Electroweak EFT(s) the on-shell way

Yael Shadmi

Technion

YS, Yaniv Weiss 1809.09644
Gauthier Durieux, Teppei Kitahara, YS, Yaniv Weiss 1909.10551
Gauthier Durieux, Teppei Kitahara, Camila Machado, YS, Yaniv Weiss 2008.09652

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develop an on-shell approach to EFT (=EFfective Theory*) extensions of the standard model

2 main themes:

• toolbox for EFT computations, analyses

• an on-shell understanding of EWSB/Higgs mechanism: how does SU(2)$\times$U(1) emerge? relations between different masses?

* credit: Baratella Fernandez von Harling Pomarol

Which EFTs?

- Tools: apply to any EFT: SM, SM+X, ..
- Focus: applied to electroweak: any EFT with the electroweak particle content

SMEFT: impose SU(2)$\times$U(1) at high scales

bottom-up (bootstrap) approach: can explore general EFT extensions of standard model: weakly/strongly coupled

Outline

- Motivation for going on-shell
- Amplitude essentials   following up on Henriette Elvang's talk, LianTao Wang
  - massless and massive spinors
  - massless $\leftrightarrow$ massive: high-energy limit, (un)bolding
- Some warm-up examples to demonstrate the construction of amplitudes: $X+3$ gluons; $X=$ massive spin-0,1 (SM or new particle)
- The electroweak EFT:
  - The high-energy behavior of 3- and 4-point massive amplitudes: smooth HE limit and perturbative unitarity: gauge symmetry and Higgsing
    - 3 massive vectors
    - fermion-fermion-vector
    - fermion-fermion-Z-Higgs
  - Bases (in terms of LG covariant massive spinor formalism): constructing bases for all 4-point amplitudes, contact-terms

Our current (massive) toolbox

general 3- and 4-point amplitudes of massive/massless particles of various spins:

Durieux Kitahara YS Weiss; Durieux Machado; Durieux Kitahara Machado YS Weiss

electroweak: spins 0, 1/2, 1

- detailed gluing prescription (massive spin 1/2, 1)
- bases for all massive 3-points
- bases for all massive 4-points (any dimension): generic amplitudes, contact-terms
- matching to broken-phase SMEFT 3-point (+) couplings

beyond electroweak:

- bases for all massive 3-point spin≤ 3
- bases for all 4-point massless contact terms spin ≤ 2; higher points for spins 0,1/2,1,2, dim ≤ 8

See also: Christensen Field; Christensen Field Moore Pinto

Herderschee Koren Trott; Aoude Machado; Bachu Yelleshpur..

On-shell Effective theories: motivation

The (main) reason we're all here:

our complete ignorance about BSM (whether it's there); in particular: EWSB, Naturalness

$\rightarrow$ **bottom-up EFTs**
a well-defined framework for quantifying this ignorance
= parametrizing deviations from SM

On-shell Effective theories: motivation

input:

- SM fields [+ possibly: some BSM fields]
- SM symmetry (global, gauge)
- + Lorentz, locality

$\rightarrow$ most general Lagrangian

$$\mathcal{L}_{\rm EFT} = \sum_i \frac{c_i}{\Lambda^{n_i}} \mathcal{O}_i$$

On-shell Effective theories: motivation

$\rightarrow$ **bottom-up** (bootstrap) on-shell approach is very natural:

input:

- SM particles [+ possibly: some BSM particles]
- symmetry (global, gauge)
- + Lorentz, locality

$\rightarrow$ most general amplitudes:

starting with 3-point amplitudes & gluing these into higher-point/higher-order amplitudes

gauge symmetry emerges from the requirement of consistent interactions of spin-1 particles (Bose statistics, factorization..)

On-shell Effective theories: motivation

1st step of EFT construction: identify basis of operators $\mathcal{O}_i$ (complete, independent)

modulo field redefinitions, EOMs, gauge redundancies

mathematically: polynomials of operators subject to a set of constraints
beautifully solved using Hilbert Series Jenkins Manohar; Lehman Martin; Henning Lu Melia Murayama

On-shell Effective theories: motivation

on-shell construction: only deal with physical quantities
$\rightarrow$ this 1st step is mostly circumvented (field redefinitions, gauge redundancies)

the remainder simplifies considerably:

polynomials of operators subject to a set of constraints (EOM, momentum conservation)

maps to:
polynomials in the Mandelstam invariants (just numbers..) subject to a set of kinematical constraints, e.g., $s + t + u = \sum m^2$

also used in Henning Lu Melia Murayama

by-product: counting & classification of EFT operators

On-shell Effective theories: motivation

- SM applications:

$$\mathcal{L}_{\rm EFT} = \sum_i \frac{c_i}{\Lambda^{n_i}} \mathcal{O}_i$$

typically: start with full SU(2)$\times$U(1) (SMEFT)

EWSB: $\mathcal{O}_i$ modify masses, SM couplings
$\rightarrow$redefine input parameters

on-shell construction: relate physical observables (largely), directly measurable
future work

- [computation: avoid Feynman diagrams, particularly complicated with EFT vertices]

on-shell EFTs for the SM

How on-shell do you want to be? or: can you be half on-shell?

yes, and it's useful: work in unbroken electroweak symmetry phase
impose full SU(2)$\times$U(1) massless particles only

- good approximation at high-energies
- all you care about for running, anomalous dimensions
- mostly used, many powerful results: classification of operators, selection rules, operator mixings, anomalous dimensions, positivity..

Chueng Shen; Azatov Contino Machado Riva; Bern Parra-Martinez Sawyer; Ma Shu Xiao; Baratella Fernandez von Harling Pomarol; Gu Wang Zhang; Falkowski; ..

on-shell EFTs for the SM

focus here: fully on-shell:

work with SM particles: massive W, Z, h..

want to exploit full power of on-shell approach in relating purely physical observables

also motivated by aforementioned ignorance: back to basics: work with what we know is there

(MASSIVE) EFT AMPLITUDE BASICS

EFT AMPLITUDE BASICS

- Little Group + spin-statistics determine all n-point Contact Terms (CT) (≥ 3)

  3-points: generically complex momenta

- Remaining parts of amplitudes determined by factorization/generalized unitarity

$\rightarrow$ bootstrap higher point amplitudes by gluing together lower-point amplitudes + adding CTs

so: derive 3-point amplitudes
get 4-point amplitudes by gluing two 3-points + adding 4-pt CT $\cdots$

CTs $\leftrightarrow$ EFT operators

unknown coefficients of CTs $\leftrightarrow$ unknown Wilson coefficients

0th order task: find CTs

Writing amplitudes: massless

Write amplitudes in terms of massless spinors

- momenta:

$$k_{\alpha\dot{\alpha}} \equiv k_\mu \sigma^\mu_{\alpha\dot{\alpha}} = |k\rangle\,[k|\,, \qquad \bar{k}^{\dot{\alpha}\alpha} \equiv k_\mu \bar{\sigma}^{\mu\dot{\alpha}\alpha} = |k]\,\langle k|,$$

- external polarizations: spin 1/2:

$$|k\rangle \quad = \lambda_\alpha(k)\,, \qquad |k] \quad = \tilde{\lambda}^{\dot{\alpha}}(k)$$

- external polarizations: spin-1:

$$\varepsilon^+_{\alpha\dot{\alpha}}(k) = \sqrt{2}\frac{|r\rangle\,[k|}{\langle kr\rangle} \quad \varepsilon^-_{\alpha\dot{\alpha}}(k) = \sqrt{2}\frac{|k\rangle\,[r|}{[kr]}$$

r is reference momentum

Writing amplitudes: massless

$\rightarrow$ amplitude written in terms of $\langle ij\rangle$, $[ij]$

can use the power of the little-group:

for a (massless) particle of momentum k:
k is invariant under little group [U(1)], polarizations are not:

assigning LG charges: $|k\rangle \;\; (1)$
$\rightarrow \quad |k] \;\; (-1) \quad \varepsilon^+ \;\; (+2) \quad \varepsilon^- \;\; (-2)$

$\rightarrow$ selection rules: external particle helicities determine LG weights of amplitude

note:
each massless fermion i: one spinor $|i]$ (or $|i\rangle$)
each massless vector i: two such spinors: $|i]\,|i]$ (or $|i\rangle|i\rangle$)

Writing amplitudes: massive

LG covariant massive formalism Arkani-Hamed Huang Huang; Conde Marzolla

- momenta: decompose in terms of two massless momenta:

$$p = p^{I=1} + p^{I=2} \qquad 2p^1 \cdot p^2 = m^2$$

LG [SU(2)] rotates:

$$|\boldsymbol{p}^I\rangle \to W^I_{\ J}|\boldsymbol{p}^J\rangle\,, \quad \text{and} \quad [\boldsymbol{p}_I| \to \left(W^{-1}\right)_I^{\ J}[\boldsymbol{p}_J|\,.$$

$$I, J = 1, 2$$

- polarizations: spin 1/2:

$$u^I(p) = \begin{pmatrix} |p^I\rangle \\ |p^I] \end{pmatrix} \quad \ldots \tag{1}$$

$$\mathbf{p}|p^I\rangle = m\,|p^I] \tag{2}$$

Writing amplitudes: massive

- polarizations: spin 1:

$$\varepsilon^{\{IJ\}}_{\alpha\dot{\alpha}} = \sqrt{2}\frac{|\boldsymbol{p}\rangle\,[\boldsymbol{p}|}{m} \qquad I{=}J{=}1\!:\, +1\,,\;\; I{=}J{=}2\!:\, -1\,,\;\; I{\neq}J\!:\, 0$$

bold denotes symmetrization over LG indices, *e.g.* $|\boldsymbol{p}\rangle\,[\boldsymbol{p}| = |p^{\{I}\rangle\left[p^{J\}}\right|$

will use boldface for massive momenta too to distinguish them from massive ones

$\rightarrow$ amplitudes are expressed in terms of spinor products $\langle\boldsymbol{ij}\rangle$, $[\boldsymbol{ij}]$
(possibly with momentum insertions)

note:
each massive fermion i: one spinor $|\boldsymbol{i}]$ (or: $|\boldsymbol{i}\rangle$)
each massive vector i: two such spinors: $|\boldsymbol{i}]\,|\boldsymbol{i}]$ (or: $|\boldsymbol{i}\rangle|\boldsymbol{i}\rangle$, $|\boldsymbol{i}]\,|\boldsymbol{i}\rangle$:)

Writing amplitudes: massive

more generally: a spin s rep can be obtained as a symmetric combination of spin-1/2 reps

$\rightarrow$ external leg of spin s:

$$\mathcal{M}^{I_1 \ldots I_{2s}} = |\boldsymbol{p}\rangle^{I_1}_{\alpha_1} \ldots |\boldsymbol{p}\rangle^{I_{2s}}_{\alpha_{2s}} M^{\{\alpha_1 \ldots \alpha_{2s}\}} \; = |\boldsymbol{p}]^{I_1}_{\dot{\alpha}_1} \ldots |\boldsymbol{p}]^{I_{2s}}_{\dot{\alpha}_{2s}} \tilde{M}^{\{\dot{\alpha}_1 \ldots \dot{\alpha}_{2s}\}}$$

Why have we suffered through this?

with the amplitudes written in terms of the LG covariant massive spinor formalism:

- can use the power of the LG: now SU(2) selection rules
- beautiful connection between massive and massless amplitudes via unbolding (and bolding)

High-energy limit: massive $\to$ massless (unbolding)

$p = p^1 + p^2$

choosing the direction of p^1 = choosing spin polarization axis

convenient to recover helicity states in high-energy limit:

$$p^1_\mu = \frac{E+p}{2}(1,0,0,1) \equiv k_\mu\,, \quad p^2_\mu = \frac{E-p}{2}(1,0,0,-1) \equiv q_\mu$$

[$k = \mathcal{O}(E), \quad q = \mathcal{O}(m^2/E)$]

High-energy limit: massive $\to$ massless (unbolding)

in HE limit: $p \to k \quad q \to 0$

then *e.g.*, for a vector of positive polarization $I = J = 1$:

$$|\boldsymbol{p}]\,|\boldsymbol{p}] \to |k]\,|k]$$

(all others vanish, *e.g.* $|\boldsymbol{p}]\,|\boldsymbol{p}\rangle \to 0$)

for a vector of zero polarization $(I, J) = (1, 2)$:

$$|\boldsymbol{p}]\,|\boldsymbol{p}\rangle \to |k]\,|k\rangle$$

$\to$get massless amplitudes by unbolding

will see that inverse works too in some cases: bold massless amplitudes to get massive ones

Let's put this to work

Example: $h + 3g$ amplitude

YS Weiss 2018

1st example: SM singlet + 3 gluons   effectively massless
h is spin-0 SM singlet
all $+$ helicities:

LG $\rightarrow [12][23][13]$ up to function of invariants   1,2,3 label gluon momenta

Contact Terms: no poles, any polynomial of s_{12}, s_{23}, s_{13} subject to
$s_{12} + s_{23} + s_{13} = m_h^2$

$$\mathcal{M}\big(h; g^{a+}(p_1) g^{b+}(p_2) g^{c+}(p_3)\big) = \frac{[12][13][23]}{\Lambda} \Bigg[f^{abc} \Bigg(+ \frac{c_7}{\Lambda^2} + \frac{c_{11}}{\Lambda^6}(s_{12}s_{23} + s_{13}s_{23} + s_{12}s_{13}) + \frac{c_{13}}{\Lambda^8} s_{12}s_{13}s_{23} \Bigg) + d^{abc} \frac{c'_{13}}{\Lambda^8} (s_{12} - s_{13})(s_{12} - s_{23})(s_{13} - s_{23}) \Bigg]$$

$$+ \cdots$$

Example: $h + 3g$ amplitude

$$\mathcal{M}\big(h; g^{a+}(p_1) g^{b+}(p_2) g^{c+}(p_3)\big) = \frac{[12][13][23]}{\Lambda} \Bigg[$$

$$f^{abc}\left(\qquad + \frac{c_7}{\Lambda^2} + \frac{c_{11}}{\Lambda^6}(s_{12}s_{23} + s_{13}s_{23} + s_{12}s_{13}) + \frac{c_{13}}{\Lambda^8} s_{12}s_{13}s_{23} \right)$$

$$+ d^{abc} \frac{c'_{13}}{\Lambda^8} (s_{12} - s_{13})(s_{12} - s_{23})(s_{13} - s_{23}) \Bigg]$$

$$+ \cdots$$

here shown for dim≤ 13 but trivial to extend: polynomials in s_{ij}'s (symmeric or antisymmetric) — derivative expansion

Example: $h + 3g$ amplitude

adding in factorizable part (from gluing ggg and hgg):

$$\mathcal{M}\big(h; g^{a+}(p_1) g^{b+}(p_2) g^{c+}(p_3)\big) = \frac{[12][13][23]}{\Lambda} \Bigg[$$

$$f^{abc} \left(-i \frac{m^4 \, g_s \, c_5^{hgg}}{s_{12} s_{13} s_{23}} + \frac{c_7}{\Lambda^2} + \frac{c_{11}}{\Lambda^6} (s_{12} s_{23} + s_{13} s_{23} + s_{12} s_{13}) + \frac{c_{13}}{\Lambda^8} s_{12} s_{13} s_{23} \right)$$

$$+ d^{abc} \frac{c'_{13}}{\Lambda^8} \, (s_{12} - s_{13}) \, (s_{12} - s_{23}) \, (s_{13} - s_{23}) \Bigg]$$

$$+ \cdots$$

dim	operators	operators
	$\mathcal{M}(+++)$	$\mathcal{M}(++-)$
5	—	—
7	$h\, G_{\rm SD}^3\ [1, f^{abc}]$	—
9	—	$\mathcal{D}^2\, G_{\rm SD}^2\, G_{\rm ASD}\, h\ [1, f^{abc}]$
11	$\mathcal{D}^4\, G_{\rm SD}^3\, h\ [1, f^{abc}]$	$\mathcal{D}^4\, G_{\rm SD}^2\, G_{\rm ASD}\, h\ [1, f^{abc}; 1, d^{abc}]$
13	$\mathcal{D}^6\, G_{\rm SD}^3\, h\ [1, f^{abc}; 1, d^{abc}]$	$\mathcal{D}^6\, G_{\rm SD}^2\, G_{\rm ASD}\, h\ [2, f^{abc}; 1, d^{abc}]$

check with Mathematica notebook of Henning Lu Melia Murayama

Example: vector + 3-gluon amplitude

$$\mathcal{M}\big(Z'; g^{a-}(p_1); g^{b-}(p_2); g^{c+}(p_3)\big)$$
$$= d^{abc}\,\langle 12\rangle^2 \times \Big[[34]^2\,A + [13][23]\langle 14\rangle\langle 24\rangle\,B + [34]\Big([31]\langle 14\rangle - [32]\langle 24\rangle\Big)\,C\Big]$$
$$+ f^{abc}\,\langle 12\rangle^2 \times \Big[[34]^2\,D + [13][23]\langle 14\rangle\langle 24\rangle\,E + [34]\Big([31]\langle 14\rangle - [32]\langle 24\rangle\Big)\,F\Big]$$

up to dim ≤ 12:

$$A = \frac{d_8}{\Lambda^4} + \frac{d_{10}^{(1)}}{\Lambda^6}\,s_{12} + \frac{d_{12}^{(1)}\,s_{12}^2 + d_{12}^{(2)}\,s_{13}s_{23}}{\Lambda^8} \qquad B = \frac{m^2\,s_{12}\,d_{12}^{(6)}}{\Lambda^8}$$

$$C = (s_{13} - s_{23})\frac{m\,d_{12}^{(5)}}{\Lambda^8} \qquad D = (s_{23} - s_{13})\left(\frac{d_{10}^{(3)}}{\Lambda^6} + \frac{d_{12}^{(4)}}{\Lambda^8}\,s_{12}\right)$$

$$E = 0 \qquad F = \frac{m\,d_{10}^{(2)}}{\Lambda^6} + \frac{m\,d_{12}^{(3)}}{\Lambda^8}\,s_{12}$$

Example: vector + 3-gluon amplitude

unbolding →massless amplitudes:

$$\mathcal{M}\big(Z'; g^{a-}(p_1); g^{b-}(p_2); g^{c+}(p_3)\big)$$
$$= d^{abc}\,\langle 12\rangle^2 \times \Big[[34]^2\,A + [13][23]\langle 14\rangle\langle 24\rangle\,B + [34]\Big([31]\langle 14\rangle - [32]\langle 24\rangle\Big)\,C\Big]$$
$$+ f^{abc}\,\langle 12\rangle^2 \times \Big[[34]^2\,D + [13][23]\langle 14\rangle\langle 24\rangle\,E + [34]\Big([31]\langle 14\rangle - [32]\langle 24\rangle\Big)\,F\Big]$$

POSITIVE NEGATIVE ZERO

ELECTROWEAK EFTs

1st thing we need: bases of massive spinor-structures which span generic amplitudes, contact term pieces

• A construction of 3-point bases was given by Arkani-Hamed Huang and Huang: gives over-complete bases in some cases (*e.g.*, 3 vectors)

$\rightarrow$ Derived independent bases for electroweak 3-points (+ for spins up to 3), 4-points

postpone discussion of bases, start by looking at a few interesting amplitudes Durieux Kitahara YS Weiss

The WWZ coupling

Three nearly-degenerate spin-1 particles: $M_Z^2 - M_W^2 \ll M_W^2$

What do we learn from the WWZ coupling?
Let's first neglect the mass difference
The most general 3-point amplitude (for three degenerate spin-1):

The WWW coupling

$$\mathcal{M}(\mathbf{1}_W^a, \mathbf{2}_W^b, \mathbf{3}_W^c)$$

$$\frac{g}{m_W^2}\Big(\langle\mathbf{12}\rangle\langle\mathbf{13}\rangle[\mathbf{23}] + \langle\mathbf{12}\rangle[\mathbf{13}]\langle\mathbf{23}\rangle + [\mathbf{12}]\langle\mathbf{13}\rangle\langle\mathbf{23}\rangle$$

$$+ \langle\mathbf{12}\rangle[\mathbf{13}][\mathbf{23}] + [\mathbf{12}]\langle\mathbf{13}\rangle[\mathbf{23}] + [\mathbf{12}][\mathbf{13}]\langle\mathbf{23}\rangle\Big)$$

$$+\frac{c_6^L}{\bar{\Lambda}^2}\langle\mathbf{12}\rangle\langle\mathbf{13}\rangle\langle\mathbf{23}\rangle + \frac{c_6^R}{\bar{\Lambda}^2}[\mathbf{12}][\mathbf{13}][\mathbf{23}]$$

completely antisymmetric $\rightarrow$ completely antisymmetric prefactor

The WWW coupling

$$\mathcal{M}(\mathbf{1}_W^a, \mathbf{2}_W^b, \mathbf{3}_W^c) = \epsilon^{abc} \Big\{$$

$$\frac{g}{m_W^2}\Big(\langle\mathbf{12}\rangle\langle\mathbf{13}\rangle[\mathbf{23}] + \langle\mathbf{12}\rangle[\mathbf{13}]\langle\mathbf{23}\rangle + [\mathbf{12}]\langle\mathbf{13}\rangle\langle\mathbf{23}\rangle$$

$$+ \langle\mathbf{12}\rangle[\mathbf{13}][\mathbf{23}] + [\mathbf{12}]\langle\mathbf{13}\rangle[\mathbf{23}] + [\mathbf{12}][\mathbf{13}]\langle\mathbf{23}\rangle\Big)$$

$$+ \frac{c_6^L}{\bar{\Lambda}^2}\langle\mathbf{12}\rangle\langle\mathbf{13}\rangle\langle\mathbf{23}\rangle + \frac{c_6^R}{\bar{\Lambda}^2}[\mathbf{12}][\mathbf{13}][\mathbf{23}]\Big\}$$

SU(2) structure emerges! with n degenerate particles: $\epsilon^{abc} \rightarrow$ completely AS tensor

to see full Lie group structure need to consider proper factorization of 4-point amplitude $\rightarrow$ Jacobi identity (as in massless case)

- complete (anti)symmetry of massive 3-vector amplitude is more transparent than massless case, *e.g.*, $[12]^3/[13][23]$ of $++-$
- masses smooth the singular massless behavior

The WWW coupling

EM charge conservation follows: since the WWW spinor structure is fully antisymmetric $\rightarrow$ vanishes for identical W's:
W^+W^+ amplitude vanishes

can see full SU(2)$\times$U(1) emerge in high-energy limit:
WWZ and $WW\gamma$:
define combination B of Z and γ for which $WWB = 0$
couplings of $3V$, Vff, VSS related by requirement of consistent factorization.

Renormalizable vs Non-renormalizable (WWW)

identify NR 3-point couplings via "high-energy" behavior (complex momenta)
$\mathcal{O}(E)$ is good $\quad \mathcal{O}(E^2)$ or higher is bad

$$\frac{g}{m_W^2}\Big(\langle\mathbf{12}\rangle\langle\mathbf{13}\rangle[\mathbf{23}] + \langle\mathbf{12}\rangle[\mathbf{13}]\langle\mathbf{23}\rangle + [\mathbf{12}]\langle\mathbf{13}\rangle\langle\mathbf{23}\rangle + \langle\mathbf{12}\rangle[\mathbf{13}][\mathbf{23}] + [\mathbf{12}]\langle\mathbf{13}\rangle[\mathbf{23}] + [\mathbf{12}][\mathbf{13}]\langle\mathbf{23}\rangle\Big)$$
$$+\frac{c_6^L}{\bar\Lambda^2}\langle\mathbf{12}\rangle\langle\mathbf{13}\rangle\langle\mathbf{23}\rangle + \frac{c_6^R}{\bar\Lambda^2}[\mathbf{12}][\mathbf{13}][\mathbf{23}]$$

at HE:

$\langle\mathbf{12}\rangle\langle\mathbf{13}\rangle[\mathbf{23}] \sim m_W^2 E \to$ renormalizable
and has smooth limit with $1/m_W^2$ normalization

$[\mathbf{12}][\mathbf{13}][\mathbf{23}] \sim E^3 \to$ requires cutoff $\to 1/\bar\Lambda^2$ normalization

Renormalizable vs Non-renormalizable

identify NR 3-point couplings via "high-energy" behavior (complex momenta)
$\mathcal{O}(E)$ is good $\mathcal{O}(E^2)$ or higher is bad

3-point version of perturbative unitarity:

Llewellyn Smith; Joglekar; Cornwall Levin Tiktopoulos; Lee Quigg Thacker; Chanowitz Gaillard '70s

$$\mathcal{M}_4 \sim \mathcal{M}_3 \frac{1}{E^2} \mathcal{M}_3$$

$\rightarrow$ 3-point amplitude can grow like E, but E^2 requires cutoff

equivalently: want smooth behavior as $m \to 0$: E/m is bad, $E/\bar{\Lambda}$ is ok

implicitly assume a hierarchy between m and $\bar{\Lambda}$

note: 3-point amplitudes are exact expressions: kinematics is fixed
($p_i \cdot p_j$ determined by masses) $\rightarrow$ in matching to field theory computation:
coefficients sum all loop orders and v/Λ expansion

fermion-fermion-Z coupling

$$\mathcal{M}(\mathbf{1}_{\psi^c}, \mathbf{2}_{\psi}, \mathbf{3}_Z) = \frac{g_R}{m_Z}\langle\mathbf{13}\rangle[\mathbf{23}] + \frac{g_L}{m_Z}[\mathbf{13}]\langle\mathbf{23}\rangle + \frac{c_R}{\bar{\Lambda}}[\mathbf{13}][\mathbf{23}] + \frac{c_L}{\bar{\Lambda}}\langle\mathbf{13}\rangle\langle\mathbf{23}\rangle$$

$$\text{high energy}: \quad \mathcal{O}(m_Z E) \qquad\qquad \mathcal{O}(E^2)$$

Relearn some QFT (Higgsing)

or really the interactions of massive fermions and vectors

requiring a smooth high-energy limit of separate helicity amplitudes:

- fermions with a vector-like coupling to the Z do not couple to its longitudinal component
- the (chiral) coupling of a fermion to the longitudinal Z is proportional to the fermion mass
- the coupling of a massive fermion to a massless vector is vector-like
- The mass of a fermion with chiral couplings to a vector must tend to 0 at least as fast as the vector mass   see now
- ...

Relearn some QFT (Higgsing)

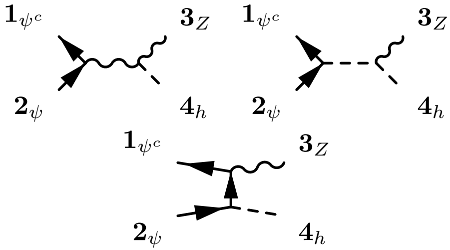
$$\mathcal{M}(\mathbf{1}^-_{\psi^c}, \mathbf{2}^-_{\psi}, \mathbf{3}^+_Z) \sim (g_L - g_R)\,\frac{m_\psi}{m_Z}\,\langle 12 \rangle$$

so:

- either: $g_L - g_R = 0$ vectorlike fermion
- or: $m_\psi \to 0$ as $m_Z \to 0$ chiral fermion mass has same origin as vector mass and goes to zero at least as fast refine when talk about 4-point amplitude

Relearn some QFT (Higgsing): fermion-fermion-Z-higgs amplitude

factorizable part featuring 3-point couplings:



$$+\cdots$$

+ new contact terms

fermion-fermion-Z-higgs amplitude

$$\mathcal{M}^{\text{contact}}(\mathbf{1}_{\psi^c}, \mathbf{2}_{\psi}, \mathbf{3}_Z, \mathbf{4}_h) = \frac{C_6^{(1)}}{\bar{\Lambda}^2}[\mathbf{13}][\mathbf{23}] + \frac{C_6^{(2)}}{\bar{\Lambda}^2}[\mathbf{13}]\langle\mathbf{23}\rangle + \frac{C_7^{(1)}}{\bar{\Lambda}^3}[\mathbf{312}\rangle[\mathbf{13}] + \frac{C_7^{(2)}}{\bar{\Lambda}^3}[\mathbf{321}\rangle[\mathbf{23}] + \text{angle} \leftrightarrow \text{square}$$

most general form

each coefficient is a series, *e.g.*,

$$C_6^{(1)} = c_6^{(1)} + c_8^{(1,1)}\frac{\tilde{s}_{12}}{\bar{\Lambda}^2} + c_8^{(1,2)}\frac{\tilde{s}_{13}}{\bar{\Lambda}^2} + \cdots \qquad [\tilde{s}_{ij} \equiv 2p_i \cdot p_j]$$

this expression captures all orders in $v/\bar{\Lambda}$

will come back to this but for now look at high-energy behavior

high-energy behavior

require good high-energy behavior: perturbative unitarity: $\mathcal{O}(E)$ growth accompanied by $1/\bar{\Lambda}$

4-point: genuine energy growth

$\rightarrow$coefficient of $\mathcal{O}(E)$ term vanishes

$\rightarrow$relates fermion-fermion-Z coupling (gauge); fermion-fermion-higgs coupling (Yukawa); ZZh coupling:

$$(g_L - g_R)\left(g_{ZZh}\frac{m_\psi}{m_Z} - y\right) = 0 + \mathcal{O}(m/\bar{\Lambda})$$

- for a vectorlike fermion $g_L = g_R$: no constraint on mass
- for a chiral fermion $g_L \neq g_R$:

$$m_\psi = 2\frac{y}{g_{ZZh}}m_Z$$

fermion mass from Higgs mechanism!

see also Maltoni Mantani Mimasu '19

ON-SHELL EFTs: BASES

Durieux Kitahara Machado YS Weiss

bases of $n \geq 4$-amplitudes

3 types of bases:

1) Spinor structure basis: minimal set of spinor structures $\boldsymbol{S}^{\{I\}}$ spanning the amplitude

spinor structure $\boldsymbol{S}^{\{I\}}$: product of *e.g.*, $[\boldsymbol{ij}]$,$\langle \boldsymbol{ijk}]$,$[\boldsymbol{ijki}]$, .. $\{I\}$= all LG indices
no prefactors of invariants s_{ij}

analogous constructions in terms of usual polarizations

eg, Bonifacio Hinetrbichler

a structure is redundant if it can be written as a linear combination of other $\boldsymbol{S}^{\{I\}}$'s
with coefficients = rational functions of the invariants

number of elements: $\Pi_i(2s_i + 1)$ Schomerus Sobko Isachenkov; Kravchuk Simmons-Duffin

to test independence: can construct inner products of spinor-structures:
$(\boldsymbol{S}_1, \boldsymbol{S}_2) \equiv \sum_{\{I\}} \boldsymbol{S}_1^{\{I\}^*} \boldsymbol{S}_2^{\{I\}}$= function of the invariants

bases of $n \geq 4$-amplitudes

EFT amplitude: contact terms only: no poles

to get a manifestly local amplitude: don't want negative powers of the invariants

$\rightarrow$ a structure is redundant if it can be written as a linear combination of other $\boldsymbol{S}^{\{I\}}$'s with coefficients = polynomials of the invariants (as opposed to rational functions)

$\rightarrow$ 2) Stripped Contact Term (SCT) basis: minimal set of spinor structures of this type

From this it's easy to construct the 3) Contact Term basis: (SCT)$\times$ polynomials of s_{ij}'s

bases

simple example: massless 4-fermion $++++$ amplitude:

spinor-structure basis: $[14][23]$

spans generic amplitude:

$$[13][24] = -\frac{s_{13}}{s_{14}}\,[14][23] \qquad + \text{Schouten } [12][34] = [13][24] - [14][23]$$

$$------------------------------$$

stripped-contact term basis: $[14][23]$, $[13][24]$

spans the contact-term (EFT) amplitude): manifestly local

$$------------------------------$$

contact-term basis:

$$\text{dim} - 6 : [14][23]\,, [13][24]$$

$$\text{dim} - 8 : s_{13}\,[14][23]\,, s_{14}\,[14][23]\,, s_{13}\,[13][24] \qquad \text{but no } s_{14}\,[13][24]$$

massless $\leftrightarrow$ massive

the massless case is easy, so let's use it:

• classify spinor structures according to "helicity categories" = helicities of high-energy limits = unbolded versions

e.g., in $ffvs$:

$$[\mathbf{13}][\mathbf{23}] \text{ is } (++ +0)$$
$$[\mathbf{13}]\langle\mathbf{23}\rangle \text{ is } (+-00)$$

number of helicity categories: $\Pi_i(2s_i+1)$

• massless limit: spinor-structures of different helicities: inner-product=0

massless $\leftrightarrow$ massive

$\rightarrow$ simple prescription for obtaining the spinor-structure basis:
take the massless amplitude in each helicity category and bold it

bolding = covariantizing wrt massive little group

note: adding extra scalars doesn't change basis: so have a simple prescription for finding bases of $\mathcal{M}(s_1, s_2, s_3, s_4, 0, 0, \ldots, 0)$

massless $\leftrightarrow$ massive

but we are mainly interested in EFT amplitudes: Stripped Contact Term basis:
not as simple, but can still use may elements of the massless limit

• each massive spinor-structure identity $\leftrightarrow$ massless spinor-structure identity
massless identity: $\langle 123][24] = \langle 124][23]$ $\quad\quad$ $\langle 12\rangle[23][24] = \langle 12\rangle[24][23]$

bolds to $\langle \mathbf{123}][\mathbf{24}] = \langle \mathbf{124}][\mathbf{23}] + \mathcal{O}(m)$

specifically $\langle \mathbf{123}][\mathbf{24}] = \langle \mathbf{124}][\mathbf{23}] + m_2\langle \mathbf{12}\rangle[\mathbf{34}]$

[and often don't need precise form of $\mathcal{O}(m)$]

• if a spinor-structure is redundant in massless case: it is also redundant in massive case $\quad$ so can consider unbolded versions

(stripped) contact term bases

- + a final stage of reductions:

recall: don't allow negative powers of invariants in spinor relations
but: allow negative powers of masses when longitudinal vectors are involved or higher spins

e.g. many a slide ago.. in $ffZh$ amplitude
used the spinor-structure identity:

$$[\mathbf{12}]\langle\mathbf{3123}\rangle = [\mathbf{12}]\left[\tilde{s}_{23}\,\langle\mathbf{313}] - \tilde{s}_{13}\,\langle\mathbf{323}]\right]/m_3 - \tilde{s}_{12}[\mathbf{13}][\mathbf{23}] - m_1[\mathbf{321}\rangle[\mathbf{23}] - m_2[\mathbf{312}\rangle[\mathbf{13}]$$

inverse mass in $\langle\mathbf{3}\cdots\mathbf{3}]$: longitudinal vector

(stripped) contact term bases

so: $|\boldsymbol{i}\rangle\,[\boldsymbol{i}| \to |\boldsymbol{i}\rangle\,[\boldsymbol{i}|/m_i$

different ways to see this:

• massive polarization is $|\boldsymbol{i}\rangle\,[\boldsymbol{i}|/m_i$: so with $1/m_i$ correctly identify dimension of operator

• these are the Goldstone amplitudes $\to$ scale as p_i/m_i

massless limit: $p_i = |i\rangle\,[i|$

covariantizing wrt massive little-group: $\to |\boldsymbol{i}\rangle\,[\boldsymbol{i}|$

RESULTS:

- Stripped Contact Term bases for all 4-point amplitudes of massive spin 0, 1/2, 1 (all dimensions)
- Stripped Contact Term bases for all 4-point amplitudes of massless spin 0, 1/2, 1 (all dimensions)
  complementing results of Durieux, Machado '19 for dim ≤ 8
- + all 3-point amplitudes for spin ≤ 3
  complementing results of Durieux, Kitahara, YS, Weiss '18 for spin ≤ 1

4-point SCT basis

spins	$n_{\rm SCT}$	n_s	hel. cat.	spinor structures	$n_{\rm perm}$	$\min\{d_{\rm op}\}$
$ssss$	1	1	(0000)	constant	1	4
$vsss$	$4 \to 3$	3	(0000)	$[121\rangle, [131\rangle$	1	5
			(+000)	$[1231] \to [1231] - \langle 1231\rangle$	$\cancel{2} \to 1$	7
$ffss$	4	4	(++00)	$[12]$	2	5
			(+−00)	$[132\rangle$	2	6
$vvss$	$10 \to 9$	9	(0000)	$[12]\langle 12\rangle, [131\rangle[232\rangle$	1	4,6
			(+000)	$[12][132\rangle$	4	6
			(++00)	$[12]^2$	2	6
			(+−00)	$[132\rangle^2 \to [132\rangle^2 - \langle 132]^2$	$\cancel{2} \to 1$	8
$ffvs$	$14 \to 12$	12	(++00)	$[12]\{[313\rangle, [323\rangle\}$	2	6
			(+−00)	$[13]\langle 23\rangle$	2	5
			(+++0)	$[13][23]$	2	6
			(++−0)	$[12]\langle 3123\rangle \to \emptyset$	$\cancel{2} \to 0$	8
			(+−+0)	$[13][312\rangle$	4	7
$ffff$	18	16	(++++)	$[12][34], [13][24]$	2	6
			(++−−)	$[12]\langle 34\rangle$	6	6
			(+++−)	$[12][324\rangle$	8	7
$vvvs$	$35 \to 29$	27	(0000)	$[12][343\rangle\langle 12\rangle, [13][242\rangle\langle 13\rangle, [23][141\rangle\langle 23\rangle$	1	5
			(+000)	$[12][13]\langle 23\rangle$	6	5
			(++00)	$[12]^2\left\{[313\rangle, [323\rangle\right\}$	6	7
			(+−00)	$[13][132\rangle\langle 23\rangle$	6	7
			(+++0)	$[12][13][23]$	2	7
			(++−0)	$[12]^2\langle 3123\rangle \to \emptyset$	$\cancel{6} \to 0$	9
$vvff$	$46 \to 38$	36	(00++)	$\langle 12\rangle \times \{[12][34], [13][24]\}$	2	5
			(00+−)	$\langle 14\rangle\langle 231][23], \langle 24\rangle\langle 132][13]$	2	6
			(0−++)	$\langle 12\rangle[34]\langle 241] \to \langle 12\rangle[34](\langle 241]/m_1 - \langle 142]/m_2)$	$\cancel{4} \to 2$	7
			(0+++)	$\langle 132] \times \{[12][34], [13][24]\}$	4	7
			(0++−)	$\langle 14\rangle[12][23]$	8	6
			(+++−)	$[12]^2[314\rangle$	4	8
			(++++)	$[12] \times \{[12][34], [13][24]\}$	2	7
			(−+++)	$\langle 1231\rangle[23][24] \to \emptyset$	$\cancel{4} \to 0$	9
			(++−−)	$[12]^2\langle 34\rangle$	2	7
			(+−−+)	$[14][132\rangle\langle 23\rangle \to [14][132\rangle\langle 23\rangle - [24][231\rangle\langle 13\rangle$	$\cancel{4} \to 2$	8
$vvvv$	$116 \to 85$	81	(0000)	$\{[12][34], [13][24]\} \times \{\langle 12\rangle\langle 34\rangle, \langle 13\rangle\langle 24\rangle\}$	1	4
			(+000)	$\{[12][34], [13][24]\} \times [142\rangle\langle 34\rangle \to \cdots$	$\cancel{8} \to 6$	6
			(++00)	$\{[12][34], [13][24]\} \times [12]\langle 34\rangle$	12	6
			(+−00)	$[13][14]\langle 23\rangle\langle 24\rangle$	12	6
			(+++0)	$\{[12][34], [13][24]\} \times [23][134\rangle$	8	8
			(++−0)	$[12]^2\langle 34\rangle\langle 324] \to [12]^2\langle 34\rangle(\langle 324]/m_4 - \langle 423]/m_3) \to \cdots$	$\cancel{24} \to \cancel{12} \to 5$	8
			(++++)	$[12]^2[34]^2, [12][13][24][34], [13]^2[24]^2$	2	8
			(+++−)	$[12][13][23]\langle 4124\rangle \to \emptyset$	$\cancel{8} \to 0$	10
			(++−−)	$[12]^2\langle 34\rangle^2$	6	8

Conclusions and outlook

• on-shell approach is natural for constructing bottom-up "EFT" extensions of low-energy SM

= all possible couplings of SM particles consistent with Lorentz, global symmetries, locality and unitarity

• compact analytic expressions for amplitudes: all orders in v/Λ expansion, straightforward to go to higher orders in derivative expansion (non-trivial part is the spinor structure)

• SU(2)$\times$U(1) structure & Higgs mechanism beautifully emerge from Lorentz + unitarity/good high energy limit for v hierachically smaller than Λ

• truly bottom-up: can explore extensions beyond SMEFT, extra Higgses, new particles

Conclusions and outlook

- systematic derivation of $n > 4$ SCT bases
- massive recursions Ochirov; Franken Schwinn; Falkowski Machado
- include loops—running
- global analysis in terms of on-shell quantities

s_1	s_2	s_3	$n^{3\text{-pt}}$	$n_{\rm rel}$	spinor structures
0	0	0	1		constant
0	0	1	1		$[3(1-2)3\rangle$
0	0	2	1		$[3(1-2)3\rangle^2$
0	0	3	1		$[3(1-2)3\rangle^3$
0	1/2	1/2	2		$([23], \langle 23\rangle)$
0	1/2	3/2	2		$[3(1-2)3\rangle \otimes ([23], \langle 23\rangle)$
0	1/2	5/2	2		$[3(1-2)3\rangle^2 \otimes ([23], \langle 23\rangle)$
0	1	1	3		$([23]^2, \langle 23\rangle[23], \langle 23\rangle^2)$
0	1	2	3		$[3(1-2)3\rangle \otimes ([23]^2, \langle 23\rangle[23], \langle 23\rangle^2)$
0	1	3	3		$[3(1-2)3\rangle^2 \otimes ([23]^2, \langle 23\rangle[23], \langle 23\rangle^2)$
0	3/2	3/2	4		$([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3)$
0	3/2	5/2	4		$[3(1-2)3\rangle \otimes ([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3)$
0	2	2	5		$([23]^4, \langle 23\rangle[23]^3, \langle 23\rangle^2[23]^2, \langle 23\rangle^3[23], \langle 23\rangle^4)$
0	2	3	5		$[3(1-2)3\rangle \otimes ([23]^4, \langle 23\rangle[23]^3, \langle 23\rangle^2[23]^2, \langle 23\rangle^3[23], \langle 23\rangle^4)$
0	5/2	5/2	6		$([23]^5, \langle 23\rangle[23]^4, \langle 23\rangle^2[23]^3, \langle 23\rangle^3[23]^2, \langle 23\rangle^4[23], \langle 23\rangle^5)$
0	3	3	7		$([23]^6, \langle 23\rangle[23]^5, \langle 23\rangle^2[23]^4, \langle 23\rangle^3[23]^3, \langle 23\rangle^4[23]^2, \langle 23\rangle^5[23], \langle 23\rangle^6)$
1/2	1/2	1	4		$([23], \langle 23\rangle) \otimes ([13], \langle 13\rangle)$
1/2	1/2	2	4		$[3(1-2)3\rangle \otimes ([23], \langle 23\rangle) \otimes ([13], \langle 13\rangle)$
1/2	1/2	3	4		$[3(1-2)3\rangle^2 \otimes ([23], \langle 23\rangle) \otimes ([13], \langle 13\rangle)$
1/2	1	3/2	6		$([23]^2, \langle 23\rangle[23], \langle 23\rangle^2) \otimes ([13], \langle 13\rangle)$
1/2	1	5/2	6		$[3(1-2)3\rangle \otimes ([23]^2, \langle 23\rangle[23], \langle 23\rangle^2) \otimes ([13], \langle 13\rangle)$
1/2	3/2	2	8		$([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3) \otimes ([13], \langle 13\rangle)$
1/2	3/2	3	8		$[3(1-2)3\rangle \otimes ([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3) \otimes ([13], \langle 13\rangle)$
1/2	2	5/2	10		$([23]^4, \langle 23\rangle[23]^3, \langle 23\rangle^2[23]^2, \langle 23\rangle^3[23], \langle 23\rangle^4) \otimes ([13], \langle 13\rangle)$
1/2	5/2	3	12		$([23]^5, \langle 23\rangle[23]^4, \langle 23\rangle^2[23]^3, \langle 23\rangle^3[23]^2, \langle 23\rangle^4[23], \langle 23\rangle^5) \otimes ([13], \langle 13\rangle)$
1	1	1	7	1	$([12], \langle 12\rangle) \otimes ([23], \langle 23\rangle) \otimes ([13], \langle 13\rangle)$
1	1	2	9		$([23]^2, \langle 23\rangle[23], \langle 23\rangle^2) \otimes ([13]^2, \langle 13\rangle[13], \langle 13\rangle^2)$
1	1	3	9		$[3(1-2)3\rangle \otimes ([23]^2, \langle 23\rangle[23], \langle 23\rangle^2) \otimes ([13]^2, \langle 13\rangle[13], \langle 13\rangle^2)$
1	3/2	3/2	10	2	$([12], \langle 12\rangle) \otimes ([23]^2, \langle 23\rangle[23], \langle 23\rangle^2) \otimes ([13], \langle 13\rangle)$
1	3/2	5/2	12		$([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3) \otimes ([13]^2, \langle 13\rangle[13], \langle 13\rangle^2)$
1	2	2	13	3	$([12], \langle 12\rangle) \otimes ([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3) \otimes ([13], \langle 13\rangle)$
1	2	3	15		$([23]^4, \langle 23\rangle[23]^3, \langle 23\rangle^2[23]^2, \langle 23\rangle^3[23], \langle 23\rangle^4) \otimes ([13]^2, \langle 13\rangle[13], \langle 13\rangle^2)$
1	5/2	5/2	16	4	$([12], \langle 12\rangle) \otimes ([23]^4, \langle 23\rangle[23]^3, \langle 23\rangle^2[23]^2, \langle 23\rangle^3[23], \langle 23\rangle^4) \otimes ([13], \langle 13\rangle)$
1	3	3	19	5	$([12], \langle 12\rangle) \otimes ([23]^5, \langle 23\rangle[23]^4, \langle 23\rangle^2[23]^3, \langle 23\rangle^3[23]^2, \langle 23\rangle^4[23], \langle 23\rangle^5) \otimes ([13], \langle 13\rangle)$
3/2	3/2	2	14	4	$([12], \langle 12\rangle) \otimes ([23]^2, \langle 23\rangle[23], \langle 23\rangle^2) \otimes ([13]^2, \langle 13\rangle[13], \langle 13\rangle^2)$
3/2	3/2	3	16		$([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3) \otimes ([13]^3, \langle 13\rangle[13]^2, \langle 13\rangle^2[13], \langle 13\rangle^3)$
3/2	2	5/2	18	6	$([12], \langle 12\rangle) \otimes ([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3) \otimes ([13]^2, \langle 13\rangle[13], \langle 13\rangle^2)$
3/2	5/2	3	22	8	$([12], \langle 12\rangle) \otimes ([23]^4, \langle 23\rangle[23]^3, \langle 23\rangle^2[23]^2, \langle 23\rangle^3[23], \langle 23\rangle^4) \otimes ([13]^2, \langle 13\rangle[13], \langle 13\rangle^2)$
2	2	2	19	8	$([12]^2, \langle 12\rangle[12], \langle 12\rangle^2) \otimes ([23]^2, \langle 23\rangle[23], \langle 23\rangle^2) \otimes ([13]^2, \langle 13\rangle[13], \langle 13\rangle^2)$
2	2	3	23	9	$([12], \langle 12\rangle) \otimes ([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3) \otimes ([13]^3, \langle 13\rangle[13]^2, \langle 13\rangle^2[13], \langle 13\rangle^3)$
2	5/2	5/2	24	12	$([12]^2, \langle 12\rangle[12], \langle 12\rangle^2) \otimes ([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3) \otimes ([13]^2, \langle 13\rangle[13], \langle 13\rangle^2)$
2	3	3	29	16	$([12]^2, \langle 12\rangle[12], \langle 12\rangle^2) \otimes ([23]^4, \langle 23\rangle[23]^3, \langle 23\rangle^2[23]^2, \langle 23\rangle^3[23], \langle 23\rangle^4) \otimes ([13]^2, \langle 13\rangle[13], \langle 13\rangle^2)$
5/2	5/2	3	30	18	$([12]^2, \langle 12\rangle[12], \langle 12\rangle^2) \otimes ([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3) \otimes ([13]^3, \langle 13\rangle[13]^2, \langle 13\rangle^2[13], \langle 13\rangle^3)$
3	3	3	37	27	$([12]^3, \langle 12\rangle[12]^2, \langle 12\rangle^2[12], \langle 12\rangle^3) \otimes ([23]^3, \langle 23\rangle[23]^2, \langle 23\rangle^2[23], \langle 23\rangle^3) \otimes ([13]^3, \langle 13\rangle[13]^2, \langle 13\rangle^2[13], \langle 13\rangle^3)$

THANK YOU