{2\pi xp^+} e^{-ixp^+\omega^-} \int \frac{d^4q}{(2\pi)^4} e^{i(p^+-q^+)\omega^-} ixp^+ g^j_\mu \epsilon^{*\mu} (-i)(xp^+ g^j_\nu - (p-q)^j n_{+\nu}) f^{\alpha\gamma\beta'} \\
&\times \frac{-i}{-q^+} (-g_s n^\rho_+) g_s f^{\beta\gamma\beta'} [(p+q)^\nu g^{\rho\sigma} + (-q+p-q)^\sigma g^{\nu\rho} + (-p+q-p)^\rho g^{\nu\sigma}] \epsilon_\sigma \frac{-i}{(q-p)^2} \frac{-i}{q^2} \\
&= i g_s^2 C_2(G) \delta^{\alpha\beta} \sum_j \int \frac{d\omega^-}{2\pi} e^{-ixp^+\omega^-} \int \frac{d^4q}{(2\pi)^4} e^{i(p^+-q^+)\omega^-} \epsilon^{*j} (xp^+ g^j_\nu + q^j n_{+\nu}) \\
&\times \frac{1}{q^+} [(p+q)^\nu n^\sigma_+ + (-q+p-q)^\sigma n^\nu_+ + n_+ \cdot (-p+q-p) g^{\nu\sigma}] \epsilon_\sigma \frac{1}{(q-p)^2} \frac{1}{q^2} \\
&= i g_s^2 C_2(G) \delta^{\alpha\beta} \sum_j p^+ \int \frac{d^4q}{(2\pi)^4} \delta(xp^+ + q^+ - p^+) x \epsilon^{*j} \epsilon^j (q-2p)^+ \frac{1}{q^+} \frac{1}{(q-p)^2} \frac{1}{q^2} \\
&= i g_s^2 C_2(G) \delta^{\alpha\beta} \int \frac{dq^+ d^2 q_\perp}{(2\pi)^4} \delta(xp^+ - p^+ + q^+) xp^+ (q-2p)^+ (-2\pi i) \frac{1}{q^+} \frac{1}{2p^+ q^2_\perp} \\
&= -\frac{g_s^2}{16\pi^2} \frac{1}{\epsilon} C_2(G) \delta^{\alpha\beta} x \left(1 - \frac{2}{1-x}\right). \qquad (81)
\end{aligned}$$

The contributions from the third and fourth diagrams are the same as above.

Let us now consider the fifth diagram:

$$\begin{aligned}
f^{(1,f)}_{g/g}(x) &= \sum_j \int \frac{d\omega^-}{2\pi xp^+} e^{-ixp^+\omega^-} \int \frac{d^4q}{(2\pi)^4} e^{i(p^+-q^+)\omega^-} (-i)(xp^+ g^j_\mu - (p-q)^j n_{+\mu}) i (xp^+ g^j_\nu - (p-q)^j n_{+\nu}) \\
&\times \frac{(-i)^2}{[(q-p)^2]^2} \frac{-i}{q^2} g_s f^{\alpha\beta'\gamma} [(-p-p+q)^\lambda g^{\rho\nu} + (p-q-q)^\rho g^{\lambda\nu} + (q+p)^\nu g^{\rho\lambda}] \\
&\times g_s f^{\beta\beta'\gamma} [(p+p-q)^\lambda g^{\sigma\mu} + (-p+q+q)^\sigma g^{\mu\lambda} + (-q-p)^\mu g^{\lambda\sigma}] \epsilon^*_\rho \epsilon_\sigma \\
&= i g_s^2 C_2(G) \delta^{\alpha\beta} \sum_j \frac{1}{xp^+} \int \frac{d^4q}{(2\pi)^4} \delta(p^+ - xp^+ - q^+)(xp^+ g^j_\mu + q^j n_{+\mu})(xp^+ g^j_\nu + q^j n_{+\nu}) \\
&\times \frac{1}{[(q-p)^2]^2} \frac{1}{q^2} [(q-2p)^\lambda \epsilon^{*\nu} - 2\epsilon^* \cdot q g^{\lambda\nu} + (q+p)^\nu \epsilon^{*\lambda}] \\
&\times [(2p-q)^\lambda \epsilon^\mu + 2\epsilon \cdot q g^{\mu\lambda} + (-q-p)^\mu \epsilon^\lambda] \\
&= i g_s^2 C_2(G) \delta^{\alpha\beta} \sum_j \frac{1}{xp^+} \int \frac{d^4q}{(2\pi)^4} \delta(p^+ - xp^+ - q^+) \frac{1}{[(q-p)^2]^2} \frac{1}{q^2} \\
&\times [xp^+ (q-2p)^\lambda \epsilon^{*j} - 2xp^+ \epsilon^* \cdot q g^{\lambda j} + xp^+ q^j \epsilon^{*\lambda} - 2q^j \epsilon^* \cdot q n^\lambda_+ + q^j (q+p)^+ \epsilon^{*\lambda}] \\
&\times [xp^+ (2p-q)^\lambda \epsilon^j + 2xp^+ \epsilon \cdot q g^{j\lambda} - xp^+ q^j \epsilon^\lambda + 2q^j \epsilon \cdot q n^\lambda_+ - q^j (p^+ + q^+) \epsilon^\lambda] \\
&= i g_s^2 C_2(G) \delta^{\alpha\beta} \frac{1}{xp^+} \int \frac{dq^+ d^2 q_\perp}{(2\pi)^4} \delta(p^+ - xp^+ - q^+)(-2\pi i) \frac{(q^+)^2}{(p^+ q^2_\perp)^2} \frac{1}{2q^+} \\
&\times 2(p^+)^3 q^2_\perp (2x^3 - 3x^2 + 2x - 2) \\
&= -\frac{g_s^2}{8\pi^2} C_2(G) \delta^{\alpha\beta} \frac{1}{\epsilon} (2x^2 - 3x + 2 - \frac{2}{x}), \qquad (82)
\end{aligned}$$

where some part of simplification has relied on mathematica, and see Fig. 5.

Adding all diagram together we have

$$\begin{aligned}
f^{(1)}_{g/g}(x) &= \frac{g_s^2}{8\pi^2} \frac{1}{\epsilon} C_2(G) \delta^{\alpha\beta} \left(\delta(x-1) \int_0^1 dy \left(1 - \frac{2}{1-y}\right) - x\left(1 - \frac{2}{1-x}\right) - (2x^2 - 3x + 2 - \frac{2}{x}) \right) \\
&- \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \delta^{\alpha\beta} \left(\frac{2}{3} n_f - 5 \right) \\
&= \frac{g_s^2}{8\pi^2} \frac{1}{\epsilon} \left\{ C_2(G) \left(\frac{2x}{(1-x)_+} + \frac{2(1-x)}{x} + 2x(1-x) \right) + \frac{1}{2} \left(11 - \frac{2}{3} n_f \right) \right\}, \qquad (83)
\end{aligned}$$

real corrections: here qT2= -q^j q^j

```
ScalarProduct[ϵ, np] = 0; ScalarProduct[ϵp, np] = 0;
ScalarProduct[ϵ, p] = 0; ScalarProduct[ϵp, p] = 0;
ScalarProduct[p, p] = 0; ScalarProduct[np, np] = 0;
vertWC1 = x * ScalarProduct[np, p] * MetricTensor[j, μ] + FV[q, j] * FV[np, μ];
vertWC2 = x * ScalarProduct[np, p] * MetricTensor[jp, ν] + FV[q, jp] * FV[np, ν];
vert3g1 = (-2 FV[p, λ] + FV[q, λ]) MetricTensor[ρ, ν] + (FV[p, ρ] - 2 FV[q, ρ])
    MetricTensor[λ, ν] + (FV[q, ν] + FV[p, ν]) MetricTensor[ρ, λ];
vert3g2 = (2 FV[p, λ] - FV[q, λ]) MetricTensor[σ, μ] + (2 FV[q, σ] - FV[p, σ])
    MetricTensor[μ, λ] - (FV[q, μ] + FV[p, μ]) MetricTensor[λ, σ];
resultCont = Contract[vertWC1 * vertWC2 * vert3g1 * vert3g2 * FV[ϵp, ρ] * FV[ϵ, σ]] //.
   {Pair[LorentzIndex[jp], Momentum[p]] → 0, Pair[LorentzIndex[j], Momentum[p]] →
     0, Pair[LorentzIndex[j], Momentum[np]] → 0,
    Pair[LorentzIndex[jp], Momentum[np]] → 0, Pair[Momentum[p], Momentum[p]] → 0,
    Pair[Momentum[np], Momentum[np]] → 0, Pair[Momentum[q], Momentum[q]] → 0};

FullSimplify[
  resultSim = resultCont //. { Pair[LorentzIndex[j], LorentzIndex[jp]] → -2,
     Pair[LorentzIndex[j], Momentum[ϵ]]
       Pair[LorentzIndex[jp], Momentum[ϵp]] → 1,
     Pair[LorentzIndex[j], Momentum[ϵp]] Pair[LorentzIndex[jp], Momentum[ϵ]] → 1,
     Pair[Momentum[q], Momentum[p]] → pp * qm,
     Pair[Momentum[np], Momentum[p]] → pp,
     Pair[Momentum[np], Momentum[q]] → qp,
     Pair[Momentum[q], Momentum[ϵ]] Pair[Momentum[q], Momentum[ϵp]] → -1 / 2 qT2,
     Pair[LorentzIndex[jp], Momentum[q]] Pair[Momentum[q], Momentum[ϵ]] →
      Pair[LorentzIndex[jp], Momentum[ϵ]] * qT2 / 2,
     Pair[LorentzIndex[jp], Momentum[q]] Pair[Momentum[q], Momentum[ϵp]] →
      Pair[LorentzIndex[jp], Momentum[ϵp]] * qT2 / 2,
     Pair[LorentzIndex[j], Momentum[q]] Pair[Momentum[q], Momentum[ϵp]] →
      Pair[LorentzIndex[j], Momentum[ϵp]] * qT2 / 2,
     Pair[LorentzIndex[j], Momentum[q]] Pair[Momentum[q], Momentum[ϵ]] →
      Pair[LorentzIndex[j], Momentum[ϵ]] * qT2 / 2,
     Pair[Momentum[ϵ], Momentum[ϵp]] → -1,
     Pair[LorentzIndex[j], Momentum[q]]
       Pair[LorentzIndex[jp], Momentum[q]] → -qT2 }];
Simplify[resultSim * qp //. {qm → -qT2 / (2 qp), qp → pp * (1 - x)}]
```

Out[15]= $2\,\mathrm{pp}^3\,\mathrm{qT2}\left(2x^3-3x^2+2x-2\right)$

FIG. 5: gluon PDF simplication

Thus the evolution kernel is given as

$$
\begin{aligned}
P_{g/g} &= \frac{g_s^2}{4\pi^2}\left\{C_2(G)\left(\frac{2x}{(1-x)_+}+\frac{2(1-x)}{x}+2x(1-x)\right)+\frac{1}{2}\left(11-\frac{2}{3}n_f\right)\right\} \\
&= \frac{\alpha_s}{\pi}\left\{C_2(G)\left(\frac{2x}{(1-x)_+}+\frac{2(1-x)}{x}+2x(1-x)\right)+\frac{1}{2}\left(11-\frac{2}{3}n_f\right)\right\} \equiv P_{g\to g}\frac{\alpha_s}{\pi}. \qquad (84)
\end{aligned}
$$

IV. LANGE-NEUBERT EVOLUTION

In HQET, the leading twist LCDA of B meson $\phi_B^+(\omega)$ is defined as

$$
\tilde{f}_B m_B \phi_B^+(\omega) = \int \frac{d\tau}{2\pi} e^{i\omega\tau}\langle 0|\bar{q}(\tau n_+)W_c[\tau n_+,0]\not{n}_+\gamma_5 h_v(0)|\overline{B}(m_B v)\rangle. \qquad (85)
$$

The renormalized operator is given as

$$
O_+^{\mathrm{ren}}(\omega,\mu) = \int d\omega' Z_+(\omega,\omega',\mu) O_+^{\mathrm{bare}}(\omega'), \qquad (86)
$$

where $Z_+(\omega,\omega',\mu)=\delta(\omega-\omega')$ at tree level. The renormalization group equation is given as

$$
\frac{d}{d\ln\mu}\phi_B^+(\omega) = -\int_0^\infty d\omega' \gamma_+(\omega,\omega',\mu)\phi_+^B(\omega',\mu), \qquad (87)
$$

Next, we will deduce two useful formulas in order to get the expression of the anomalous dimension. First, we define

$$\int d\omega' Z_+(\omega,\omega',\mu) Z_+^{-1}(\omega,\omega'',\mu) = \delta(\omega-\omega''). \tag{88}$$

Second, we have

$$\begin{aligned} O_+^{\rm ren}(\omega,\mu) &= \int d\omega'' \delta(\omega-\omega'') O_+^{\rm ren}(\omega'',\mu) \\ &= \int d\omega'' \Big[\int d\omega' Z_+(\omega,\omega',\mu) Z_+^{-1}(\omega',\omega'',\mu) \Big] O_+^{\rm ren}(\omega'',\mu) \\ &= \int d\omega' Z_+(\omega,\omega',\mu) \int d\omega'' Z_+^{-1}(\omega',\omega'',\mu) O_+^{\rm ren}(\omega'',\mu)\,, \end{aligned} \tag{89}$$

So, we have

$$O_+^{\rm bare}(\omega') = \int d\omega'' Z_+^{-1}(\omega',\omega'',\mu) O_+^{\rm ren}(\omega'',\mu)\,. \tag{90}$$

In order to get the expression of the anomalous dimension, we start from beginning,

$$O_+^{\rm ren}(\omega,\mu) = \int d\tilde{\omega} Z_+(\omega,\tilde{\omega},\mu) O_+^{\rm bare}(\tilde{\omega})\,, \tag{91}$$

so,

$$\mu \frac{dO_+^{\rm ren}(\omega,\mu)}{d\mu} = \mu \int d\tilde{\omega} \frac{dZ_+(\omega,\tilde{\omega},\mu)}{d\mu} O_+^{\rm bare}(\tilde{\omega})\,, \tag{92}$$

so,

$$\begin{aligned} \int d\omega' \Gamma_+(\omega,\omega',\mu) O_+^{\rm ren}(\omega',\mu) &= -\int d\tilde{\omega} \frac{dZ_+(\omega,\tilde{\omega},\mu)}{d\ln\mu} O_+^{\rm bare}(\tilde{\omega}) \\ &= -\int d\tilde{\omega} \int d\tilde{\omega} \frac{dZ_+(\omega,\tilde{\omega},\mu)}{d\ln\mu} \int d\omega' Z_+^{-1}(\tilde{\omega},\omega',\mu) O_+^{\rm ren}(\omega',\mu) \\ &= \int d\omega' \Big[-\int d\tilde{\omega} \frac{dZ_+(\omega,\tilde{\omega},\mu)}{d\ln\mu} Z_+^{-1}(\tilde{\omega},\omega',\mu) \Big] O_+^{\rm ren}(\omega',\mu)\,. \end{aligned} \tag{93}$$

Finally, we have

$$\Gamma_+(\omega,\omega',\mu) = -\int d\tilde{\omega} \frac{dZ_+(\omega,\tilde{\omega},\mu)}{d\ln\mu} Z_+^{-1}(\tilde{\omega},\omega',\mu)\,. \tag{94}$$

We know that $O_+^{\rm ren}(\omega,\mu)$ is given by the product $\tilde{f}_B\phi_B^+(\omega)$, so, we can define γ_F, which is the universal anomalous dimension of local heavy-light currents in HQET, determines the scale dependence of $\tilde{f}_B$. Actually, this quantity γ_F can be found in (3.24) in "Heavy Quark Effective Theory". So, the anomalous dimension:

$$\gamma_+(\omega,\omega',\mu) = -\int d\tilde{\omega} \frac{dZ_+(\omega,\tilde{\omega},\mu)}{d\ln\mu} Z_+^{-1}(\tilde{\omega},\omega',\mu) - \gamma_F(\alpha_s)\delta(\omega-\omega'). \tag{95}$$

We will calculate the one-loop corrections with the B meson in the initial state.

For the quark-level Feynman diagrams, we will choose the free quark states: the light anti-quark carries the momentum ω_0.

First, we write down the notations we need,

$$n_{+\mu} = \frac{1}{\sqrt{2}}(1,0,0,1), \quad n_{-\mu} = \frac{1}{\sqrt{2}}(1,0,0,-1), \quad v^\mu = (1,0,0,0). \tag{96}$$

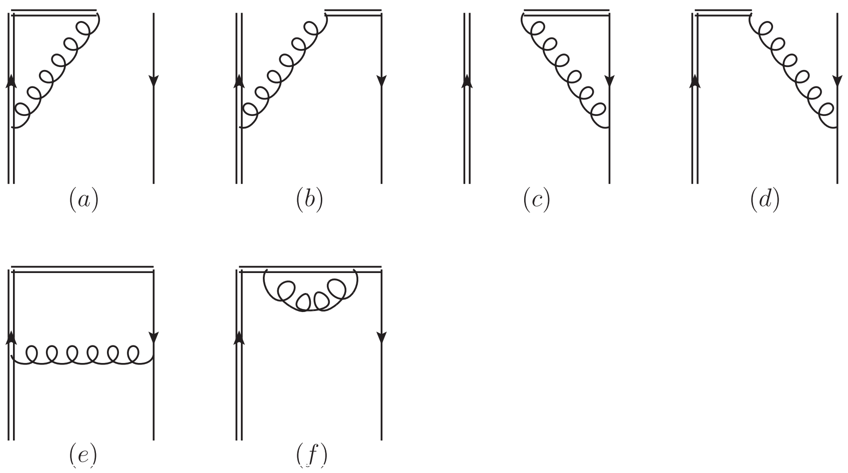

FIG. 6: Lange-Neubert evolution kernel at 1 loop. The heavy and light quarks have the incoming momentum. The quark double line denotes a heavy quark, while the gluon double line corresponds to the gauge link/Wilson line.

So,

$$n_+ \cdot v = \frac{1}{\sqrt{2}}. \qquad v_\mu = \frac{1}{\sqrt{2}}(n_{+\mu} + n_{-\mu})\,. \tag{97}$$

We calculate this matrix element in tree level

$$\begin{aligned}
&\int \frac{d\tau}{2\pi} e^{i\omega\tau} \langle 0|\bar{q}(\tau n_+)\not{n}_+\gamma_5 h_v(0)|\bar{d}(\omega_0)b(p_b)\rangle \\
&= \int \frac{d\tau}{2\pi} e^{i\omega\tau} e^{-i\tau n_+\omega_0}\bar{v}(\omega_0)\not{n}_+\gamma_5 u_v(p_b) \\
&= \int \frac{d\tau}{2\pi} e^{-i\tau(n_+\omega_0-\omega)}\bar{v}(\omega_0)\not{n}_+\gamma_5 u_v(p_b) \\
&= \delta(\omega - \omega_0^+)\bar{v}(\omega_0)\not{n}_+\gamma_5 u_v(p_b)\,.
\end{aligned} \tag{98}$$

Next, we will move on to one-loop diagrams, the amplitude of (c) diagram is

$$\begin{aligned}
\langle\Phi\rangle^{(c)} \;=\;& \int \frac{d\tau}{2\pi} e^{i\omega\tau}\langle 0|\bar{q}(\tau n_+)[ig_s\int_{-\infty}^{0} ds n_+ A(\tau n_+ + sn_+)]/n_+\gamma_5 h_v(0)\int d^4z_1 ig_s[\bar{d}T^b \not{A}^b d](z_1)|\bar{d}(\omega_0)b(p_b)\rangle \\
&= -g_s^2 C_F \int \frac{d\tau}{2\pi} e^{i\omega\tau}\int_{-\infty}^{0} ds \int d^4z_1 \int \frac{d^4k}{(2\pi)^4} e^{-ik(z_1-\tau n_+)} \int \frac{d^4q}{(2\pi)^4} e^{-iq(z_1-\tau n_+ - sn_+)} \\
&\times \frac{-i}{q^2} e^{-i\omega_0 z_1} e^{-ip_b\cdot 0}\,[\bar{v}(\omega_0)\not{n}_+\not{k}\not{n}_+\gamma_5 u_v(p_b)] \\
&= ig_s^2 C_F \delta(w + n_+k + n_+q)\int \frac{d^4q}{(2\pi)^4}\int \frac{d^4k}{(2\pi)^4}(2\pi)^4\delta(k+q+\omega_0)\frac{1}{n_+q}\frac{1}{q^2}\frac{1}{k^2}\,[\bar{v}(\omega_0)\not{n}_+\not{k}\not{n}_+\gamma_5 u_v(p_b)] \\
&= ig_s^2 C_F \delta(w - n_+w_0)\int \frac{d^4q}{(2\pi)^4}\frac{1}{n_+q}\frac{1}{q^2}\frac{1}{(q+\omega_0)^2}\,[\bar{v}(\omega_0)\not{n}_+(-\not{q} - \not{\omega}_0)\not{n}_+\gamma_5 u_v(p_b)]\;.
\end{aligned} \tag{99}$$

This tells us the Feynman Rules we want. So, we will calculate these diagrams directly writing the amplitude.

$$
\begin{aligned}
\langle\Phi\rangle^{(c)} &= C_F\bar{v}(\omega_0)\int\frac{d^4q}{(2\pi)^4}ig_s\gamma^\mu\frac{-i(\not{q}+\not{\omega}_0)}{(q+\omega_0)^2}(ig_sn_{+\mu})\not{n}_+\gamma_5\frac{-i}{q^2}\frac{1}{in_+q}u_v(p_b)\delta(\omega-\omega_0^+) \\
&= ig_s^2C_F\delta(\omega-\omega_0^+)\frac{1}{(2\pi)^4}\int dq^+dq^-d^2q_\perp\frac{1}{q^+}\frac{1}{2q^+q^-+q_\perp^2}\frac{1}{2q^-(q^++\omega_0^+)+q_\perp^2}(-2)n_+\cdot(q+\omega_0)\bar{v}(\omega_0)\not{n}_+\gamma_5u_v(p_b) \\
&= -ig_s^2C_F\delta(\omega-\omega_0^+)\frac{1}{(2\pi)^4}\int dq^+d^2q_\perp(-2\pi i)\frac{q^++\omega_0^+}{\omega_0^+}\frac{1}{q^+}\frac{1}{q_\perp^2}\bar{v}(\omega_0)\not{n}_+\gamma_5u_v(p_b) \\
&= ig_s^2C_F\delta(\omega-\omega_0^+)i\frac{1}{(2\pi)^3}\int dq^+\frac{q^++\omega_0^+}{\omega_0^+}\frac{1}{q^+}\int d^2q_\perp\frac{1}{q_\perp^2}\bar{v}(\omega_0)\not{n}_+\gamma_5u_v(p_b) \\
&= \frac{g_s^2}{8\pi^2}C_F\delta(\omega-\omega_0^+)\frac{1}{\varepsilon}\int_{-1}^0\omega_0^+dy\frac{\omega_0^+(y+1)}{\omega_0^+}\frac{1}{y\omega_0^+}\bar{v}(\omega_0)\not{n}_+\gamma_5u_v(p_b) \\
&= -\frac{\alpha_s}{2\pi}C_F\delta(\omega-\omega_0^+)\frac{1}{\varepsilon}\int_0^1dy\frac{1-y}{y}\bar{v}(\omega_0)\not{n}_+\gamma_5u_v(p_b)\,. \qquad (100)
\end{aligned}
$$

The amplitude of (d) diagram is

$$
\begin{aligned}
\langle\Phi\rangle^{(d)} &= C_F\bar{v}(\omega_0)\int\frac{d^4q}{(2\pi)^4}ig_s\gamma^\mu\frac{-i(\not{q}+\not{\omega}_0)}{(q+\omega_0)^2}(-ig_sn_{+\mu})\not{n}_+\gamma_5\frac{-i}{q^2}\frac{1}{in_+q}u_v(p_b)\delta(\omega-\omega_0^+-q^+) \\
&= ig_s^2C_F\delta(\omega-\omega_0^+-q^+)\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q+\omega_0)^2}\frac{1}{q^2}\frac{1}{n_+q}\bar{v}(\omega_0)\not{n}_+(\not{q}+\not{\omega}_0)\not{n}_+\gamma_5u_v(p_b) \\
&= ig_s^2C_F\delta(\omega-\omega_0^+-q^+)\frac{1}{(2\pi)^4}(-2\pi i)(-\frac{\pi}{\varepsilon})\int dq^+\frac{q^++\omega_0^+}{\omega_0^+}\frac{1}{q^+}\bar{v}(\omega_0)\not{n}_+\gamma_5u_v(p_b) \\
&= -\frac{g_s^2}{8\pi}C_F\frac{1}{\varepsilon}\frac{\omega-\omega_0^++\omega_0^+}{\omega_0^+}\frac{1}{\omega-\omega_0^+}\theta(\omega_0^+-\omega)\bar{v}(\omega_0)\not{n}_+\gamma_5u_v(p_b) \\
&= \frac{\alpha_s}{2\pi}C_F\frac{1}{\varepsilon}\frac{\omega}{\omega_0^+(\omega_0^+-\omega)}\theta(\omega_0^+-\omega)\bar{v}(\omega_0)\not{n}_+\gamma_5u_v(p_b)\,. \qquad (101)
\end{aligned}
$$

The amplitude of (a) diagram has so called "Cusp Divergence", now we tackle it. Note that divergence looks like $\frac{1}{\varepsilon^2}$ would show up, so, some quantities which did not contribute divergence to (b), (c) and (d) would also contribute to divergence here, they should be considered too.

$$
\begin{aligned}
\langle\Phi\rangle^{(a)} &= C_F\bar{v}(\omega_0)\int\frac{d^dq}{(2\pi)^d}\not{n}_+\gamma_5(-ig_sn_{+\mu}\mu^\varepsilon)\frac{1}{in_+q}\frac{i}{v\cdot q}\frac{-i}{q^2}(ig_sv^\mu\mu^\varepsilon)u_v(p_b)\delta(\omega-\omega_0^+) \\
&= -ig_s^2C_F\delta(\omega-\omega_0^+)\mu^{2\varepsilon}\underbrace{\int\frac{d^dq}{(2\pi)^d}\frac{n_+v}{n_+q}\frac{1}{vq}\frac{1}{q^2}}_{(1)}\bar{v}(\omega_0)\not{n}_+\gamma_5u_v(p_b) \qquad (102)
\end{aligned}
$$

$$
\begin{aligned}
(1) &= \int\frac{d^dq}{(2\pi)^d}\frac{1}{\sqrt{2}}\frac{1}{q^+}\frac{1}{\frac{1}{\sqrt{2}}(q^++q^-)}\frac{1}{2q^+q^-+q_\perp^2} \\
&= \frac{1}{(2\pi)^d}\int dq^+d^{2-2\varepsilon}q_\perp(2\pi i)\frac{1}{q^+}\frac{2q^+}{2(2q^+)^2-q_\perp^2}\frac{1}{2q^+}\theta(-q^+) \\
&= \frac{2\pi i}{(2\pi)^d}\theta(-q^+)\int dq^+\frac{1}{q^+}\frac{(2\pi)^{2-2\varepsilon}}{(4\pi)^{-\varepsilon}}\Gamma(\varepsilon)(2(q^+)^2)^{-\varepsilon} \\
&= -\frac{i}{8\pi}(4\pi)^\varepsilon\Gamma(\varepsilon)\int_0^\infty dq^+\frac{1}{q^+}(\frac{1}{2(q^+)^2})^\varepsilon\,. \qquad (103)
\end{aligned}
$$

So,

$$
\langle\Phi\rangle^{(a)} = -\frac{\alpha_s}{2\pi}C_F\delta(\omega-\omega_0^+)(4\pi\mu^2)^\varepsilon\Gamma(\varepsilon)\bar{v}(\omega_0)\not{n}_+\gamma_5u_v(p_b)\left(\frac{1}{2}\right)^\varepsilon\underbrace{\int_0^\infty dq^+\frac{1}{q^+}(\frac{1}{(q^+)^2})^\varepsilon}_{(2)}\,. \qquad (104)
$$

$$
\begin{aligned}
(2) &= \int_0^{\omega} dq^+ (q^+)^{-1-2\varepsilon} + \int_{\omega}^{\infty} dq^+ (q^+)^{-1-2\varepsilon} \\
&= \int_0^{\omega} dq^+ (q^+)^{-1-2\varepsilon} + \frac{\omega^{-2\varepsilon}}{2\varepsilon} \\
&= \frac{\omega^{-2\varepsilon}}{2\varepsilon} + \int_0^{\omega} \frac{dt}{\omega - t}\,.
\end{aligned} \tag{105}
$$

The first term in the last line of equation above is "uv pole", and the other is "non-uv pole".

The amplitude of (b) is

$$
\begin{aligned}
\langle \Phi \rangle^{(b)} &= C_F \bar{v}(\omega_0) \int \frac{d^4 q}{(2\pi)^4} \not{n}_+ \gamma_5 \frac{1}{i n_+ q} (i g_s n_{+\mu}) \frac{-i}{q^2} \frac{i}{vq} (i g_s v^{\mu}) u_v(p_b) \delta(\omega - \omega_0^+ + q^+) \\
&= i g_s^2 C_F \delta(\omega - \omega_0^+ + q^+) 2\pi^2 i \int dq^+ \frac{1}{q^+} \frac{1}{\varepsilon} \\
&= -\frac{g_s^2}{16\pi^4} 2\pi^2 C_F \frac{1}{\varepsilon} \int_{-\infty}^{0} dq^+ \frac{1}{q^+} \delta(\omega - \omega_0^+ + q^+) \\
&= \frac{\alpha_s}{2\pi} C_F \frac{1}{\varepsilon} \frac{\theta(\omega - \omega_0^+)}{\omega - \omega_0^+}\,.
\end{aligned} \tag{106}
$$

The diagram (e) has no divergence.

After calculating these four diagrams, let's go back to the work of renormalization. The renormalized operator is given as

$$
O_+^{\rm ren}(\omega, \mu) = \int d\omega' Z_+(\omega, \omega', \mu) O_+^{\rm bare}(\omega'), \tag{107}
$$

At tree level, $O_+^{\rm ren}(\omega, \mu)$ and $O_+^{\rm bare}(\omega')$ have same structure, actually, we have already calculated it.

$$
\delta(\omega - \omega_0^+) \bar{v}(\omega_0) \not{n}_+ \gamma_5 u_v(p_b) = \int d\omega' Z_+(\omega, \omega', \mu) \delta(\omega' - \omega_0^+) \bar{v}(\omega_0) \not{n}_+ \gamma_5 u_v(p_b)\,, \tag{108}
$$

so, we have

$$
Z_+(\omega, \omega', \mu) = \delta(\omega - \omega')\,. \tag{109}
$$

Next, we do the same thing to the one-loop level.

$$
\langle O_+^{\rm ren}(\omega, \mu) \rangle = \int d\omega' Z_+(\omega, \omega', \mu) \sqrt{z_q} \sqrt{z_h} \langle \bar{q}_s^{\rm ren} \cdots h^{\rm ren} \rangle \tag{110}
$$

We know that

$$
\begin{aligned}
\sqrt{z_q} &= 1 - \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon} \frac{1}{2}\,, \\
\sqrt{z_h} &= 1 + \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon}\,.
\end{aligned} \tag{111}
$$

For the compaction of formulations, we define four quantities from the calculations in the former four one-loop diagrams,

$$
\begin{aligned}
&-\frac{\alpha_s}{4\pi}C_F\delta(\omega'-\omega_0^+)\frac{2}{\varepsilon}\underbrace{\int_0^1 dy\frac{1-y}{y}}_{A1}\,,\\
&\frac{\alpha_s}{4\pi}C_F\frac{2}{\varepsilon}\underbrace{\frac{\omega'}{\omega_0^+(\omega_0^+-\omega')}\theta(\omega_0^+-\omega')}_{A2}\,,\\
&-\frac{\alpha_s}{4\pi}C_F\delta(\omega'-\omega_0^+)\underbrace{\Big[(\frac{1}{\varepsilon^2}+\frac{2}{\varepsilon}\ln\frac{\mu}{\omega'})+\frac{2}{\varepsilon}\int_0^{\omega'}\frac{dt}{\omega'-t}\Big]}_{A3}\,,\\
&\frac{\alpha_s}{4\pi}C_F\frac{2}{\varepsilon}\underbrace{\frac{\theta(\omega'-\omega_0^+)}{\omega'-\omega_0^+}}_{A4}\,.
\end{aligned}
\tag{112}
$$

So, Eq. (110) can be re-expressed as (we will set up $\frac{\alpha_s}{4\pi}C_F$ as 1 for simplicity)

$$
\begin{aligned}
\langle O_+^{\rm ren}(\omega,\mu)\rangle &= \int d\omega' Z_+(\omega,\omega',\mu)\left(1+\frac{1}{\varepsilon}\right)\left(1-\frac{1}{2\varepsilon}\right)\\
&\quad\times\left\{\delta(\omega'-\omega_0^+)+\frac{2}{\varepsilon}\big[-\delta(\omega'-\omega_0^+)A1+A2+A4\big]-\delta(\omega'-\omega_0^+)A3\right\}
\end{aligned}
\tag{113}
$$

We define

$$
Z_+(\omega,\omega',\mu)=\delta(\omega-\omega')-N(\omega,\omega',\mu)\,. \tag{114}
$$

At one-loop level, the divergences on the R.H.S. of Eq. (113) should cancel, so, we have

$$
\begin{aligned}
\int d\omega' N(\omega,\omega',\mu)\delta(\omega'-\omega_0^+) &= \int d\omega'\left\{\delta(\omega-\omega')\delta(\omega'-\omega_0^+)\frac{1}{2}\right.\\
&\quad\left.+\delta(\omega-\omega')\left[\frac{2}{\varepsilon}\Big(-\delta(\omega'-\omega_0^+)A1+A2+A4\Big)-\delta(\omega'-\omega_0^+)A3\right]\right\}
\end{aligned}
\tag{115}
$$

This means

$$
N(\omega'\to\omega_0^+)=\left\{\frac{1}{2\varepsilon}\delta(\omega-\omega_0^+)+\frac{2}{\varepsilon}\Big(-\delta(\omega-\omega_0^+)A1+A2+A4\Big)-\delta(\omega-\omega_0^+)A3\right\}\Big|_{\omega'\to\omega}\,. \tag{116}
$$

After converting $\omega_0^+\to\omega'$ in the R.H.S. of the quation above, we have the $N(\omega,\omega',\mu)$ we want,

$$
\begin{aligned}
N(\omega,\omega',\mu) &= \frac{\alpha_s}{4\pi}C_F\left\{\delta(\omega-\omega')\Big[\frac{1}{2\varepsilon}\underbrace{-\frac{2}{\varepsilon}\int_0^1 dy\frac{1-y}{y}}_{\rm I}-(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}2\ln\frac{\mu}{\omega})\underbrace{-\frac{2}{\varepsilon}\int_0^{\omega}\frac{dt}{\omega-t}}_{\rm II}\Big]\right.\\
&\quad\left.+\frac{2}{\varepsilon}\Big[\frac{\omega}{\omega'(\omega'-\omega)}\theta(\omega'-\omega)+\frac{\theta(\omega-\omega')}{\omega-\omega'}\Big]\right\}.
\end{aligned}
\tag{117}
$$

We can rearrange the last line of equation above using plus function, when a plus function integrated with a function $g(\omega')$, one must replace $g(\omega')\to g(\omega')-g(\omega)$ under the integral. So, we have

$$
f(\omega')=[f(\omega')]_+ +\int_0^\infty dt f(t)\delta(\omega'-\omega)\,. \tag{118}
$$

So the last line of Eq. (117) can be re-expressed as

$$\begin{aligned}
\frac{2}{\varepsilon}\Big(\frac{\omega\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}+\frac{\theta(\omega-\omega')}{\omega-\omega'}\Big) &= \frac{2}{\varepsilon}\Big[\frac{\omega\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}+\frac{\theta(\omega-\omega')}{\omega-\omega'}\Big]_+ \\
&\quad +\frac{2}{\varepsilon}\int dt\Big[\frac{\omega}{t}\frac{\theta(t-\omega)}{t-\omega}+\frac{\theta(\omega-t)}{\omega-t}\Big]\delta(\omega'-\omega) \\
&= \frac{2}{\varepsilon}\Big[\frac{\omega\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}+\frac{\theta(\omega-\omega')}{\omega-\omega'}\Big]_+ \\
&\quad +\underbrace{\frac{2}{\varepsilon}\int_\omega^\infty dt\frac{\omega}{t}\frac{1}{t-\omega}\delta(\omega'-\omega)}_{\mathrm{III}}+\underbrace{\frac{2}{\varepsilon}\int_0^\omega dt\frac{1}{\omega-t}\delta(\omega'-\omega)}_{\mathrm{IV}}\,. \qquad (119)
\end{aligned}$$

We take this expression back to Eq. (117), one can find that $\mathrm{II}+\mathrm{IV}=0$, and $\mathrm{I}+\mathrm{III}=\frac{2}{\varepsilon}$. Finally, we get

$$\begin{aligned}
N(\omega,\omega',\mu) &= \frac{\alpha_s}{4\pi}C_F\Big\{\delta(\omega-\omega')\Big[\frac{1}{2\varepsilon}+\frac{2}{\varepsilon}-(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}2\ln\frac{\mu}{\omega})\Big] \\
&\quad +\frac{2}{\varepsilon}\Big[\frac{\omega\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}+\frac{\theta(\omega-\omega')}{\omega-\omega'}\Big]_+\Big\}. \qquad (120)
\end{aligned}$$

So,

$$\begin{aligned}
Z_+(\omega,\omega',\mu) &= \delta(\omega-\omega')-\frac{\alpha_s}{4\pi}C_F\Big\{\delta(\omega-\omega')\Big[\frac{5}{2\varepsilon}-\frac{1}{\varepsilon^2}-\frac{2}{\varepsilon}\ln\frac{\mu}{\omega}\Big] \\
&\quad +\frac{2}{\varepsilon}\Big[\frac{\omega\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}+\frac{\theta(\omega-\omega')}{\omega-\omega'}\Big]_+\Big\}. \qquad (121)
\end{aligned}$$

Next, we will deduce γ_+, we have already known that

$$\gamma_+(\omega,\omega',\mu) = -\int d\tilde{\omega}\frac{dZ_+(\omega,\tilde{\omega},\mu)}{d\ln\mu}Z_+^{-1}(\tilde{\omega},\omega',\mu)-\gamma_F(\alpha_s)\delta(\omega-\omega'). \qquad (122)$$

$$\begin{aligned}
\frac{dZ_+(\omega,\tilde{\omega},\mu)}{d\ln\mu} &= \frac{C_F}{4\pi}\frac{d\alpha_s}{d\ln\mu}\Big\{\delta(\omega-\omega')\Big[-\frac{5}{2\varepsilon}+\frac{1}{\varepsilon^2}+\frac{2}{\varepsilon}\ln\frac{\mu}{\omega}\Big] \\
&\quad -\frac{2}{\varepsilon}\Big[\frac{\omega\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}+\frac{\theta(\omega-\omega')}{\omega-\omega'}\Big]_+\Big\} \\
&\quad +\frac{d\ln\frac{\mu}{\omega}}{d\ln\mu}\frac{2}{\varepsilon}\frac{\alpha_s}{4\pi}C_F\delta(\omega-\tilde{\omega}) \\
&= \frac{\alpha_s}{4\pi}C_F\left\{\left(-\frac{2}{\varepsilon}-4\ln\frac{\mu}{\omega}+5\right)\delta(\omega-\tilde{\omega})+4\Big[\frac{\omega}{\tilde{\omega}}\frac{\theta(\tilde{\omega}-\omega)}{\tilde{\omega}-\omega}+\frac{\theta(\omega-\tilde{\omega})}{\omega-\tilde{\omega}}\Big]_++\frac{2}{\varepsilon}\delta(\omega-\tilde{\omega})\right\}. \qquad (123)
\end{aligned}$$

So, we have

$$-\int d\tilde{\omega}\frac{dZ_+(\omega,\tilde{\omega},\mu)}{d\ln\mu}Z_+^{-1}(\tilde{\omega},\omega',\mu)=\frac{\alpha_s}{4\pi}C_F\left\{\left(4\ln\frac{\mu}{\omega}-5\right)\delta(\omega-\omega')-4\omega\left[\frac{\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}+\frac{\theta(\omega-\omega')}{\omega(\omega-\omega')}\right]_+\right\}. \qquad (124)$$

So far, we have all the blocks we need.

The explicit expression is

$$\gamma_+(\omega,\omega',\mu) = \Big[\Gamma_{\rm cusp}(\alpha_s)\ln\frac{\mu}{\omega}+\gamma(\alpha_s)\Big]\delta(\omega-\omega')+\omega\Gamma(\omega,\omega',\alpha_s). \qquad (125)$$

In terms of $C_F\alpha_s/(4\pi)$, one has

$$\Gamma^{(1)}_{\rm cusp}=4,\quad \gamma^{(1)}=-2,\ \ \Gamma^{(1)}(\omega,\omega')=-\Gamma^{(1)}_{\rm cusp}\left[\frac{\theta(\omega-\omega')}{\omega'(\omega'-\omega)}+\frac{\theta(\omega-\omega')}{\omega(\omega-\omega')}\right]_+. \qquad (126)$$