ALL *THINGS* EFT…

An On-shell Formulation of Chiral Perturbation Theory

(The Fall and Rise of S-matrix Theory)

Ian Low

Argonne/Northwestern

February 17, 2021

Based on works collaborated with L. Dai, D. Liu
T. Mehen, A. Mohapatra and Z. Yin.

The Mother of Modern EFTs

- Chiral Perturbation Theory (ChPT) is a low-energy effective theory of QCD and an integral component of our understanding of many nuclear processes.

- It describes interactions of mesons and baryons at the energy scale ~ 1 GeV or below and, in my view, is the most elegant example employing ”modern” techniques of effective field theories.

- Modern EFTs:
  - In most cases, symmetry consideration alone is sufficient to capture the long wavelength dynamics of a physical system.

    For ChPT the symmetry is $SU(N_f)_L$ x $SU(N_f)_R$ chiral symmetry.
  - Short wavelength fluctuations are encoded in uncalculable “Wilson coefficients.”
  - A power counting scheme must be supplied to organized the relative importance of different effective operators.

    In ChPT the power counting is the “derivative expansion.”

- The history of ChPT is described in the inaugural lecture of this seminar series:

All Things EFT Inaugural Lecture: On the Development of Effective Field Theory

Steven Weinberg (U. Texas, Austin (main))

30 September 2020, 09:00 AM - 10:00 AM

2101.04241

  - “…phenomenological Lagrangians were developed…as merely labor-saving devices…guaranteed to give the same results as current algebras…”
  - “No one took these theories seriously as true quantum field theories at the time.“
  - “The soft pion theorems had been successful, not only in agreeing with experiment, but also in killing off a competitor of quantum field theory known as S-matrix theory.”

The birth of ChPT contributed to the downfall of S-matrix theory!

- "Modern" EFT means more than half-a-century ago:

PHYSICAL REVIEW VOLUME 166, NUMBER 5 25 FEBRUARY 1968

Nonlinear Realizations of Chiral Symmetry*

STEVEN WEINBERG†
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts
(Received 25 September 1967)

We explore possible realizations of chiral symmetry, based on isotopic multiplets of fields whose transformation rules involve only isotopic-spin matrices and the pion field. The transformation rules are unique, up to possible redefinitions of the pion field. Chiral-invariant Lagrangians can be constructed by forming isotopic-spin-conserving functions of a covariant pion derivative, plus other fields and their covariant derivatives. The resulting models are essentially equivalent to those that have been derived by treating

- ChPT is an EFT about Nambu-Goldstone bosons, which has an even longer history:

PHYSICAL REVIEW VOLUME 117, NUMBER 3 FEBRUARY 1, 1960

Quasi-Particles and Gauge Invariance in the Theory of Superconductivity*

YOICHIRO NAMBU
The Enrico Fermi Institute for Nuclear Studies and the Department of Physics, The University of Chicago, Chicago, Illinois
(Received July 23, 1959)

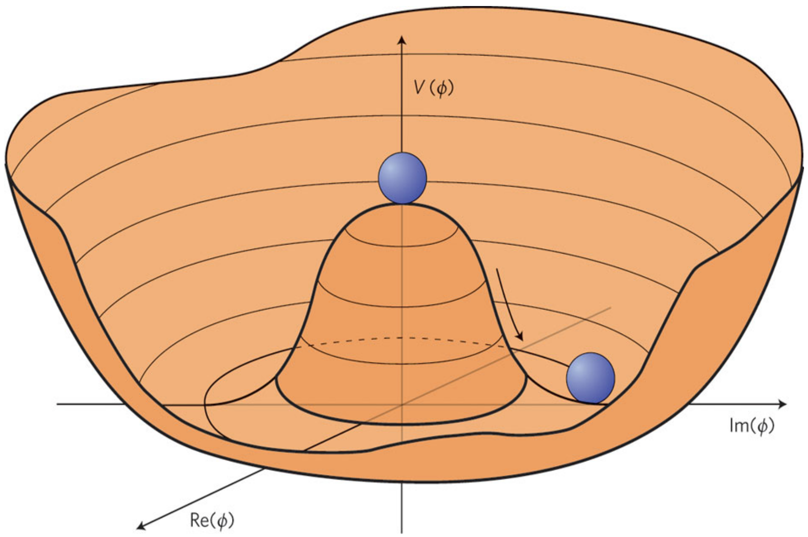
Field Theories with «Superconductor» Solutions.

J. GOLDSTONE
CERN - Geneva

(ricevuto l'8 Settembre 1960)

The “textbook” example of Nambu-Goldstone bosons starts with the following potential energy:

$$\phi = \phi_1 + i\phi_2$$
$$V(\phi) = -\mu^2|\phi|^2 + \lambda|\phi|^4$$



- There is an infinite number of ground state, labelled by the polar angle :

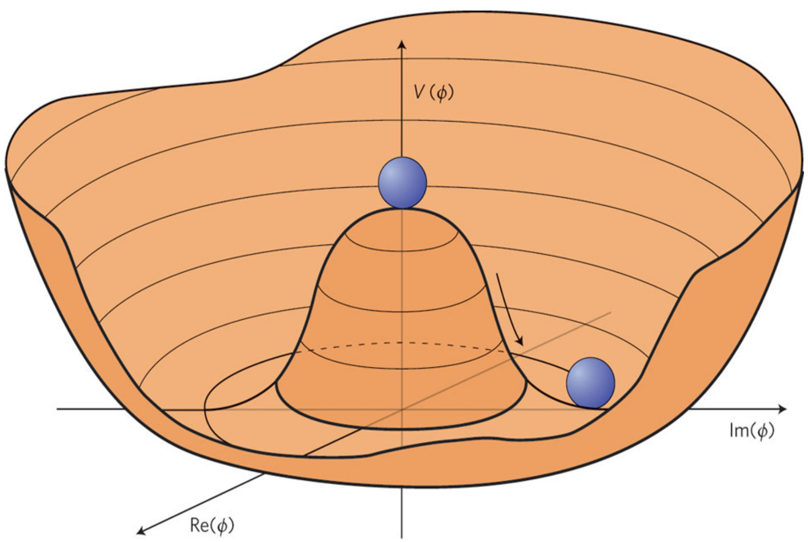
$$|\mathrm{VAC}\rangle = \{|0\rangle_\alpha;\ \alpha \in [0, 2\pi)\}$$

- Once an “alpha” is chosen, the rotational invariance is hidden.

To see the NGB explicitly, let's go to "polar coordinate:"

$$\phi(x) = \frac{\rho(x)}{\sqrt{2}} \; e^{i\frac{1}{\sqrt{2}v}\pi(x)}$$

In this parameterization, the ground state is $\langle \rho \rangle_\alpha = v \; , \quad \langle \pi \rangle_\alpha = \alpha$



Expanding with respect to the ground state:

$$\rho \to \rho + \langle \rho \rangle_\alpha$$
$$\pi \to \pi + \langle \pi \rangle_\alpha$$

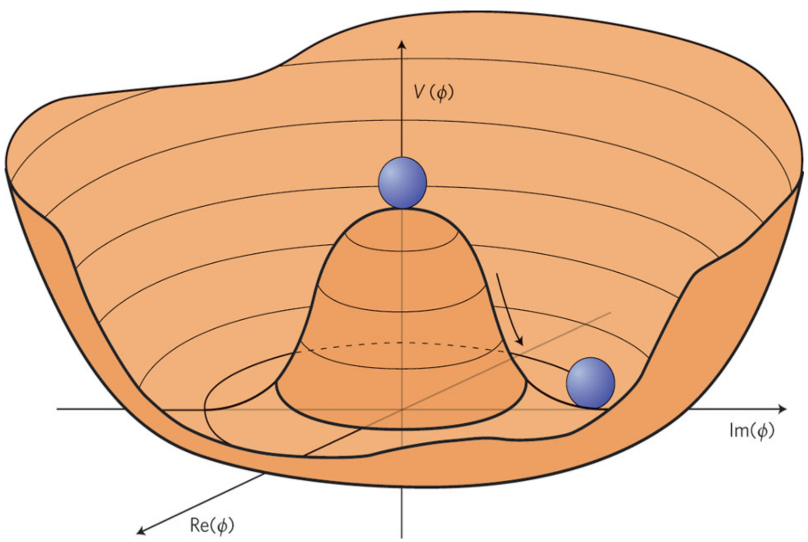
Under rotation by theta-angle,

$$\langle \pi \rangle_\alpha \to \langle \pi \rangle_{\alpha+\theta} = \alpha + \theta$$

To see the NGB explicitly, let's go to "polar coordinate:"

$$\phi(x) = \frac{\rho(x)}{\sqrt{2}}\ e^{i \frac{1}{\sqrt{2}v} \pi(x)}$$

In this parameterization, the ground state is $\langle \rho \rangle_\alpha = v \ , \quad \langle \pi \rangle_\alpha = \alpha$



This is the equivalent to shifting the Pi-mode by a constant:

$$\pi \to \pi + \theta$$

Rotational symmetry implies the dynamics must be independent of the constant shift!

This is a "shift symmetry," which forbids a "mass" term for the pi-mode!

We could generalize to more complicated cases.

For example, let's consider n real scalars:

$$\vec{\phi} = (\phi_1, \cdots, \phi_n)$$

$$V(\vec{\phi}) = -\mu^2 \vec{\phi} \cdot \vec{\phi} + \lambda \left(\vec{\phi} \cdot \vec{\phi} \right)^2$$

$$\langle \vec{\phi} \rangle = (v, 0, \cdots, 0)$$

Then

$$\text{Broken symmetry } G = O(n)$$

$$\text{Unbroken symmetry } H = O(n-1)$$

The NGB mode can be parameterized by

$$\vec{\phi} = v(\sigma, \pi^1, \cdots, \pi^{n-1}) \ , \quad \sigma = \sqrt{1-\pi^2}$$

$$\frac{1}{2}\left(\partial_\mu \vec{\phi}\right)^2 \to \frac{1}{2}\left[(\partial_\mu \vec{\pi})^2 + \frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{1-\pi^2}\right]$$

Key observations:

- Interactions of NGB, in general, are horribly nonlinear.
- NGBs are always “derivatively coupled,” due to the shift symmetry:

$$\vec{\pi} \to \vec{\pi} + \vec{\theta} + \cdots$$

More generally, there is a well-defined procedure to write down NGB effective actions for arbitrary symmetry breaking pattern.

(CCWZ: Coleman, Callan, Wess and Zumino, Phys. Rev. 1969.)

One picks a nonlinearly realized group G, and a subgroup H of G that is linearly realized.

We say G is the broken group and H the unbroken group:

$$\xi = e^{i\pi^a X^a}$$

$$g\xi = \xi' U \ , \quad U \in H \ , \quad \xi' = e^{i\pi'^a X^a}$$

The "pions" are the coordinates on the coset manifold G/H.

When $$g = e^{i\varepsilon^a X^a} \ , \quad \pi'^a = \pi^a + \varepsilon^a + \cdots$$

This is the shift symmetry!

Rarely discussed. Because it's not needed in CCWZ.

CCWZ looked for objects that have "simple" transformation properties under the action of G.

These are contained in the Cartan-Maurer one-form:

$$\xi^\dagger \partial_\mu \xi \;=\; i\mathcal{D}_\mu^a X^a + i\mathcal{E}_\mu^i T^i \equiv i\mathcal{D}_\mu + i\mathcal{E}_\mu$$

They are the "Goldstone covariant derivative" and the "associated gauge field",

$$\mathcal{D}_\mu \to U\mathcal{D}_\mu U^{-1}\;, \quad \mathcal{E}_\mu \to U\mathcal{E}_\mu U^{-1} - (\partial_\mu U)U^{-1}$$

upon which the complete effective Lagrangian can be built (apart from the topological terms)

$$\mathcal{L}_{eff} = \frac{f^2}{2}\mathrm{Tr}\mathcal{D}_\mu\mathcal{D}^\mu + \cdots$$

In CCWZ, NGB interactions depends both on the full symmetry G in the UV and the unbroken symmetry H in the IR.

Consider two different G's and H's, which both contain a complex NGB charged under (a U(1) subgroup of) H

$$G_1 = SU(2);\ H_1 = U(1)$$

$$|\partial_\mu \phi|^2 - \frac{1}{3f^2}\left|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right|^2 + \frac{8}{45f^4}\left|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right|^2|\phi|^2$$
$$-\frac{16}{315f^6}\left|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right|^2|\phi|^4 + \cdots\ ,$$

$$G_2 = SU(5);\ H_2 = SO(5)$$

$$|\partial_\mu \Phi|^2 - \frac{1}{48f^2}\left|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*\right|^2 + \frac{1}{1440f^4}\left|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*\right|^2|\Phi|^2$$
$$-\frac{1}{80640f^6}\left|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*\right|^2|\Phi|^4 + \cdots\ ,$$

Indeed, the NGB effective interactions look different.

The conventional wisdom from the last half-a-century:

- SSB occurs when the ground state is not invariant under the full symmetry of the system.
- Nambu-Goldstone modes are long wavelength, “gapless” excitations over the degenerate ground states.
- NGBs are “derivatively coupled,” due to a shift symmetry.
- Effective interactions of NGB are dependent on both the full symmetry group G in the UV and the unbroken group H in the IR.

- CCWZ has been applied to ChPT to very high orders. Some heroic efforts went into constructing the effective Lagrangian up to O(p^8), which is completed only recently:

The Mesonic Chiral Lagrangian of Order p^6 *

Johan Bijnens

Dept. of Theor. Phys. 2, Lund University,
Sölvegatan 14A, S-22362 Lund, Sweden

Gilberto Colangelo

Inst. Theor. Physik, Univ. Zürich, Winterthurerstr. 190,
CH-8057 Zürich-Irchel, Switzerland

Gerhard Ecker

Inst. Theor. Phys., Univ. Wien, Boltzmanng. 5,
A-1090 Wien, Austria

Abstract: We construct the effective chiral Lagrangian for chiral perturbation theory in the mesonic even-intrinsic-parity sector at order p^6. The Lagrangian contains 112 in principle measurable + 3 contact terms for the general case of n light flavours, 90+4 for three and 53+4 for two flavours. The equivalence between equations of motion and field redefinitions to remove spurious terms in the Lagrangians is shown to all orders in the chiral expansion. We also discuss and implement other methods for reducing the number of terms to a minimal set.

- CCWZ has been applied to ChPT to very high orders. Some heroic efforts went into constructing the effective Lagrangian up to $O(p^8)$, which is completed only recently:

The order p^8 mesonic chiral Lagrangian

Johan Bijnens, Nils Hermansson-Truedsson and Si Wang

Department of Astronomy and Theoretical Physics,
Lund University, Sölvegatan 14A, SE 223-62 Lund, Sweden

Abstract

We derive the chiral Lagrangian at next-to-next-to-next-to-leading order (NNNLO) for a general number N_f of light quark flavours as well as for $N_f = 2, 3$. We enumerate the contact terms separately. We also discuss the cases where some of the external fields are not included. An example of a choice of Lagrangian is given in the supplementary material.

- CCWZ has been applied to ChPT to very high orders. Some heroic efforts went into constructing the effective Lagrangian up to O(p^8), which is completed only recently:

-ph] 24 Jan 2019

The order p^8 mesonic chiral Lagrangian

	N_f		$N_f = 3$		$N_f = 2$	
	Total	Contact	Total	Contact	Total	Contact
p^2	2	0	2	0	2	0
p^4	13	2	12	2	10	3
p^6	115	3	94	4	56	4
p^8	1862	22	1254	21	475	23

Table 3: Number of monomials in the minimal basis for the case with all external fields included. Also listed is how many of them are contact terms. Our results agree with the known ones for p^2, p^4, p^6.

contact terms separately. We also discuss the cases where some of the external fields are not included. An example of a choice of Lagrangian is given in the supplementary material.

In spite of the tremendous success of CCWZ in a wide range of topics, there's something odd about it, in my view.

Its best summarized as the following question:

Nambu-Goldstone bosons are long-range degrees of freedom interpolating different vacua, why would its interactions know anything about the "broken group" G in the UV?

In other words, NGB's should be all about the IR physics, not the UV.

Indeed, NGB's have a very peculiar IR property that has been known for (again) more than half-a-century.

First let's talk about something that is not usually emphasized in the textbook. (One exception is Weinberg's QFT Vol. 2.)

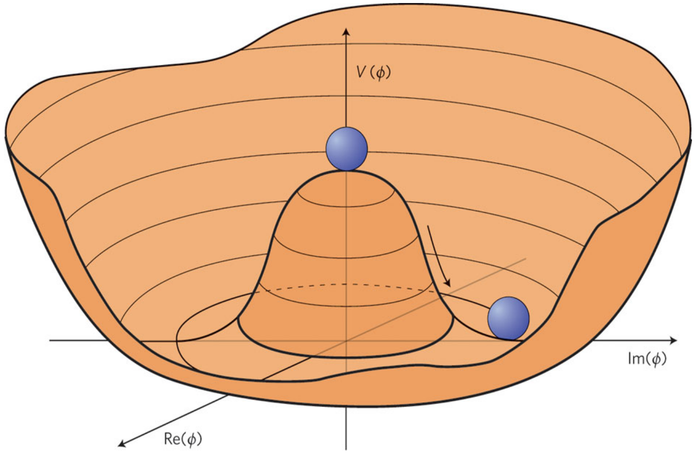
Recall the ground state is characterized by

$$|\mathrm{VAC}\rangle = \{|0\rangle_\alpha;\ \alpha \in [0, 2\pi)\}$$

Now let's bring in Quantum Mechanics...

$$|\tilde{0}\rangle = \int \frac{d\alpha}{2\pi} |0\rangle_\alpha \ , \qquad R(\theta)|\tilde{0}\rangle = |\tilde{0}\rangle$$

This superposition of alpha-state is invariant under rotation. Why couldn't it be the "ground state"?



This is because an important ingredient for SSB to occur is the "superselection rule",

$$ {}_{\alpha}\langle 0|\mathcal{O}|0\rangle_{\alpha'} = 0 $$

for any Hermitian local operator "O".

Then a stable ground state must carry a definite "alpha" and cannot be a superposition of alpha-states.

The superselection rule has an important implication for the S-matrix elements involving a soft pion:

$$ \lim_{p^{\mu}\to 0} \, {}_{\alpha}\langle f|i+\pi(p)\rangle_{\alpha} $$

Recall in quantum mechanics that a momentum eigenstate has the wave function:

$$|\vec{k}\rangle = e^{ik\cdot x}$$

Then a zero-momentum eigenstate has a constant wave function.

Since the pion interpolates between different vacua, a zero-momentum pion flips the direction of the ground state *uniformly* in the system:

$$\lim_{p^\mu \to 0} \pi(p)|0\rangle_\alpha \sim |0\rangle_{\alpha'}$$

This implies

$$\lim_{p^\mu \to 0} |i + \pi(p)\rangle_\alpha \ \sim \ |i\rangle_{\alpha'}$$

The superselection rule then tells us

$$\lim_{p^\mu \to 0} \ {}_\alpha\langle f|i + \pi(p)\rangle_\alpha \ \sim \ {}_\alpha\langle f|i\rangle_{\alpha'} = 0$$

IL and Z. Yin: 1804.08629

This IR property of the NGB scattering amplitudes was first derived in the context of pions in low-energy QCD by Adler in 1960's.

It is now known as the **Adler's zero condition**.

More importantly, it is a universal behavior of NGBs, independent of the symmetry breaking pattern G/H.

In QFT, one can show that the Adler's zero condition,

$$\lim_{p^\mu \to 0} {}_\alpha\langle f | i + \pi(p) \rangle_\alpha = 0$$

follows directly from the shift symmetry acting on the NGB:

$$\pi \to \pi + \epsilon + \cdots$$

This is hardly surprising, as the shift symmetry is an indication of the existence of other degenerate ground states!

IL: 1512.01232

It turns out that the Adler's zero condition allows for an entirely on-shell formulation of ChPT.

It all started from an obscure paper by Susskind and Frye from (again) half-a-century ago:

PHYSICAL REVIEW D VOLUME 1, NUMBER 6 15 MARCH 1970

Algebraic Aspects of Pionic Duality Diagrams

LEONARD SUSSKIND* AND GRAHAM FRYE

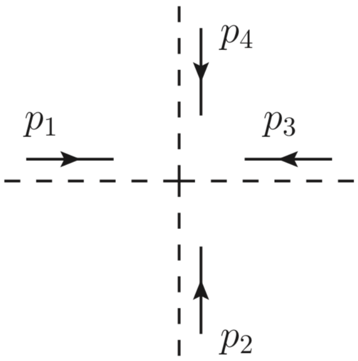
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

(Received 9 May 1969)

Certain algebraic aspects are abstracted from the duality principle and are incorporated in a simple model of pion n-point functions. An algorithm for constructing the n-point function in the tree-graph approximation is based on the duality assumption and the Adler condition which states that the amplitudes vanishes if any pion four-momentum vanishes, all others remaining on shell. The resulting amplitudes satisfy the constraints of current algebra and partial conservation of axial-vector current for $n=4$, 6, and 8, and (we conjecture) for all n. In addition, duality specifies a definite form for chiral symmetry breaking.

This is what they did, schematically.

Use the Adler's zero condition to fix the 4-pt amplitude of pions,



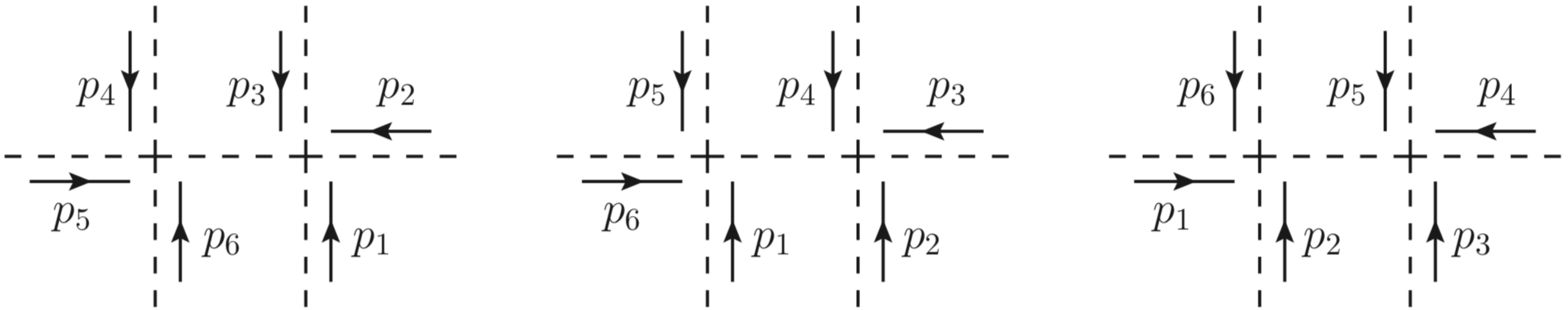
$$p_i^2 = 0 \; ; \sum_{i=1}^{4} p_i = 0$$

$$M_4(p_1, p_2, p_3, p_4) = c\frac{p_2 \cdot p_4}{f^2} = c\frac{p_1 \cdot p_3}{f^2}$$

Some comments:

- They worked directly with "flavor-ordered" amplitudes.
- There is no constant term in the amplitude.
- "f" is a dimensionful parameter, while "c" is an arbitrary number, which could be absorbed into the normalization of "f".

Once we have the 4-pt amplitude, we can build up the 6-pt amplitude from the 4-pt amplitude:

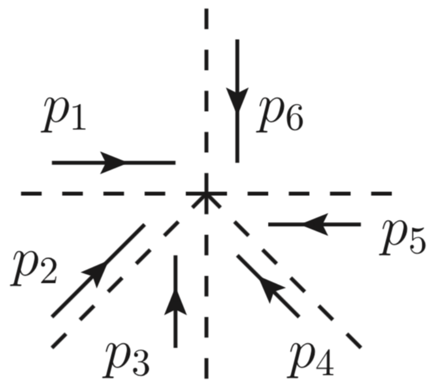


$$\frac{1}{f^2}\left(\frac{s_{13}s_{46}}{P_{123}^2}+\frac{s_{24}s_{15}}{P_{234}^2}+\frac{s_{35}s_{26}}{P_{345}^2}\right)$$

$$s_{ij}=(p_i+p_j)^2$$
$$P_{ijk}^2=(p_i+p_j+p_k)^2$$

This expression doesn't satisfy the Adler's zero condition!

The resolution is to introduce an additional contribution, the “contact interaction,”



It turns out imposing the Adler’s condition also uniquely fixes this 6-pt contact interaction:

$$M_6 = \frac{1}{f^2}\left(\frac{s_{13}s_{46}}{P_{123}^2} + \frac{s_{24}s_{15}}{P_{234}^2} + \frac{s_{35}s_{26}}{P_{345}^2}\right) - \frac{1}{f^2}P_{135}^2$$

Susskind and Frye constructed up to 8-pt amplitudes this way, and conjectured that this can be extended to arbitrary multiplicity "n".

This is quite striking because they only invoked

- A notion of "flavor ordering," which arises due to some discrete quantum numbers, given by the "unbroken group" H.

- The vanishing "soft limit" in the scattering amplitudes.

In other words, only IR data are used; They made no reference to the group "G" in the UV.

How general is this approach?

The answer is affirmative, but didn't arrive until much later, when the S-matrix theory strikes back!

The Rise of Modern S-Matrix Theory

As a motivation for developing the techniques of modern EFT, Weinberg asked: (2101.04241)

"...the question naturally arose, is there a way of avoiding the machinery of current algebra by just writing down a field theory that would automatically produce the same results with much greater ease and perhaps physical clarity?"

Similarly, modern S-matrix programmers asked:

"...the question naturally arose, is there a way of avoiding the machinery of quantum field theories by just writing down a simple set of algorithms that would automatically produce the same results with much greater ease and perhaps physical clarity?"

- On-shell Soft Bootstrap

What Susskind and Frye did was the precursor to modern S-matrix program of "Soft Bootstrap,"

For NGB's, all interaction vertices are determined by recursively requiring Adler's zero on tree-level amplitudes.

In particular, the progress is based on the "soft recursion relation" proposed by Cheung, Kampf, Novotny and Trnka in 1412.4095 and 1509.03309.

There's a parallel algebraic approach seeking to reconstruct the CCWZ Lagrangian using only IR data, by recursively imposing the shift symmetry. (IL: 1412.2145; 1412.2146; IL and Z. Yin: 1804.08629)

Soft recursion -- an all-leg shift in external momenta:

$$p_i \to \hat{p}_i = (1 - a_i z)\, p_i \qquad \sum_{i=1}^{n} a_i\, p_i^{\mu} = 0$$

Taking z → $1/a_i$ is equivalent to taking the soft limit of p_i. Then

$$\oint \frac{dz}{z} \frac{\hat{M}_n(z)}{F_n(z)} = 0 \qquad F_n(z) \equiv \prod_{i=1}^{n} (1 - a_i z)$$

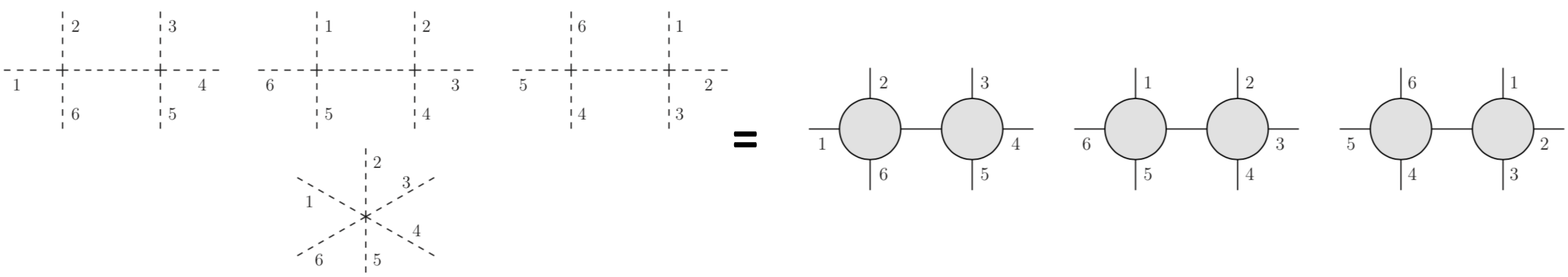
- The integrand vanishes like $1/z^{n-1}$ and the residue at infinity vanishes.
- There is no pole at $z = 1/a_i$ because of Adler's zero condition.
- The only poles are at $z = 0$ and when the internal propagators go on-shell (ie factorization channel).

Internal propagators go on-shell at

$$\hat{P}_I^2(z_I^{\pm}) = 0 \qquad\qquad \hat{P}_I(z) = \sum_{i\in I} p_i - z\left(\sum_{i\in I} a_i p_i\right)$$

Cauchy's theorem then gives

$$M_n = \hat{M}_n(0) = -\sum_{I,\pm} \frac{1}{P_I^2} \frac{\hat{M}_L^{(I)}(z_I^{\pm})\hat{M}_R^{(I)}(z_I^{\pm})}{F_n(z_I^{\pm})(1 - z_I^{\pm}/z_I^{\mp})}$$



LHS includes contact term.

RHS includes only factorization channel!

A comment on the all-leg-shift:

$$p_i \to \hat{p}_i = (1 - a_i z)\, p_i \qquad\qquad \sum_{i=1}^{n} a_i\, p_i^{\mu} = 0$$

- Nontrivial solutions for a_i don't always exist.

- For D=4, the number of non-trivial solutions is (n-5), n=number of external momenta.

- The general solution is only defined "projectively" and has a "shift symmetry":

$$\{a_i\} = \sum_{r=1}^{n-5} A^{(r)} \{a_i^{(r)}\} + B$$

$A^{(r)}$ and B are arbitrary constants!

When soft-bootstrapping the amplitudes,

$$M_n = \hat{M}_n(0) = -\sum_{I,\pm} \frac{1}{P_I^2} \frac{\hat{M}_L^{(I)}(z_I^\pm)\hat{M}_R^{(I)}(z_I^\pm)}{F_n(z_I^\pm)(1 - z_I^\pm/z_I^\mp)}$$

it is a non-trivial check that the outcome is independent of $\boldsymbol{A^{(r)}}$ and $\boldsymbol{B}$.

We will see that this doesn't automatically happen. Thus we define a consistent EFT in soft bootstrap when

$\boxed{\text{\textit{The amplitude } } M_n \text{\textit{ obtained from the soft recursion relation is independent of the arbitrary constants }} A^{(r)} \text{\textit{ and }} B \text{\textit{ for all }} n.}$

Soft-recursion relation allows one to generalize Susskind and Frye, to all orders in the multiplicity “n”.

But the discussion so far has been confined to the leading two-derivative operators.

In ChPT and/or a general EFT of NGB’s

- There exist higher derivative operators which become increasingly important as the energy becomes higher.
- Each higher derivative operator comes with an uncalculable Wilson coefficient, called the low-energy constant in ChPT.
- There also exists topological operators, the Wess-Zumino-Witten terms, at higher derivatives.
- There are more complicated “flavor structure,” beyond the single trace operator, at higher derivatives.

How does on-shell soft bootstrap incorporate these features?

Introducing "Soft blocks"

- A soft block $\mathcal{S}^{(k)}(p_1, \cdots, p_n)$ is a contact interaction carrying n scalars and k derivatives that satisfies the Adler's zero condition when all external legs are on-shell.

For four-derivative or less, one can show the soft blocks exist only for 4-pt and 5-pt contact terms.

IL and Z. Yin: 1904.12859

- Soft blocks are "seeds" for soft recursion relations.
- Soft blocks are in 1-to-1 correspondence with the number of independent operators at a particular order in the derivative expansion.
- Practically speaking, each soft block represents the lowest order interaction vertex from a particular higher-derivative operator.

Introducing “Soft blocks”

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For four-derivative or less, one can show the soft blocks exist only for 4-pt and 5-pt contact terms.

IL and Z. Yin: 1904.12859

At O(p^2), there are two “flavor-ordered“ 4-pt soft blocks:

Single-trace

$$\mathcal{S}^{(2)}(1,2,3,4) = c_0\,\frac{s_{13}}{f^2}$$

Invariant under cyclic Permutations of (1234)

Double-trace

$$\mathcal{S}^{(2)}(1,2|3,4) = \frac{d_0}{f^2} s_{12}$$

Invariant under separate cyclic Permutations of (12) and (34)

At this order in derivative expansion, one can construct two EFT's using these two soft blocks, separately.

At 6-pt amplitudes,

Single-trace

$$M^{(2)}(1,2,3,4,5,6) = -\frac{c_0^2}{f^4}\left[\frac{s_{13}s_{46}}{P_{123}^2} + \frac{s_{24}s_{15}}{P_{234}^2} + \frac{s_{35}s_{26}}{P_{345}^2} - P_{135}^2\right]$$

- Invariant under cyclic permutations of (123456).
- Coincides with pions amplitudes worked out by Susskind and Frye.

Multi-trace

$$M^{(2)}(1,2|3,4|5,6) = -\frac{d_0^2}{f^4}\left[s_{12}s_{56}\left(\frac{1}{P_{124}^2} + \frac{1}{P_{123}^2}\right) + s_{12}s_{34}\left(\frac{1}{P_{125}^2} + \frac{1}{P_{126}^2}\right)\right.$$
$$\left. + s_{34}s_{56}\left(\frac{1}{P_{134}^2} + \frac{1}{P_{234}^2}\right) - s_{12} - s_{34} - s_{56}\right] ,$$

- At 6-pt this is a "triple-trace" amplitudes.
- Not previously known in amplitudes community.

Can there be a "mixed" EFT involving both single-trace and double-trace soft blocks at $O(p^2)$?

The answer is NO, as the resulting amplitude is dependent on the arbitrary coefficient *B* in the general solutions for a_i :

$$M^{(2),c}(1,2|3,4,5,6) = -\frac{c_0 d_0}{f^4}\left(\frac{32}{9} - \frac{100}{3}B - \frac{100}{3}B^2 - \frac{275}{36}B^3\right)$$

The two EFT's from single- and double-trace soft blocks are mutually exclusive!

Single-trace

$$\mathcal{S}^{(2)}(1,2,3,4) = c_0\,\frac{s_{13}}{f^2}$$

NGBs in the adjoint of SU(N) = SU(N)xSU(N)/SU(N)

Double-trace

$$\mathcal{S}^{(2)}(1,2|3,4) = \frac{d_0}{f^2}s_{12}$$

NGBs in the fundamental of SO(N) = SO(N+1)/SO(N)

4-pt Soft blocks at O(p^4):

$$\text{Single-trace:} \quad \mathcal{S}_1^{(4)}(1,2,3,4) = \frac{c_1}{\Lambda^2 f^2}\, s_{13}^2 \,, \qquad \mathcal{S}_2^{(4)}(1,2,3,4) = \frac{c_2}{\Lambda^2 f^2}\, s_{12} s_{23} \,,$$

$$\text{Double-trace:} \quad \mathcal{S}_1^{(4)}(1,2|3,4) = \frac{d_1}{\Lambda^2 f^2}\, s_{12}^2 \,, \qquad \mathcal{S}_2^{(4)}(1,2|3,4) = \frac{d_1}{\Lambda^2 f^2}\, s_{13} s_{23} \,,$$

All four soft blocks appear in ChPT, where the NGB transforms as the adjoint of the unbroken SU(N) group:

$$\mathcal{L}^{(2)} = \frac{f^2}{2} \mathrm{tr}\left(d_\mu d^\mu\right)$$

$$\mathcal{L}^{(4)} = \sum_{i=1}^{4} L_{4,i}\, O_i$$

$$\begin{aligned} O_1 &= \left[\mathrm{tr}(d_\mu d^\mu)\right]^2 \,, \\ O_2 &= \left[\mathrm{tr}(d_\mu d_\nu)\right]^2 \,, \\ O_3 &= \mathrm{tr}([d_\mu, d_\nu]^2) \,, \\ O_4 &= \mathrm{tr}(\{d_\mu, d_\nu\}^2) \,, \end{aligned}$$

$$\boxed{c_1 = L_{4,3} + 3L_{4,4}, \quad c_2 = 2(L_{4,3} - L_{4,4}), \quad d_1 \;=\; 2L_{4,1} + L_{4,2}, \quad d_2 = 2L_{4,2}.}$$

For the fundamental of SO(N), only two operators exist at O(p^4):

O(p^2) $$\mathcal{S}^{(2)}(1,2|3,4) = \frac{d_0}{f^2} s_{12}$$

O(p^4) $$\mathcal{S}_1^{(4)}(1,2|3,4) = \frac{d_1}{\Lambda^2 f^2}\, s_{12}^2\ , \qquad \mathcal{S}_2^{(4)}(1,2|3,4) = \frac{d_1}{\Lambda^2 f^2}\, s_{13} s_{23}\ ,$$

These soft blocks generate all tree-amplitudes from the symmetry breaking pattern SO(N+1)/SO(N) up to O(p^4).

At $O(p^4)$ and n=5, there is only 1 soft block,

$$\mathcal{S}_-^{(4)}(1,2,3,4,5) = \frac{c_-}{\Lambda^2 f^3}\, \varepsilon(1234)\ , \qquad \varepsilon(ijkl) \equiv \varepsilon_{\mu\nu\rho\sigma} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma$$

Clearly, this corresponds to the "parity-odd" WZW term. It is NOT invariant under the shift symmetry. Instead, it varies by a total derivative.

A few comments:

- The WZW soft blocks is non-zero only when the number of flavors N_f >=5, due to Bose symmetry.
- When introducing WZW soft block to SU(N>2) EFT, all amplitudes are consistent.
- When introducing WZW soft block to SO(N) EFT, we can't get a sensible solution for the soft-bootstrapped 7-pt amplitude; the resulting EFT is *inconsistent*.

IL and Z. Yin: 1904.12859

However, there is a subtlety for N_f=5 in SO(N) EFT.

In this case three of the 7 external particles must have identical flavors

→ Need to symmetrize the 7-pt amplitude with respect to these three particles due to Bose symmetry.

Miraculously, after symmetrization a consistent 7-pt amplitude now emerges!

In the end,

- For the adjoin of SU(N), the WZW term exists for $N_f > 3$.
  Since $N_f = N^2 - 1$. → $N >= 3$.
- For the fundamental of SO(N), the WZW is absent except for $N_f = 5$.
  Since $N_f = N$ → WZW only for SO(5) fundamental!

IL and Z. Yin: 1904.12859

These results agree completely with those from group-theoretic considerations based on the 5th de Rham cohomology group!

D'Hoker and Weinberg: hep-ph/9409402;
D'Hoker: hep-th/9502162

Mystery:

How do (Adler's zero + Bose symmetry) know about the 5th de Rham cohomology of the G/H coset?

Let's pause and reflect on what we have learned:

Interactions of NGB can be deduced entirely from IR data (Adler's zero + IR quantum numbers), without reference to the broken group G.

An important corollary:

Self-interactions of NGBs with identical IR quantum numbers must be *universal*, independent of the G/H coset.

The universality has important implications for composite Higgs models where the Higgs boson arises as a pseudo-NGB. (D. Liu, IL and Z, Yin: 1805.00489; 1809.09126)

This universality can be checked explicitly:

Going back to the earlier example of a complex NGB charged under the U(1) (sub)group of H:

$$G_1 = SU(2);\ \ H_1 = U(1)$$

$$|\partial_\mu \phi|^2 - \frac{1}{3f^2}\left|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right|^2 + \frac{8}{45f^4}\left|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right|^2|\phi|^2$$
$$-\frac{16}{315f^6}\left|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right|^2|\phi|^4 + \cdots\ ,$$

$$G_2 = SU(5);\ \ H_2 = SO(5)$$

$$|\partial_\mu \Phi|^2 - \frac{1}{48f^2}\left|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*\right|^2 + \frac{1}{1440f^4}\left|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*\right|^2|\Phi|^2$$
$$-\frac{1}{80640f^6}\left|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*\right|^2|\Phi|^4 + \cdots\ ,$$

They have identical IR quantum numbers (under U(1) group), and the self-interactions should be universal!

This universality can be checked explicitly:

Going back to the earlier example of a complex NGB charged under the U(1) (sub)group of H:

$$G_1 = SU(2);\ \ H_1 = U(1)$$

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$$-\frac{16}{315f^6}\left|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right|^2|\phi|^4 + \cdots\ ,$$

$$G_2 = SU(5);\ \ H_2 = SO(5)$$

$$|\partial_\mu \Phi|^2 - \frac{1}{48f^2}\left|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*\right|^2 + \frac{1}{1440f^4}\left|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*\right|^2|\Phi|^2$$
$$-\frac{1}{80640f^6}\left|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*\right|^2|\Phi|^4 + \cdots\ ,$$

Indeed, if we make f → 4f in the first case, the two Lagrangians are identical!

Now let's apply the on-shell method to counting operators in ChPT:

- As is well-known, construction of independent operators in EFT is notoriously difficult, due to the complicated operator relations such as integration-by-parts, equation-of-motion and etc.

  For example, in ChPT the leading-order E.O.M. is

$$\nabla_\mu d^\mu = 0 \qquad\qquad \nabla_\mu d_\nu \equiv \partial_\mu d_\nu + i[E_\mu, d_\nu]$$

- Making things worse, there are additional relations imposed by the "symmetry" of the coset in ChPT.

$$\nabla_{[\mu} d_{\nu]} = 0 \ , \qquad E_{\mu\nu} \ \equiv \ -i[\nabla_\mu, \nabla_\nu] = -i[d_\mu, d_\nu]$$

- It turns out that these complicated operator relations manifest themselves trivially in soft blocks.

  a) Integration-by-parts = total momentum conservation
  b) Equation-of-motion = on-shell conditions for external momenta
  c) Symmetry relations are automatically incorporated in soft-bootstrap.

In the end, soft blocks are an efficient way to count the number of independent operators at each order in the derivative expansion!

Moreover, all tree amplitudes can be generated recursively once the soft blocks are obtained.

- We enumerated all soft blocks in ChPT up to $O(p^{10})$, which correspond to pure mesonic operators in ChPT, ie turning off spin-1/2 and spin-1 fields.
- For simplicity we work in general spacetime dimension D and general flavor N_f :

	$\mathcal{O}(p^4)$	$\mathcal{O}(p^6)$	$\mathcal{O}(p^8)$	$\mathcal{O}(p^{10})$
No. of Soft Blocks	4	19	135	1451

TABLE VI: A summary table for the number of soft blocks, including all trace structures.

Dai, IL, Mehen and Mohapatra: 2009.01819

- The outcome agree with existing literature up to $O(p^8)$ and makes a prediction for $O(p^{10})$.

There are other mysteries waiting to be explored for ChPT.

Ex1: Subleading single-soft theorem:

- In both QED and Gravity, S-matrix elements factorize universally:

$$\lim_{\tau\to 0} M_{n+1}(p_1,\cdots,p_n;\tau p_{n+1}) = \left(\frac{1}{\tau} + \tau^0 + \cdots\right) M_n(p_1,\cdots,p_n)$$

- For NGBs, the Adler's zero condition states:

$$\lim_{\tau\to 0} M_{n+1}(p_1,\cdots,p_n;\tau p_{n+1}) = (\tau + \cdots)\, M_n(p_1,\cdots,p_n)$$

However, in ChPT only even-point amplitudes exist due to parity.

What is M_n then ?

The subleading single soft limit in ChPT wasn't computed until a few years ago. Using the Cachazo-He-Yuan formulation of scattering equations.

Cachazo, Cha and Mizera found, for ChPT,

M_n = Scattering amplitudes of an extended theory containing cubic biadjoint scalars interacting with the pions!

1604.03893

The same result can be derived in QFT using Ward identity. (IL and Z. Yin: 1709.08639 and 1804.08629.)

What are the phenomenological implications of this observation?

Can we observe the extended theory experimentally?

Ex2: "double-copy" structure in ChPT:

- At the leading O(p^2), there's a "trivial" double-copy:

$$\text{NLSM}^{(2)} = \text{NLSM}^{(2)} \overset{\text{KLT}}{\otimes} \phi^3$$

$$\mathcal{L}_{\phi^3} = \frac{1}{2}\partial_\mu \phi^{a\tilde{a}} \partial^\mu \phi^{a\tilde{a}} - \frac{\lambda}{6}\phi^{a\tilde{a}}\phi^{b\tilde{b}}\phi^{c\tilde{c}} f^{abc} \tilde{f}^{\tilde{a}\tilde{b}\tilde{c}},$$

  This is a trivial relation because the Kawai-Lowellen-Tye (KLT) kernel is $[\phi^3]^{-1}$.

- Among the four O(p^4) operators, one of them has a non-trivial double-copy:

$$\text{NLSM}^{(4)} = \text{NLSM}^{(2)} \overset{\text{KLT}}{\otimes} \left(\text{YM} + \phi^3\right)$$

$$\mathcal{L}_{\text{YM}+\phi^3} = \tilde{\text{tr}}\left(\frac{1}{4}\underline{F}^{\mu\nu}\underline{F}_{\mu\nu} + \frac{1}{2}D_\mu \underline{\phi}^a D^\mu \underline{\phi}^a - \frac{g^2}{4}[\underline{\phi}^a, \underline{\phi}^b]^2\right) - \frac{\lambda}{6}\phi^{a\tilde{a}}\phi^{b\tilde{b}}\phi^{c\tilde{c}} f^{abc} \tilde{f}^{\tilde{a}\tilde{b}\tilde{c}}$$

Again what's the implications for ChPT?

IL, L. Rodina, Z. Yin: 2009.00008

Concluding Remarks:

- Interactions of NGB's can be determined (almost) entirely from the IR – using the Adler's zero condition as the defining property.

- The nonlinearity in the NGB interactions arises entirely from IR physics. What's being "broken" in the UV is irrelevant, for the most part.

- The complete effective Lagrangian for ChPT can be formulated in an entirely on-shell perspective, when external sources and fermions are neglected.

- What happens when we put back the photon and the nucleon in ChPT??