Classification of Effective Operators

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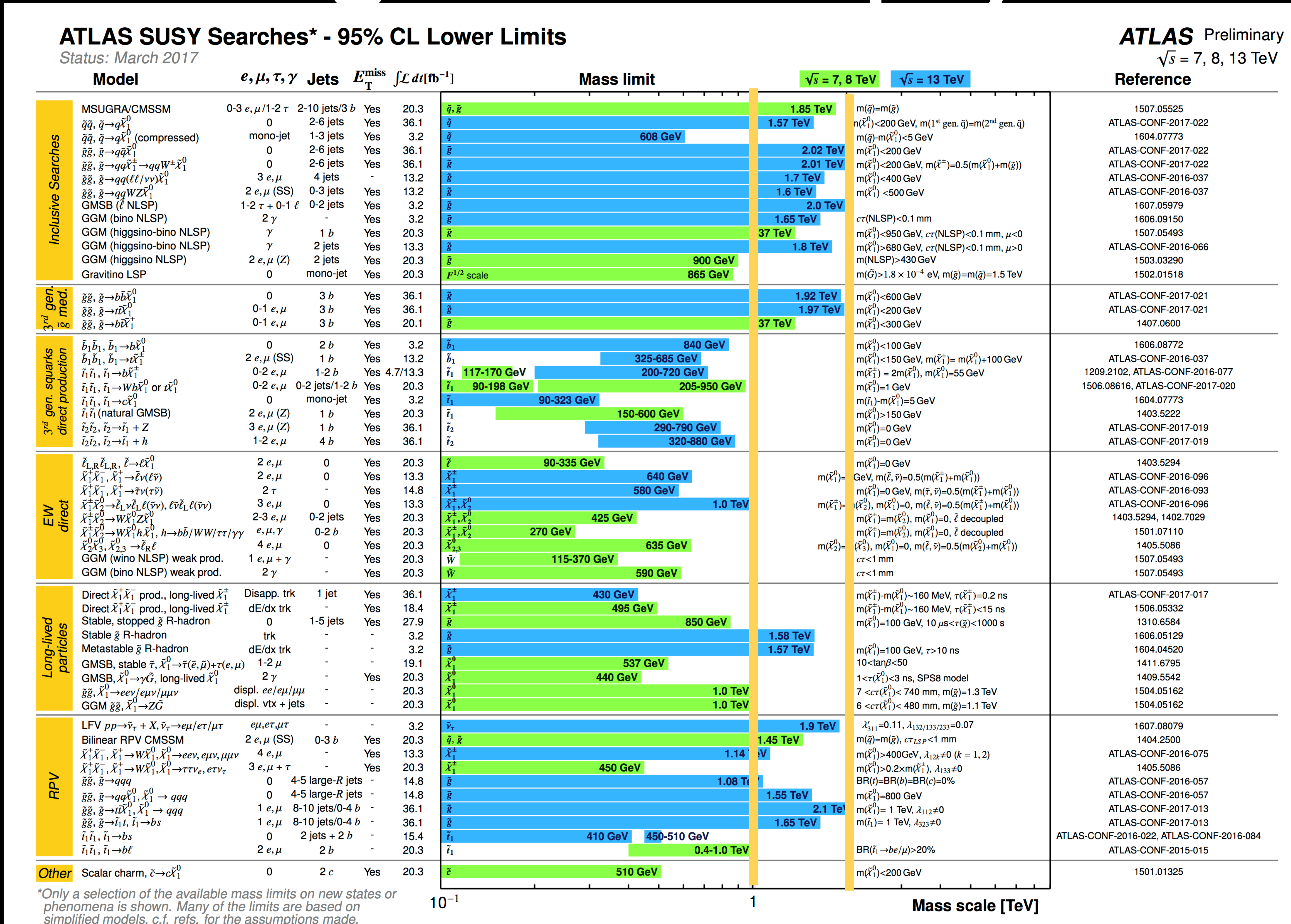
Effective Operators

- how do we classify effective operators?
  - long-standing problem since 80's
- conformal field theory helps in SM and any other renormalizable field theory
- also chiral Lagrangian using Hodge theory
- and connect to phenomenology!

Introduction

no sign of new physics



"The 2 TeV line has been reached for some scenarios"

ENGINEERING
Machines That Change Shape

MEDICINE
An Off Switch for Cancer

NEUROSCIENCE
How to Reach "Vegetative" Patients

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A CRISIS IN PHYSICS?

IF SUPERSYMMETRY DOESN'T PAN OUT, SCIENTISTS NEED A NEW WAY TO EXPLAIN THE UNIVERSE

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why effective operators

- No signal of BSM @ LHC so far
- use effective operators to parametrize physics at higher energies
  - precision electroweak
  - precision Higgs
  - precision flavor
  - *B*, *L* violation
- once deviation ⇒ BSM theory
- similar to four-fermion operators in weak interactions ⇒ Standard Model

Rare effects from high energies

- Effects of high-energy physics as effective operators added to the standard model

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \frac{1}{\Lambda}\mathcal{L}_5 + \frac{1}{\Lambda^2}\mathcal{L}_6 + \cdots$$

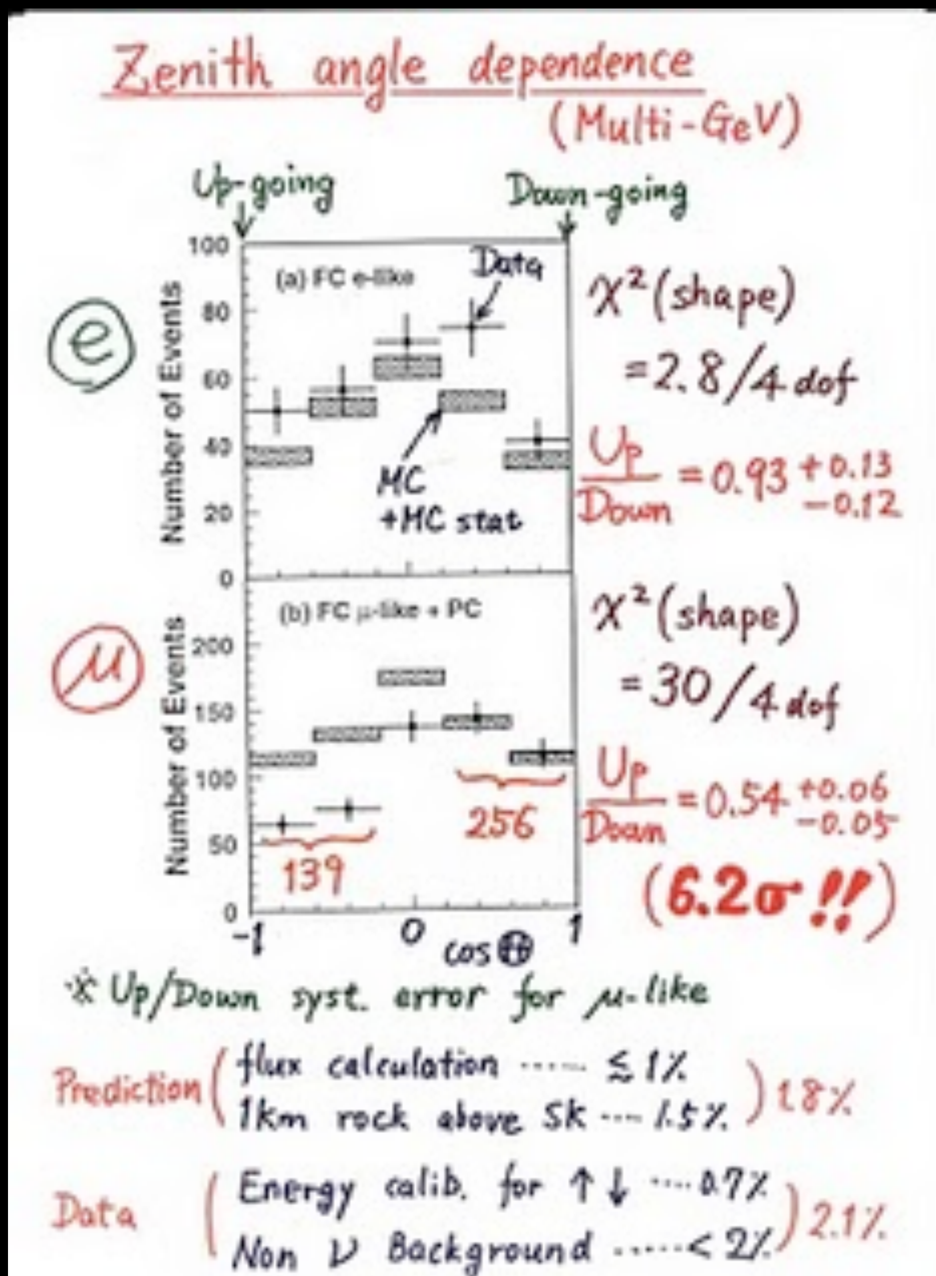
- can be classified systematically

$$\mathcal{L}_5 = (LH)(LH) \to \frac{1}{\Lambda}(L\langle H\rangle)(L\langle H\rangle) = m_\nu \nu\nu$$

$$\mathcal{L}_6 = QQQL, \bar{L}\sigma^{\mu\nu}W_{\mu\nu}Hl, \epsilon_{abc}W_\nu^{a\mu}W_\lambda^{b\nu}W_\mu^{c\lambda},$$
$$(H^\dagger D_\mu H)(H^\dagger D^\mu H), B_{\mu\nu}H^\dagger W^{\mu\nu}H, \cdots$$

nospheric neutrinos

1998



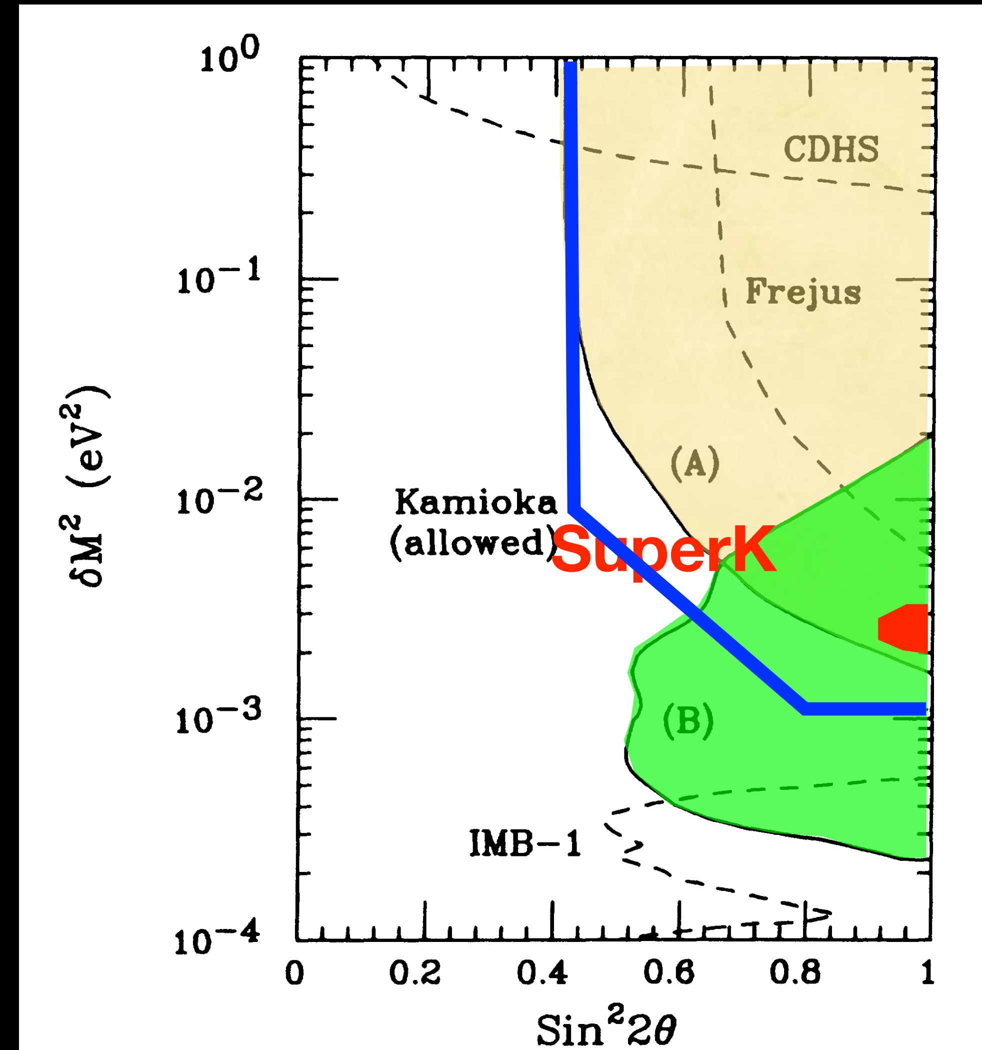


FIG. 2. 90% C.L. limits on ν_μ to ν_τ oscillations from rate (A) and stopping fraction (B). Dashed curves show limits from IMB-1 [14], Frejus [3], and CERN-Dortmund-Heidelberg-Saclay (CDHS) [15]. Dotted curve shows the allowed region from Kamiokande [16]. The Frejus limit is 95% C.L.; others are 90%.

IMB, PRL 69, 1010 (1992)

unique role of m_ν

- Lowest order effect of physics at short distances
- tiny effect: $(m_\nu/E_\nu)^2 \approx (0.1\text{eV}/\text{GeV})^2 \approx 10^{-20}$!
- interferometry (e.g. Michaelson-Morley)
  - need a coherent source
  - need a long baseline
  - need interference (i.e. large mixing angle)
- Nature was kind to provide them all!
- neutrino interferometry (a.k.a. oscillation) a unique tool to study physics at very high E
- probing up to $\Lambda \approx 10^{15}$ GeV

Effective Operators

- Surprisingly difficult question
- In the case of the Standard Model
  - Weinberg (1980) on D=6 $\not{B}$, D=5 $\not{L}$
  - Buchmüller-Wyler (1986) on D=6 ops
    - 80 operators for N_f=1, B, L conserving
  - Grzadkowski et al (2010) removed redundancies and discovered one missed
    - 59 operators for N_f=1, B, L conserving
  - Mahonar et al (2013) general N_f
  - Lehman-Martin (2014,15) D=7 for general N_f, D=8 for N_f=1 (incomplete)

$$
\begin{aligned}
\widehat{H}_6 \;&= H^3H^{\dagger\,3} + u^\dagger Q^\dagger H H^{\dagger\,2} + 2Q^2Q^{\dagger\,2} + Q^{\dagger\,3}L^\dagger + Q^3L + 2QQ^\dagger LL^\dagger + L^2L^{\dagger\,2} + uQH^2H^\dagger \\
&+ 2uu^\dagger QQ^\dagger + uu^\dagger LL^\dagger + u^2u^{\dagger\,2} + e^\dagger u^\dagger Q^2 + e^\dagger L^\dagger H^2H^\dagger + 2e^\dagger u^\dagger Q^\dagger L^\dagger + eLHH^{\dagger\,2} + euQ^{\dagger\,2} \\
&+ 2euQL + ee^\dagger QQ^\dagger + ee^\dagger LL^\dagger + ee^\dagger uu^\dagger + e^2e^{\dagger\,2} + d^\dagger Q^\dagger H^2H^\dagger + 2d^\dagger u^\dagger Q^{\dagger\,2} + d^\dagger u^\dagger QL \\
&+ d^\dagger e^\dagger u^{\dagger\,2} + d^\dagger eQ^\dagger L + dQHH^{\dagger\,2} + 2duQ^2 + duQ^\dagger L^\dagger + de^\dagger QL^\dagger + deu^2 + 2dd^\dagger QQ^\dagger + dd^\dagger LL^\dagger \\
&+ 2dd^\dagger uu^\dagger + dd^\dagger ee^\dagger + d^2d^{\dagger\,2} + u^\dagger Q^\dagger H^\dagger G_R + d^\dagger Q^\dagger HG_R + HH^\dagger G_R^2 + G_R^3 + uQHG_L \\
&+ dQH^\dagger G_L + HH^\dagger G_L^2 + G_L^3 + u^\dagger Q^\dagger H^\dagger W_R + e^\dagger L^\dagger HW_R + d^\dagger Q^\dagger HW_R + HH^\dagger W_R^2 + W_R^3 \\
&+ uQHW_L + eLH^\dagger W_L + dQH^\dagger W_L + HH^\dagger W_L^2 + W_L^3 + u^\dagger Q^\dagger H^\dagger B_R + e^\dagger L^\dagger HB_R \\
&+ d^\dagger Q^\dagger HB_R + HH^\dagger B_RW_R + HH^\dagger B_R^2 + uQHB_L + eLH^\dagger B_L + dQH^\dagger B_L + HH^\dagger B_LW_L \\
&+ HH^\dagger B_L^2 + 2QQ^\dagger HH^\dagger \mathcal{D} + 2LL^\dagger HH^\dagger \mathcal{D} + uu^\dagger HH^\dagger \mathcal{D} + ee^\dagger HH^\dagger \mathcal{D} + d^\dagger uH^2\mathcal{D} + du^\dagger H^{\dagger\,2}\mathcal{D} \\
&+ dd^\dagger HH^\dagger \mathcal{D} + 2H^2H^{\dagger\,2}\mathcal{D}^2 \,. \qquad (3.16)
\end{aligned}
$$

$\mathcal{D}$: space time derivative

Repeating this at order ϵ^6 we obtain the Hilbert series for dimension-six operators of the SM EFT:

$\not{B}, \not{L}$

$$
\begin{aligned}
\widehat{H}_6 \; = \; & H^3 H^{\dagger\,3} + u^\dagger Q^\dagger H H^{\dagger\,2} + 2Q^2 Q^{\dagger\,2} + \boxed{Q^{\dagger\,3} L^\dagger + Q^3 L} + 2QQ^\dagger LL^\dagger + L^2 L^{\dagger\,2} + uQH^2H^\dagger \\
& + 2uu^\dagger QQ^\dagger + uu^\dagger LL^\dagger + u^2 u^{\dagger\,2} + \boxed{e^\dagger u^\dagger Q^2} + e^\dagger L^\dagger H^2 H^\dagger + 2e^\dagger u^\dagger Q^\dagger L^\dagger + eLHH^{\dagger\,2} + \boxed{euQ^{\dagger\,2}} \\
& + 2euQL + ee^\dagger QQ^\dagger + ee^\dagger LL^\dagger + ee^\dagger uu^\dagger + e^2 e^{\dagger\,2} + d^\dagger Q^\dagger H^2 H^\dagger + 2d^\dagger u^\dagger Q^{\dagger\,2} + \boxed{d^\dagger u^\dagger QL} \\
& + \boxed{d^\dagger e^\dagger u^{\dagger\,2}} + d^\dagger e Q^\dagger L + dQHH^{\dagger\,2} + 2duQ^2 + \boxed{duQ^\dagger L^\dagger} + de^\dagger QL^\dagger + \boxed{deu^2} + 2dd^\dagger QQ^\dagger + dd^\dagger LL^\dagger \\
& + 2dd^\dagger uu^\dagger + dd^\dagger ee^\dagger + d^2 d^{\dagger\,2} + u^\dagger Q^\dagger H^\dagger G_R + d^\dagger Q^\dagger H G_R + HH^\dagger G_R^2 + G_R^3 + uQHG_L \\
& + dQH^\dagger G_L + HH^\dagger G_L^2 + G_L^3 + u^\dagger Q^\dagger H^\dagger W_R + e^\dagger L^\dagger H W_R + d^\dagger Q^\dagger H W_R + HH^\dagger W_R^2 + W_R^3 \\
& + uQHW_L + eLH^\dagger W_L + dQH^\dagger W_L + HH^\dagger W_L^2 + W_L^3 + u^\dagger Q^\dagger H^\dagger B_R + e^\dagger L^\dagger H B_R \\
& + d^\dagger Q^\dagger H B_R + HH^\dagger B_R W_R + HH^\dagger B_R^2 + uQHB_L + eLH^\dagger B_L + dQH^\dagger B_L + HH^\dagger B_L W_L \\
& + HH^\dagger B_L^2 + 2QQ^\dagger HH^\dagger \mathcal{D} + 2LL^\dagger HH^\dagger \mathcal{D} + uu^\dagger HH^\dagger \mathcal{D} + ee^\dagger HH^\dagger \mathcal{D} + d^\dagger u H^2 \mathcal{D} + du^\dagger H^{\dagger\,2} \mathcal{D} \\
& + dd^\dagger HH^\dagger \mathcal{D} + 2H^2 H^{\dagger\,2} \mathcal{D}^2 \, .
\end{aligned}
\tag{3.16}
$$

Setting all of the spurions equal to unity gives $\widehat{H}_6 = 84$, the total number of independent local operators at dimension 6, but more information is contained in eq.(3.16). For instance, the counting can easily be further decomposed by baryon number violation, $76 + 8$. The perhaps more familiar '$59 + 4$' counting is one in which hermitian conjugates of fermionic operators are not counted separately (such counting can of course also be obtained from eq. (3.16)).

Repeating this at order ϵ^6 we obtain the Hilbert series for dimension-six operators of the SM EFT:

$$\begin{aligned}
\widehat{H}_6 \;= \;& H^3 H^{\dagger\,3} + u^\dagger Q^\dagger H H^{\dagger\,2} + 2Q^2 Q^{\dagger\,2} + \text{[redacted]} + 2QQ^\dagger LL^\dagger + L^2 L^{\dagger\,2} + uQH^2H^\dagger \\
& + 2uu^\dagger QQ^\dagger + uu^\dagger LL^\dagger + u^2 u^{\dagger\,2} + \text{[redacted]} + e^\dagger L^\dagger H^2 H^\dagger + 2e^\dagger u^\dagger Q^\dagger L^\dagger + eLHH^{\dagger\,2} + \text{[redacted]} \\
& + 2euQL + ee^\dagger QQ^\dagger + ee^\dagger LL^\dagger + ee^\dagger uu^\dagger + e^2 e^{\dagger\,2} + d^\dagger Q^\dagger H^2 H^\dagger + 2d^\dagger u^\dagger Q^{\dagger\,2} + \text{[redacted]} \\
& + \text{[redacted]} + d^\dagger e Q^\dagger L + dQHH^{\dagger\,2} + 2duQ^2 + \text{[redacted]} + de^\dagger QL^\dagger + \text{[redacted]} + 2dd^\dagger QQ^\dagger + dd^\dagger LL^\dagger \\
& + 2dd^\dagger uu^\dagger + dd^\dagger ee^\dagger + d^2 d^{\dagger\,2} + u^\dagger Q^\dagger H^\dagger G_R + d^\dagger Q^\dagger H G_R + HH^\dagger G_R^2 + G_R^3 + uQHG_L \\
& + dQH^\dagger G_L + HH^\dagger G_L^2 + G_L^3 + u^\dagger Q^\dagger H^\dagger W_R + e^\dagger L^\dagger H W_R + d^\dagger Q^\dagger H W_R + HH^\dagger W_R^2 + W_R^3 \\
& + uQHW_L + eLH^\dagger W_L + dQH^\dagger W_L + HH^\dagger W_L^2 + W_L^3 + u^\dagger Q^\dagger H^\dagger B_R + e^\dagger L^\dagger H B_R \\
& + d^\dagger Q^\dagger H B_R + HH^\dagger B_R W_R + HH^\dagger B_R^2 + uQHB_L + eLH^\dagger B_L + dQH^\dagger B_L + HH^\dagger B_L W_L \\
& + HH^\dagger B_L^2 + 2QQ^\dagger HH^\dagger \mathcal{D} + 2LL^\dagger HH^\dagger \mathcal{D} + uu^\dagger HH^\dagger \mathcal{D} + ee^\dagger HH^\dagger \mathcal{D} + d^\dagger u H^2 \mathcal{D} + du^\dagger H^{\dagger\,2}\mathcal{D} \\
& + dd^\dagger HH^\dagger \mathcal{D} + 2H^2 H^{\dagger\,2}\mathcal{D}^2 \,.
\end{aligned} \tag{3.16}$$

Hermitian conjugates

Setting all of the spurions equal to unity gives $\widehat{H}_6 = 84$, the total number of independent local operators at dimension 6, but more information is contained in eq.(3.16). For instance, the counting can easily be further decomposed by baryon number violation, $76 + 8$. The perhaps more familiar '$59 + 4$' counting is one in which hermitian conjugates of fermionic operators are not counted separately (such counting can of course also be obtained from eq. (3.16)).

Repeating this at order ϵ^6 we obtain the Hilbert series for dimension-six operators of the SM EFT:

59 operators

$$
\begin{aligned}
\widehat{H}_6 = {} & H^3 H^{\dagger\,3} + u^\dagger Q^\dagger H H^{\dagger\,2} + 2Q^2 Q^{\dagger\,2} + \text{[redacted]} + 2QQ^\dagger LL^\dagger + L^2 L^{\dagger\,2} + \text{[redacted]} \\
& + 2uu^\dagger QQ^\dagger + uu^\dagger LL^\dagger + u^2 u^{\dagger\,2} + \text{[redacted]} + e^\dagger L^\dagger H^2 H^\dagger + 2e^\dagger u^\dagger Q^\dagger L^\dagger + \text{[redacted]} + \text{[redacted]} \\
& + \text{[redacted]} + ee^\dagger QQ^\dagger + ee^\dagger LL^\dagger + ee^\dagger uu^\dagger + e^2 e^{\dagger\,2} + d^\dagger Q^\dagger H^2 H^\dagger + 2d^\dagger u^\dagger Q^{\dagger\,2} + \text{[redacted]} \\
& + \text{[redacted]} + d^\dagger e Q^\dagger L + \text{[redacted]} + \text{[redacted]} + \text{[redacted]} + \text{[redacted]} + 2dd^\dagger QQ^\dagger + dd^\dagger LL^\dagger \\
& + 2dd^\dagger uu^\dagger + dd^\dagger ee^\dagger + d^2 d^{\dagger\,2} + u^\dagger Q^\dagger H^\dagger G_R + d^\dagger Q^\dagger H G_R + HH^\dagger G_R^2 + G_R^3 + \text{[redacted]} \\
& + \text{[redacted]} + HH^\dagger G_L^2 + G_L^3 + u^\dagger Q^\dagger H^\dagger W_R + e^\dagger L^\dagger H W_R + d^\dagger Q^\dagger H W_R + HH^\dagger W_R^2 + W_R^3 \\
& + \text{[redacted]} + HH^\dagger W_L^2 + W_L^3 + u^\dagger Q^\dagger H^\dagger B_R + e^\dagger L^\dagger H B_R \\
& + d^\dagger Q^\dagger H B_R + HH^\dagger B_R W_R + HH^\dagger B_R^2 + \text{[redacted]} + HH^\dagger B_L W_L \\
& + HH^\dagger B_L^2 + 2QQ^\dagger HH^\dagger \mathcal{D} + 2LL^\dagger HH^\dagger \mathcal{D} + uu^\dagger HH^\dagger \mathcal{D} + ee^\dagger HH^\dagger \mathcal{D} + d^\dagger u H^2 \mathcal{D} + \text{[redacted]} \\
& + dd^\dagger HH^\dagger \mathcal{D} + 2H^2 H^{\dagger\,2} \mathcal{D}^2 \,.
\end{aligned}
\tag{3.16}
$$

Setting all of the spurions equal to unity gives $\widehat{H}_6 = 84$, the total number of independent local operators at dimension 6, but more information is contained in eq.(3.16). For instance, the counting can easily be further decomposed by baryon number violation, $76 + 8$. The perhaps more familiar '$59 + 4$' counting is one in which hermitian conjugates of fermionic operators are not counted separately (such counting can of course also be obtained from eq. (3.16)).

D=8 operators

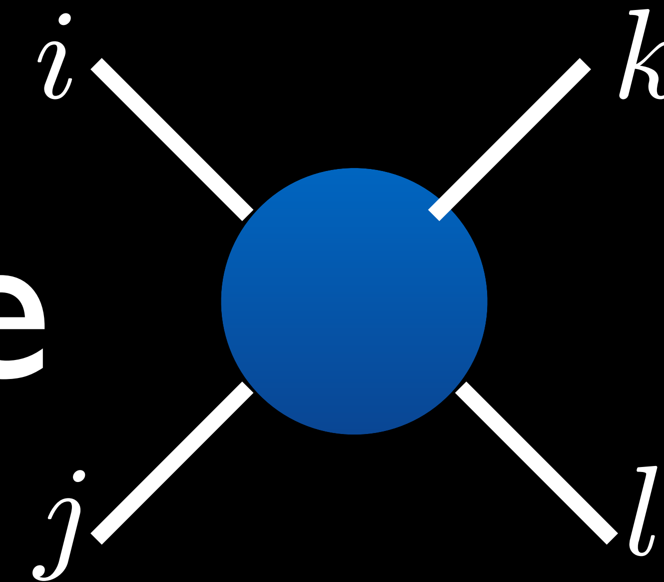
```
f =
2*L^2*Ld^2*t^2 + 2*ee*ed*L*Ld*t^2 + ee^2*ed^2*t^2 + 2*d*dd*L*Ld*t^2 + 2*
d*dd*ee*ed*t^2 + 2*d^2*dd^2*t^2 + ud^2*dd*ed*t^2 + 2*u*ud*L*Ld*t^2 + 2*u
*ud*ee*ed*t^2 + 4*u*ud*d*dd*t^2 + u^2*d*ee*t^2 + 2*u^2*ud^2*t^2 + 2*Qd*
dd*ee*L*t^2 + 3*Qd*ud*ed*Ld*t^2 + 2*Qd*u*d*Ld*t^2 + 3*Qd^2*ud*dd*t^2 +
Qd^2*u*ee*t^2 + Qd^3*Ld*t^2 + 2*Q*d*ed*Ld*t^2 + 2*Q*ud*dd*L*t^2 + 3*Q*u*
ee*L*t^2 + 4*Q*Qd*L*Ld*t^2 + 2*Q*Qd*ee*ed*t^2 + 4*Q*Qd*d*dd*t^2 + 4*Q*Qd
*u*ud*t^2 + Q^2*ud*ed*t^2 + 3*Q^2*u*d*t^2 + 4*Q^2*Qd^2*t^2 + Q^3*L*t^2
 + Wr*L^2*Ld^2 + Wr*ee*ed*L*Ld + Wr*d*dd*L*Ld + Wr*u*ud*L*Ld + Wr*Qd*dd*
ee*L + 3*Wr*Qd*ud*ed*Ld + Wr*Qd*u*d*Ld + 3*Wr*Qd^2*ud*dd + Wr*Qd^2*u*ee
 + 2*Wr*Qd^3*Ld + Wr*Q*d*ed*Ld + Wr*Q*ud*dd*L + 3*Wr*Q*Qd*L*Ld + Wr*Q*Qd
*ee*ed + 2*Wr*Q*Qd*d*dd + 2*Wr*Q*Qd*u*ud + 2*Wr*Q^2*Qd^2 + Wr^2*L*Ld*t
 + Wr^2*Q*Qd*t + 2*Wr^4 + Wl*L^2*Ld^2 + Wl*ee*ed*L*Ld + Wl*d*dd*L*Ld +
Wl*u*ud*L*Ld + Wl*Qd*dd*ee*L + Wl*Qd*u*d*Ld + Wl*Q*d*ed*Ld + Wl*Q*ud*dd*
L + 3*Wl*Q*u*ee*L + 3*Wl*Q*Qd*L*Ld + Wl*Q*Qd*ee*ed + 2*Wl*Q*Qd*d*dd + 2*
Wl*Q*Qd*u*ud + Wl*Q^2*ud*ed + 3*Wl*Q^2*u*d + 2*Wl*Q^2*Qd^2 + 2*Wl*Q^3*L
 + 2*Wl*Wr*L*Ld*t + Wl*Wr*ee*ed*t + Wl*Wr*d*dd*t + Wl*Wr*u*ud*t + 2*Wl*
Wr*Q*Qd*t + Wl^2*L*Ld*t + Wl^2*Q*Qd*t + 2*Wl^2*Wr^2 + 2*Wl^4 + Gr*d*dd*L
*Ld + Gr*d*dd*ee*ed + Gr*d^2*dd^2 + 3*Gr*ud^2*dd*ed + Gr*u*ud*L*Ld + Gr*
u*ud*ee*ed + 4*Gr*u*ud*d*dd + Gr*u^2*ud^2 + Gr*Qd*dd*ee*L + 3*Gr*Qd*ud*
ed*Ld + 2*Gr*Qd*u*d*Ld + 6*Gr*Qd^2*ud*dd + Gr*Qd^2*u*ee + 2*Gr*Qd^3*Ld
 + Gr*Q*d*ed*Ld + 2*Gr*Q*ud*dd*L + 2*Gr*Q*Qd*L*Ld + Gr*Q*Qd*ee*ed + 4*Gr
*Q*Qd*d*dd + 4*Gr*Q*Qd*u*ud + Gr*Q^2*ud*ed + 2*Gr*Q^2*Qd^2 + Gr*Wr*Q*Qd*
t + Gr*Wl*Q*Qd*t + Gr^2*d*dd*t + Gr^2*u*ud*t + Gr^2*Q*Qd*t + 2*Gr^2*Wr^2
 + Gr^2*Wl^2 + 3*Gr^4 + Gl*d*dd*L*Ld + Gl*d*dd*ee*ed + Gl*d^2*dd^2 + Gl*
u*ud*L*Ld + Gl*u*ud*ee*ed + 4*Gl*u*ud*d*dd + 3*Gl*u^2*d*ee + Gl*u^2*ud^2
 + Gl*Qd*dd*ee*L + 2*Gl*Qd*u*d*Ld + Gl*Qd^2*u*ee + Gl*Q*d*ed*Ld + 2*Gl*Q
*ud*dd*L + 3*Gl*Q*u*ee*L + 2*Gl*Q*Qd*L*Ld + Gl*Q*Qd*ee*ed + 4*Gl*Q*Qd*d*
dd + 4*Gl*Q*Qd*u*ud + Gl*Q^2*ud*ed + 6*Gl*Q^2*u*d + 2*Gl*Q^2*Qd^2 + 2*Gl
*Q^3*L + Gl*Wr*Q*Qd*t + Gl*Wl*Q*Qd*t + Gl*Gr*L*Ld*t + Gl*Gr*ee*ed*t + 3*
Gl*Gr*d*dd*t + 3*Gl*Gr*u*ud*t + 3*Gl*Gr*Q*Qd*t + Gl*Gr*Wl*Wr + Gl^2*d*dd
*t + Gl^2*u*ud*t + Gl^2*Q*Qd*t + Gl^2*Wr^2 + 2*Gl^2*Wl^2 + 3*Gl^2*Gr^2
 + 3*Gl^4 + Br*ee*ed*L*Ld + Br*d*dd*L*Ld + Br*d*dd*ee*ed + 2*Br*ud^2*dd*
ed + Br*u*ud*L*Ld + Br*u*ud*ee*ed + 2*Br*u*ud*d*dd + Br*Qd*dd*ee*L + 3*
Br*Qd*ud*ed*Ld + Br*Qd*u*d*Ld + 3*Br*Qd^2*ud*dd + Br*Qd^3*Ld + Br*Q*d*ed
*Ld + Br*Q*ud*dd*L + 2*Br*Q*Qd*L*Ld + Br*Q*Qd*ee*ed + 2*Br*Q*Qd*d*dd + 2
*Br*Q*Qd*u*ud + Br*Q^2*ud*ed + Br*Wr*L*Ld*t + Br*Wr*Q*Qd*t + Br*Wl*L*Ld*
t + Br*Wl*Q*Qd*t + Br*Gr*d*dd*t + Br*Gr*u*ud*t + Br*Gr*Q*Qd*t + Br*Gr^3
 + Br*Gl*d*dd*t + Br*Gl*u*ud*t + Br*Gl*Q*Qd*t + Br*Gl^2*Gr + 2*Br^2*Wr^2
 + Br^2*Wl^2 + 2*Br^2*Gr^2 + Br^2*Gl^2 + Br^4 + Bl*ee*ed*L*Ld + Bl*d*dd*
L*Ld + Bl*d*dd*ee*ed + Bl*u*ud*L*Ld + Bl*u*ud*ee*ed + 2*Bl*u*ud*d*dd + 2
*Bl*u^2*d*ee + Bl*Qd*dd*ee*L + Bl*Qd*u*d*Ld + Bl*Qd^2*u*ee + Bl*Q*d*ed*
Ld + Bl*Q*ud*dd*L + 3*Bl*Q*u*ee*L + 2*Bl*Q*Qd*L*Ld + Bl*Q*Qd*ee*ed + 2*
Bl*Q*Qd*d*dd + 2*Bl*Q*Qd*u*ud + 3*Bl*Q^2*u*d + Bl*Q^3*L + Bl*Wr*L*Ld*t
 + Bl*Wr*Q*Qd*t + Bl*Wl*L*Ld*t + Bl*Wl*Q*Qd*t + Bl*Gr*d*dd*t + Bl*Gr*u*
ud*t + Bl*Gr*Q*Qd*t + Bl*Gl*d*dd*t + Bl*Gl*u*ud*t + Bl*Gl*Q*Qd*t + Bl*Gl
*Gr^2 + Bl*Gl^3 + Bl*Br*L*Ld*t + Bl*Br*ee*ed*t + Bl*Br*d*dd*t + Bl*Br*u*
ud*t + Bl*Br*Q*Qd*t + Bl*Br*Wl*Wr + Bl*Br*Gl*Gr + Bl^2*Wr^2 + 2*Bl^2*
Wl^2 + Bl^2*Gr^2 + 2*Bl^2*Gl^2 + Bl^2*Br^2 + Bl^4 + 3*Hd*ee*L^2*Ld*t +
Hd*ee^2*ed*L*t + 3*Hd*d*dd*ee*L*t + 3*Hd*ud*d*ed*Ld*t + 2*Hd*ud^2*dd*L*t
 + 2*Hd*u*d^2*Ld*t + 3*Hd*u*ud*ee*L*t + 6*Hd*Qd*ud*L*Ld*t + 3*Hd*Qd*ud*
ee*ed*t + 6*Hd*Qd*ud*d*dd*t + 3*Hd*Qd*u*d*ee*t + 3*Hd*Qd*u*ud^2*t + 3*Hd
*Qd^2*d*Ld*t + Hd*Qd^3*ee*t + 6*Hd*Q*d*L*Ld*t + 3*Hd*Q*d*ee*ed*t + 3*Hd*
Q*d^2*dd*t + 2*Hd*Q*ud^2*ed*t + 6*Hd*Q*u*ud*d*t + 6*Hd*Q*Qd*ee*L*t + 6*
Hd*Q*Qd^2*ud*t + 3*Hd*Q^2*ud*L*t + 6*Hd*Q^2*Qd*d*t + Hd*Wr*ee*L*t^2 + 2*
Hd*Wr*Qd*ud*t^2 + Hd*Wr*Q*d*t^2 + Hd*Wr^2*ee*L + 2*Hd*Wr^2*Qd*ud + Hd*
Wr^2*Q*d + 2*Hd*Wl*ee*L*t^2 + Hd*Wl*Qd*ud*t^2 + 2*Hd*Wl*Q*d*t^2 + 2*Hd*
Wl^2*ee*L + Hd*Wl^2*Qd*ud + 2*Hd*Wl^2*Q*d + 2*Hd*Gr*Qd*ud*t^2 + Hd*Gr*Q*
d*t^2 + 2*Hd*Gr*Wr*Qd*ud + Hd*Gr*Wr*Q*d + Hd*Gr^2*ee*L + 3*Hd*Gr^2*Qd*ud
 + 2*Hd*Gr^2*Q*d + Hd*Gl*Qd*ud*t^2 + 2*Hd*Gl*Q*d*t^2 + Hd*Gl*Wl*Qd*ud +
2*Hd*Gl*Wl*Q*d + Hd*Gl^2*ee*L + 2*Hd*Gl^2*Qd*ud + 3*Hd*Gl^2*Q*d + Hd*Br*
ee*L*t^2 + 2*Hd*Br*Qd*ud*t^2 + Hd*Br*Q*d*t^2 + Hd*Br*Wr*ee*L + 2*Hd*Br*
Wr*Qd*ud + Hd*Br*Wr*Q*d + 2*Hd*Br*Gr*Qd*ud + Hd*Br*Gr*Q*d + Hd*Br^2*ee*L
 + Hd*Br^2*Qd*ud + Hd*Br^2*Q*d + 2*Hd*Bl*ee*L*t^2 + Hd*Bl*Qd*ud*t^2 + 2*
Hd*Bl*Q*d*t^2 + 2*Hd*Bl*Wl*ee*L + Hd*Bl*Wl*Qd*ud + 2*Hd*Bl*Wl*Q*d + Hd*
Bl*Gl*Qd*ud + 2*Hd*Bl*Gl*Q*d + Hd*Bl^2*ee*L + Hd*Bl^2*Qd*ud + Hd*Bl^2*Q*
d + Hd^2*ee^2*L^2 + Hd^2*ud*d*t^3 + Hd^2*ud*d*L*Ld + Hd^2*Qd*ud*ee*L + 2
*Hd^2*Qd^2*ud^2 + 2*Hd^2*Q*d*ee*L + 2*Hd^2*Q*Qd*ud*d + 2*Hd^2*Q^2*d^2 +
Hd^2*Wr*ud*d*t + Hd^2*Wl*ud*d*t + Hd^2*Gr*ud*d*t + Hd^2*Gl*ud*d*t + Hd^2
*Br*ud*d*t + Hd^2*Bl*ud*d*t + 3*H*ed*L*Ld^2*t + H*ee*ed^2*Ld*t + 3*H*d*
dd*ed*Ld*t + 2*H*ud*dd^2*L*t + 3*H*u*dd*ee*L*t + 3*H*u*ud*ed*Ld*t + 2*H*
u^2*d*Ld*t + 6*H*Qd*dd*L*Ld*t + 3*H*Qd*dd*ee*ed*t + 3*H*Qd*d*dd^2*t + 6*
H*Qd*u*ud*dd*t + 2*H*Qd*u^2*ee*t + 3*H*Qd^2*u*Ld*t + 3*H*Q*ud*dd*ed*t +
6*H*Q*u*L*Ld*t + 3*H*Q*u*ee*ed*t + 6*H*Q*u*d*dd*t + 3*H*Q*u^2*ud*t + 6*H
*Q*Qd*ed*Ld*t + 6*H*Q*Qd^2*dd*t + 3*H*Q^2*dd*L*t + 6*H*Q^2*Qd*u*t + H*
Q^3*ed*t + 2*H*Wr*ed*Ld*t^2 + 2*H*Wr*Qd*dd*t^2 + H*Wr*Q*u*t^2 + 2*H*Wr^2
*ed*Ld + 2*H*Wr^2*Qd*dd + H*Wr^2*Q*u + H*Wl*ed*Ld*t^2 + H*Wl*Qd*dd*t^2
 + 2*H*Wl*Q*u*t^2 + H*Wl^2*ed*Ld + H*Wl^2*Qd*dd + 2*H*Wl^2*Q*u + 2*H*Gr*
Qd*dd*t^2 + H*Gr*Q*u*t^2 + 2*H*Gr*Wr*Qd*dd + H*Gr*Wr*Q*u + H*Gr^2*ed*Ld
 + 3*H*Gr^2*Qd*dd + 2*H*Gr^2*Q*u + H*Gl*Qd*dd*t^2 + 2*H*Gl*Q*u*t^2 + H*
Gl*Wl*Qd*dd + 2*H*Gl*Wl*Q*u + H*Gl^2*ed*Ld + 2*H*Gl^2*Qd*dd + 3*H*Gl^2*Q
*u + 2*H*Br*ed*Ld*t^2 + 2*H*Br*Qd*dd*t^2 + H*Br*Q*u*t^2 + 2*H*Br*Wr*ed*
Ld + 2*H*Br*Wr*Qd*dd + H*Br*Wr*Q*u + 2*H*Br*Gr*Qd*dd + H*Br*Gr*Q*u + H*
Br^2*ed*Ld + H*Br^2*Qd*dd + H*Br^2*Q*u + H*Bl*ed*Ld*t^2 + H*Bl*Qd*dd*t^2
 + 2*H*Bl*Q*u*t^2 + H*Bl*Wl*ed*Ld + H*Bl*Wl*Qd*dd + 2*H*Bl*Wl*Q*u + H*Bl
*Gl*Qd*dd + 2*H*Bl*Gl*Q*u + H*Bl^2*ed*Ld + H*Bl^2*Qd*dd + H*Bl^2*Q*u + 4
*H*Hd*L*Ld*t^3 + 2*H*Hd*L^2*Ld^2 + 2*H*Hd*ee*ed*t^3 + 2*H*Hd*ee*ed*L*Ld
 + H*Hd*ee^2*ed^2 + 2*H*Hd*d*dd*t^3 + 2*H*Hd*d*dd*L*Ld + H*Hd*d*dd*ee*ed
 + H*Hd*d^2*dd^2 + H*Hd*ud^2*dd*ed + 2*H*Hd*u*ud*t^3 + 2*H*Hd*u*ud*L*Ld
 + H*Hd*u*ud*ee*ed + 2*H*Hd*u*ud*d*dd + H*Hd*u^2*d*ee + H*Hd*u^2*ud^2 +
2*H*Hd*Qd*dd*ee*L + 4*H*Hd*Qd*ud*ed*Ld + 2*H*Hd*Qd*u*d*Ld + 4*H*Hd*Qd^2*
ud*dd + H*Hd*Qd^2*u*ee + 2*H*Hd*Qd^3*Ld + 2*H*Hd*Q*d*ed*Ld + 2*H*Hd*Q*ud
*dd*L + 4*H*Hd*Q*u*ee*L + 4*H*Hd*Q*Qd*t^3 + 5*H*Hd*Q*Qd*L*Ld + 2*H*Hd*Q*
Qd*ee*ed + 4*H*Hd*Q*Qd*d*dd + 4*H*Hd*Q*Qd*u*ud + H*Hd*Q^2*ud*ed + 4*H*Hd
*Q^2*u*d + 3*H*Hd*Q^2*Qd^2 + 2*H*Hd*Q^3*L + 6*H*Hd*Wr*L*Ld*t + 2*H*Hd*Wr
*ee*ed*t + 2*H*Hd*Wr*d*dd*t + 2*H*Hd*Wr*u*ud*t + 6*H*Hd*Wr*Q*Qd*t + 2*H*
Hd*Wr^2*t^2 + H*Hd*Wr^3 + 6*H*Hd*Wl*L*Ld*t + 2*H*Hd*Wl*ee*ed*t + 2*H*Hd*
Wl*d*dd*t + 2*H*Hd*Wl*u*ud*t + 6*H*Hd*Wl*Q*Qd*t + 2*H*Hd*Wl*Wr*t^2 + 2*H
*Hd*Wl^2*t^2 + H*Hd*Wl^3 + 2*H*Hd*Gr*d*dd*t + 2*H*Hd*Gr*u*ud*t + 4*H*Hd*
Gr*Q*Qd*t + H*Hd*Gr^2*t^2 + H*Hd*Gr^3 + 2*H*Hd*Gl*d*dd*t + 2*H*Hd*Gl*u*
ud*t + 4*H*Hd*Gl*Q*Qd*t + H*Hd*Gl*Gr*t^2 + H*Hd*Gl^2*t^2 + H*Hd*Gl^3 + 4
*H*Hd*Br*L*Ld*t + 2*H*Hd*Br*ee*ed*t + 2*H*Hd*Br*d*dd*t + 2*H*Hd*Br*u*ud*
t + 4*H*Hd*Br*Q*Qd*t + 2*H*Hd*Br*Wr*t^2 + H*Hd*Br*Wr^2 + H*Hd*Br*Wl*t^2
 + H*Hd*Br^2*t^2 + 4*H*Hd*Bl*L*Ld*t + 2*H*Hd*Bl*ee*ed*t + 2*H*Hd*Bl*d*dd
*t + 2*H*Hd*Bl*u*ud*t + 4*H*Hd*Bl*Q*Qd*t + H*Hd*Bl*Wr*t^2 + 2*H*Hd*Bl*Wl
*t^2 + H*Hd*Bl*Wl^2 + H*Hd*Bl*Br*t^2 + H*Hd*Bl^2*t^2 + 6*H*Hd^2*ee*L*t^2
 + 6*H*Hd^2*Qd*ud*t^2 + 6*H*Hd^2*Q*d*t^2 + 2*H*Hd^2*Wr*Qd*ud + 2*H*Hd^2*
Wl*ee*L + 2*H*Hd^2*Wl*Q*d + H*Hd^2*Gr*Qd*ud + H*Hd^2*Gl*Q*d + H*Hd^2*Br*
Qd*ud + H*Hd^2*Bl*ee*L + H*Hd^2*Bl*Q*d + H*Hd^3*ud*d*t + H^2*ed^2*Ld^2
 + H^2*u*dd*t^3 + H^2*u*dd*L*Ld + 2*H^2*Qd*dd*ed*Ld + 2*H^2*Qd^2*dd^2 +
H^2*Q*u*ed*Ld + 2*H^2*Q*Qd*u*dd + 2*H^2*Q^2*u^2 + H^2*Wr*u*dd*t + H^2*Wl
*u*dd*t + H^2*Gr*u*dd*t + H^2*Gl*u*dd*t + H^2*Br*u*dd*t + H^2*Bl*u*dd*t
 + 6*H^2*Hd*ed*Ld*t^2 + 6*H^2*Hd*Qd*dd*t^2 + 6*H^2*Hd*Q*u*t^2 + 2*H^2*Hd
*Wr*ed*Ld + 2*H^2*Hd*Wr*Qd*dd + 2*H^2*Hd*Wl*Q*u + H^2*Hd*Gr*Qd*dd + H^2*
Hd*Gl*Q*u + H^2*Hd*Br*ed*Ld + H^2*Hd*Br*Qd*dd + H^2*Hd*Bl*Q*u + 3*H^2*
Hd^2*t^4 + 4*H^2*Hd^2*L*Ld*t + H^2*Hd^2*ee*ed*t + H^2*Hd^2*d*dd*t + H^2*
Hd^2*u*ud*t + 4*H^2*Hd^2*Q*Qd*t + 2*H^2*Hd^2*Wr*t^2 + 2*H^2*Hd^2*Wr^2 +
2*H^2*Hd^2*Wl*t^2 + 2*H^2*Hd^2*Wl^2 + H^2*Hd^2*Gr^2 + H^2*Hd^2*Gl^2 +
H^2*Hd^2*Br*t^2 + H^2*Hd^2*Br*Wr + H^2*Hd^2*Br^2 + H^2*Hd^2*Bl*t^2 + H^2
*Hd^2*Bl*Wl + H^2*Hd^2*Bl^2 + H^2*Hd^3*ee*L + H^2*Hd^3*Qd*ud + H^2*Hd^3*
Q*d + H^3*Hd*u*dd*t + H^3*Hd^2*ed*Ld + H^3*Hd^2*Qd*dd + H^3*Hd^2*Q*u + 2
*H^3*Hd^3*t^2 + H^4*Hd^4;
```

993 of them for N_f=1

redundancies

- effective operators are invariants under the gauge group, Lorentz group, etc
- their classifications go back to Hilbert, Weyl
- applied to superpotentials, Standard Model
- but so far no general discussions on operators with derivatives
- two sources of redundancies
  - equation of motion (EOM)
  - integration by parts (IBP)

Simplest Example



- scalars four-point at $O(\partial^2)$: 4(4+1)/2=10

  $(\partial_\mu \partial_\mu \varphi_i)\varphi_j \varphi_k \varphi_l \qquad (\partial_\mu \varphi_i)(\partial_\mu \varphi_j)\varphi_k \varphi_l$

- $\partial^2\phi_i = m_i^2$ removes the first class: 4
- We know only 2 out of 6 are independent
  - s, t, u, $s+t+u=m_1^2+m_2^2+m_3^2+m_4^2$

$$(\partial_\mu\varphi_i)(\partial_\mu\varphi_j)\varphi_k\varphi_l - \varphi_i\varphi_j(\partial_\mu\varphi_k)(\partial_\mu\varphi_l) = \frac{1}{2}\partial^2(\varphi_i\varphi_j)(\varphi_k\varphi_l) - \frac{1}{2}(\varphi_i\varphi_j)\partial^2(\varphi_k\varphi_l) \approx 0$$

$$\partial_\mu\varphi_i\partial_\mu\varphi_j\varphi_k\varphi_l + \partial_\mu\varphi_i\varphi_j\partial_\mu\varphi_k\varphi_l + \partial_\mu\varphi_i\varphi_j\varphi_k\partial_\mu\varphi_l = \partial_\mu\varphi_i\partial_\mu(\varphi_j\varphi_k\varphi_l) \approx 0$$

- In addition, there are only *d* linearly independent momenta in *d*-dimensions for higher-point functions

1D QFT = QM

- only one derivative ∂
- EOM: $\partial^2\phi=0$
- building blocks of operators: $(\phi, \partial\phi)$
- $O(\phi_1, \phi_2)$:

$$\begin{pmatrix} \varphi_1 \\ \partial\varphi_1 \end{pmatrix} \otimes \begin{pmatrix} \varphi_2 \\ \partial\varphi_2 \end{pmatrix} \to \begin{pmatrix} \varphi_1\varphi_2 \\ \varphi_1\partial\varphi_2 + \partial\varphi_1\varphi_2 \\ \partial\varphi_1\partial\varphi_2 \\ \hline \varphi_1\partial\varphi_2 - \partial\varphi_1\varphi_2 \end{pmatrix} \begin{matrix} \to \text{highest weight} \\ \to \text{IBP} \\ \to \text{IBP+EOM} \\ \to \text{highest weight} \end{matrix}$$

$$\mathbf{2} \otimes 2 = \mathbf{3}\oplus\mathbf{1}$$

$SL(2,\mathbb{R}) \simeq SO(1,2) = SO(1,d+1)$

conformal group!

Main idea

- Take kinetic terms as the zeroth order Lagrangian $(\partial\phi)^2, \; \bar{\psi} i \partial\!\!\!/ \psi, \; (F_{\mu\nu})^2$
- Classically, it is conformally invariant under SO(4,2)≃SO(6,$\mathbb{C}$)
- Operator-State correspondence in CFT tells us that operators fall into representations of the conformal group
  - equation of motion: short multiplets
  - remove total derivatives: primary states

Master formula

- Define a multi-variate Hilbert series

$$H(p,\phi_1,\cdots,\phi_n) = \int d\mu_{\rm conformal} d\mu_{\rm gauge} \sum_{n=1}^{\infty} p^n \chi^*_{[n;0]} \prod_i PE[\phi_i \chi_i(q,\alpha,\beta)]$$

- PE are (anti-)symmetric products of characters for each field ϕ_i of dimension d_i
- integration over the gauge groups pick up gauge invariants
- integration over the conformal group picks only the primary states and Lorentz scalars
- expand it in power series in ϕ_i and p to find operators at given order in them

*There are corrections for operators d≤4 due to lack of orthonormality among characters for short multiplets

Machinery

Lorentz invariance

- Lagrangian obviously should be Lorentz-inv
- Wick rotation:
  - $SO(3,1) \to SO(4) \simeq SU(2)\times SU(2)/Z_2$
  - irrep: (j_1, j_2)
  - $\phi(0,0)$
  - $\psi_L(\frac{1}{2},0)$, $\psi_R(0,\frac{1}{2})$
  - $A_\mu(\frac{1}{2},\frac{1}{2})$
  - $F_{\mu\nu}+\tilde{F}_{\mu\nu}(1,0)$, $F_{\mu\nu}+\tilde{F}_{\mu\nu}(0,1)$

characters

- character $\chi(y_1, y_2, \ldots, y_r) = \mathrm{Tr}_R\, g$
- e.g., SU(2)

$$e^{i\theta T_3} = \mathrm{diag}(e^{ij\theta}, e^{i(j-1)\theta}, \cdots, e^{i(-j)\theta}) = (y^{2j}, y^{2j-2}, \cdots y^{-2j})$$

$$y = e^{i\theta/2}$$

$$\chi = y^{2j} + y^{2j-2} + \cdots + y^{-2j} = y^{2j}\frac{1-y^{-4j-2}}{1-y^{-2}} = \frac{y^{2j+1} - y^{-2j-1}}{y - y^{-1}}$$

- orthonormality on Haar measure

$$\delta_{R_i,R_j} = \int d\mu_{SU(2)}\chi^*_{R_i}\chi_{R_j} = \oint_{|y|=1}\frac{dy}{2\pi i}\frac{(1-y^2)(1-y^{-2})}{y}\chi^*_{R_i}\chi_{R_j}$$

Lorentz characters

- SU(2)×SU(2) $\alpha = e^{i\theta_R/2}$

  $\beta = e^{i\theta_L/2}$

$$d\mu_{\text{Lorentz}} = \frac{1}{4}\oint_{|\alpha|=1}\frac{d\alpha}{2\pi i\alpha}\oint_{|\beta|=1}\frac{d\beta}{2\pi i\beta}(1-\alpha^2)(1-\alpha^{-2})(1-\beta^2)(1-\beta^{-2})$$

- characters
  - ϕ(0,0) $\chi(\alpha,\beta) = 1$
  - ψ_L(½,0), ψ_R(0,½) $\chi_L(\alpha,\beta) = \frac{\alpha^2-\alpha^{-2}}{\alpha-\alpha^{-1}} = \alpha+\alpha^{-1}$
  - A_μ(½,½) $\chi_R(\alpha,\beta) = \beta+\beta^{-1}$
  - $F_{\mu\nu}+\tilde{F}_{\mu\nu}$(1,0), $F_{\mu\nu}-\tilde{F}_{\mu\nu}$(0,1) $\chi_R(\alpha,\beta) = \beta^2+1+\beta^{-2}$

$$\chi_L(\alpha,\beta) = \frac{\alpha^3-\alpha^{-3}}{\alpha-\alpha^{-1}} = \alpha^2+1+\alpha^{-2}$$

conformal symmetry

- largest symmetry of Minkowski spacetime
- $SO(d-1,1) \to SO(d,2)$
- Lorentz $M_{\mu\nu}$: $d(d-1)/2$
- translation P_μ: d
- dilation D $x^\mu \to e^t x^\mu$: 1
- special conformal K_μ: d

$$x^\mu \to -\frac{x^\mu}{x^2} \to -\frac{x^\mu}{x^2} + a^\mu$$

- total: $(d+1)(d+2)/2$

$$\to \frac{x^\mu - a^\mu x^2}{1 - 2a \cdot x + a^2 x^2}$$

$$= x^\mu + (2x_\mu x_\nu - x^2 g_{\mu\nu})a^\nu + O(a^2)$$

conformal symmetry

- Wick rotation: SO(d+1,1)
- $K_\mu = (P_\mu)^\dagger$
- "ground state"
- unitarity limit: i.e., (j,0)

  $$\Delta \geq j + 1$$

- build "excited states"

$$[P_\mu, P_\nu] = 0$$
$$[D, K_\mu] = -K_\mu$$
$$[D, P_\mu] = P_\mu$$
$$[K_\mu, K_\nu] = 0$$
$$[K_\mu, P_\nu] = \eta_{\mu\nu} D - i M_{\mu\nu}$$

$$|0\rangle = |\Delta, (j_1, j_2)\rangle$$
$$D|0\rangle = \Delta|0\rangle$$
$$K_\mu|0\rangle = 0$$

$$P_{\mu_1} P_{\mu_2} \cdots P_{\mu_n} |0\rangle = |\Delta + n, (j_k, j_l)\rangle$$

Building blocks

- Under Euclidean SO(4)≃SU(2)$_R$×SU(2)$_L$

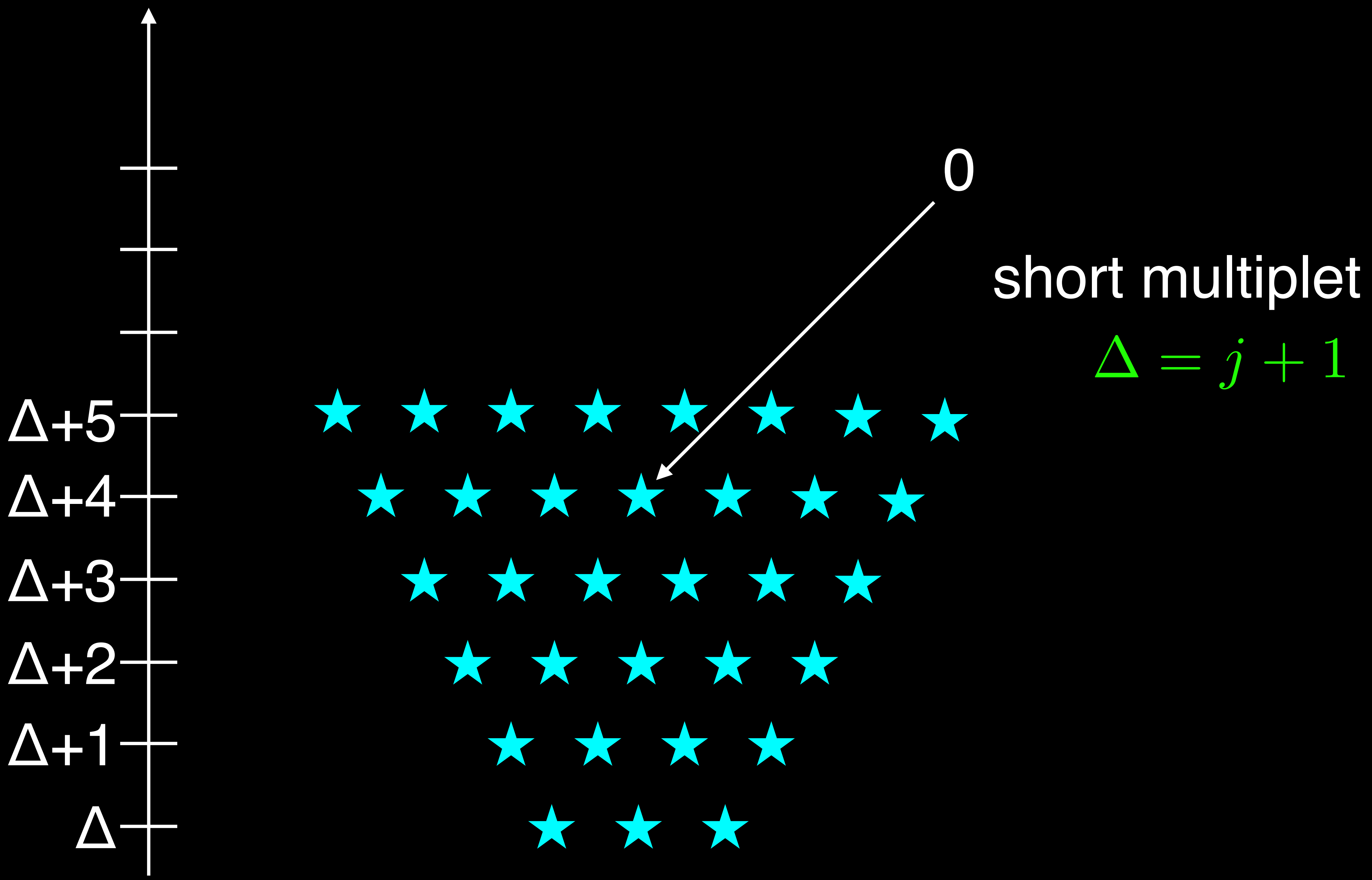
$$\phi(0,0)$$

$$\psi_R(\tfrac{1}{2},0) \quad \psi_L(0,\tfrac{1}{2}) \quad \bar{\psi}_R(0,\tfrac{1}{2}) \quad \bar{\psi}_L(\tfrac{1}{2},0)$$

$$F_R = F_{\mu\nu} + \tilde{F}_{\mu\nu}(1,0) \quad F_L = F_{\mu\nu} - \tilde{F}_{\mu\nu}(0,1)$$

- single particle module

$$R_\phi = \begin{pmatrix} \phi \\ \partial_\mu \phi \\ \partial_{\{\mu_1}\partial_{\mu_2\}}\phi \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}}\phi \\ \vdots \end{pmatrix} \qquad \partial_\mu(\tfrac{1}{2},\tfrac{1}{2})$$



0
short multiplet
$\Delta = j+1$
$\Delta+5$
$\Delta+4$
$\Delta+3$
$\Delta+2$
$\Delta+1$
Δ

conformal characters

- Primary field characterized by its spin $s=(j_1,j_2)$ and conformal weight Δ

$$\alpha = e^{i\theta_R/2}$$

$$\chi_{[\Delta,s]}(q,\alpha,\beta) = q^\Delta P(q;\alpha,\beta)\chi_s(\alpha,\beta) \qquad \beta = e^{i\theta_L/2}$$

$$P(q;\alpha,\beta) = \frac{1}{(1-q\alpha\beta)(1-q\alpha\beta^{-1})(1-q\alpha^{-1}\beta)(1-q\alpha^{-1}\beta^{-1})}$$

- $\Delta=1+j_1+j_2$ ($j_1j_2=0$) saturates the unitarity bound, there are "short multiplets" for EoM

$$\chi_0(\alpha,\beta) = 1 - q^2$$

$$\phi \qquad \Box\phi$$

$$\chi_{(\frac{1}{2},0)}(\alpha,\beta) = \alpha + \alpha^{-1} - q(\beta+\beta^{-1})$$

$$\psi_R \qquad i\gamma^\mu\partial_\mu\psi_R$$

$$\chi_{(1,0)}(\alpha,\beta) = \alpha^2 + 1 + \alpha^{-2} - q(\alpha+\alpha^{-1})(\beta+\beta^{-1}) + q^2$$

$$F_L^{\mu\nu} \qquad \partial_\mu F_L^{\mu\nu} \qquad \partial_\mu\partial_\nu F_L^{\mu\nu}$$

Hilbert series

- ring freely generated by ϕ:
  - 1, ϕ, ϕ^2, ϕ^3, ϕ^4, …

$$H(\varphi) = \frac{1}{1-\varphi}$$

- mod out by ideal, e.g., $\phi^2=0$

$$H(\varphi) = \frac{1-\varphi^2}{1-\varphi} = 1+\varphi$$

- convenient way to encode all possible operators in a given theory
- basically a "generating function"

multi-boson operator

- "plethystic exponential"
- symmetric tensor product R^n of R

$$PE[u\chi_R](x_1, x_2, \cdots, x_r) \equiv \frac{1}{\det_R(1-ug)}$$

$$= \sum_n u^n \chi_{R^n} = \exp\left[-\mathrm{Tr}_R \log(1-ug)\right]$$

$$= \exp\left[\sum_{n=1}^{\infty} \frac{u^n}{n} \chi_R(x_1^n, \cdots, x_r^n)\right]$$

$$PE[u\chi_{1/2}] = \frac{1}{\det\begin{pmatrix} 1-uy & 0 \\ 0 & 1-uy^{-1} \end{pmatrix}}$$

$$= \frac{1}{(1-uy)(1-uy^{-1})} = 1 + u(y+y^{-1}) + u^2(y^2+1+y^{-2}) + u^3(y^3+y+y^{-1}+y^{-3}) + \cdots$$

multi-fermion operator

- "plethystic exponential"
- anti-symmetric tensor product *R^n* of *R*

$$PE[u\chi_R](x_1, x_2, \cdots, x_r) \equiv \det{}_R(1+ug)$$

$$= \sum_n u^n \chi_{R^n} = \exp\left[\mathrm{Tr}_R \log(1+ug)\right]$$

$$= \exp\left[-\sum_{n=1}^{\infty} \frac{(-u)^n}{n} \chi_R(x_1^n, \cdots, x_r^n)\right]$$

$$PE[u\chi_{1/2}] = \det\begin{pmatrix} 1+uy & 0 \\ 0 & 1+uy^{-1} \end{pmatrix}$$

$$= (1+uy)(1+uy^{-1}) = 1 + u(y+y^{-1}) + u^2$$

Hear measures

- conformal group

$$d\mu_{\rm conformal} = \oint \frac{dq}{2\pi i q}\mu_{\rm Lorentz}\frac{1}{P(q;\alpha,\beta)P(q^{-1};\alpha,\beta)}$$

- Lorentz group

$$d\mu_{\rm Lorentz} = \frac{1}{4}\oint_{|\alpha|=1}\frac{d\alpha}{2\pi i\alpha}\oint_{|\beta|=1}\frac{d\beta}{2\pi i\beta}(1-\alpha^2)(1-\alpha^{-2})(1-\beta^2)(1-\beta^{-2})$$

- U(1)

$$d\mu_{\rm U(1)} = \oint_{|x|=1}\frac{dx}{2\pi i x}$$

- SU(2)

$$d\mu_{\rm SU(2)} = \frac{1}{2}\oint_{|y|=1}\frac{dy}{2\pi i y}(1-y^2)(1-y^{-2})$$

- SU(3)

$$d\mu_{\rm SU(3)} = \frac{1}{6}\oint_{|z_1|=1}\oint_{|z_2|=1}\frac{dz_1}{2\pi i z_1}\frac{dz_2}{2\pi i z_2}(1-z_1z_2)(1-z_1^2z_2^{-1})(1-z_2^2z_1^{-1})(1-z_1^{-1}z_2^{-1})(1-z_2z_1^{-2})(1-z_1z_2^{-2})$$

Master formula

- Define a multi-variate Hilbert series

$$H(p, \phi_1, \cdots, \phi_n) = \int d\mu_{\text{conformal}} d\mu_{\text{gauge}} \sum_{n=1}^{\infty} p^n \chi^*_{[n;0]} \prod_i PE[\phi_i \chi_i(q, \alpha, \beta)]$$

- integration over the gauge groups pick up gauge invariants
- integration over the conformal group picks only the primary states and Lorentz scalars
- expand it in power series in ϕ_i and p to find operators at given order in them
- Possible for any Lorentz-inv "free" QFT

*There are corrections for operators d≤4 due to lack of orthonormality among characters for short multiplets

Applications

Standard Model

```
χH[t_, α_, β_, x_, y_, z1_, z2_] := χscal[t, α, β] * u1[3, x] * su2f[y];
χHd[t_, α_, β_, x_, y_, z1_, z2_] := χscal[t, α, β] * u1[-3, x] * su2fb[y];
χQ[t_, α_, β_, x_, y_, z1_, z2_] := χfermL[t, α, β] * u1[1, x] * su2f[y] * su3f[z1, z2];
χQd[t_, α_, β_, x_, y_, z1_, z2_] :=
  χfermR[t, α, β] * u1[-1, x] * su2fb[y] * su3fb[z1, z2];
χu[t_, α_, β_, x_, y_, z1_, z2_] := χfermL[t, α, β] * u1[-4, x] * su3fb[z1, z2];
χud[t_, α_, β_, x_, y_, z1_, z2_] := χfermR[t, α, β] * u1[4, x] * su3f[z1, z2];
χd[t_, α_, β_, x_, y_, z1_, z2_] := χfermL[t, α, β] * u1[2, x] * su3fb[z1, z2];
χdd[t_, α_, β_, x_, y_, z1_, z2_] := χfermR[t, α, β] * u1[-2, x] * su3f[z1, z2];
χL[t_, α_, β_, x_, y_, z1_, z2_] := χfermL[t, α, β] * u1[-3, x] * su2f[y];
χLd[t_, α_, β_, x_, y_, z1_, z2_] := χfermR[t, α, β] * u1[3, x] * su2fb[y];
χe[t_, α_, β_, x_, y_, z1_, z2_] := χfermL[t, α, β] * u1[6, x];
χed[t_, α_, β_, x_, y_, z1_, z2_] := χfermR[t, α, β] * u1[-6, x];
χBl[t_, α_, β_, x_, y_, z1_, z2_] := χfsL[t, α, β];
χBr[t_, α_, β_, x_, y_, z1_, z2_] := χfsR[t, α, β];
χWl[t_, α_, β_, x_, y_, z1_, z2_] := χfsL[t, α, β] * su2ad[y];
χWr[t_, α_, β_, x_, y_, z1_, z2_] := χfsR[t, α, β] * su2ad[y];
χGl[t_, α_, β_, x_, y_, z1_, z2_] := χfsL[t, α, β] * su3ad[z1, z2];
χGr[t_, α_, β_, x_, y_, z1_, z2_] := χfsR[t, α, β] * su3ad[z1, z2];
```

$$H(p, \phi_1, \cdots, \phi_n) = \int d\mu_{\text{conformal}} d\mu_{\text{gauge}} \sum_{n=1}^{\infty} p^n \chi^*_{[n;0]} \prod_i PE[\phi_i \chi_i(q, \alpha, \beta)]$$

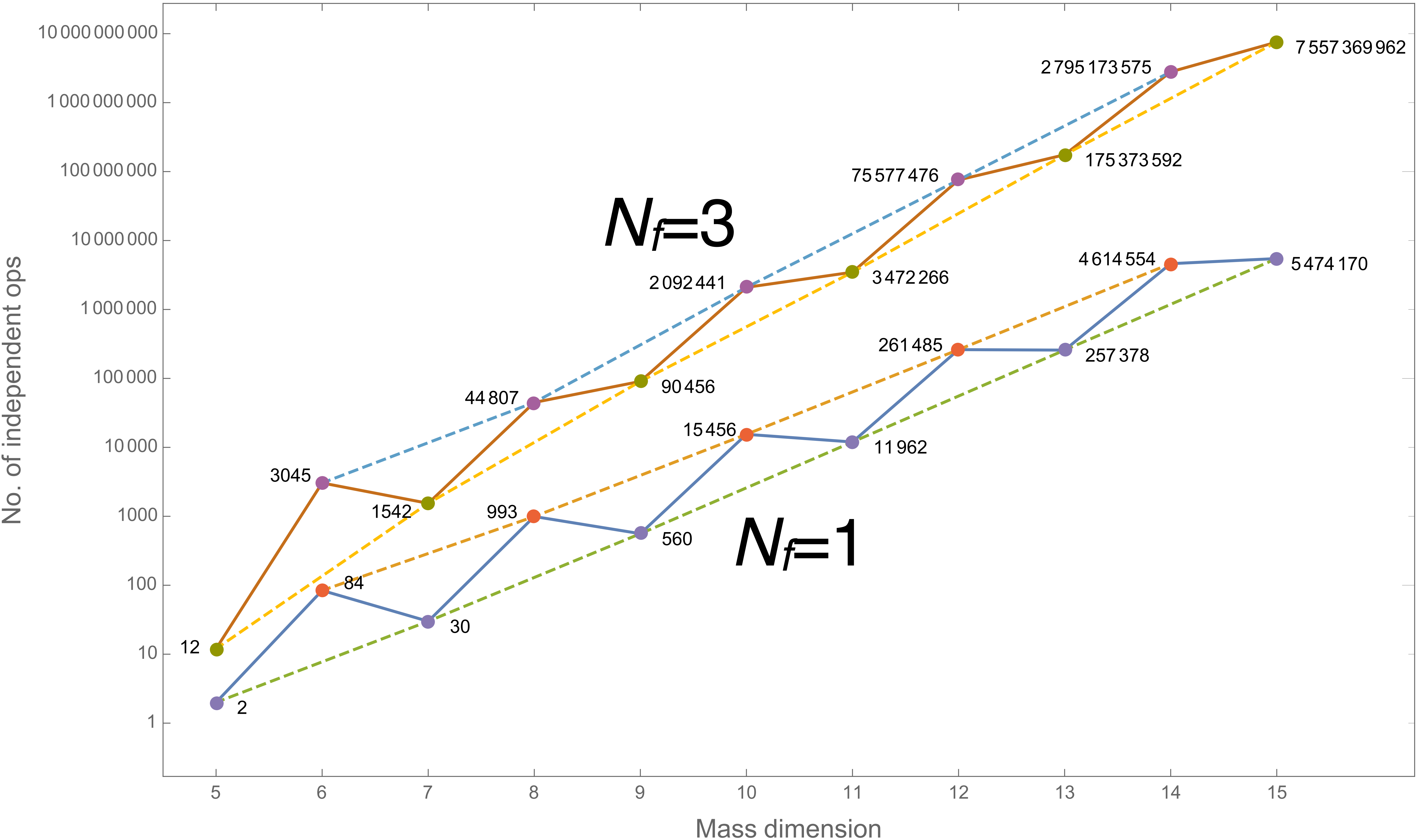
```
Hitoshi-no-MacBook-Pro.local 44: form hssm6.frm
```

```
Hitoshi-no-MacBook-Pro.local 49: form hssm8.frm
```

D=8 operators

```
f =
    2*L^2*Ld^2*t^2 + 2*ee*ed*L*Ld*t^2 + ee^2*ed^2*t^2 + 2*d*dd*L*Ld*t^2 + 2*
    d*dd*ee*ed*t^2 + 2*d^2*dd^2*t^2 + ud^2*dd*ed*t^2 + 2*u*ud*L*Ld*t^2 + 2*u
    *ud*ee*ed*t^2 + 4*u*ud*d*dd*t^2 + u^2*d*ee*t^2 + 2*u^2*ud^2*t^2 + 2*Qd*
    dd*ee*L*t^2 + 3*Qd*ud*ed*Ld*t^2 + 2*Qd*u*d*Ld*t^2 + 3*Qd^2*ud*dd*t^2 +
    Qd^2*u*ee*t^2 + Qd^3*Ld*t^2 + 2*Q*d*ed*Ld*t^2 + 2*Q*ud*dd*L*t^2 + 3*Q*u*
    ee*L*t^2 + 4*Q*Qd*L*Ld*t^2 + 2*Q*Qd*ee*ed*t^2 + 4*Q*Qd*d*dd*t^2 + 4*Q*Qd
    *u*ud*t^2 + Q^2*ud*ed*t^2 + 3*Q^2*u*d*t^2 + 4*Q^2*Qd^2*t^2 + Q^3*L*t^2
     + Wr*L^2*Ld^2 + Wr*ee*ed*L*Ld + Wr*d*dd*L*Ld + Wr*u*ud*L*Ld + Wr*Qd*dd*
    ee*L + 3*Wr*Qd*ud*ed*Ld + Wr*Qd*u*d*Ld + 3*Wr*Qd^2*ud*dd + Wr*Qd^2*u*ee
     + 2*Wr*Qd^3*Ld + Wr*Q*d*ed*Ld + Wr*Q*ud*dd*L + 3*Wr*Q*Qd*L*Ld + Wr*Q*Qd
    *ee*ed + 2*Wr*Q*Qd*d*dd + 2*Wr*Q*Qd*u*ud + 2*Wr*Q^2*Qd^2 + Wr^2*L*Ld*t
     + Wr^2*Q*Qd*t + 2*Wr^4 + Wl*L^2*Ld^2 + Wl*ee*ed*L*Ld + Wl*d*dd*L*Ld +
    Wl*u*ud*L*Ld + Wl*Qd*dd*ee*L + Wl*Qd*u*d*Ld + Wl*Q*d*ed*Ld + Wl*Q*ud*dd*
    L + 3*Wl*Q*u*ee*L + 3*Wl*Q*Qd*L*Ld + Wl*Q*Qd*ee*ed + 2*Wl*Q*Qd*d*dd + 2*
    Wl*Q*Qd*u*ud + Wl*Q^2*ud*ed + 3*Wl*Q^2*u*d + 2*Wl*Q^2*Qd^2 + 2*Wl*Q^3*L
     + 2*Wl*Wr*L*Ld*t + Wl*Wr*ee*ed*t + Wl*Wr*d*dd*t + Wl*Wr*u*ud*t + 2*Wl*
    Wr*Q*Qd*t + Wl^2*L*Ld*t + Wl^2*Q*Qd*t + 2*Wl^2*Wr^2 + 2*Wl^4 + Gr*d*dd*L
    *Ld + Gr*d*dd*ee*ed + Gr*d^2*dd^2 + 3*Gr*ud^2*dd*ed + Gr*u*ud*L*Ld + Gr*
    u*ud*ee*ed + 4*Gr*u*ud*d*dd + Gr*u^2*ud^2 + Gr*Qd*dd*ee*L + 3*Gr*Qd*ud*
    ed*Ld + 2*Gr*Qd*u*d*Ld + 6*Gr*Qd^2*ud*dd + Gr*Qd^2*u*ee + 2*Gr*Qd^3*Ld
     + Gr*Q*d*ed*Ld + 2*Gr*Q*ud*dd*L + 2*Gr*Q*Qd*L*Ld + Gr*Q*Qd*ee*ed + 4*Gr
    *Q*Qd*d*dd + 4*Gr*Q*Qd*u*ud + Gr*Q^2*ud*ed + 2*Gr*Q^2*Qd^2 + Gr*Wr*Q*Qd*
    t + Gr*Wl*Q*Qd*t + Gr^2*d*dd*t + Gr^2*u*ud*t + Gr^2*Q*Qd*t + 2*Gr^2*Wr^2
     + Gr^2*Wl^2 + 3*Gr^4 + Gl*d*dd*L*Ld + Gl*d*dd*ee*ed + Gl*d^2*dd^2 + Gl*
    u*ud*L*Ld + Gl*u*ud*ee*ed + 4*Gl*u*ud*d*dd + 3*Gl*u^2*d*ee + Gl*u^2*ud^2
     + Gl*Qd*dd*ee*L + 2*Gl*Qd*u*d*Ld + Gl*Qd^2*u*ee + Gl*Q*d*ed*Ld + 2*Gl*Q
    *ud*dd*L + 3*Gl*Q*u*ee*L + 2*Gl*Q*Qd*L*Ld + Gl*Q*Qd*ee*ed + 4*Gl*Q*Qd*d*
    dd + 4*Gl*Q*Qd*u*ud + Gl*Q^2*ud*ed + 6*Gl*Q^2*u*d + 2*Gl*Q^2*Qd^2 + 2*Gl
    *Q^3*L + Gl*Wr*Q*Qd*t + Gl*Wl*Q*Qd*t + Gl*Gr*L*Ld*t + Gl*Gr*ee*ed*t + 3*
    Gl*Gr*d*dd*t + 3*Gl*Gr*u*ud*t + 3*Gl*Gr*Q*Qd*t + Gl*Gr*Wl*Wr + Gl^2*d*dd
    *t + Gl^2*u*ud*t + Gl^2*Q*Qd*t + Gl^2*Wr^2 + 2*Gl^2*Wl^2 + 3*Gl^2*Gr^2
     + 3*Gl^4 + Br*ee*ed*L*Ld + Br*d*dd*L*Ld + Br*d*dd*ee*ed + 2*Br*ud^2*dd*
    ed + Br*u*ud*L*Ld + Br*u*ud*ee*ed + 2*Br*u*ud*d*dd + Br*Qd*dd*ee*L + 3*
    Br*Qd*ud*ed*Ld + Br*Qd*u*d*Ld + 3*Br*Qd^2*ud*dd + Br*Qd^3*Ld + Br*Q*d*ed
    *Ld + Br*Q*ud*dd*L + 2*Br*Q*Qd*L*Ld + Br*Q*Qd*ee*ed + 2*Br*Q*Qd*d*dd + 2
    *Br*Q*Qd*u*ud + Br*Q^2*ud*ed + Br*Wr*L*Ld*t + Br*Wr*Q*Qd*t + Br*Wl*L*Ld*
    t + Br*Wl*Q*Qd*t + Br*Gr*d*dd*t + Br*Gr*u*ud*t + Br*Gr*Q*Qd*t + Br*Gr^3
     + Br*Gl*d*dd*t + Br*Gl*u*ud*t + Br*Gl*Q*Qd*t + Br*Gl^2*Gr + 2*Br^2*Wr^2
     + Br^2*Wl^2 + 2*Br^2*Gr^2 + Br^2*Gl^2 + Br^4 + Bl*ee*ed*L*Ld + Bl*d*dd*
    L*Ld + Bl*d*dd*ee*ed + Bl*u*ud*L*Ld + Bl*u*ud*ee*ed + 2*Bl*u*ud*d*dd + 2
    *Bl*u^2*d*ee + Bl*Qd*dd*ee*L + Bl*Qd*u*d*Ld + Bl*Qd^2*u*ee + Bl*Q*d*ed*
    Ld + Bl*Q*ud*dd*L + 3*Bl*Q*u*ee*L + 2*Bl*Q*Qd*L*Ld + Bl*Q*Qd*ee*ed + 2*
    Bl*Q*Qd*d*dd + 2*Bl*Q*Qd*u*ud + 3*Bl*Q^2*u*d + Bl*Q^3*L + Bl*Wr*L*Ld*t
     + Bl*Wr*Q*Qd*t + Bl*Wl*L*Ld*t + Bl*Wl*Q*Qd*t + Bl*Gr*d*dd*t + Bl*Gr*u*
    ud*t + Bl*Gr*Q*Qd*t + Bl*Gl*d*dd*t + Bl*Gl*u*ud*t + Bl*Gl*Q*Qd*t + Bl*Gl
    *Gr^2 + Bl*Gl^3 + Bl*Br*L*Ld*t + Bl*Br*ee*ed*t + Bl*Br*d*dd*t + Bl*Br*u*
    ud*t + Bl*Br*Q*Qd*t + Bl*Br*Wl*Wr + Bl*Br*Gl*Gr + Bl^2*Wr^2 + 2*Bl^2*
    Wl^2 + Bl^2*Gr^2 + 2*Bl^2*Gl^2 + Bl^2*Br^2 + Bl^4 + 3*Hd*ee*L^2*Ld*t +
    Hd*ee^2*ed*L*t + 3*Hd*d*dd*ee*L*t + 3*Hd*ud*d*ed*Ld*t + 2*Hd*ud^2*dd*L*t
     + 2*Hd*u*d^2*Ld*t + 3*Hd*u*ud*ee*L*t + 6*Hd*Qd*ud*L*Ld*t + 3*Hd*Qd*ud*
    ee*ed*t + 6*Hd*Qd*ud*d*dd*t + 3*Hd*Qd*u*d*ee*t + 3*Hd*Qd*u*ud^2*t + 3*Hd
    *Qd^2*d*Ld*t + Hd*Qd^3*ee*t + 6*Hd*Q*d*L*Ld*t + 3*Hd*Q*d*ee*ed*t + 3*Hd*
    Q*d^2*dd*t + 2*Hd*Q*ud^2*ed*t + 6*Hd*Q*u*ud*d*t + 6*Hd*Q*Qd*ee*L*t + 6*
    Hd*Q*Qd^2*ud*t + 3*Hd*Q^2*ud*L*t + 6*Hd*Q^2*Qd*d*t + Hd*Wr*ee*L*t^2 + 2*
    Hd*Wr*Qd*ud*t^2 + Hd*Wr*Q*d*t^2 + Hd*Wr^2*ee*L + 2*Hd*Wr^2*Qd*ud + Hd*
    Wr^2*Q*d + 2*Hd*Wl*ee*L*t^2 + Hd*Wl*Qd*ud*t^2 + 2*Hd*Wl*Q*d*t^2 + 2*Hd*
    Wl^2*ee*L + Hd*Wl^2*Qd*ud + 2*Hd*Wl^2*Q*d + 2*Hd*Gr*Qd*ud*t^2 + Hd*Gr*Q*
    d*t^2 + 2*Hd*Gr*Wr*Qd*ud + Hd*Gr*Wr*Q*d + Hd*Gr^2*ee*L + 3*Hd*Gr^2*Qd*ud
     + 2*Hd*Gr^2*Q*d + Hd*Gl*Qd*ud*t^2 + 2*Hd*Gl*Q*d*t^2 + Hd*Gl*Wl*Qd*ud +
    2*Hd*Gl*Wl*Q*d + Hd*Gl^2*ee*L + 2*Hd*Gl^2*Qd*ud + 3*Hd*Gl^2*Q*d + Hd*Br*
    ee*L*t^2 + 2*Hd*Br*Qd*ud*t^2 + Hd*Br*Q*d*t^2 + Hd*Br*Wr*ee*L + 2*Hd*Br*
    Wr*Qd*ud + Hd*Br*Wr*Q*d + 2*Hd*Br*Gr*Qd*ud + Hd*Br*Gr*Q*d + Hd*Br^2*ee*L
     + Hd*Br^2*Qd*ud + Hd*Br^2*Q*d + 2*Hd*Bl*ee*L*t^2 + Hd*Bl*Qd*ud*t^2 + 2*
    Hd*Bl*Q*d*t^2 + 2*Hd*Bl*Wl*ee*L + Hd*Bl*Wl*Qd*ud + 2*Hd*Bl*Wl*Q*d + Hd*
    Bl*Gl*Qd*ud + 2*Hd*Bl*Gl*Q*d + Hd*Bl^2*ee*L + Hd*Bl^2*Qd*ud + Hd*Bl^2*Q*
    d + Hd^2*ee^2*L^2 + Hd^2*ud*d*t^3 + Hd^2*ud*d*L*Ld + Hd^2*Qd*ud*ee*L + 2
    *Hd^2*Qd^2*ud^2 + 2*Hd^2*Q*d*ee*L + 2*Hd^2*Q*Qd*ud*d + 2*Hd^2*Q^2*d^2 +
    Hd^2*Wr*ud*d*t + Hd^2*Wl*ud*d*t + Hd^2*Gr*ud*d*t + Hd^2*Gl*ud*d*t + Hd^2
    *Br*ud*d*t + Hd^2*Bl*ud*d*t + 3*H*ed*L*Ld^2*t + H*ee*ed^2*Ld*t + 3*H*d*
    dd*ed*Ld*t + 2*H*ud*dd^2*L*t + 3*H*u*dd*ee*L*t + 3*H*u*ud*ed*Ld*t + 2*H*
    u^2*d*Ld*t + 6*H*Qd*dd*L*Ld*t + 3*H*Qd*dd*ee*ed*t + 3*H*Qd*d*dd^2*t + 6*
    H*Qd*u*ud*dd*t + 2*H*Qd*u^2*ee*t + 3*H*Qd^2*u*Ld*t + 3*H*Q*ud*dd*ed*t +
    6*H*Q*u*L*Ld*t + 3*H*Q*u*ee*ed*t + 6*H*Q*u*d*dd*t + 3*H*Q*u^2*ud*t + 6*H
    *Q*Qd*ed*Ld*t + 6*H*Q*Qd^2*dd*t + 3*H*Q^2*dd*L*t + 6*H*Q^2*Qd*u*t + H*
    Q^3*ed*t + 2*H*Wr*ed*Ld*t^2 + 2*H*Wr*Qd*dd*t^2 + H*Wr*Q*u*t^2 + 2*H*Wr^2
    *ed*Ld + 2*H*Wr^2*Qd*dd + H*Wr^2*Q*u + H*Wl*ed*Ld*t^2 + H*Wl*Qd*dd*t^2
     + 2*H*Wl*Q*u*t^2 + H*Wl^2*ed*Ld + H*Wl^2*Qd*dd + 2*H*Wl^2*Q*u + 2*H*Gr*
    Qd*dd*t^2 + H*Gr*Q*u*t^2 + 2*H*Gr*Wr*Qd*dd + H*Gr*Wr*Q*u + H*Gr^2*ed*Ld
     + 3*H*Gr^2*Qd*dd + 2*H*Gr^2*Q*u + H*Gl*Qd*dd*t^2 + 2*H*Gl*Q*u*t^2 + H*
    Gl*Wl*Qd*dd + 2*H*Gl*Wl*Q*u + H*Gl^2*ed*Ld + 2*H*Gl^2*Qd*dd + 3*H*Gl^2*Q
    *u + 2*H*Br*ed*Ld*t^2 + 2*H*Br*Qd*dd*t^2 + H*Br*Q*u*t^2 + 2*H*Br*Wr*ed*
    Ld + 2*H*Br*Wr*Qd*dd + H*Br*Wr*Q*u + 2*H*Br*Gr*Qd*dd + H*Br*Gr*Q*u + H*
    Br^2*ed*Ld + H*Br^2*Qd*dd + H*Br^2*Q*u + H*Bl*ed*Ld*t^2 + H*Bl*Qd*dd*t^2
     + 2*H*Bl*Q*u*t^2 + H*Bl*Wl*ed*Ld + H*Bl*Wl*Qd*dd + 2*H*Bl*Wl*Q*u + H*Bl
    *Gl*Qd*dd + 2*H*Bl*Gl*Q*u + H*Bl^2*ed*Ld + H*Bl^2*Qd*dd + H*Bl^2*Q*u + 4
    *H*Hd*L*Ld*t^3 + 2*H*Hd*L^2*Ld^2 + 2*H*Hd*ee*ed*t^3 + 2*H*Hd*ee*ed*L*Ld
     + H*Hd*ee^2*ed^2 + 2*H*Hd*d*dd*t^3 + 2*H*Hd*d*dd*L*Ld + H*Hd*d*dd*ee*ed
     + H*Hd*d^2*dd^2 + H*Hd*ud^2*dd*ed + 2*H*Hd*u*ud*t^3 + 2*H*Hd*u*ud*L*Ld
     + H*Hd*u*ud*ee*ed + 2*H*Hd*u*ud*d*dd + H*Hd*u^2*d*ee + H*Hd*u^2*ud^2 +
    2*H*Hd*Qd*dd*ee*L + 4*H*Hd*Qd*ud*ed*Ld + 2*H*Hd*Qd*u*d*Ld + 4*H*Hd*Qd^2*
    ud*dd + H*Hd*Qd^2*u*ee + 2*H*Hd*Qd^3*Ld + 2*H*Hd*Q*d*ed*Ld + 2*H*Hd*Q*ud
    *dd*L + 4*H*Hd*Q*u*ee*L + 4*H*Hd*Q*Qd*t^3 + 5*H*Hd*Q*Qd*L*Ld + 2*H*Hd*Q*
    Qd*ee*ed + 4*H*Hd*Q*Qd*d*dd + 4*H*Hd*Q*Qd*u*ud + H*Hd*Q^2*ud*ed + 4*H*Hd
    *Q^2*u*d + 3*H*Hd*Q^2*Qd^2 + 2*H*Hd*Q^3*L + 6*H*Hd*Wr*L*Ld*t + 2*H*Hd*Wr
    *ee*ed*t + 2*H*Hd*Wr*d*dd*t + 2*H*Hd*Wr*u*ud*t + 6*H*Hd*Wr*Q*Qd*t + 2*H*
    Hd*Wr^2*t^2 + H*Hd*Wr^3 + 6*H*Hd*Wl*L*Ld*t + 2*H*Hd*Wl*ee*ed*t + 2*H*Hd*
    Wl*d*dd*t + 2*H*Hd*Wl*u*ud*t + 6*H*Hd*Wl*Q*Qd*t + 2*H*Hd*Wl*Wr*t^2 + 2*H
    *Hd*Wl^2*t^2 + H*Hd*Wl^3 + 2*H*Hd*Gr*d*dd*t + 2*H*Hd*Gr*u*ud*t + 4*H*Hd*
    Gr*Q*Qd*t + H*Hd*Gr^2*t^2 + H*Hd*Gr^3 + 2*H*Hd*Gl*d*dd*t + 2*H*Hd*Gl*u*
    ud*t + 4*H*Hd*Gl*Q*Qd*t + H*Hd*Gl*Gr*t^2 + H*Hd*Gl^2*t^2 + H*Hd*Gl^3 + 4
    *H*Hd*Br*L*Ld*t + 2*H*Hd*Br*ee*ed*t + 2*H*Hd*Br*d*dd*t + 2*H*Hd*Br*u*ud*
    t + 4*H*Hd*Br*Q*Qd*t + 2*H*Hd*Br*Wr*t^2 + H*Hd*Br*Wr^2 + H*Hd*Br*Wl*t^2
     + H*Hd*Br^2*t^2 + 4*H*Hd*Bl*L*Ld*t + 2*H*Hd*Bl*ee*ed*t + 2*H*Hd*Bl*d*dd
    *t + 2*H*Hd*Bl*u*ud*t + 4*H*Hd*Bl*Q*Qd*t + H*Hd*Bl*Wr*t^2 + 2*H*Hd*Bl*Wl
    *t^2 + H*Hd*Bl*Wl^2 + H*Hd*Bl*Br*t^2 + H*Hd*Bl^2*t^2 + 6*H*Hd^2*ee*L*t^2
     + 6*H*Hd^2*Qd*ud*t^2 + 6*H*Hd^2*Q*d*t^2 + 2*H*Hd^2*Wr*Qd*ud + 2*H*Hd^2*
    Wl*ee*L + 2*H*Hd^2*Wl*Q*d + H*Hd^2*Gr*Qd*ud + H*Hd^2*Gl*Q*d + H*Hd^2*Br*
    Qd*ud + H*Hd^2*Bl*ee*L + H*Hd^2*Bl*Q*d + H*Hd^3*ud*d*t + H^2*ed^2*Ld^2
     + H^2*u*dd*t^3 + H^2*u*dd*L*Ld + 2*H^2*Qd*dd*ed*Ld + 2*H^2*Qd^2*dd^2 +
    H^2*Q*u*ed*Ld + 2*H^2*Q*Qd*u*dd + 2*H^2*Q^2*u^2 + H^2*Wr*u*dd*t + H^2*Wl
    *u*dd*t + H^2*Gr*u*dd*t + H^2*Gl*u*dd*t + H^2*Br*u*dd*t + H^2*Bl*u*dd*t
     + 6*H^2*Hd*ed*Ld*t^2 + 6*H^2*Hd*Qd*dd*t^2 + 6*H^2*Hd*Q*u*t^2 + 2*H^2*Hd
    *Wr*ed*Ld + 2*H^2*Hd*Wr*Qd*dd + 2*H^2*Hd*Wl*Q*u + H^2*Hd*Gr*Qd*dd + H^2*
    Hd*Gl*Q*u + H^2*Hd*Br*ed*Ld + H^2*Hd*Br*Qd*dd + H^2*Hd*Bl*Q*u + 3*H^2*
    Hd^2*t^4 + 4*H^2*Hd^2*L*Ld*t + H^2*Hd^2*ee*ed*t + H^2*Hd^2*d*dd*t + H^2*
    Hd^2*u*ud*t + 4*H^2*Hd^2*Q*Qd*t + 2*H^2*Hd^2*Wr*t^2 + 2*H^2*Hd^2*Wr^2 +
    2*H^2*Hd^2*Wl*t^2 + 2*H^2*Hd^2*Wl^2 + H^2*Hd^2*Gr^2 + H^2*Hd^2*Gl^2 +
    H^2*Hd^2*Br*t^2 + H^2*Hd^2*Br*Wr + H^2*Hd^2*Br^2 + H^2*Hd^2*Bl*t^2 + H^2
    *Hd^2*Bl*Wl + H^2*Hd^2*Bl^2 + H^2*Hd^3*ee*L + H^2*Hd^3*Qd*ud + H^2*Hd^3*
    Q*d + H^3*Hd*u*dd*t + H^3*Hd^2*ed*Ld + H^3*Hd^2*Qd*dd + H^3*Hd^2*Q*u + 2
    *H^3*Hd^3*t^2 + H^4*Hd^4;
```

993 of them for N_f=1



see discussion on the asymptotic behavior by Melia and Pal, 2010.08560

discrete spacetime symmetries: *P* & *C*

Parity

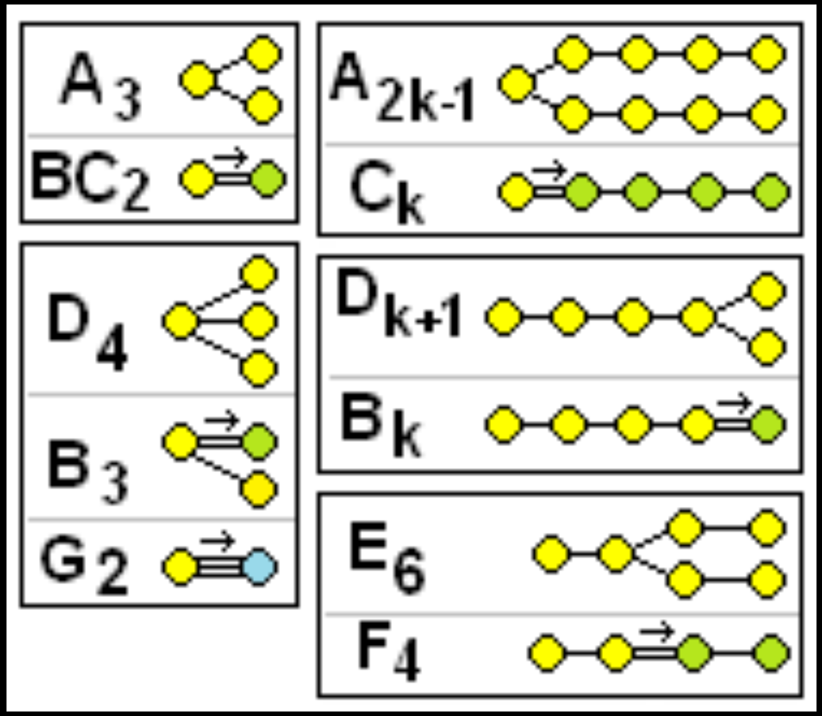
- For SO(2r+1)=B_r, parity is equivalent to a Z_2 external discrete symmetry

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{2r} \\ \phi_{2r+1} \end{pmatrix} \xrightarrow{P} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{2r} \\ -\phi_{2r+1} \end{pmatrix} \xrightarrow{\text{rot}} \begin{pmatrix} -\phi_1 \\ -\phi_2 \\ \vdots \\ -\phi_{2r} \\ -\phi_{2r+1} \end{pmatrix} = - \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{2r} \\ \phi_{2r+1} \end{pmatrix}$$

$$H(\phi) \longrightarrow \frac{1}{2}\left(H(\phi) + H(-\phi)\right)$$

Parity

- For SO(2*r*) = D_r, parity is an outer automorphism
- invariant subgroup is SO(2*r*–1) = B_{r-1}
- But the characters for the odd elements are Sp(2*r*–2)= C_{r-1} characters!
- dual of B_{r-1}
- “twining character” Fuchs, Schellekens, Schweigert, *Commun. Math. Phys.* 180, 39 (1996)



Wikipedia

SO(2*r*) notation

$$O_{2r} \xrightarrow{\text{diag}} \begin{pmatrix} e^{+i\theta_1} & & & & & & \\ & e^{-i\theta_1} & & & & & \\ & & e^{+i\theta_2} & & & & \\ & & & e^{-i\theta_2} & & & \\ & & & & \ddots & & \\ & & & & & e^{+i\theta_r} & \\ & & & & & & e^{-i\theta_r} \end{pmatrix}$$

$$= \text{diag}(x_1, x_1^{-1}, x_2, x_2^{-1}, \cdots, x_r, x_r^{-1})$$

SO(2r) adjoint

$$\chi_{SO(2r)}(\text{adj}) = r + \sum_{j>i=1}^{r}\left(x_i x_j + \frac{x_i}{x_j} + \frac{x_j}{x_i} + \frac{1}{x_i x_j}\right)$$

- under invariant subgroup SO(2r–1), x_r=1

$$\chi_{SO(2r)}(\text{adj}, x_r = 1) = \chi_{SO(2r-1)}(\text{adj}) + \chi_{SO(2r-1)}(\text{vec})$$

$$\chi_{SO(2r-1)}(\text{adj}) = r - 1 + \sum_{j>i=1}^{r-1}\left(x_i x_j + \frac{x_i}{x_j} + \frac{x_j}{x_i} + \frac{1}{x_i x_j}\right) + \sum_{i=1}^{r-1}\left(x_i + \frac{1}{x_i}\right)$$

$$\chi_{SO(2r-1)}(\text{vec}) = 1 + \sum_{i=1}^{r-1}\left(x_i + \frac{1}{x_i}\right)$$

- odd elements

$$\chi_{SO(2r-1)}(\text{adj}) - \chi_{SO(2r-1)}(\text{vec}) = r - 2 + \sum_{j>i=1}^{r-1}\left(x_i x_j + \frac{x_i}{x_j} + \frac{x_j}{x_i} + \frac{1}{x_i x_j}\right)$$

$$= \chi_{Sp(2r-2)}(A_{ij} = -A_{ji})$$

Charge Conjugation

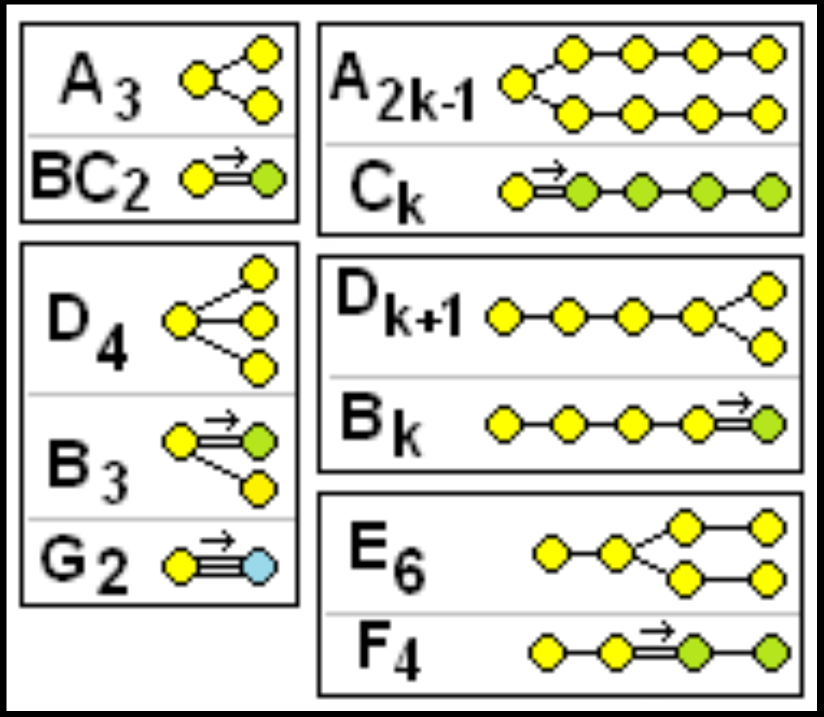
- For SU(2k) = A_{2k-1}, charge conjugation is an outer automorphism

$$J^2 = 1, \quad {}^tJ = -J$$

$$T^a_C \equiv J\,{}^tT^a J$$

$$\begin{aligned}[T^a_C, T^b_C] &= [J\,{}^tT^a J, J\,{}^tT^b J] \\ &= -J[{}^tT^a, {}^tT^b]J \\ &= J\,{}^t[T^a, T^b]J \\ &= i f^{abc} J\,{}^tT^c J = T^c_C\end{aligned}$$

- invariant subgroup is Sp(2k) = C_k
- its dual is SO(2k+1) = B_k



Wikipedia

SU(2*k*) notation

$$U_{2k} \xrightarrow{\text{diag}} \begin{pmatrix} e^{i\theta_1} & & & & \\ & e^{i\theta_2} & & & \\ & & \ddots & & \\ & & & e^{i\theta_{2k-1}} & \\ & & & & e^{i\theta_{2k}} \end{pmatrix}$$

$$= \text{diag}(x_1, x_2, \cdots, x_{2k-1}, x_{2k})$$

$$x_1 x_2 \cdots x_{2k-1} x_{2k} = 1$$

SU(2*k*) adjoint

$$\chi_{SU(2k)}(\mathrm{adj}) = 2k - 1 + \sum_{j>i=1}^{2k} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i}\right)$$

- under invariant subgroup Sp(2*k*), $x_m = x_{2k-m}^{-1}$

$$\chi_{SU(2k)}(\mathrm{adj}) = \chi_{Sp(2k)}(S_{ij} = S_{ji}) + \chi_{Sp(2k)}(A_{ij} = -A_{ji})$$

$$\chi_{Sp(2k)}(S) = k + \sum_{i=1}^{k} \left(x_i^2 + \frac{1}{x_i^2}\right) + \sum_{j>i=1}^{k} \left(x_i x_j + \frac{1}{x_i x_j}\right)$$

$$\chi_{Sp(2k)}(A) = k - 1 + + \sum_{j>i=1}^{k} \left(x_i x_j + \frac{1}{x_i x_j}\right)$$

- odd elements

$$\chi_{Sp(2k)}(S) - \chi_{Sp(2k)}(A) = 1 + \sum_{i=1}^{k} \left(x_i^2 + \frac{1}{x_i^2}\right)$$

$$= \chi_{SO(2k+1)}(\mathrm{vec}, x_i^2)$$

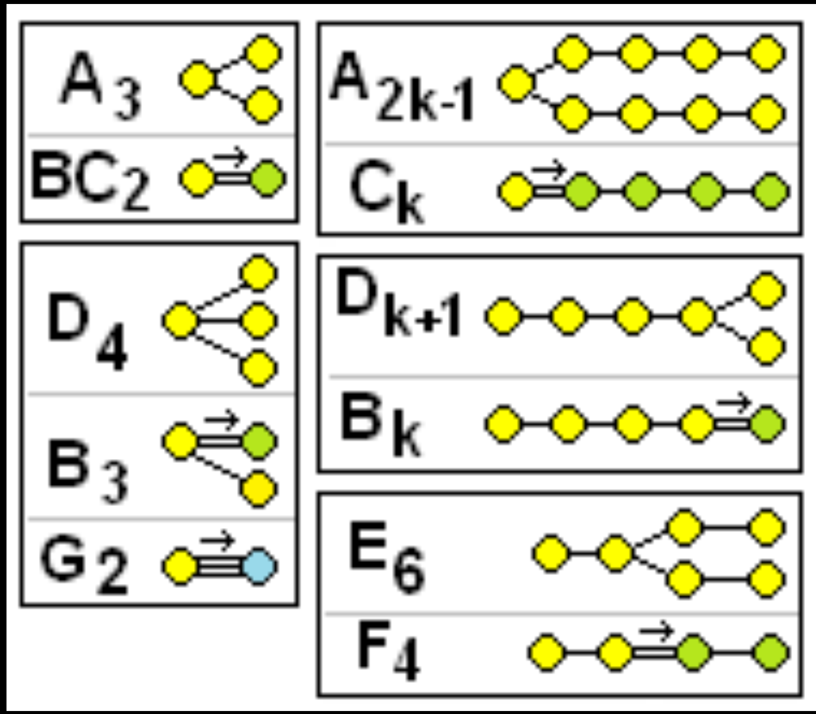
Charge Conjugation

- For SU(2*k*+1) = A_{2k}, charge conjugation is an outer automorphism

$$T_C^a = -{}^tT^a$$

$$\begin{aligned}[T_C^a, T_C^b] &= [-{}^tT^a, -{}^tT^b] \\ &= {}^t[T^b, T^a] \\ &= {}^t(-if^{abc}T^c) \\ &= if^{abc}T_C^c\end{aligned}$$

- invariant subgroup is SO(2*k*+1) = B_k
- its dual is Sp(2*k*) = C_k



Wikipedia

The automorphism of A_{2n} does not yield a folding because the middle two nodes are connected by an edge, but in the same orbit.

wrong!

SU(2k) adjoint

$$\chi_{SU(2k+1)}(\text{adj}) = 2k + \sum_{j>i=1}^{2k+1} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} \right)$$

- under invariant subgroup SO(2k+1), $x_m = (x_{2k+1-m})^{-1}$, $x_k = 1$

$$\chi_{SU(2k+1)}(\text{adj}) = \chi_{SO(2k+1)}(S_{ij} = S_{ji}) + \chi_{SO(2k+1)}(A_{ij} = -A_{ji})$$

$$\chi_{SO(2k+1)}(S) = k + \sum_{i=1}^{k} \left(x_i^2 + \frac{1}{x_i^2} + x_i + \frac{1}{x_i} \right) + \sum_{j>i=1}^{k} \left(x_i x_j + \frac{x_i}{x_j} + \frac{x_j}{x_i} + \frac{1}{x_i x_j} \right)$$

$$\chi_{SO(2k+1)}(A) = k + \sum_{i=1}^{k} \left(x_i + \frac{1}{x_i} \right) + \sum_{j>i=1}^{k} \left(x_i x_j + \frac{x_i}{x_j} + \frac{x_j}{x_i} + \frac{1}{x_i x_j} \right)$$

- odd elements

$$\chi_{SO(2k+1)}(S) - \chi_{SO(2k+1)}(A) = \sum_{i=1}^{k} \left(x_i^2 + \frac{1}{x_i^2} \right) = \chi_{Sp(2k)}(\text{fund}, x_i^2)$$

nutshell

- P in SO($2r$) is implemented by Sp($2r$–2) characters and Haar measure
- C in SU($2k$) is implemented by SO($2k$+1) characters and Haar measure
- C in SU($2k$+1) is implemented by Sp($2k$) characters and Haar measure
- Hilbert series is the average of even and odd elements

Chiral Lagrangian

chiral Lagrangian

- can't use CFT: $\xi = e^{i\pi/f}$ $\qquad [f] = 1$
- extend the UV Lagrangian

$$\mathcal{L}_{\rm UV} = \mathcal{L}_{\rm QCD} + \bar{q}(\gamma^\mu \ell_\mu P_L + \gamma^\mu r_\mu P_R - s + ip\gamma^5)q$$

$$\xi = e^{i\pi^a T^a/f_\pi} \to g_R \xi h^{-1} = h\xi g_L^{-1}$$

- covariant objects:

$$u_\mu = u_\mu^a T^a \qquad \equiv i\left[\xi^\dagger\left(\partial_\mu - ir_\mu\right)\xi - \xi\left(\partial_\mu - i\ell_\mu\right)\xi^\dagger\right]$$

$$\Sigma_\pm + \langle\Sigma_\pm\rangle = \Sigma_\pm^a T^a + \langle\Sigma_\pm\rangle \mathbf{1} \qquad \equiv \xi^\dagger\,\Sigma\,\xi^\dagger \pm \xi\,\Sigma^\dagger\,\xi$$

$$f_\pm^{\mu\nu} = f_\pm^{\mu\nu,a} T^a \qquad \equiv \xi\,F_L^{\mu\nu}\,\xi^\dagger \pm \xi^\dagger\,F_R^{\mu\nu}\,\xi$$

$$F_L^{\mu\nu} = \partial^\mu \ell^\nu - \partial^\nu \ell^\mu - i[\ell^\mu \ell^\nu - \ell^\nu \ell^\mu]$$

$$F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu r^\nu - r^\nu r^\mu]$$

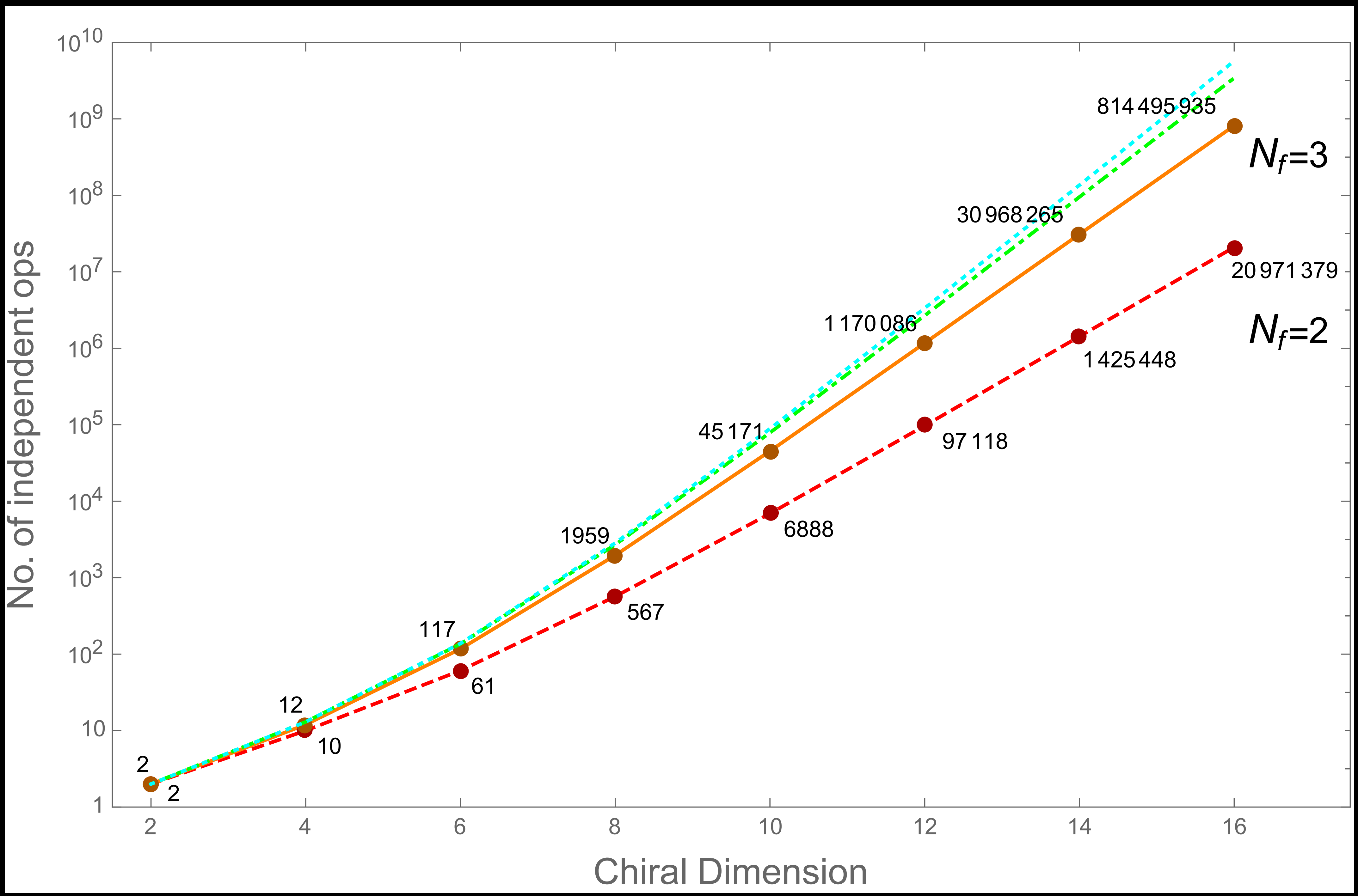
IBP = translational inv

$$\int d\mu_{\text{spacetime}}(x) \frac{1}{P(p,x)} = \int d\mu_{SO(4)}(x) \frac{1}{P_+(p,x)}$$

$$P_+(p,x) = \frac{1}{(1-px_1)(1-px_1^{-1})(1-px_2)(1-px_2^{-1})}$$

- together with all the internal symmetries as before
- write down Hilbert series using characters integrated over Haar measures
- implement *P* and *C* as needed

	$N_f=2$				$N_f=3$			
chiral dim	Total	C-even	P-even	CP-even	Total	C-even	P-even	CP-even
p^6	151	103	82	88	315	206	165	178
p^8	1834	1050	943	975	6882	3768	3479	3553



Obtaining & Using EFT

Effective Field Theory

$$\mathcal{L}_{\rm BSM}$$

integrate out

covariant derivative expansion

$$\mathcal{L}_{\rm SM} + \sum_n \frac{c}{\Lambda^n} \mathcal{O}_{n+4}$$

RGE

$$\mathcal{L}_{\rm SM} + \sum_n \frac{c'}{\Lambda^n} \mathcal{O}_{n+4}$$

$$\mathcal{L}_{\rm SM}$$

observables

$$Z_{\mu\nu} W^{+\mu} W^{-\nu}$$
$$\subset |H|^2 W^a_{\mu\nu} W^{a,\mu\nu},$$
$$H^\dagger \tau^a H W^a_{\mu\nu} B^{\mu\nu},$$
$$H^\dagger \tau^a \overleftrightarrow{D}^\mu H D^\nu W^a_{\mu\nu}$$

interference corrections
residue corrections
parametric corrections

+ matching

F. del Aguila, Z. Kunszt, and J. Santiago
arXiv:1602.00126

covariant derivative expansion

$$\Delta L_{\text{eff,1-loop}} = -ic_s \int dq \int dm^2$$

$$\text{tr} \frac{1}{\Delta^{-1}\left[1 - \Delta\left(- \{q_\mu, \widetilde{G}_{\nu\mu}\}\partial_\nu - \widetilde{G}_{\mu\sigma}\widetilde{G}_{\nu\sigma}\partial_\mu\partial_\nu + \widetilde{U}\right)\right]}$$

$$\Delta \equiv \frac{1}{q^2 - m^2}$$

$$\widetilde{G}_{\nu\mu} = \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!}\left([D_{\alpha_1}, [\ldots, [D_{\alpha_n}, G'_{\nu\mu}] \cdots]]\right)\partial^n_{\alpha_1 \ldots \alpha_n}$$

$$\widetilde{U} = \sum_{n=0}^{\infty} \frac{1}{n!}\left([D_{\alpha_1}, [\ldots, [D_{\alpha_n}, U] \cdots]]\right)\partial^n_{\alpha_1 \ldots \alpha_n}$$

matching

- specific issue with mixed heavy-light diagram in dim reg

$$
\begin{aligned}
\Gamma_{\mathrm{L,UV}}[\phi] &= S_{\mathrm{UV}}[\phi, \Phi_c[\phi]] \\
&+ \frac{i}{2} \log\det\left(- \frac{\delta^2 S_{\mathrm{UV}}[\phi, \Phi_c[\phi]]}{\delta\phi^2} \right) \\
&+ \frac{i}{2} \log\det\left(- \left.\frac{\delta^2 S_{\mathrm{UV}}[\phi, \Phi]}{\delta\Phi^2}\right|_{\Phi_c} \right)
\end{aligned}
$$

See more recent progress
1610.00710, 2006.16260, 2011.02484, 2012.07851

$$\epsilon_{ggF,I} = \frac{(4\pi)^2}{\mathrm{Re}(A^{\mathrm{SM}}_{hgg})}\frac{16v^2}{\Lambda^2}c_{GG}$$

$$\epsilon_{WWh,I}(s) = \Big[-f_b(s)-f_c(s)\Big]\frac{2m_W^2}{\Lambda^2}c_{2W} + \Big[-f_a(s)+2f_c(s)\Big]\frac{8m_W^2}{\Lambda^2}c_{WW} + \Big[f_b(s)+2f_c(s)\Big]\frac{2m_W^2}{\Lambda^2}c_W + f_c(s)\frac{2v^2}{\Lambda^2}c_R + \frac{2m_h^2}{\Lambda^2}c_D$$

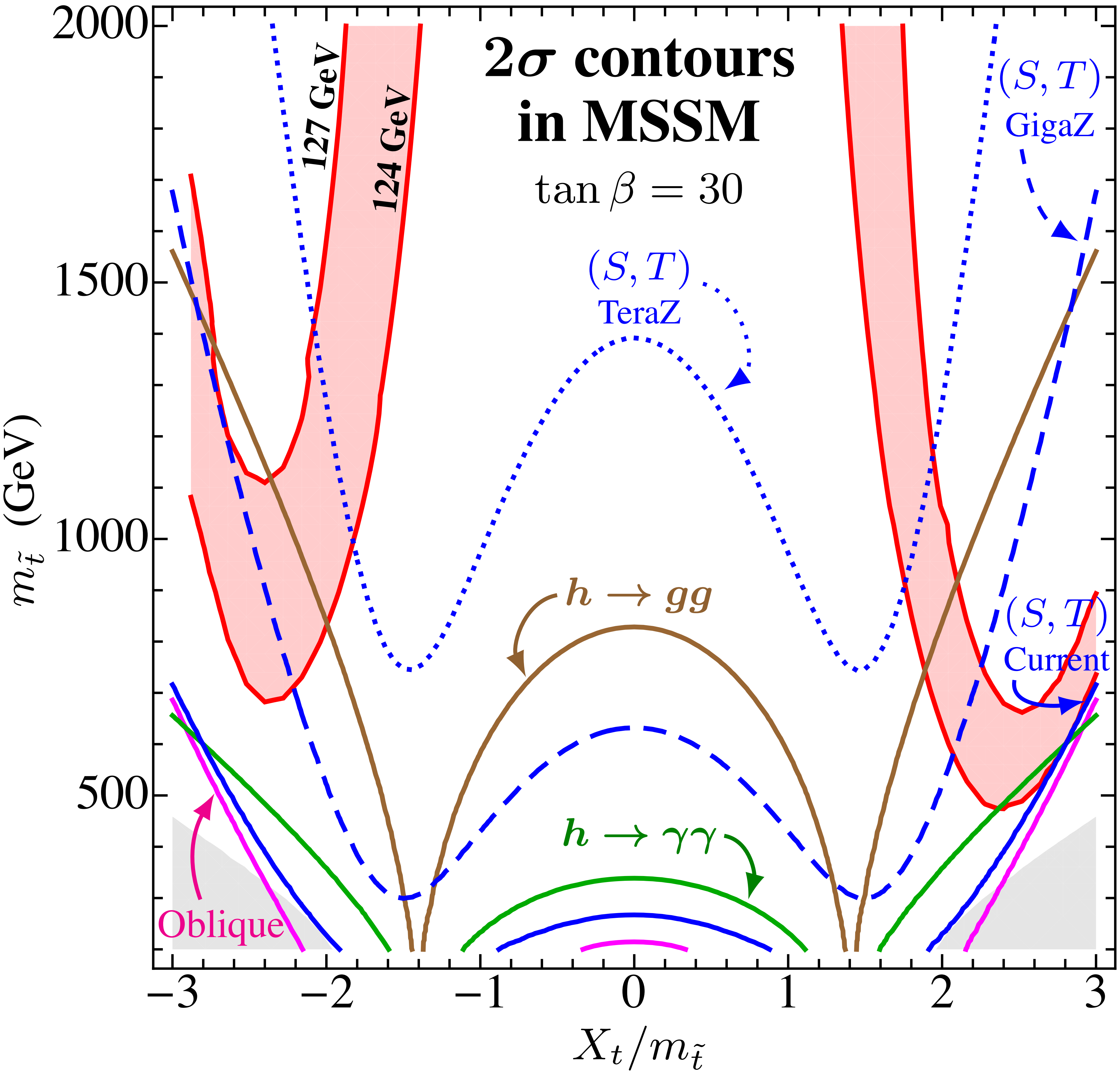
$$\epsilon_{Wh,I} = \frac{1}{1-\eta_W^2}\left[\begin{array}{l} -\frac{2s}{\Lambda^2}c_{2W} + I_{VH}(\eta_h,\eta_W)\frac{16m_W^2}{\Lambda^2}c_{WW} \\ +(1+2\eta_W^2-\eta_W^4)\frac{2s}{\Lambda^2}c_W + (2-\eta_W^2)\frac{2v^2}{\Lambda^2}c_R \end{array}\right] + \frac{2m_h^2}{\Lambda^2}c_D$$

$$\epsilon_{Zh,I} = \frac{1}{1-\eta_Z^2}\left[\begin{array}{l} -\frac{2s}{\Lambda^2}\left(c_Z^2c_{2W}+s_Z^2c_{2B}\right) \\ +I_{VH}(\eta_h,\eta_Z)\frac{16m_Z^2}{\Lambda^2}\left(c_Z^4c_{WW}+s_Z^4c_{BB}+c_Z^2s_Z^2c_{WB}\right) \\ +\left(1+2\eta_Z^2-\eta_Z^4\right)\frac{2s}{\Lambda^2}\left(c_Z^2c_W+s_Z^2c_B\right) \\ +\left(2-\eta_Z^2\right)\frac{2v^2}{\Lambda^2}\left(-2c_T+c_R\right) \end{array}\right] + \frac{2m_h^2}{\Lambda^2}c_D$$
$$+\frac{2eQ_fc_Z^2s_Z}{g(T_f^3-s_Z^2Q_f)}\left\{\begin{array}{l} -\frac{s}{\Lambda^2}\left(c_{2W}-c_{2B}-c_W+c_B\right) \\ +I_{VH}(\eta_h,\eta_Z)\frac{4m_Z^2}{\Lambda^2}\Big[2c_Z^2c_{WW}-2s_Z^2c_{BB}-\left(c_Z^2-s_Z^2\right)c_{WB}\Big] \end{array}\right\}$$

Table 11. Interference corrections ϵ_I to Higgs production cross sections, with $\eta_h \equiv \frac{m_h}{\sqrt{s}}$, $\eta_Z \equiv \frac{m_Z}{\sqrt{s}}$, and the auxiliary function defined as $I_{VH}(\eta_h,\eta_V) \equiv 1 + \frac{6(1-\eta_h^2+\eta_V^2)(1-\eta_V^2)}{(1-\eta_h^2+\eta_V^2)^2+8\eta_V^2}$. The numerical results of the auxiliary functions $f_a(s)$, $f_b(s)$, and $f_c(s)$ in $\epsilon_{WWh,I}(s)$ are shown in Fig. 4.

	ϵ_R	ϵ_P
σ_{ggF}	0	0
σ_{WWh}	Δr_h	$4\Delta w_{g^2} + \Delta w_{v^2}$
σ_{Wh}	$\Delta r_h + \Delta r_W$	$3\Delta w_{g^2} + \Delta w_{v^2}$
σ_{Zh}	$\Delta r_h + \Delta r_Z$	$3\Delta w_{g^2} + \Delta w_{v^2} + \left(3\frac{s_Z^2}{c_Z^2} - \frac{2s_Z^2 Q_f}{T_f^3 - s_Z^2 Q_f}\right)\Delta w_{s_Z^2}$

Table 12. Residue corrections ϵ_R and parametric corrections ϵ_P to Higgs production cross sections. The results of residue modifications and parameter modifications are listed in Tables 15 and 16 of Appendix C.



2σ contours
in MSSM
tan β = 30
127 GeV
124 GeV
(S, T)
GigaZ
(S, T)
TeraZ
h → gg
(S, T)
Current
h → γγ
Oblique
2000
1500
1000
500
−3
−2
−1
0
1
2
3
$m_{\tilde{t}}$ (GeV)
$X_t/m_{\tilde{t}}$

Conclusions

- Nailed the question of classifying effective operators in any given Lorentz-inv theory
- Also for chiral Lagrangians
- useful techniques for matching
- careful mapping to observables
- now working on Higgs EFT
- hope for deviations from Standard Model
- inverse problem to identify BSM physics