Reduction and calculation of multiloop multiscale Feynman integrals

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Based on:

Xin Guan, Xiao Liu and Yan-Qing Ma, Chin.Phys.C 44 (2020)

Xiao Liu and Yan-Qing Ma, in preparation

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Outline

I. Introduction

II. Reduction of Feynman integrals

III. Calculation of master integrals

IV. Summary and outlook

Era of precision physics

➢ Physics at Large Hadron Collider

Amoroso et al, 2003.01700

- Increasing integrated luminosity → decreasing statistical errors
- Better understanding of detectors → decreasing systematic errors
- Improving requirements on theoretical predictions

➢ State-of-the-art

- $2 \to 2$ processes

  $pp \to \gamma\gamma$ at $\mathrm{NNLO_{QCD}}$ Catani et al, JHEP (2018); Cieri et al, JHEP(2015); Campbell et al, JHEP(2016)

  $pp \to t\bar{t}$ at $\mathrm{NNLO_{QCD}}$ Czakon et al, Phys.Rev.Lett.(2016); Czakon et al, JHEP(2016)

  $pp \to \gamma + j$ at $\mathrm{NNLO_{QCD}}$ Campbell, Phys.Rev.Lett.(2017)

  …

- $2 \to 3$ process at $\mathrm{NNLO_{QCD}}$: $pp \to \gamma\gamma\gamma$ (leading color) Chawdhry et al, JHEP(2020)

➢ Scattering amplitudes

- amplitudes generation

  Feynman diagrams, unitarity cuts

  Ossola et al, Nucl.Phys.B(2007)
  Giele et al, JHEP(2008)
  Kosower, Larsen, Phys.Rev.D 85(2012)

$$\mathcal{M} = \sum_{i=1}^{\mathcal{O}(10^4)} f_i \times I_i$$

- scalar integrals reduction

  integration-by-parts

  Chetyrkin and Tkachov, Nucl.Phys.B (1981)
  Laporta, Int.J.Mod.Phys.A(2000)

$$I_i = \sum_{j=1}^{\mathcal{O}(10^2)} c_{ij} \times \mathcal{I}_j$$

- master integrals calculation

  sector decomposition, differential equation

  Binoth, Heinrich, Nucl.Phys.B(2000)

  Kotikov, Phys. Lett. B(1991)
  Henn, Phys.Rev.Lett. (2013)

$$\mathcal{I} = \sum_{n=-2L}^{n_0} a_n\, \epsilon^n$$

Feynman integrals

➢ Multiloop multiscale Feynman integrals

- prohibitive algebraic complexity

  non-planar contribution of $pp \to \gamma\gamma\gamma$ missed — Chawdhry et al, JHEP(2020)

- "basis" of special functions not fully known

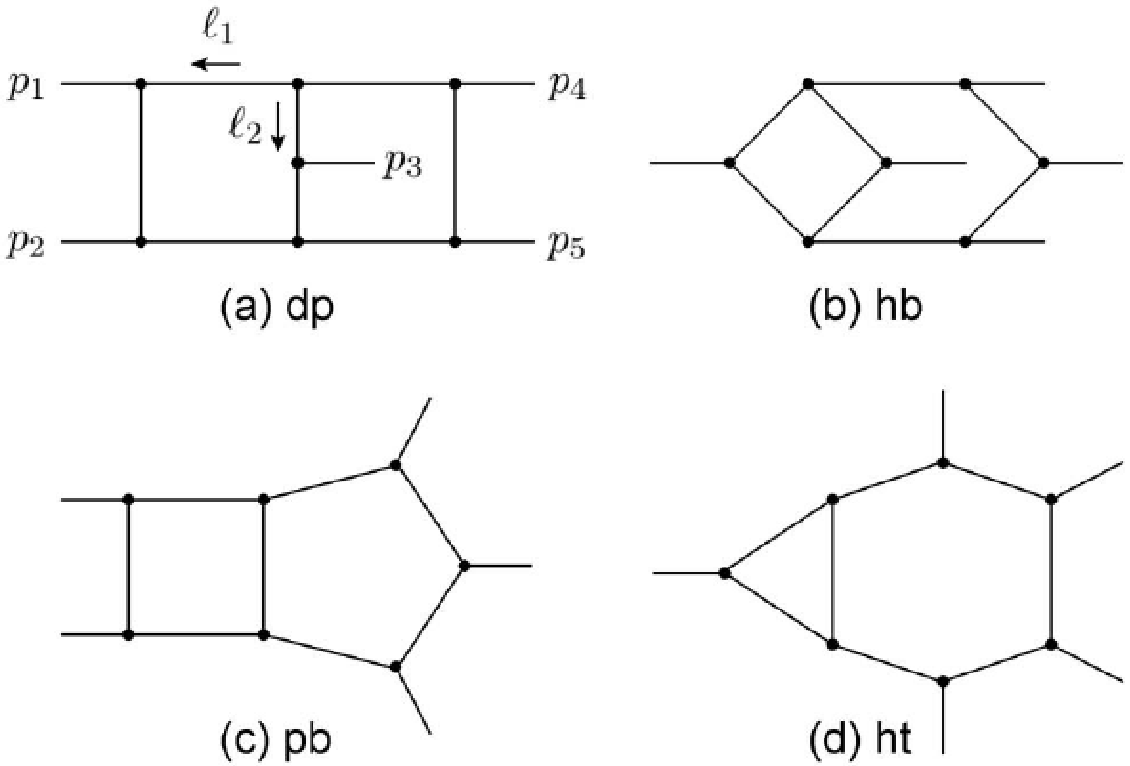
  elliptic sectors in H+jet production in QCD — Bonciani et al, JHEP(2016); Bonciani et al, JHEP(2020); Frellesvig et al, JHEP(2020)

➢ Two new methods

- search algorithm for integrals reduction
- auxiliary-mass-flow method for masters calculation

Massless two-loop five-point integrals

➢ Topologies



- 5 mass scales $\vec{s} = \{s_1, s_2, s_3, s_4, s_5\}$ and dimensional regulator ϵ
- degree-5 irreducible numerators

Reduction: overview

➢ Integration-by-parts reduction

Chetyrkin and Tkachov, Nucl.Phys.B (1981)
Laporta, Int.J.Mod.Phys.A(2000)

$$\mathbf{Q} \cdot \mathbf{I} = 0 \xrightarrow{\text{Gaussian Elimination}} I_i = \sum_{j=1}^{108} C_{ij} \times \mathcal{I}_j$$

- $Q(\epsilon, \vec{s})$: $O(10^5)$ equations, coupled
- difficult to obtain
- C_{ij}: huge expression size and hard to use

➢ Block-triangular relations

$$\mathbf{Q} \cdot \mathbf{I} = \text{simpler integrals}$$

- much smaller size: $O(10^2)$ in each block and $O(10^3)$ in total
- much better structure: each block can be solved independently

Reduction

➢ search algorithm

XL, Ma, Phys.Rev.D (2019)
Guan, XL, Ma, Chin.Phys.C (2020)

1. use IBP system to express numerically all target integrals to master integrals

$$I_i \quad \longleftrightarrow \quad \{C_{i,1}, \ldots, C_{i,108}\}$$

2. search for relations among integrals

$$\sum_{i=1}^{N} Q_i(\epsilon, \vec{s}) I_i(\epsilon, \vec{s}) = 0\,, \quad Q_i(\epsilon, \vec{s}) = \sum_{\kappa=0}^{\epsilon_{\max}} \sum_{\vec{\lambda}\in\Omega_{d_i}} \tilde{Q}_i^{\kappa\lambda_1\ldots\lambda_5}\, \epsilon^{\kappa} s_1^{\lambda_1} \cdots s_5^{\lambda_5}$$

- number of unknowns: finite
- numerical IBP at 1 point → at most 108 constraints
- $\tilde{Q}_i^{\kappa\lambda_1\ldots\lambda_5}$ can be determined after finite number of numerical IBP evaluations

Equipped with a carefully designed reduction scheme, block-triangular relations can be obtained finally.

Results

top.	#int.	#MIs	t_{search} (h)	t_{solve} (s)	size(MB)
(a)	3914	108	112	0.17	66
(b)	3584	73	31	0.090	40
(c)	3458	61	56	0.075	31
(d)	2634	28	8	0.035	11

- explicit result:

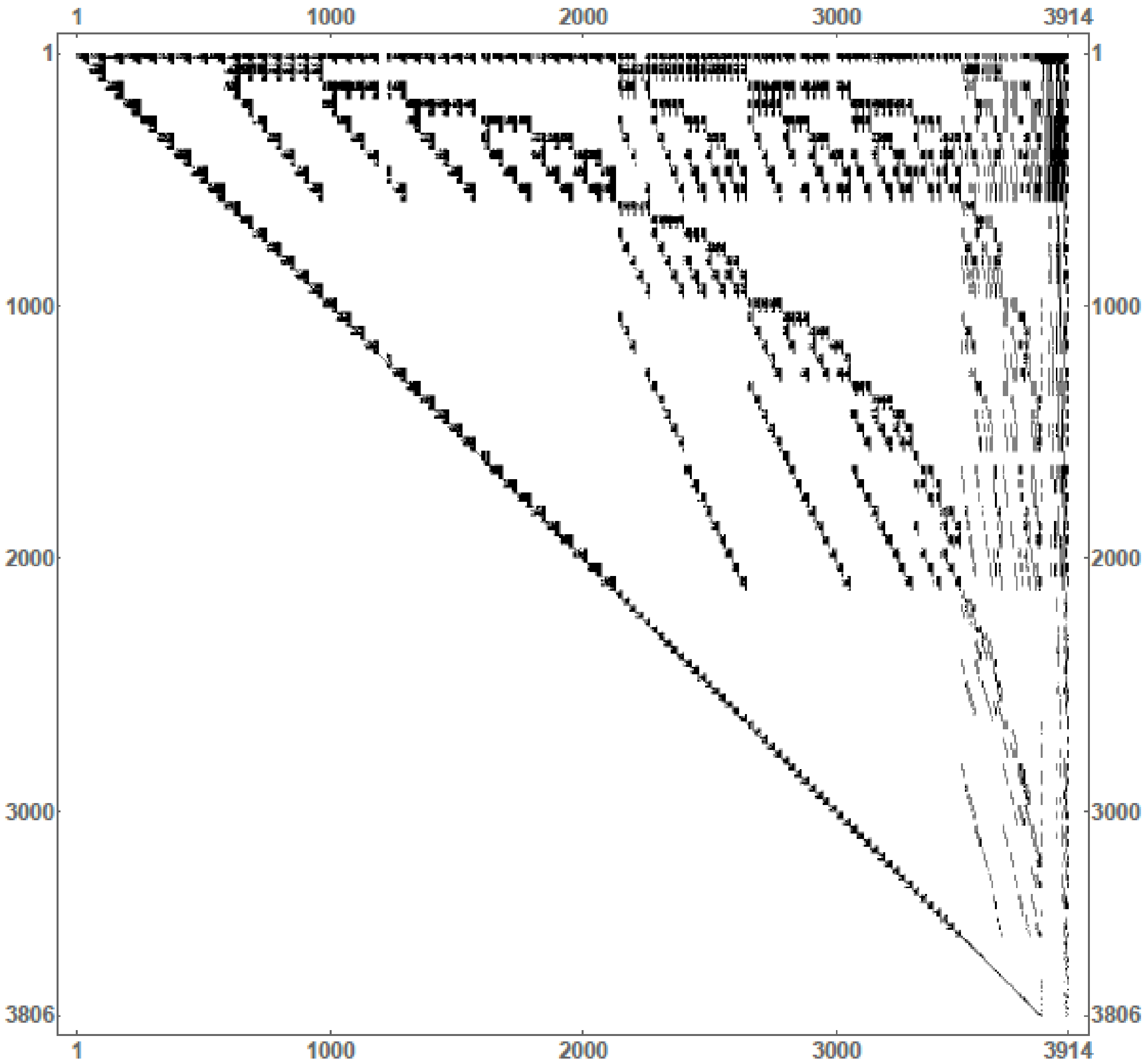
  topology (a) up to degree-4, 2GB; **Bendle et al, JHEP(2020)**

  topology (b) up to degree-4, 0.8GB; **Böhm et al, JHEP(2018)**

  all integrals in topology (c), over 20GB. **Chawdhry et al, Phys.Rev.D(2019)**

- much more efficient for practical usage

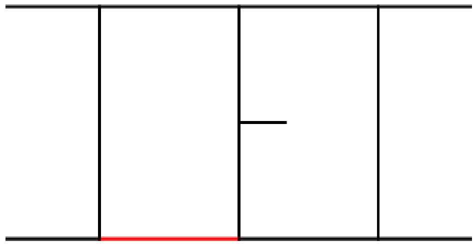
Results



Master integrals

➢ Auxiliary mass flow

XL, Ma, Wang, Phys.Lett.B(2018)
XL, Ma, in preparation



- the red line: $\frac{1}{(\ell_1+p_1+p_2)^2} \to \frac{1}{(\ell_1+p_1+p_2)^2+\eta}$
- $\vec{s} \to \vec{s}_0 = \{4, -\frac{113}{47}, \frac{281}{149}, \frac{349}{257}, -\frac{863}{541}\}$
- set up differential equation with respect to η

$$\frac{\partial}{\partial\eta}\vec{\mathcal{I}}(\epsilon, \vec{s}_0, \eta) = A(\epsilon, \eta)\vec{\mathcal{I}}(\epsilon, \vec{s}_0, \eta)$$

- boundary condition: $\eta \to \infty$
- numerically solve via power series to obtain: $\eta = \mathrm{i}\, 0^+$

Master integrals

➢ Auxiliary mass flow

- master integrals: 108 $\to$ 176
- set up differential equation (finite field interpolation): 70 CPU hours
- numerical solution: 9 CPU hours
- $I_{11111111000}(\epsilon, \vec{s}_0) = -0.0694356251726377 6\, \epsilon^{-4}$

  $+(1.16225663671128 7 + 1.41635985344671 7\, \mathrm{i})\, \epsilon^{-3}$

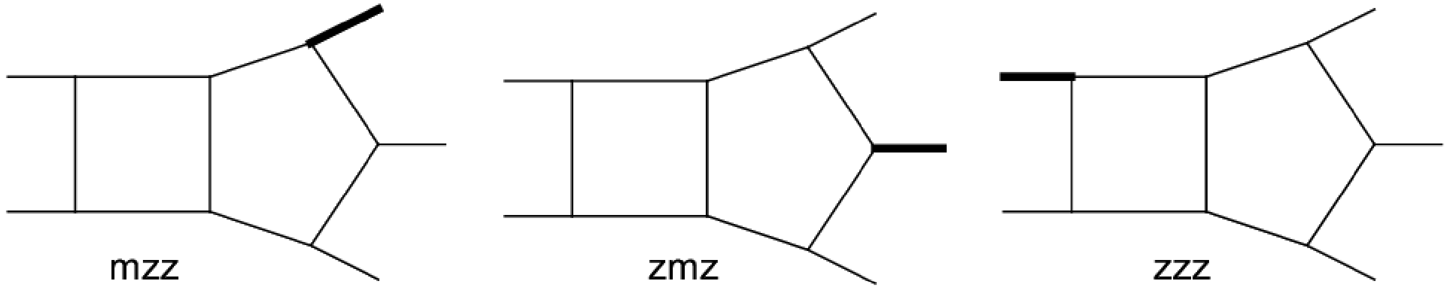
  $+(37.8247433211693 8 + 15.9191244358173 9\, \mathrm{i})\epsilon^{-2}$

  $+(86.2861798369034 + 166.897153571127 7\, \mathrm{i})\epsilon^{-1}$

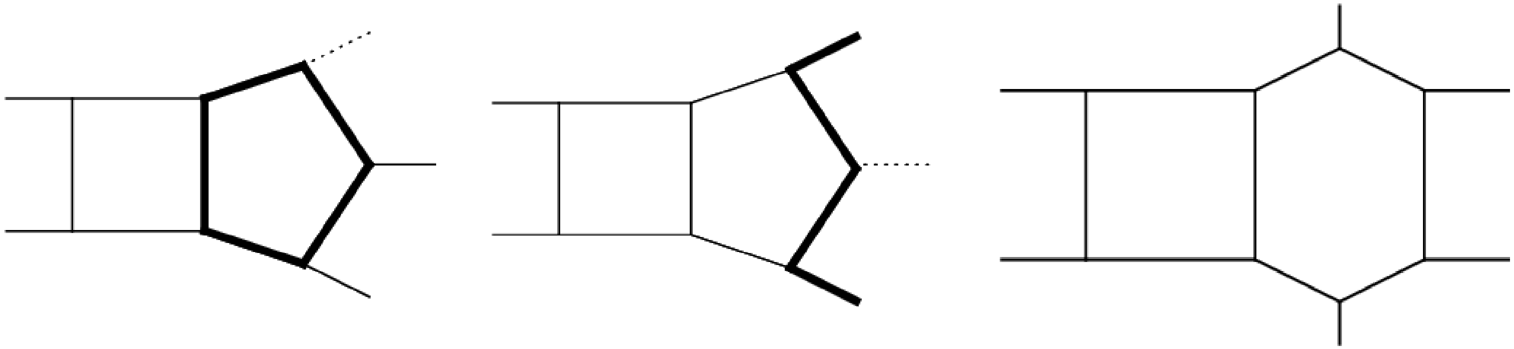
  $+(-4.1435965578662 + 333.099604007130 5\, \mathrm{i})$

  $+(-531.834114822928 + 1583.72467250214 1\, \mathrm{i})\, \epsilon + O(\epsilon^2)$
- fully agreement with analytic results **Chicherin, Gehrmann, Henn et al. Phys.Rev.Lett.123(2019)** **Chicherin, Sotnikov, 2009.07803**
- sector decomposition: cannot obtain results within reasonable time

➢ Other examples



Abreu, Ita, Moriello et al. 2005.04195



Summary and outlook

- We present search algorithm to reduce multiloop multiscale integrals to master integrals and auxiliary-mass-flow method to calculate these master integrals.
- These methods are systematic and efficient by definition.
- In future, we will apply these methods to more physically interesting processes: $pp \to t\bar{t}H, pp \to t\bar{t}j, \ldots$

Thank you!

Feynman integrals

➢ Dimensional regulated integral family

$$I(\epsilon,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^{4-2\epsilon}\ell_i}{\mathrm{i}\pi^{2-\epsilon}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha+\eta)^{\nu_\alpha}}$$

- $\{D_1,\cdots,D_N\}$ with $N=\frac{L(L+1)}{2}+L\,E$
- $\vec{s}=\{p_i^2, p_i\cdot p_j, m_i^2\}$
- η: the auxiliary mass parameter

$$I_{\mathrm{phy}}(\epsilon,\vec{s}) \equiv \lim_{\eta\to \mathrm{i}0^+} I(\epsilon,\vec{s},\eta)$$

Expansion

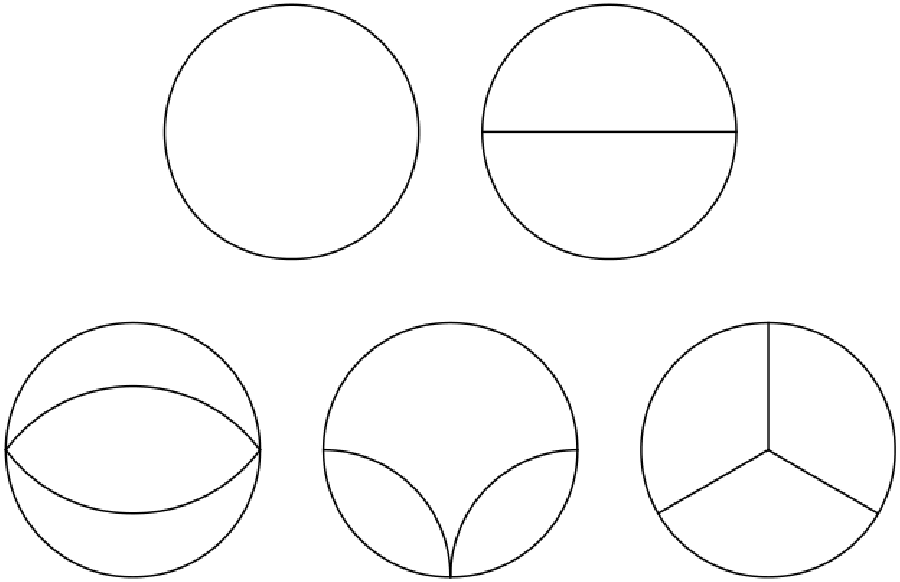
➢ Expansion near $\eta = \infty$

Smirnov, Commun. Math. Phys. 134(1990)
Smirnov, Mod. Phys. Lett. A 10 (1995)

- integration region: $\ell_i \sim \sqrt{\eta}$
- $\frac{1}{(\ell+p)^2-m^2+\eta} = \frac{1}{\ell^2+\eta} + \cdots$

$$I(\epsilon, \vec{s}, \eta) = \eta^{(2-\epsilon)L-\sum \nu_\alpha} \sum_{n=0}^{\infty} M_n(\epsilon, \vec{s})\eta^{-n}$$

XL, Ma, Phys. Rev. D 99 (2019)
Wang, Li, Basat, Phys.Rev.D 101(2020)



Davydychev,Tausk, Nucl.Phys.B 397(1993)
Broadhurst, Eur.Phys.J.C8 (1999)
Kniehl, Pikelner, Veretin, JHEP 08(2017)
Pikelner Comput.Phys.Commun. 224 (2018)

Auxiliary mass flow

➢ Differential equation method

Kotikov, Phys. Lett. B 254 (1991)
Bern, Dixon, Kosower, Phys.Lett.B302(1993)

$$\frac{\partial}{\partial \eta}\vec{\mathcal{I}}(\epsilon,\vec{s}_0,\eta) = A(\epsilon,\eta)\vec{\mathcal{I}}(\epsilon,\vec{s}_0,\eta)$$

- $\vec{s} = \vec{s}_0$
- power series expansion near singular and regular points

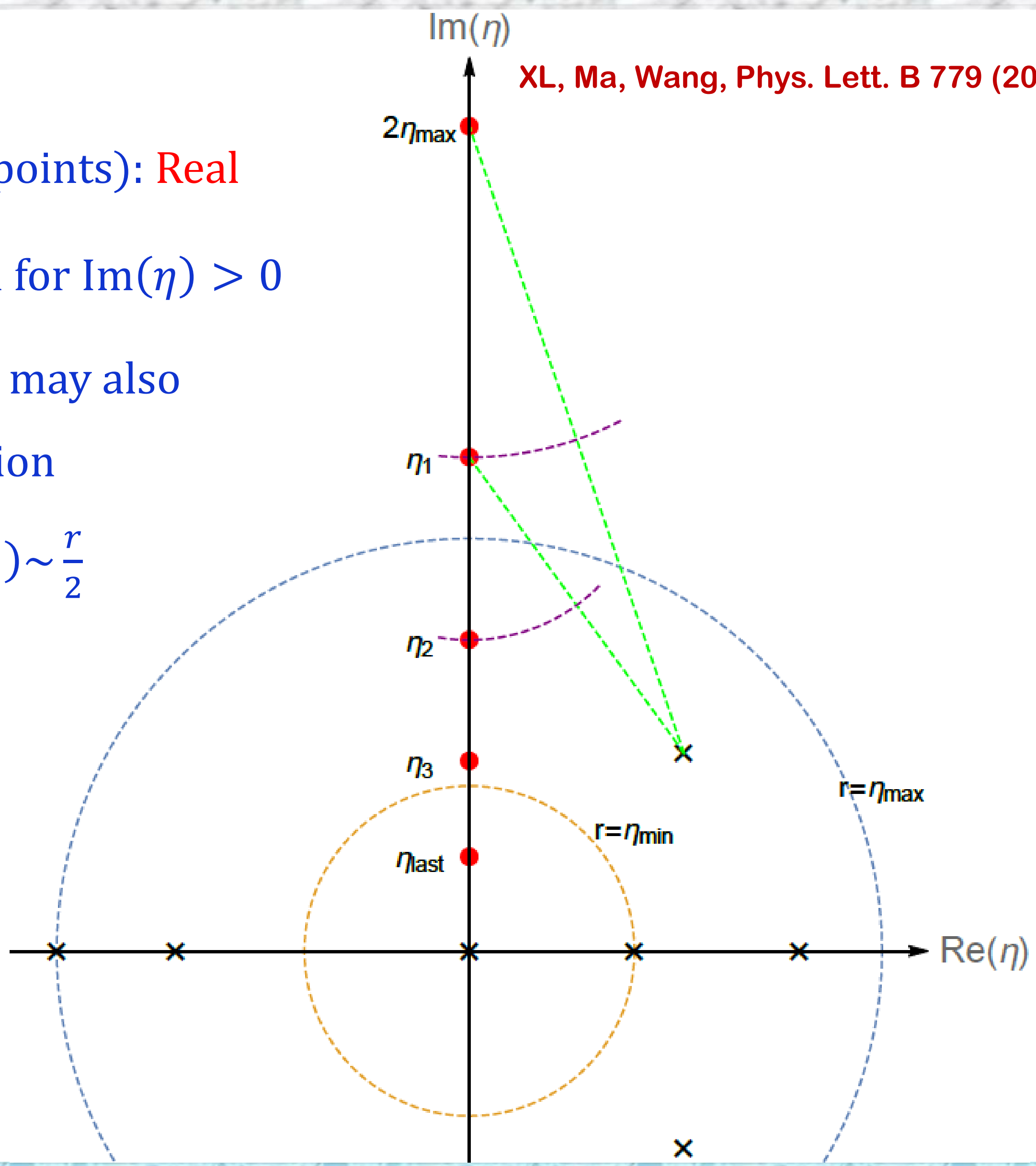
$$\mathcal{I}_i(\epsilon,\vec{s}_0,\eta) = \sum_{\mu\in S}(\eta-\eta_0)^{\mu}\sum_{k=0}^{k_\mu}\log(\eta-\eta_0)^k\sum_{n=0}^{\infty}c_{i,\mu,k,n}(\eta-\eta_0)^n$$

- regular: $S = \{0\}, k_0 = 0$
- singular: $S = \{-2\epsilon, 1+\epsilon, \cdots\}, k_\mu \geq 0$
- only valid locally

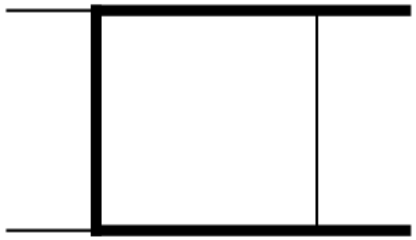
Auxiliary mass flow

- physical singularities (branch points): Real
- $\frac{1}{(\ell+p)^2-m^2+\mathrm{Re}(\eta)+i\,\mathrm{Im}(\eta)}$ off-shell for $\mathrm{Im}(\eta) > 0$
- singularities far from real axis: may also affect the convergence of solution
- estimation at η_{next}: $(\eta_{\mathrm{next}} - \eta_0) \sim \frac{r}{2}$
- error $\sim \left(\frac{1}{2}\right)^{n_0}$

XL, Ma, Wang, Phys. Lett. B 779 (2018)



➢ Example: one-loop four-point integral



$s = 10, t = -3, m^2 = 1$

- η reg: $-0.1309\ \mathrm{i}\sqrt{\eta} + (0.0665971\ - 0.101394\ \mathrm{i})\log(\eta) - (0.29347\ - 0.0201092\ \mathrm{i})$
- dim-reg: $\eta^{-\epsilon} f_1(\epsilon) + f_2(\epsilon) + \eta^{\frac{1}{2}-\epsilon} f_3(\epsilon)$
- $f_1(\epsilon) = (-0.0665971 + 0.101394\ \mathrm{i})\epsilon^{-1} + (-0.280099 - 0.267748\ \mathrm{i})$
- $f_2(\epsilon) = (0.0665971\ - 0.101394\ \mathrm{i})\epsilon^{-1} + (-0.0133705 + 0.287857\ \mathrm{i})$
- $f_1(\epsilon) = -0.1309\ \mathrm{i}$
- take $\eta \to 0$, only $f_2(\epsilon)$ survives

Discussion

➢ Advantages

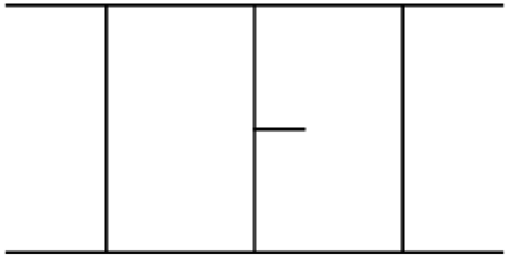
- extremely systematic
- precision is totally proportional to time consumption
- not sensitive to the choice of $\vec{s}_0$

➢ Problems

- effect of extra mass scale: many more master integrals, hard to set up differential equation for complicated problems
- develop much more powerful integral reduction method
- reduce the effect of extra mass scale

Strategy to introduce η

➢ Example: two-loop double-pentagon



- before introducing η : 108
- all: 476
- loop: 305, 319
- branch: 233, 234
- propagator: 176, 178, 220
- propagator mode seems to be the cheapest