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"All Things EFT" Seminar — October 14, 2020

Factorization at Subleading Power and Endpoint Divergences in Soft-Collinear Effective Theory

Matthias Neubert

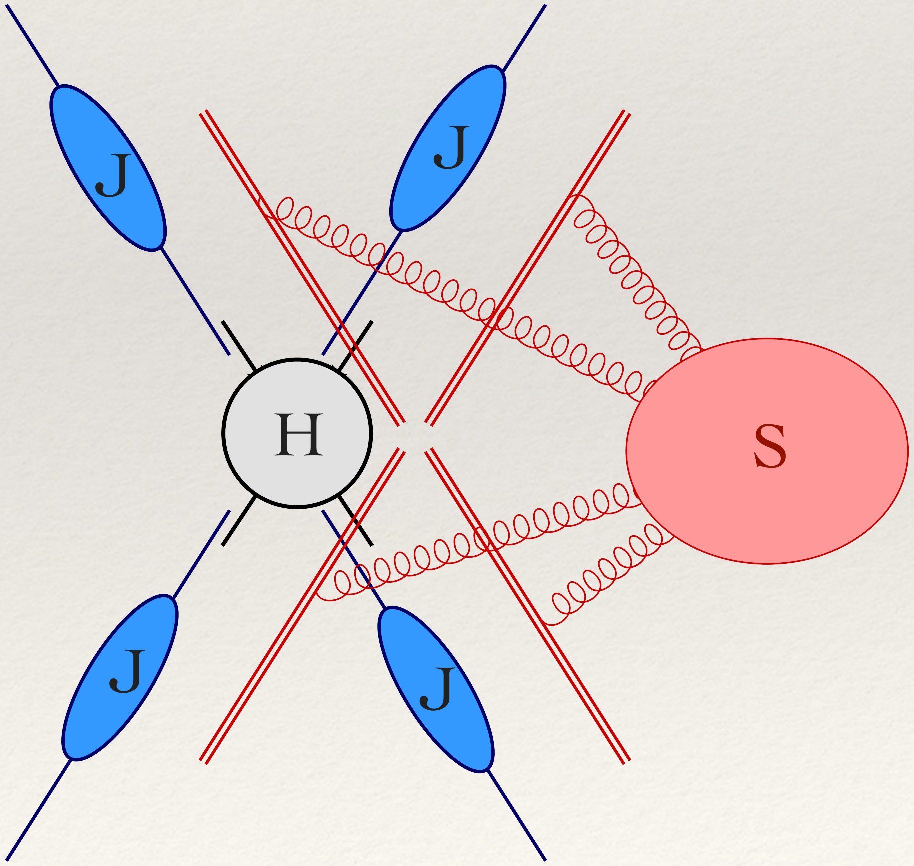
PRISMA+ Cluster of Excellence & MITP
Johannes Gutenberg University Mainz
& Cornell University & Univ. Zurich

Introduction

- Factorization of scales is a fundamental concept in HEP:
  - LHC cross section ~ $\sigma_{\text{parton}} \otimes$ PDFs
  - basis for the resummation of large logarithmic corrections

- **Soft-collinear effective theory** (SCET) provides a framework for studying factorization and resummation for processes involving light energetic particles using tools of effective field theory (EFT)

[Bauer et al. 2000, 2001; Beneke et al. 2002]



Introduction

- Conventional EFTs provide a systematic expansion in inverse powers of a large scale Q:

$$\mathcal{L}_{\text{eff}} = \sum_i C_i(Q,\mu)\,O_i(\mu) + \frac{1}{Q}\sum_j C_j^{(1)}(Q,\mu)\,O_j^{(1)}(\mu) + \frac{1}{Q^2}\sum_k C_k^{(2)}(Q,\mu)\,O_k^{(2)}(\mu) + \dots$$

- Examples: $\mathcal{H}_{\text{eff}}^{\text{weak}}$, χPT, HQET, NRQCD, SMEFT, …
- Extension to higher orders "straightforward if tedious"
    - χPT: 2, 12, 117, 1959, 45171, 1170086, … [Graf et al. 2020]
    - SMEFT: 12, 3045, 1542, 44807, 90456, 2092441, … [Henning, Lu, Melia, Murayama 2015]

Introduction

- SCET is more complicated in several aspects:
  - operators contain **non-local products of fields** (unavoidable consequence of $E \sim Q$ but $p^2 \ll Q^2$), need to introduce **Wilson lines** for gauge invariance
  - Wilson coefficients depend on large momentum components in addition to heavy masses of particles integrated out
  - fields are split up in **momentum modes** (method of regions):

[Beneke, Smirnov 1997]

$$\phi(x) \to \underbrace{\phi_{n_1}(x) + \phi_{n_2}(x) + \cdots}_{\substack{\text{collinear} \\ \text{(regions of large momentum flow)}}} + \underbrace{\phi_s(x) + \phi_{us}(x)}_{\text{soft}} + \dots$$

Introduction

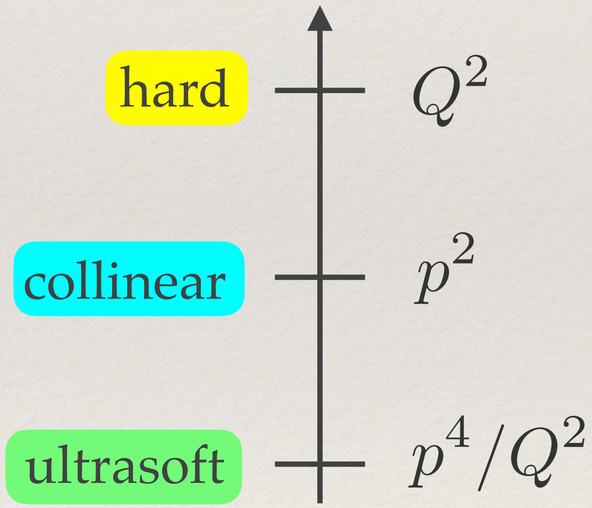
- SCET is more complicated in several aspects:
  - hard modes are integrated out
    (Wilson coefficients = hard matching coefficients)
  - different collinear sectors appear decoupled in the effective Lagrangian except for soft interactions
  - soft interactions can be decoupled by means of field redefinitions
    → **factorization theorems**
  - large logarithms can be resummed systematically by solving RGEs

Introduction

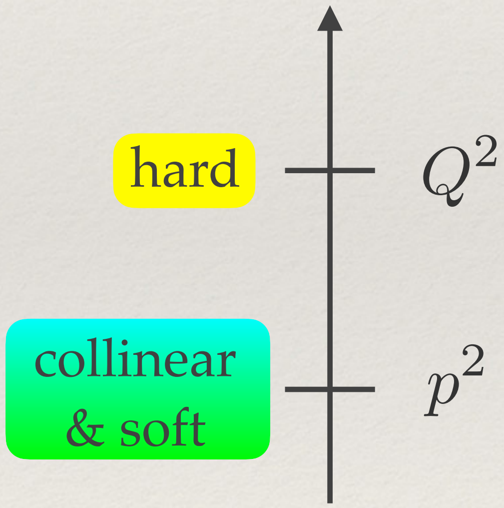
- Typical SCET factorization theorem:

$$\sigma \sim \underbrace{H}_{\text{hard}} \int \underbrace{J \otimes J}_{\text{collinear}} \otimes \underbrace{S}_{\text{soft}}$$

- Two common scale hierarchies:



SCET-1



SCET-2

In SCET-2 the product $J \otimes J \otimes S$ contains an extra dependence on Q^2 due to the **collinear anomaly**.

[Becher, MN 2010]

Introduction

- Examples:
  - threshold resummation for DIS, DY, Higgs, $t\bar{t}$ production, …
  - p_T resummation, jet vetoes, event shapes, jet substructure, …
  - electroweak Sudakov resummation
  - non-global logarithms, super-leading logarithms (ongoing work)
  - high-order structure of IR divergences of scattering amplitudes, subtractions methods for N^{n}LO fixed-order calculations (e.g. based on N-jettiness)

[many distinguished authors …; Becher, MN et al. 2006-2016]

Introduction

- Extension to next-to-leading power?
  - generically (all known examples), find **endpoint-divergent convolution integrals**! [Beneke et al., Moult et al., Stewart et al., MN et al. 2018-2020; …]
  - upsets scale separation and breaks factorization
  - prevents systematic resummation of large logarithms
  - failure of standard OPE based on dimensional regularization and $\overline{\text{MS}}$ subtractions
- Questions usefulness of entire SCET framework!
  - a hard problem; many groups world-wide work on this…

SCET 2020

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Bern, Switzerland
June 8 – 11, 2020

Cancelled due to the COVID-19 pandemic

XVIIth annual workshop on Soft-Collinear Effective Theory

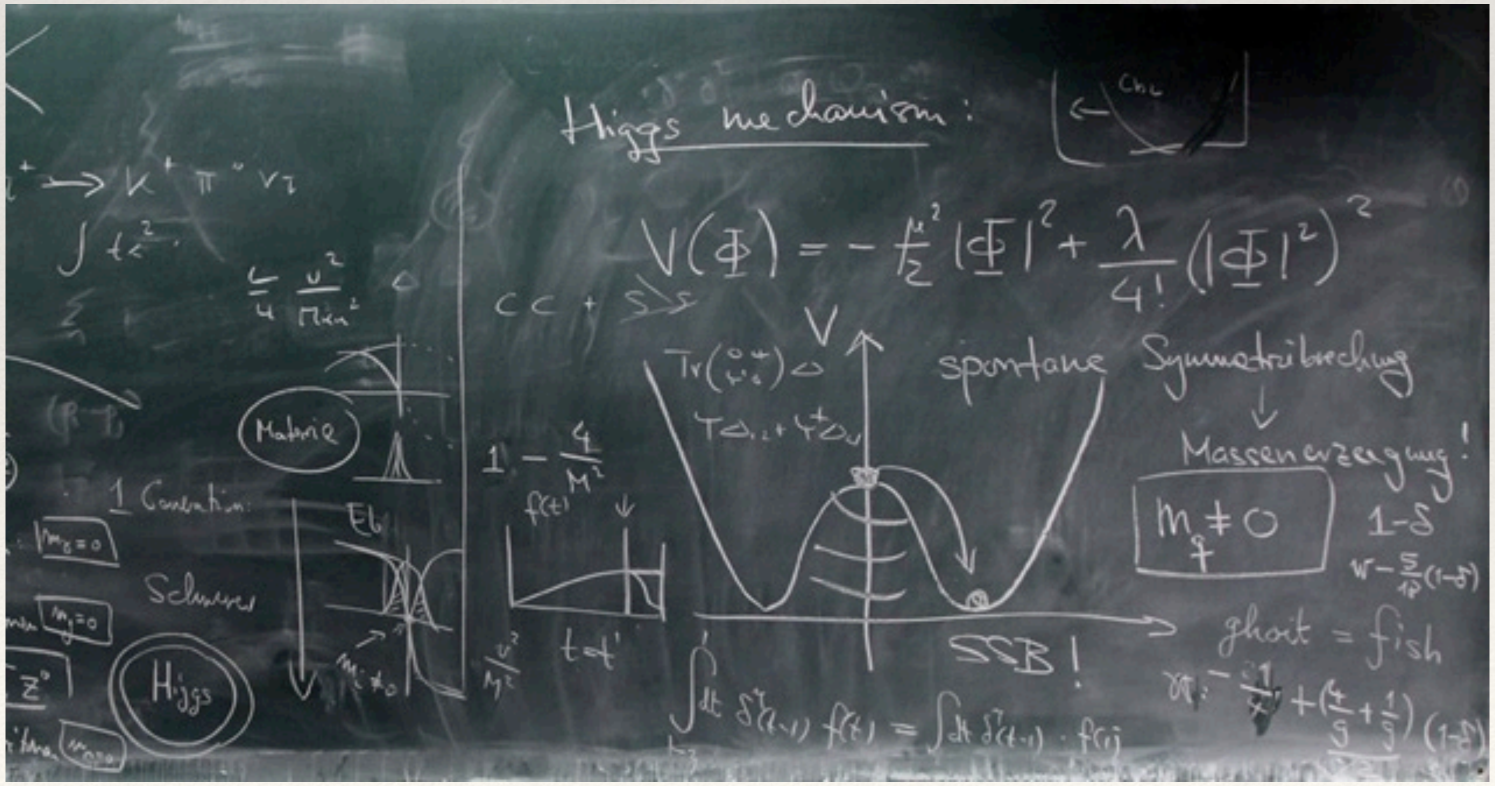
Organizers:

Thomas Becher, Christoph Greub, Thomas Rauh, Xiaofeng Xu, Marcel Balsiger, Samuel Favrod, Francesco Saturnino

http://scet.itp.unibe.ch/

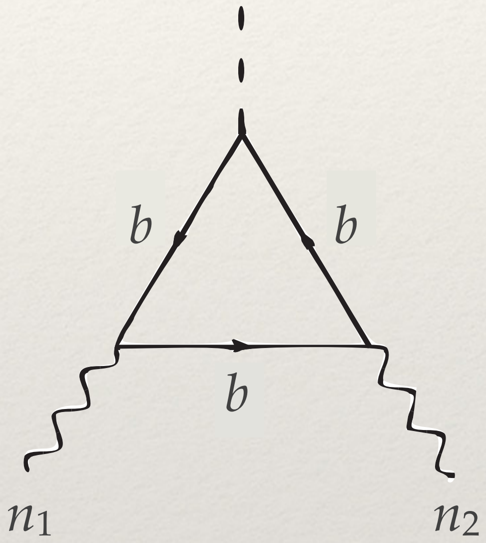
First SCET factorization theorem at subleading power

Liu, MN: 1912.08818 (JHEP)
Liu, Mecaj, MN, Wang: 2009.04456 & 2009.06779
Liu, MN: 2003.03393 (JHEP)
Liu, Mecaj, MN, Wang, Fleming: 2005.03013 (JHEP)



A subleading-power observable

- Consider b-quark induced contribution to $h \to \gamma\gamma$ decay amplitude (pseudo observable)
  - this and related $gg \to h$ process may be relevant for high-precision Higgs studies, but here are considered for academic purposes mainly
  - "sufficiently complicated but simple enough"

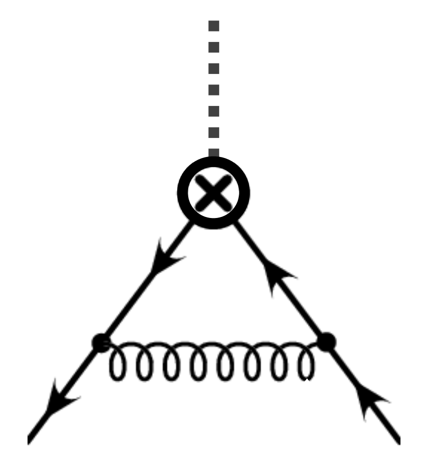


- Relevant modes are hard, collinear (n_1 and n_2) and soft, with SCET-2 scaling
- Scale hierarchy: $m_b^2 \ll M_h^2$

A subleading-power observable

- Same momentum regions appear in analysis of the Sudakov form factor (e.g. electroweak Sudakov resummation)
    - standard factorization theorem without endpoint divergences:

$$\boxed{\sigma \sim H \int J \otimes J \otimes S}$$



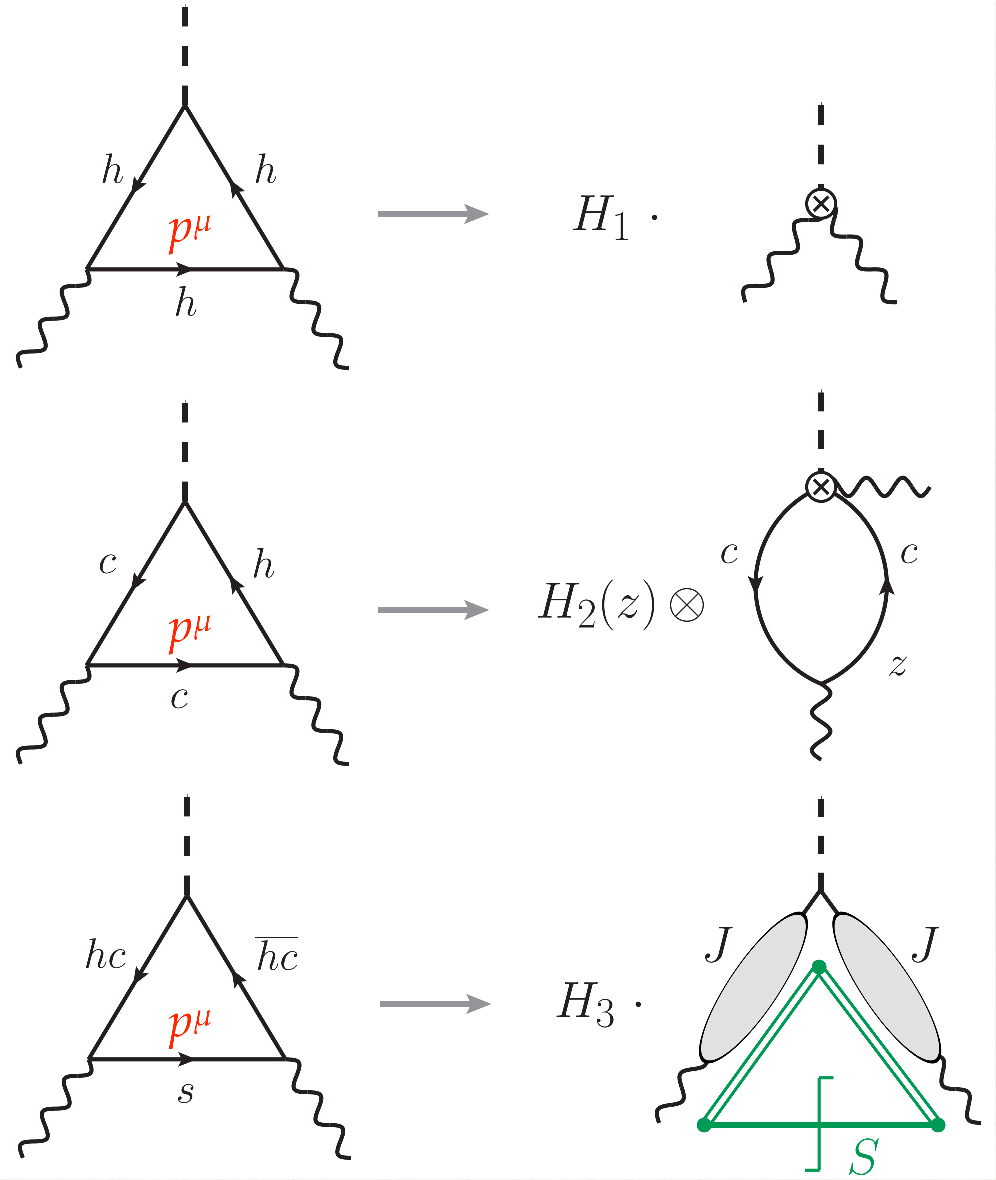
    - a single, leading-order SCET operator arises at $O(\lambda^2)$:

$$V_\mu \, \underset{\lambda}{\bar{\mathcal{X}}_{n_1}} \gamma_\perp^\mu \underset{\lambda}{\mathcal{X}_{n_2}}$$

    - crucial difference: **soft quark** can appear at subleading power

A subleading-power observable

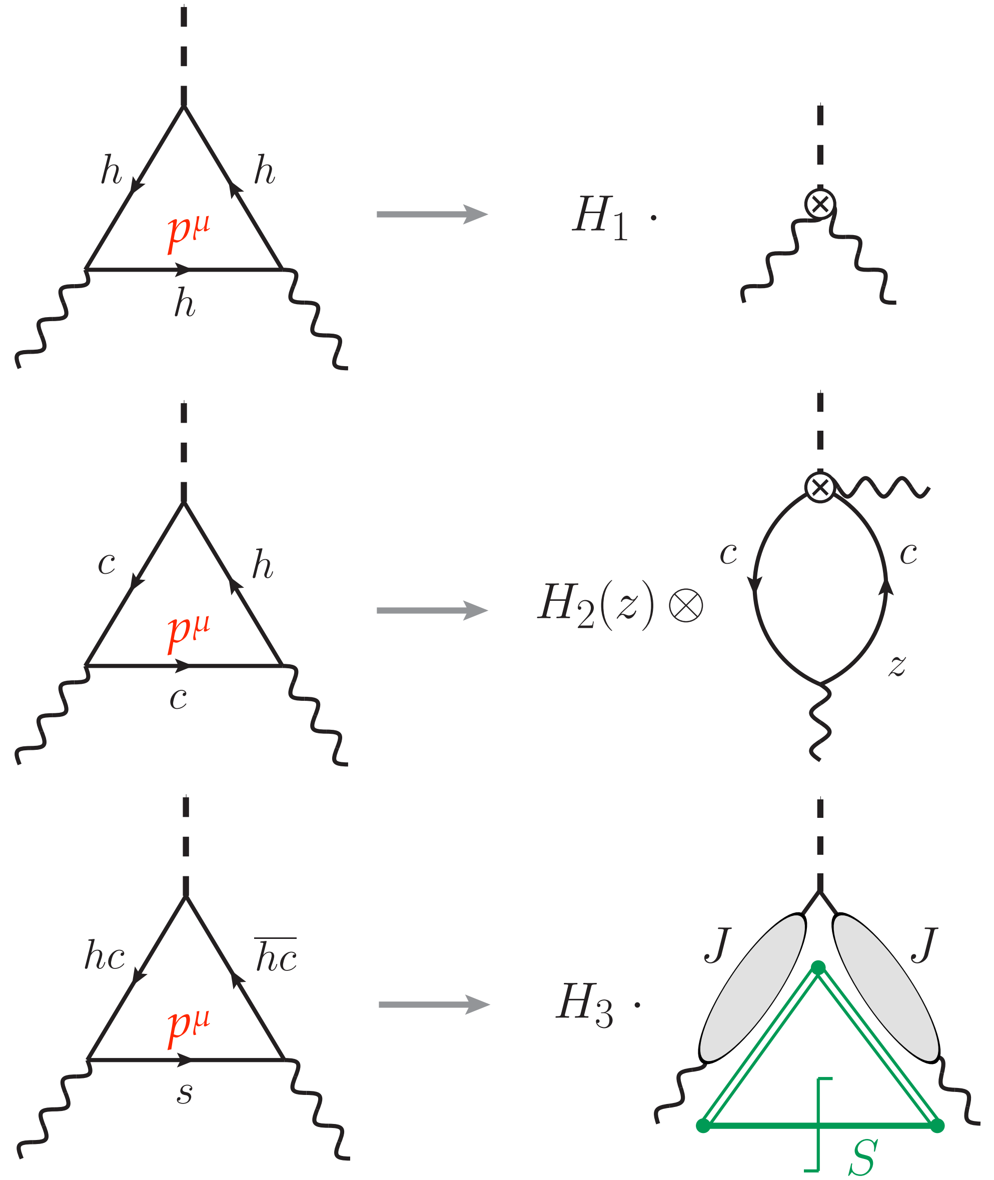
- Relevant momentum regions at 1-loop order:



$p^\mu \sim M_h$	hard
$p^\mu \parallel k_1\,, \quad p^2 \sim m_b^2$	n_1-collinear
$p^\mu \sim m_b$	soft

A subleading-power observable

- Relevant momentum regions at 1-loop order: $\lambda \sim m_b/M_h$

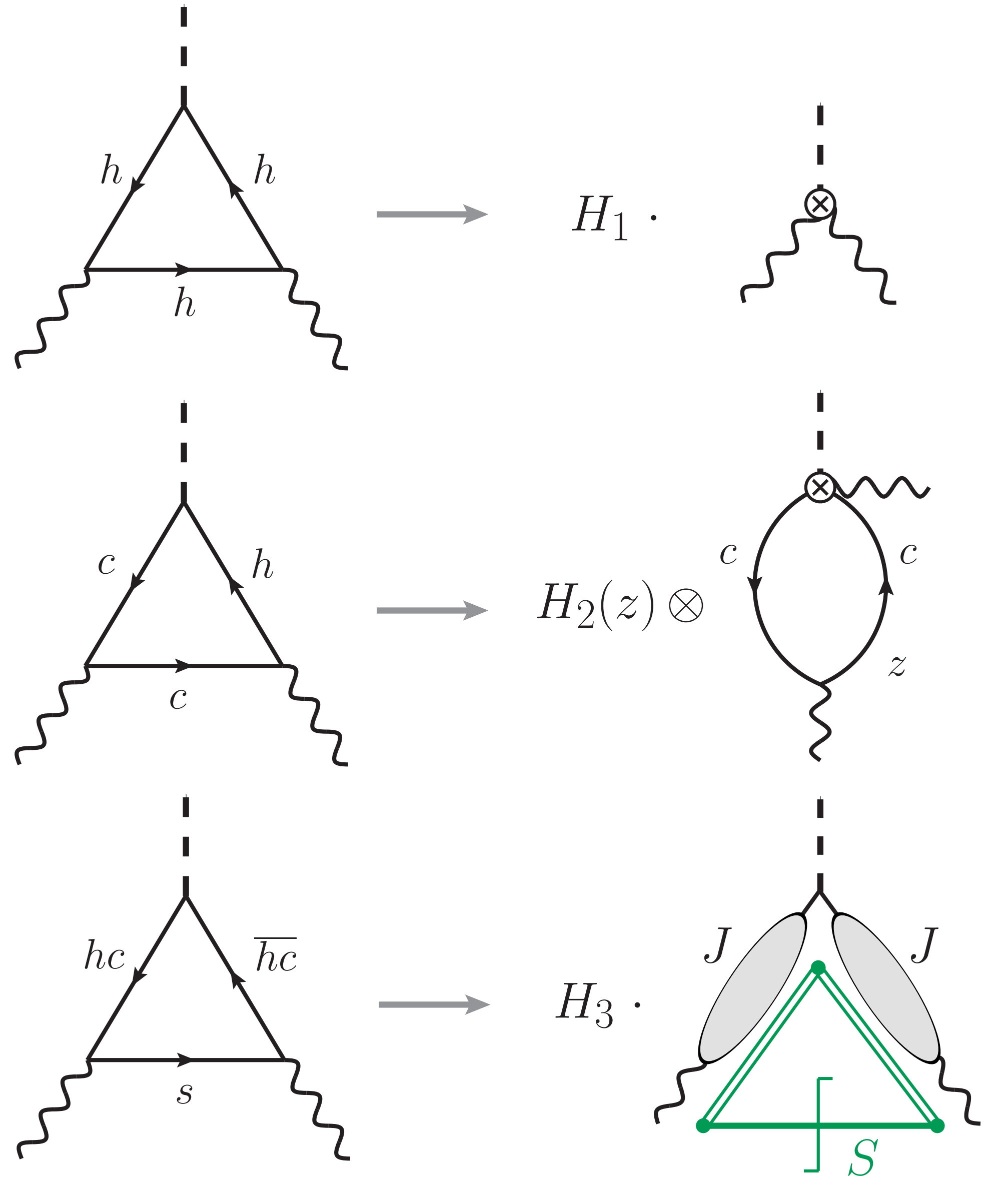


$p^\mu \sim M_h\,(1,1,1)$ $n_1\ n_2\ \perp$	hard
$p^\mu \sim M_h\,(1,\lambda^2,\lambda)$	n_1-collinear
$p^\mu \sim M_h\,(\lambda,\lambda,\lambda)$	soft

A subleading-power observable

- Corresponding SCET operators at $O(\lambda^3)$:

$\lambda \sim m_b/M_h$



$$O_1^{(0)} = \overset{\lambda}{\frac{m_b}{e_b^2}}\, h\, \overset{\lambda}{\mathcal{A}_{n_1}^{\perp\mu}}\, \overset{\lambda}{\mathcal{A}_{n_2,\mu}^{\perp}}$$

dressed collinear photon fields

$$O_2^{(0)}(z) = h \left[\overset{\lambda}{\bar{\mathcal{X}}_{n_1}} \gamma_\perp^\mu \frac{\bar{\not{n}}_1}{2}\, \delta(z\bar{n}_1 \cdot k_1 + i\bar{n}_1 \cdot \partial)\, \overset{\lambda}{\mathcal{X}_{n_1}} \right] \overset{\lambda}{\mathcal{A}_{n_2,\mu}^{\perp}}$$

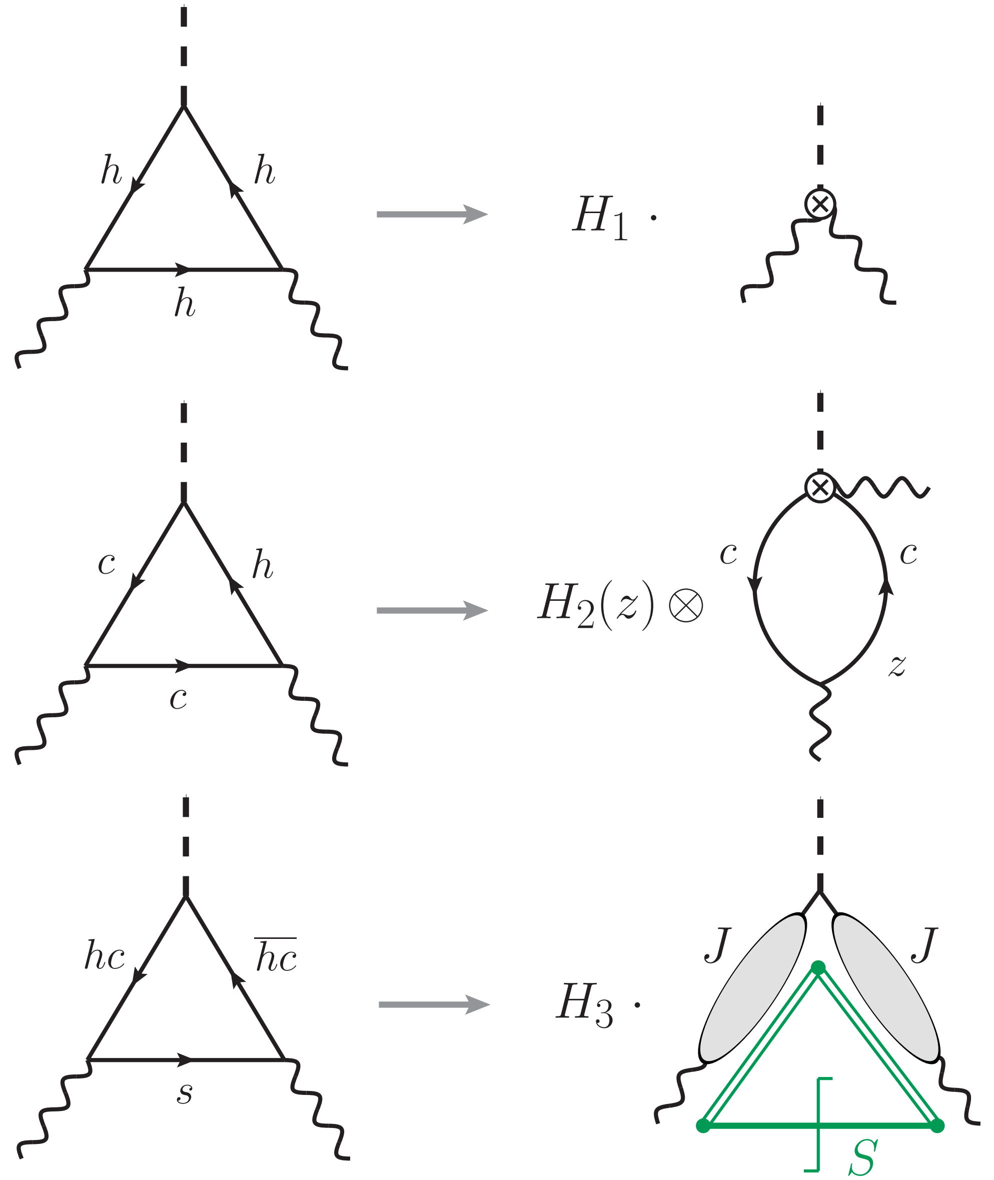
dressed collinear quark fields

$$O_3^{(0)} = T \left\{ h\, \overset{\lambda}{\bar{\mathcal{X}}_{n_1}} \overset{\lambda}{\mathcal{X}_{n_2}}, i \int d^D x\, \overset{\lambda^{1/2}}{\mathcal{L}_{q\,\xi_{n_1}}^{(1/2)}}(x), i \int d^D y\, \overset{\lambda^{1/2}}{\mathcal{L}_{\xi_{n_2} q}^{(1/2)}}(y) \right\} + \text{h.c.}$$

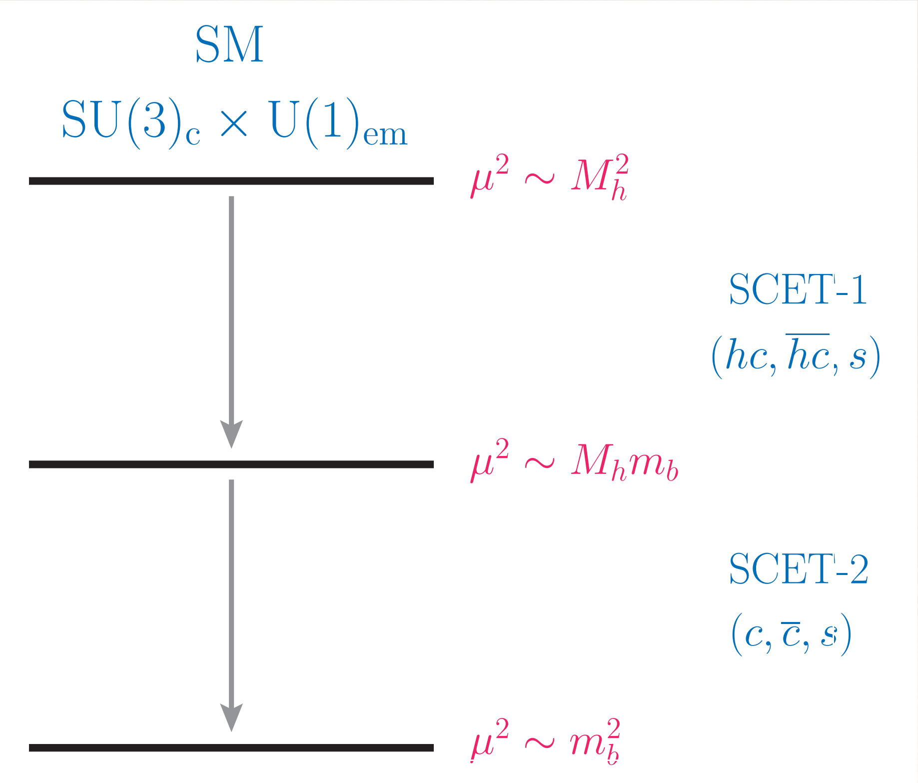
subleading SCET Lagrangian

A subleading-power observable

- Corresponding SCET operators at $O(\lambda^3)$:



Existence of only three SCET operators at $O(\lambda^3)$ ensures that these regions account for all higher-order loop graphs (see [Liu, MN 2019] for a 2-loop example)!



SM

$\mathrm{SU}(3)_c \times \mathrm{U}(1)_{\mathrm{em}}$

$\mu^2 \sim M_h^2$

SCET-1

$(hc, \overline{hc}, s)$

$\mu^2 \sim M_h m_b$

SCET-2

$(c, \overline{c}, s)$

$\mu^2 \sim m_b^2$

“Bare factorization theorem”

- Adding up the three contributions we find:

$$\mathcal{M}_b(h\to\gamma\gamma) = H_1^{(0)}\langle\gamma\gamma|\,O_1^{(0)}|h\rangle + 2\int_0^1 dz\, H_2^{(0)}(z)\,\langle\gamma\gamma|\,O_2^{(0)}(z)\,|h\rangle + H_3^{(0)}\langle\gamma\gamma|\,O_3^{(0)}|h\rangle$$

with:

$$\langle\gamma\gamma|\,O_3^{(0)}|h\rangle = \frac{g_\perp^{\mu\nu}}{2}\int_0^\infty\frac{d\ell_+}{\ell_+}\int_0^\infty\frac{d\ell_-}{\ell_-}$$

$$\times\Big[J^{(0)}(M_h\ell_+)\,J^{(0)}(-M_h\ell_-) + J^{(0)}(-M_h\ell_+)\,J^{(0)}(M_h\ell_-)\Big]\,S^{(0)}(\ell_+\ell_-)$$

- Factorization formula accomplishes a naive scale separation, but all component functions are still unrenormalized!

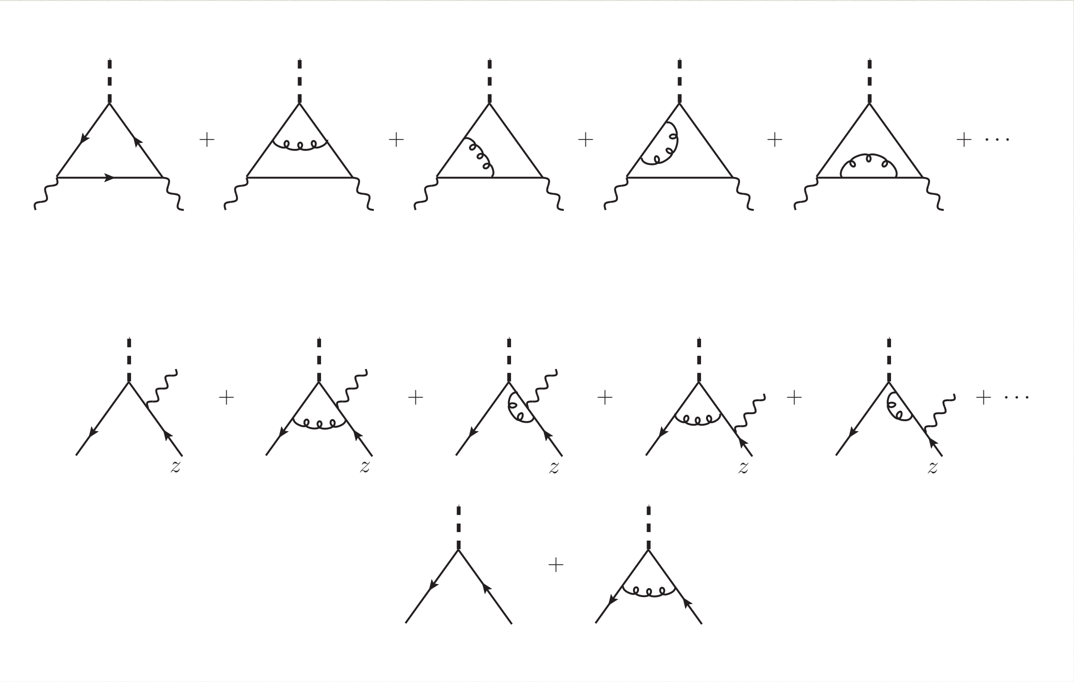
“Bare factorization theorem”

- Adding up the three contributions we find:

$$\mathcal{M}_b(h \to \gamma\gamma) = H_1^{(0)} \langle \gamma\gamma | \, O_1^{(0)} | h \rangle + 2 \int_0^1 dz \, H_2^{(0)}(z) \, \langle \gamma\gamma | \, O_2^{(0)}(z) \, | h \rangle + H_3^{(0)} \langle \gamma\gamma | \, O_3^{(0)} | h \rangle$$

- Hard matching coefficients:

$$\begin{aligned} H_1^{(0)} &= \frac{y_{b,0}}{\sqrt{2}} \frac{N_c \alpha_{b,0}}{\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon \gamma_E} \left(1 - 3\epsilon\right) \frac{2\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(3-2\epsilon)} \\ &\quad \times \bigg\{ 1 - \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon \gamma_E} \frac{\Gamma(1+2\epsilon)\,\Gamma^2(-2\epsilon)}{\Gamma(2-3\epsilon)} \\ &\qquad \times \bigg[\frac{2(1-\epsilon)(3-12\epsilon+9\epsilon^2-2\epsilon^3)}{1-3\epsilon} + \frac{8}{1-2\epsilon} \frac{\Gamma(1+\epsilon)\,\Gamma^2(2-\epsilon)\,\Gamma(2-3\epsilon)}{\Gamma(1+2\epsilon)\,\Gamma^3(1-2\epsilon)} \\ &\qquad\quad - \frac{4(3-18\epsilon+28\epsilon^2-10\epsilon^3-4\epsilon^4)}{1-3\epsilon} \frac{\Gamma(2-\epsilon)}{\Gamma(1+\epsilon)\,\Gamma(2-2\epsilon)} \bigg] \bigg\} \\ H_2^{(0)}(z) &= \frac{y_{b,0}}{\sqrt{2}} \bigg\{ \frac{1}{z} + \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon \gamma_E} \frac{\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \\ &\qquad \times \bigg[\frac{2-4\epsilon-\epsilon^2}{z^{1+\epsilon}} - \frac{2(1-\epsilon)^2}{z} - 2(1-2\epsilon-\epsilon^2) \frac{1-z^{-\epsilon}}{1-z} \bigg] \bigg\} + (z \to 1-z) \\ H_3^{(0)} &= \frac{y_{b,0}}{\sqrt{2}} \bigg[-1 + \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon \gamma_E}\, 2(1-\epsilon)^2 \frac{\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \bigg] \end{aligned}$$



“Bare factorization theorem”

- Adding up the three contributions we find:

$$\mathcal{M}_b(h \to \gamma\gamma) = H_1^{(0)} \langle\gamma\gamma|\, O_1^{(0)} |h\rangle + 2 \int_0^1 dz\, H_2^{(0)}(z)\, \langle\gamma\gamma|\, O_2^{(0)}(z)\, |h\rangle + H_3^{(0)} \langle\gamma\gamma|\, O_3^{(0)} |h\rangle$$

- Operator matrix elements:

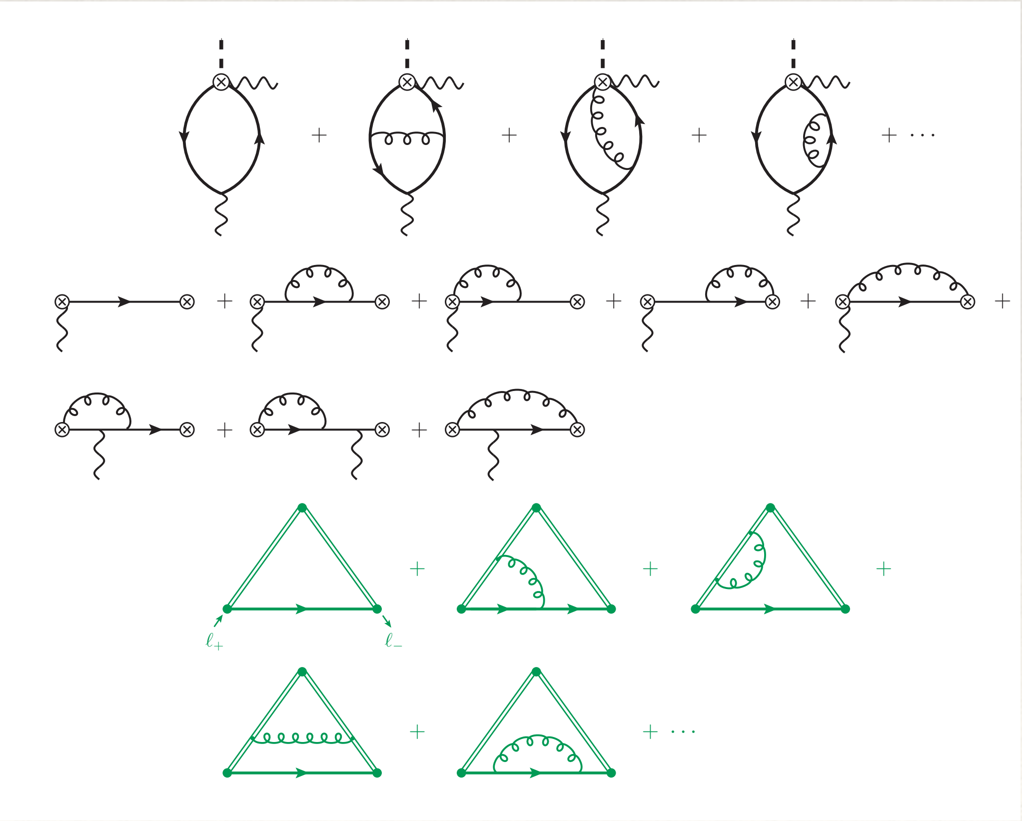
$$\langle\gamma\gamma|\, O_1^{(0)}\, |h\rangle = m_{b,0}\, g_\perp^{\mu\nu}$$

$$\langle\gamma\gamma|\, O_2^{(0)}(z)\, |h\rangle = \frac{N_c \alpha_{b,0}}{2\pi}\, m_{b,0}\, g_\perp^{\mu\nu} \left[e^{\epsilon\gamma_E}\, \Gamma(\epsilon) \left(m_{b,0}^2\right)^{-\epsilon} + \frac{C_F \alpha_{s,0}}{4\pi} \left(m_{b,0}^2\right)^{-2\epsilon} \big[K(z) + K(1-z)\big] \right]$$

$$J^{(0)}(p^2) = 1 + \frac{C_F \alpha_{s,0}}{4\pi} \left(-p^2 - i0\right)^{-\epsilon} e^{\epsilon\gamma_E}\, \frac{\Gamma(1+\epsilon)\, \Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\, (2 - 4\epsilon - \epsilon^2)$$

$$S^{(0)}(w) = -\frac{N_c \alpha_{b,0}}{\pi}\, m_{b,0} \left[S_a^{(0)}(w)\, \theta(w - m_{b,0}^2) + S_b^{(0)}(w)\, \theta(m_{b,0}^2 - w) \right]$$

$$\dots$$



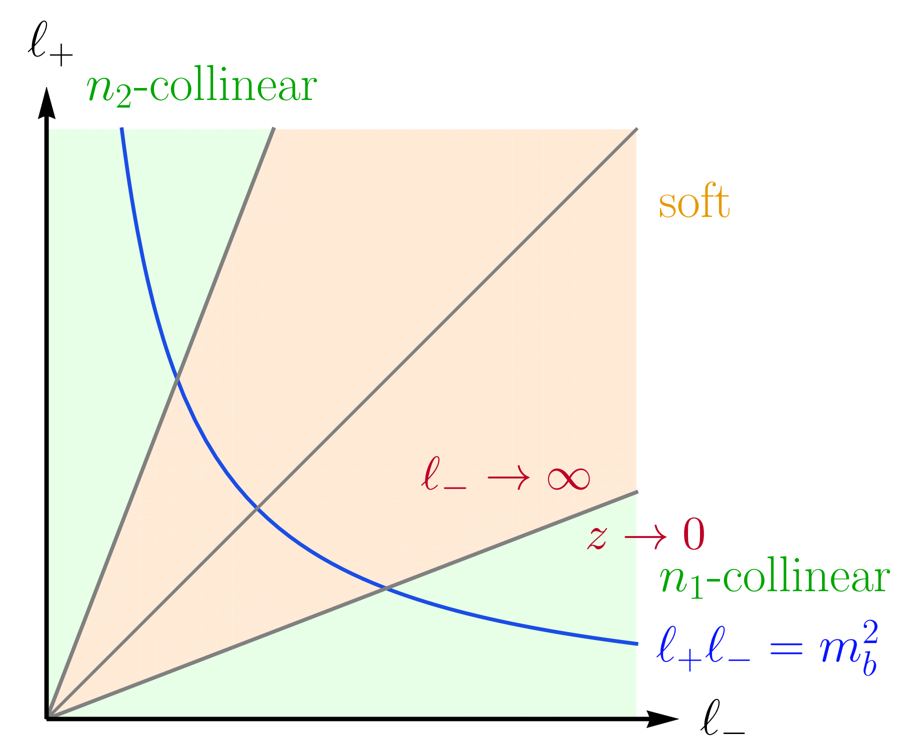
Endpoint divergences

- Closer inspection shows that the convolution integrals in the factorization formula are divergent for $z \to 0, 1$ (second term) and $\ell_\pm \to \infty$ (third term)

- Second term is symmetric under $z \leftrightarrow (1-z)$ and it suffices to study the singularity at $z \to 0$:

$$H_2^{(0)}(z) = \frac{\bar{H}_2^{(0)}(z)}{z(1-z)}$$

- Physical origin: **overlap of soft and collinear regions**, whose boundaries are not separated by the dimensional regulator



Endpoint divergences

- Things are, in fact, even more subtle. For example, in higher orders one finds that:

$$\bar{H}_2^{(0)}(z) \sim z^{-n\epsilon} \qquad \text{but} \qquad \langle O_2^{(0)}(z)\rangle \sim z^{+m\epsilon}$$

- Terms with $m = n$ require the rapidity regulator when integrated over $\int_0^1 \frac{dz}{z}$, while those with $m \neq n$ are regularized by the dimensional regulator

- In simpler examples based on SCET-1, the dimensional regulator regularizes all endpoint divergences, but this still leaves the problem of how to deal with the $1/\epsilon$ poles from the endpoint singularities, which spoil factorization

[Beneke et al., Moult et al. 2018-2020]

Endpoint divergences

- In order to define the two convolutions properly one needs to introduce a **rapidity regulator** under the integrals:

$$
\begin{aligned}
\mathcal{M}_b(h\to\gamma\gamma) = \lim_{\eta\to 0}\, & H_1^{(0)}\langle\gamma\gamma|\,O_1^{(0)}|h\rangle + 4\int_0^1 \frac{dz}{z}\left(\frac{-zM_h^2-i0}{\nu^2}\right)^{\eta} \bar{H}_2^{(0)}(z)\,\langle\gamma\gamma|\,O_2^{(0)}(z)\,|h\rangle \\
& + g_\perp^{\mu\nu}\, H_3^{(0)}\int_0^\infty \frac{d\ell_-}{\ell_-}\int_0^{\ell_-}\frac{d\ell_+}{\ell_+}\, S^{(0)}(\ell_+\ell_-) \\
& \times\Bigg[\left(\frac{\bar{n}_2\cdot k_2\,\ell_- - i0}{\nu^2}\right)^{\eta} J^{(0)}(\bar{n}_1\cdot k_1\,\ell_+)\, J^{(0)}(-\bar{n}_2\cdot k_2\,\ell_-) \\
& \quad + \left(\frac{-\bar{n}_2\cdot k_2\,\ell_- - i0}{\nu^2}\right)^{\eta} J^{(0)}(-\bar{n}_1\cdot k_1\,\ell_+)\, J^{(0)}(\bar{n}_2\cdot k_2\,\ell_-)\Bigg]
\end{aligned}
$$

- Endpoint divergences lead to $1/\eta$ poles, which cancel in the sum of all terms!

Endpoint divergences

- All-order cancellation of $1/\eta$ poles requires that the integrands of the second and third term are the same when evaluated in the singular regions!

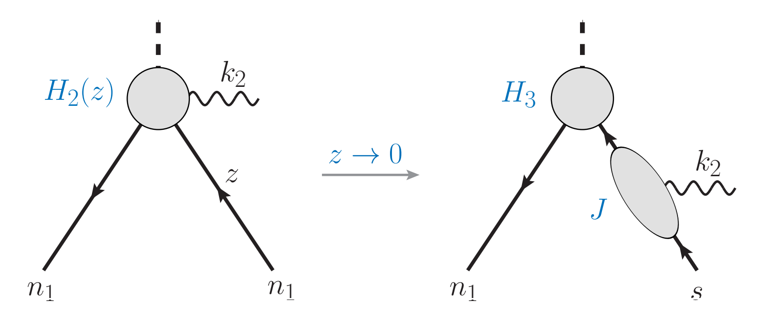
- This is ensured by the *D*-dim. **refactorization conditions**:

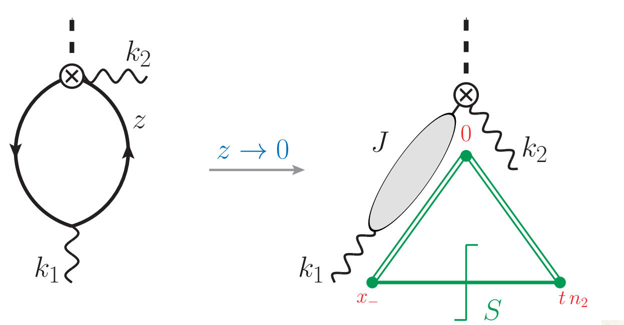
[Liu, MN 2019]

$$
\begin{aligned}
[\![\bar{H}_2^{(0)}(z)]\!] &= -H_3^{(0)} J^{(0)}(zM_h^2) \\
[\![\langle\gamma\gamma|\, O_2^{(0)}(z)\, |h\rangle]\!] &= -\frac{g_\perp^{\mu\nu}}{2} \int_0^\infty \frac{d\ell_+}{\ell_+}\, J^{(0)}(-M_h\ell_+)\, S^{(0)}(zM_h\ell_+)
\end{aligned}
$$

- We have recently proved these relations using SCET tools:

[2009.06779]





Removing endpoint divergences

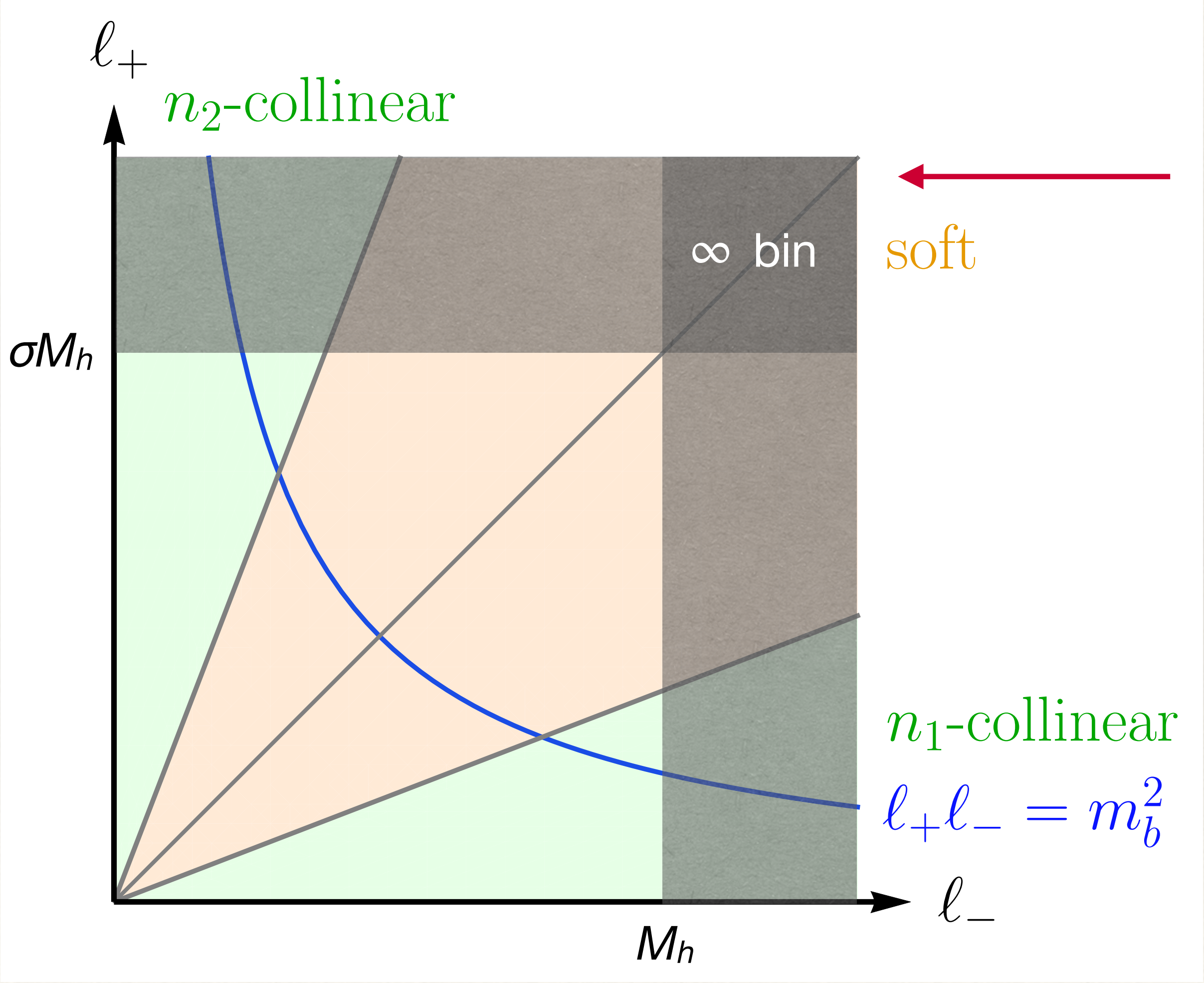
- Using these relations, the bare factorization formula can be rearranged in such a way that all endpoint divergences are removed and the limit $\eta \to 0$ can be taken. We find:

integrand for $z \to 0$

$$
\begin{aligned}
\mathcal{M}_b = {} & \left(H_1^{(0)} + \Delta H_1^{(0)} \right) \langle \gamma\gamma | O_1^{(0)} | h \rangle \\
& + 2 \lim_{\delta \to 0} \int_\delta^{1-\delta} dz \left[H_2^{(0)}(z) \, \langle \gamma\gamma | \, O_2^{(0)}(z) \, | h \rangle - \frac{\llbracket \bar{H}_2^{(0)}(z) \rrbracket}{z} \, \llbracket \langle \gamma\gamma | \, O_2^{(0)}(z) \, | h \rangle \rrbracket \right. \\
& \qquad \left. - \frac{\llbracket \bar{H}_2^{(0)}(1-z) \rrbracket}{1-z} \, \llbracket \langle \gamma\gamma | \, O_2^{(0)}(1-z) \, | h \rangle \rrbracket \right] \\
& + g_\perp^{\mu\nu} \lim_{\sigma \to -1} H_3^{(0)} \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} \, J^{(0)}(M_h \ell_-) \, J^{(0)}(-M_h \ell_+) \, S^{(0)}(\ell_+ \ell_-) \Big|_{\text{leading power}}
\end{aligned}
$$

Removing endpoint divergences

- In the space of momentum modes, this amount to the following subtractions in the third term:



"infinity bin" is subtracted twice and must be added back as a hard contribution $\Delta H_1^{(0)}$ to the coefficient of the first term

Renormalized factorization theorem

- So far, the factorization formula is still expressed in terms of bare quantities, but we wish to establish a corresponding renormalized formula:

$$
\begin{aligned}
\mathcal{M}_b = {} & H_1(\mu)\,\langle O_1(\mu)\rangle \\
& + 2\int_0^1 dz \left[H_2(z,\mu)\,\langle O_2(z,\mu)\rangle - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}\,[\![\langle O_2(z,\mu)\rangle]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}\,[\![\langle O_2(\bar{z},\mu)\rangle]\!] \right] \\
& + g_\perp^{\mu\nu}\,H_3(\mu)\,\lim_{\sigma\to -1}\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\,J(M_h\ell_-,\mu)\,J(-M_h\ell_+,\mu)\,S(\ell_+\ell_-,\mu)\Big|_{\text{leading power}}
\end{aligned}
$$

- This is non-trivial, because the presence of cutoffs does not commute with renormalization!

Renormalized factorization theorem

- Renormalization conditions for the operators:

$$O_1(\mu) = Z_{11}\, O_1^{(0)}$$

$$O_2(z,\mu) = \int_0^1 dz'\, Z_{22}(z,z')\, O_2^{(0)}(z') + Z_{21}(z)\, O_1^{(0)}$$

incompatible with cutoffs

$$\llbracket O_2(z,\mu) \rrbracket = \int_0^{\boxed{\infty}} dz'\, \llbracket Z_{22}(z,z') \rrbracket\, \llbracket O_2^{(0)}(z') \rrbracket + \llbracket Z_{21}(z) \rrbracket\, O_1^{(0)}$$

[2009.06779]

incompatible with cutoffs

$$J(p^2,\mu) = \frac{1}{p^2} \int_0^{\boxed{\infty}} dp'^2\, Z_J(p^2,p'^2;\mu)\, J^{(0)}(p'^2)$$

[2003.03393]

incompatible with cutoffs

$$S(w,\mu) = \int_0^{\boxed{\infty}} dw'\, Z_S(w,w';\mu)\, S^{(0)}(w')$$

[2005.03013]

with complicated Z factors containing plus distributions

Renormalized factorization theorem

- When the cutoffs are move from the bare over to the renormalized functions, some **left-over terms** remain, which individually have a rather complicated structure and depend both on the hard scale M_h and the soft scale m_b

- The most non-trivial part of the derivation of the renormalized factorization theorem was to show that, to all orders of perturbation theory, the sum of the left-over terms takes the form of an **additional hard subtraction** $\delta H_1^{(0)}$ of the Wilson coefficient of the operator $O_1^{(0)}$

[2009.06779]

Renormalized factorization theorem

- After this crucial step had been accomplished, we could derive the renormalization conditions for the matching coefficients:

$$H_1(\mu) = \left(H_1^{(0)} + \Delta H_1^{(0)} - \delta H_1^{(0),\,\text{tot}} \right) Z_{11}^{-1}$$

$$+ 2 \lim_{\delta\to 0} \int_\delta^{1-\delta} dz \left[H_2^{(0)}(z)\, Z_{21}^{-1}(z) - \frac{[\![\bar{H}_2^{(0)}(z)]\!]}{z} [\![Z_{21}^{-1}(z)]\!] - \frac{[\![\bar{H}_2^{(0)}(\bar{z})]\!]}{\bar{z}} [\![Z_{21}^{-1}(\bar{z})]\!] \right]$$

$$H_2(z,\mu) = \int_0^1 dz'\, H_2^{(0)}(z')\, Z_{22}^{-1}(z',z)$$

$$\frac{[\![\bar{H}_2(z,\mu)]\!]}{z} = \int_0^\infty dz'\, \frac{[\![\bar{H}_2^{(0)}(z')]\!]}{z'} [\![Z_{22}^{-1}(z',z)]\!]$$

$$H_3(\mu) = H_3^{(0)} Z_{33}^{-1}$$

Renormalized factorization theorem

- Renormalized matrix elements, with $L_m = \ln(m_b^2/\mu^2)$:

$$\langle O_1(\mu)\rangle = m_b(\mu)\, g_\perp^{\mu\nu}$$

$$\langle O_2(z,\mu)\rangle = \frac{N_c\alpha_b}{2\pi}\, m_b(\mu)\, g_\perp^{\mu\nu} \bigg\{ -L_m + \frac{C_F\alpha_s}{4\pi} \bigg[L_m^2 \Big(\ln z + \ln(1-z) + 3 \Big) - L_m \left(\ln^2 z + \ln^2(1-z) - 4\ln z \ln(1-z) + 11 - \frac{2\pi^2}{3} \right) + F(z) + F(1-z) \bigg] + \mathcal{O}(\alpha_s^2) \bigg\}$$

$$J(p^2,\mu) = 1 + \frac{C_F\alpha_s}{4\pi} \left[\ln^2\!\left(\frac{-p^2 - i0}{\mu^2} \right) - 1 - \frac{\pi^2}{6} \right] + \mathcal{O}(\alpha_s^2)$$

$$S(w,\mu) = -\frac{N_c\alpha_b}{\pi}\, m_b(\mu) \Big[S_a(w,\mu)\, \theta(w - m_b^2) + S_b(w,\mu)\, \theta(m_b^2 - w) \Big]$$

$$S_a(w,\mu) = 1 + \frac{C_F\alpha_s}{4\pi} \bigg[-L_w^2 - 6L_w + 12 - \frac{\pi^2}{2} + 2\,\text{Li}_2\Big(\frac{1}{\hat{w}}\Big) - 4\ln\Big(1 - \frac{1}{\hat{w}}\Big) \left(L_m + 1 + \ln\Big(1 - \frac{1}{\hat{w}}\Big) + \frac{3}{2}\ln\hat{w} \right) \bigg] + \mathcal{O}(\alpha_s^2)$$

$L_w = \ln(w/\mu^2)$

$\hat{w} = w/m_b^2$

Renormalized factorization theorem

- Renormalized matrix elements, with $L_m = \ln(m_b^2/\mu^2)$:

$$\langle O_1(\mu)\rangle = m_b(\mu)\, g_\perp^{\mu\nu}$$

$$\langle O_2(z,\mu)\rangle = \frac{N_c\alpha_b}{2\pi}\, m_b(\mu)\, g_\perp^{\mu\nu} \bigg\{ - L_m + \frac{C_F\alpha_s}{4\pi} \bigg[L_m^2 \Big(\ln z + \ln(1-z) + 3 \Big)$$

$$- L_m \left(\ln^2 z + \ln^2(1-z) - 4 \ln z \ln(1-z) + 11 - \frac{2\pi^2}{3} \right) + F(z) + F(1-z) \bigg] + \mathcal{O}(\alpha_s^2) \bigg\}$$

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$$L_w = \ln(w/\mu^2)$$
$$\hat{w} = w/m_b^2$$

$$S_b(w,\mu) = \frac{C_F\alpha_s}{\pi}\, \ln(1-\hat{w})\, \big[L_m + \ln(1-\hat{w}) \big] + \mathcal{O}(\alpha_s^2)$$

Renormalized factorization theorem

- Renormalized matching coefficients, with $L_h = \ln(-M_h^2/\mu^2)$:

$$H_1(\mu) = \frac{N_c\alpha_b}{\pi}\,\frac{y_b(\mu)}{\sqrt{2}}\left\{-2 + \frac{C_F\alpha_s}{4\pi}\left[-\frac{\pi^2}{3}\,L_h^2 + (12+8\zeta_3)\,L_h - 36 - \frac{2\pi^2}{3} - \frac{11\pi^4}{45}\right] + \mathcal{O}(\alpha_s^2)\right\}$$

$$H_2(z,\mu) = \frac{y_b(\mu)}{\sqrt{2}}\,\frac{1}{z(1-z)}\left\{1 + \frac{C_F\alpha_s}{4\pi}\Big[2L_h\big(\ln z + \ln(1-z)\big) + \ln^2 z + \ln^2(1-z) - 3\Big] + \mathcal{O}(\alpha_s^2)\right\}$$

$$H_3(\mu) = \frac{y_b(\mu)}{\sqrt{2}}\left[-1 + \frac{C_F\alpha_s}{4\pi}\left(L_h^2 + 2 - \frac{\pi^2}{6}\right) + \mathcal{O}(\alpha_s^2)\right]$$

Resummation of large logarithms

Liu, Mecaj, MN, Wang: 2009.04456 & 2009.06779
Liu, MN: 2003.03393 (JHEP)
Liu, Mecaj, MN, Wang, Fleming: 2005.03013 (JHEP)

Resummation of large logs

- The renormalized factorization formula

$$
\begin{aligned}
\mathcal{M}_b &= H_1(\mu)\,\langle O_1(\mu)\rangle \\
&\quad + 2\int_0^1 dz\,\Bigg[H_2(z,\mu)\,\langle O_2(z,\mu)\rangle - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}\,[\![\langle O_2(z,\mu)\rangle]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}\,[\![\langle O_2(\bar{z},\mu)\rangle]\!] \Bigg] \\
&\quad + g_\perp^{\mu\nu}\,H_3(\mu)\,\lim_{\sigma\to -1}\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\,J(M_h\ell_-,\mu)\,J(-M_h\ell_+,\mu)\,S(\ell_+\ell_-,\mu)\Big|_{\text{leading power}}
\end{aligned}
$$

provides a complete scale separation and allows us to resum large logarithms in the decay amplitude to all orders of perturbation theory!

Resummation of large logs

- RG equations for matrix elements:

$$\frac{d}{d\ln\mu}\,\langle O_1(\mu)\rangle = -\gamma_{11}\,\langle O_1(\mu)\rangle$$

$$\frac{d}{d\ln\mu}\,\langle O_2(z,\mu)\rangle = -\int_0^1 dz'\,\gamma_{22}(z,z')\,\langle O_2(z',\mu)\rangle - \gamma_{21}(z)\,\langle O_1(\mu)\rangle$$

$$\frac{d}{d\ln\mu}\,J(p^2,\mu) = -\int_0^\infty dx\,\gamma_J(p^2,xp^2)\,J(xp^2,\mu)$$

$$\frac{d}{d\ln\mu}\,S(w,\mu) = -\int_0^\infty dx\,\gamma_S(w,w/x)\,S(w/x,\mu)$$

Resummation of large logs

- RG equations for matching coefficients:

inhomogeneous contribution due to cutoffs

$$\frac{d}{d\ln\mu}H_1(\mu) = D_{\text{cut}}(\mu) + \gamma_{11}\,H_1(\mu)$$

$$+\,2\int_0^1 dz\left[H_2(z,\mu)\,\gamma_{21}(z) - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}\,[\![\gamma_{21}(z)]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}\,[\![\gamma_{21}(\bar{z})]\!]\right]$$

$$\frac{d}{d\ln\mu}H_2(z,\mu) = \int_0^1 dz'\,H_2(z',\mu)\,\gamma_{22}(z',z)$$

$$\frac{d}{d\ln\mu}H_3(\mu) = \gamma_{33}\,H_3(\mu)$$

- where:

$$D_{\text{cut}}(\mu) = -\frac{N_c\alpha_b}{\pi}\,\frac{y_b(\mu)}{\sqrt{2}}\left[\frac{C_F\alpha_s}{4\pi}\,16\zeta_3 + \left(\frac{\alpha_s}{4\pi}\right)^2 d_{\text{cut},2} + \mathcal{O}(\alpha_s^3)\right] \ni \alpha_b\,(\alpha_s L_h)^n$$

Logarithms in the 3-loop amplitude

- From a perturbative solution of the RGEs, we have obtained predictions for the terms of order $\mathcal{O}(\alpha_s^2 L^k)$ with *k*=6,5,4,3 in the 3-loop decay amplitude in the on-shell scheme, finding:

$$
\begin{aligned}
\mathcal{M}_b = \frac{N_c \alpha_b}{\pi} \frac{m_b^2}{v}\, \varepsilon_\perp^*(k_1) \cdot \varepsilon_\perp^*(k_2) \\
\times \Bigg\{ \frac{L^2}{2} - 2 + \frac{C_F \alpha_s(\hat{\mu}_h)}{4\pi} \left[-\frac{L^4}{12} - L^3 - \frac{2\pi^2}{3} L^2 + \left(12 + \frac{2\pi^2}{3} + 16\zeta_3 \right) L - 20 + 4\zeta_3 - \frac{\pi^4}{5} \right] \\
+ C_F \left(\frac{\alpha_s(\hat{\mu}_h)}{4\pi} \right)^2 \left[\frac{C_F}{90} L^6 + \left(\frac{C_F}{10} - \frac{\beta_0}{30} \right) L^5 + d_4^{\mathrm{OS}} L^4 + d_3^{\mathrm{OS}} L^3 + \dots \right] \Bigg\}
\end{aligned}
$$

- Find perfect agreement with recent numerical results!

[Czakon, Niggetiedt 2020]

Series of subleading logs

- We have reproduced the series of leading double logs (LL) and obtained a **new result** for the NLL logs to all orders in α_s:

$$\mathcal{M}_b^{\mathrm{NLL}} = \frac{N_c \alpha_b}{\pi} \frac{y_b(M_h)}{\sqrt{2}} m_b \, \varepsilon_\perp^*(k_1) \cdot \varepsilon_\perp^*(k_2) \, \frac{L^2}{2} \sum_{n=0}^{\infty} (-\rho)^n \frac{2\Gamma(n+1)}{\Gamma(2n+3)}$$
$$\times \left[1 + \frac{3\rho}{2L} \frac{2n+1}{2n+3} - \frac{\beta_0}{C_F} \frac{\rho^2}{4L} \frac{(n+1)^2}{(2n+3)(2n+5)} \right],$$

[Kotsky, Yakovlev 1997]

with $\rho = \dfrac{C_F \alpha_s(M_h)}{2\pi} L^2$

- The subleading terms disagree with earlier results in the literature!

[Akhoury, Wang, Yakovlev 2001; Anastasiou, Penin 2020]

Resummation in RG-improved PT

- Ultimate goal is to resum **all large logarithms** and exponentiate them (RG-improved perturbation theory)

- Particularly important for Sudakov problems, where leading logs are formally larger than $O(1)$

- In RG-improved perturbation theory one supplies the matching conditions for all component functions in the factorization theorem at matching scales where they are free of large logs; these functions and then evolved to a common scale solving their RG equations → **all large logs exponentiate**!

Resummation in RG-improved PT

- We have not yet performed a complete resummation, but we have resummed the most difficult contribution T3 at LO in RG-improved perturbation theory, finding: [2009.04456]

$$T_3^{\rm LO} = \frac{\alpha}{3\pi}\,\frac{y_b(\mu_h)}{\sqrt{2}} \int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{M_h}\frac{d\ell_+}{\ell_+}\,m_b(\mu_s)\,e^{2\mathcal{S}(\mu_s,\mu_h)-2\mathcal{S}(\mu_-,\mu_h)-2\mathcal{S}(\mu_+,\mu_h)}\left(\frac{-M_h\ell_-}{\mu_-^2}\right)^{a_\Gamma^-}\left(\frac{-M_h\ell_+}{\mu_+^2}\right)^{a_\Gamma^+}\left(\frac{-\ell_+\ell_-}{\mu_s^2}\right)^{-a_\Gamma^s}$$
$$\times\left(\frac{\alpha_s(\mu_s)}{\alpha_s(\mu_h)}\right)^{\frac{12}{23}} e^{-2\gamma_E\,a_\Gamma^+}\,\frac{\Gamma(1-a_\Gamma^+)}{\Gamma(1+a_\Gamma^+)}\,e^{-2\gamma_E\,a_\Gamma^-}\,\frac{\Gamma(1-a_\Gamma^-)}{\Gamma(1+a_\Gamma^-)}\,e^{4\gamma_E\,a_\Gamma^s}\,G_{4,4}^{2,2}\left(\begin{matrix}-a_\Gamma^s\,, & -a_\Gamma^s\,, & 1-a_\Gamma^s\,, & 1-a_\Gamma^s\\ 0\,, & 1\,, & 0\,, & 0\end{matrix}\,\middle|\,\frac{m_b^2}{-\ell_+\ell_-}\right)$$

with:

$$a_\Gamma^i = -\frac{8}{23}\ln\frac{\alpha_s(\mu_i)}{\alpha_s(\mu_h)}\;, \qquad \mathcal{S}(\mu_i,\mu_h) = \frac{12}{529}\left[\frac{4\pi}{\alpha_s(\mu_i)}\left(1-\frac{1}{r}-\ln r\right)+\frac{58}{23}\ln^2 r+\left(\frac{2429}{207}-\pi^2\right)(1-r+\ln r)\right]$$

$$r = \alpha_s(\mu_h)/\alpha_s(\mu_i)$$

- dynamical matching scales:

$$\mu_s^2 \sim \ell_+\ell_- \qquad \mu_\pm^2 \sim M_h\ell_\pm \qquad \mu_h \sim M_h$$

Conclusions

- We have derived the first SCET factorization theorem for an observable appearing at subleading order in power counting
- Generic features:
  - several SCET operators exist → several terms in factorization formula
  - these operators mix under renormalization
  - endpoint divergences in convolutions cancel between the different terms; cancellation ensured by ***D*-dim. refactorization conditions**
  - endpoint divergences can be removed by performing subtractions and rearranging the various terms
- Our results are an important step towards establishing SCET as a complete EFT admitting a consistent power expansion!