The RG Improved Effective Potential from EFT

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Based on [2010.15806] with **Aneesh Manohar**

Introduction

The effective action $\Gamma[\hat{\phi}]$ is a fundamental object in QFT.

- Computed by doing the path integral over the fields $\{\phi\}$ in the presence of external sources *J*.

- It is a function of classical background fields,

$$\boxed{\hat{\phi}(x) = \langle \phi(x) \rangle_J}$$

Introduction

The effective potential is the value of $\Gamma[\hat{\phi}]$ for constant $\hat{\phi}(x) = \hat{\phi}$:

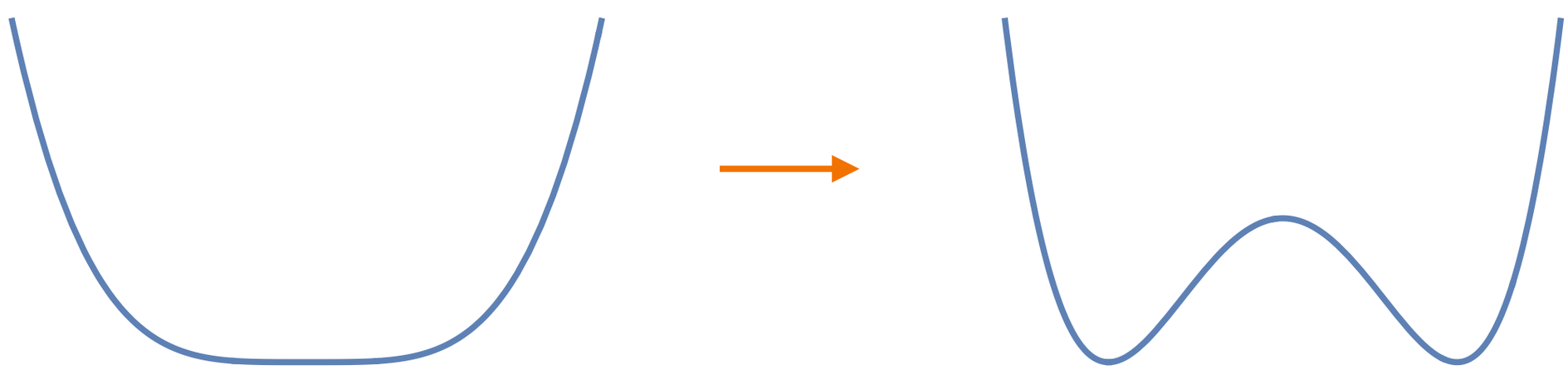
$$\Gamma[\hat{\phi}] = -\int d^4x \; V_{\text{eff}}(\hat{\phi}) + \ldots$$

$V_{\text{eff}}(\hat{\phi})$ plays an important role in determining the vacuum of a QFT.

- $\hat{\phi}_0 = \langle \phi \rangle_{J=0}$ that minimizes $V'_{\text{eff}}(\hat{\phi}_0) = 0$ is the expectation value in the **true ground state**.

  $\Rightarrow$ Important applications for spontaneous symmetry breaking!

- Foundational example [Coleman, Weinberg '73]: In massless scalar QED, radiative corrections shift the ground state from the classical vacuum.



The loop expansion

$V_{\text{eff}}(\hat{\phi})$ has a well known loop expansion. [Coleman, Weinberg '73]

$$\boxed{V_{\text{eff}}(\hat{\phi}) = V_{\text{tree}}(\hat{\phi}) + \hbar V_{\text{1-loop}}(\hat{\phi}) + \mathcal{O}(\hbar^2)}$$

$$V_{\text{1-loop}}(\hat{\phi}) = \frac{1}{64\pi^2}\left[\sum_s m_s^4(\hat{\phi})\left(\log\frac{m_s^2(\hat{\phi})}{\mu^2} - \frac{3}{2}\right) - 2\sum_f m_f^4(\hat{\phi})\left(\log\frac{m_f^2(\hat{\phi})}{\mu^2} - \frac{3}{2}\right)\right.$$
$$\left. + 3\sum_v m_v^4(\hat{\phi})\left(\log\frac{m_v^2(\hat{\phi})}{\mu^2} - \frac{5}{6}\right)\right]$$

RG scale μ

given in $\overline{\text{MS}}$

- $m_s^2(\hat{\phi}) =$ eigenvalues of $-\dfrac{\partial^2\mathcal{L}(\hat{\phi})}{\partial\phi_i\partial\phi_j}$

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RG scale μ given in $\overline{\rm MS}$

- $m_s^2(\hat{\phi}) =$ eigenvalues of $-\dfrac{\partial^2\mathcal{L}(\hat{\phi})}{\partial\phi_i\partial\phi_j}$

- E.g. massless scalar QED: $m_s^2(\hat{\phi}) \sim \lambda\hat{\phi}^2\,,\quad m_v^2 \sim e^2\hat{\phi}^2$

$\Rightarrow$ for $e^4 \sim \lambda$, the 1-loop $e^4 \log\dfrac{\hat{\phi}^2}{\mu^2}$ balances the tree-level $\dfrac{\lambda}{4!}\hat{\phi}^4$

RG Improvement

$$V_{\text{eff}} = \begin{array}{c} \begin{array}{cccccccc} & \text{LL} & & \text{NLL} & & \text{NNLL} & & \ldots \end{array} \\ \left[\begin{array}{cccccccc} & 1 & & & & & & \\ +\lambda & (L & + & 1) & & & & \\ +\lambda^2 & (L^2 & + & L & + & 1) & & \\ +\lambda^3 & (L^3 & + & L^2 & + & L & + & 1) \end{array} \right] \end{array} \begin{array}{l} \text{tree} \\ \text{1-loop} \\ \text{2-loop} \\ \vdots \end{array}$$

$$\lambda = \{\lambda_i\}$$

$$L = \left\{ \left| \log m_i^2(\hat{\phi})/\mu^2 \right| \right\}$$

- Fixed order perturbation theory breaks down when $\lambda L \gg 1$.
- One can **RG improve** by running the couplings to a scale μ where perturbation theory is reliable:

$$V_{\text{eff}}(\hat{\phi}_i, \lambda_i(\mu_0), \mu_0) = V_{\text{eff}}(e^{-\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_{\phi_i}(\mu')} \hat{\phi}_i, \lambda_i(\mu), \mu)$$

Integrating 1-loop RGE $\leftrightarrow$ sums the LL series,

Integrating 2-loop RGE $\leftrightarrow$ sums the NLL series, …

RG Improvement in multi-scale problems

- In problems with one scale $m^2(\hat{\phi})$, RG improve by running to a scale $\mu^2 \sim m^2(\hat{\phi})$, so *L* is small. [Coleman, Weinberg '73]

  ✳ E.g. in the Coleman-Weinberg model, take $\mu^2 \sim \hat{\phi}_0^2$

- In a multi-scale problem with $m_H^2(\hat{\phi}) \gg m_L^2(\hat{\phi})$, taking $\mu^2 \sim m_H^2(\hat{\phi})$ results in large logs

$$\log \frac{m_L^2}{m_H^2} \gg 1$$

  ✳ This is a typical EFT problem.

  ✳ Surprisingly, systematic RG improvement of $V_{\rm eff}$ in the multi-scale case had not been solved!

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  ✳ Surprisingly, systematic RG improvement of V_{eff} in the multi-scale case had not been solved!

Our result: sum the LL series in $L \sim \log \frac{m_L^2}{m_H^2}$ for V_{eff}.

$$V_{\text{eff}} = \begin{bmatrix} & 1 & & & & & \\ +\lambda & (L & + & 1) & & & \\ +\lambda^2 & (L^2 & + & L & + & 1) & \\ +\lambda^3 & (L^3 & + & L^2 & + & L & + \quad 1) \\ & \vdots & & & & & \end{bmatrix}$$

Our Strategy

Approach: Introduce a power counting $z \sim \frac{m_L}{m_H} \ll 1$ and construct an EFT(*)

depends on the theory

$$\mathcal{L}(\phi_H, \phi_L; \mu_H) \qquad \mu_H \sim m_H$$

integrate out ϕ_H and match

$$\mathcal{L}_{\rm EFT}(\phi_L; \mu_H)$$

run in the EFT from $\mu_H \to \mu_L$

$$\mathcal{L}_{\rm EFT}(\phi_L; \mu_L) \qquad \mu_L \sim m_L$$

integrate out ϕ_L

$$V_{\rm eff}(\mu_L) + \mathcal{O}(z^6)$$

(*) This would be straightforward, except for several important subtleties! Requires more than the standard EFT method.

For the rest of the talk:

1. RG improvement of a 2-scalar model.

   i. Explain the asterisk in the previous slide.

   ii. Walk through the RG improvement step-by-step.

   iii. Compare to exact 2-loop calculations. [Ford, Jones '92]
   ($(\lambda L)^2$ is a nontrivial check of our results!) (other examples: [Martin '02])

   iv. Show how the minimum of the potential shifts.

2. Overview of other examples.

Previous work: [Bando, Kugo, Maekawa, Nakano '92]
[Casas, Di Clemente, Quiros '98][Iso, Kawana '18][Okane '19]
…
(Also, other approaches: [Einhorn, Jones '84][Ford '94][Nakano, Yoshida '94] …)

Case Study: Two scalars

We'll study a model with a heavy field χ, and a light field ϕ, coupled via:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda_\chi}{24}\chi^4 - \frac{\lambda_3}{4}\chi^2\phi^2 - \frac{\lambda_\phi}{24}\phi^4 - \frac{m_\chi^2}{2}\chi^2 - \frac{m_\phi^2}{2}\phi^2$$

Depending on the couplings, we could study: (assume $\{\lambda_\phi, \lambda_\chi, \lambda_3\} \sim 1$)

- The unbroken phase. Power counting:

$$m_\chi^2 \sim 1 \qquad m_\phi^2 \sim z^2 \ll 1$$

- The broken phase. Power counting:

$$\chi|_{\text{classical min}} = v_\chi \sim 1 \qquad \phi|_{\text{classical min}} = v_\phi \sim z \ll 1$$

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$$\chi|_{\text{classical min}} = v_\chi \sim 1 \qquad \phi|_{\text{classical min}} = v_\phi \sim z \ll 1$$

Goal: Compute $V_{\text{eff}}(\hat{\phi}, \hat{\chi})$ in a *z*-expansion, with $\hat{\phi} = \langle \phi(\mu_0) \rangle$, etc.

On power counting

To compute $V_{\rm eff}(\hat{\phi},\hat{\chi})$, we need the second derivatives

$$W_{ab}(\hat{\phi},\hat{\chi}) \equiv -\frac{\partial^2 \mathcal{L}(\hat{\phi},\hat{\chi})}{\partial\phi_a \partial\phi_b} = \begin{cases} W_{\chi\chi} = \frac{1}{3}\lambda_\chi v_\chi^2 + \frac{1}{2}\lambda_\chi\left(\hat{\chi}^2 - v_\chi^2\right) + \frac{1}{2}\lambda_3\left(\hat{\phi}^2 - v_\phi^2\right) \\ W_{\phi\phi} = \frac{1}{6}\lambda_\phi\left(3\hat{\phi}^2 - v_\phi^2\right) + \frac{1}{2}\lambda_3\left(\hat{\chi}^2 - v_\chi^2\right) \ , \\ W_{\phi\chi} = W_{\chi\phi} = \lambda_3\hat{\phi}\hat{\chi} \ . \end{cases}$$

The eigenvalues are $m_\pm^2 = \frac{W_{\chi\chi}}{2}\left(1 + \frac{W_{\phi\phi}}{W_{\chi\chi}} \pm \sqrt{\frac{4W_{\phi\chi}^2}{W_{\chi\chi}^2} + \left(1 - \frac{W_{\phi\phi}}{W_{\chi\chi}}\right)^2}\right)$

On power counting

To compute $V_{\rm eff}(\hat{\phi},\hat{\chi})$, we need the second derivatives

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To have a mass hierarchy, take

$$\boxed{\begin{array}{ll} v_\chi \sim \hat{\chi} \sim 1 & \hat{\chi}^2 - v_\chi^2 \sim z^2 \\ v_\phi \sim \hat{\phi} \sim z & \hat{\phi}^2 - v_\phi^2 \sim z^2 \end{array}}$$

On power counting

The 1-loop fixed order effective potential is then:

$$V_{\text{tree}} = \underbrace{\frac{\lambda_\phi}{24}(\hat{\phi}^2 - v_\phi^2)^2}_{\sim z^4} + \underbrace{\frac{\lambda_\chi}{24}(\hat{\chi}^2 - v_\chi^2)^2}_{\sim z^4} + \underbrace{\frac{\lambda_3}{4}(\hat{\phi}^2 - v_\phi^2)(\hat{\chi}^2 - v_\chi^2)}_{\sim z^4} + \Lambda$$

$$V_{\text{1-loop}}(\mu) = \underbrace{\frac{m_+^4}{64\pi^2}\left[\log\frac{m_+^2}{\mu^2} - \frac{3}{2}\right]}_{\sim 1} + \underbrace{\frac{m_-^4}{64\pi^2}\left[\log\frac{m_-^2}{\mu^2} - \frac{3}{2}\right]}_{\sim z^4 \log z^2}$$

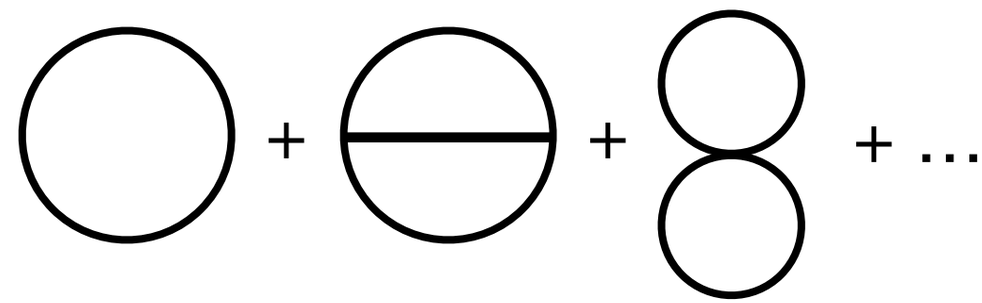
$\Rightarrow$ compute to $\mathcal{O}(z^4)$

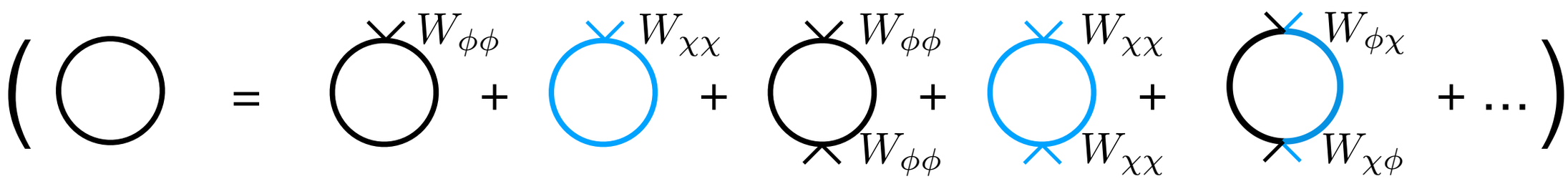
The 1PI Problem

Conceptual subtlety 1:

In the full theory,

$V_{\text{eff}} = \sum$ 1PI vacuum bubbles in a classical background $=$

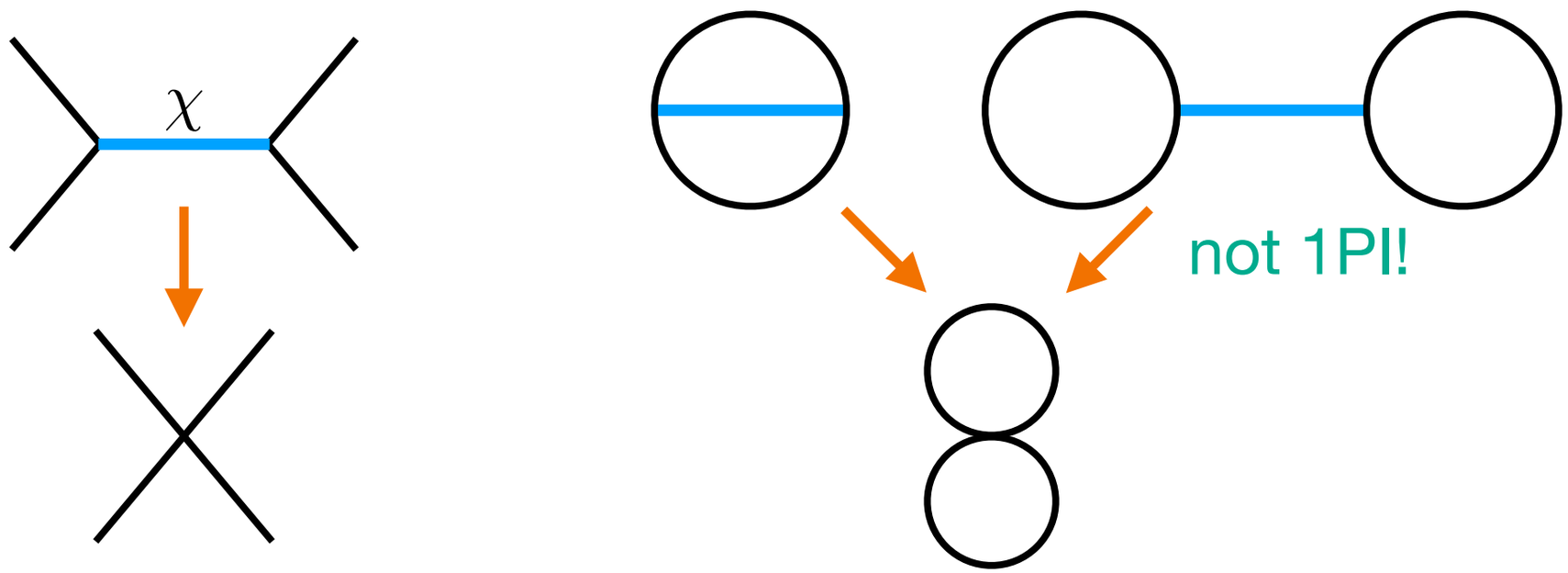




The 1PI Problem

Conceptual subtlety 1:

Once heavy states are integrated out, information about which graphs are 1PI is lost in the EFT.



Solution: The 1PI condition is equivalent to a vanishing tadpole constraint that can be implemented in the EFT. Next up: explain this.

Jackiw's Method

In order to explain this + other subtleties, let's review the standard computation of the effective potential. [Jackiw '74]

1) Shift by the Lagrangian by the vevs $\hat{\phi}, \hat{\chi}$:

$$\mathcal{L}(\phi_q + \hat{\phi}, \chi_q + \hat{\chi}) = -\hat{\Lambda} - \hat{\sigma}_\phi \phi_q - \hat{\sigma}_\chi \chi_q - \frac{1}{2} \sum_{a,b} \phi_{q,a} \left(\partial^2 \delta_{ab} + W_{ab} \right) \phi_{q,b}$$

+ (cubic) + (quartic)

- The shifted couplings $\{\hat{\Lambda}, \hat{\sigma}_\phi, W_{\phi\phi}, \dots\}$ depend on $\hat{\phi}, \hat{\chi}$

  E.g. the cosmological constant after the shift is:

$$\boxed{\hat{\Lambda} = V_{\text{tree}}(\hat{\phi}, \hat{\chi}) + \Lambda}$$

Jackiw's Method

2) Solve for the effective action in the path integral.

- The path integral after the shift is:

$$Z[J] = e^{\frac{i}{\hbar}W[J]} = \int D\chi_q D\phi_q \ \exp\frac{i}{\hbar}\int d^4x \left[\mathcal{L}(\phi_q+\hat{\phi}) + J_\phi(\phi_q+\hat{\phi}) + J_\chi(\chi_q+\hat{\chi})\right]$$

shifted Lagrangian source terms

- The 1PI effective action is the Legendre transform of *W*[*J*], the connected generating functional:

$$\Gamma[\hat{\phi},\hat{\chi}] = W[J] - \int d^4x \left(J_\phi\hat{\phi} + J_\chi\hat{\chi}\right)$$

Thus, rewrite:

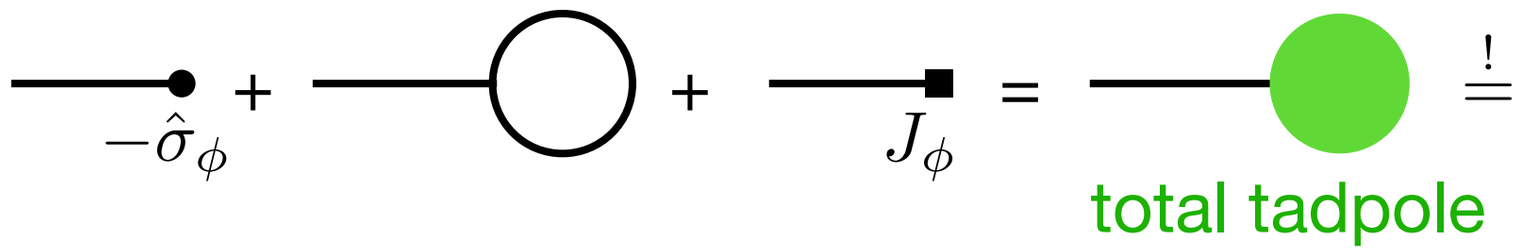
$$\boxed{e^{\frac{i}{\hbar}\Gamma[\hat{\phi},\hat{\chi}]} = \int D\chi_q D\phi_q \ \exp\frac{i}{\hbar}\int d^4x \left[\underbrace{\mathcal{L}(\phi_q+\hat{\phi}) + J_\phi\phi_q + J_\chi\chi_q}_{\equiv\,\widehat{\mathcal{L}}}\right]}$$

On Tadpoles

- In order that $\langle \phi \rangle = \hat{\phi}$, the source J_ϕ must be adjusted such that

$$\langle \phi_q \rangle = 0 \qquad \text{(and same for } \chi\text{)}$$

$\langle \phi_q \rangle =$



$\stackrel{!}{=} 0$

At tree level, set $\quad J_\phi = \hat{\sigma}_\phi$

On Tadpoles

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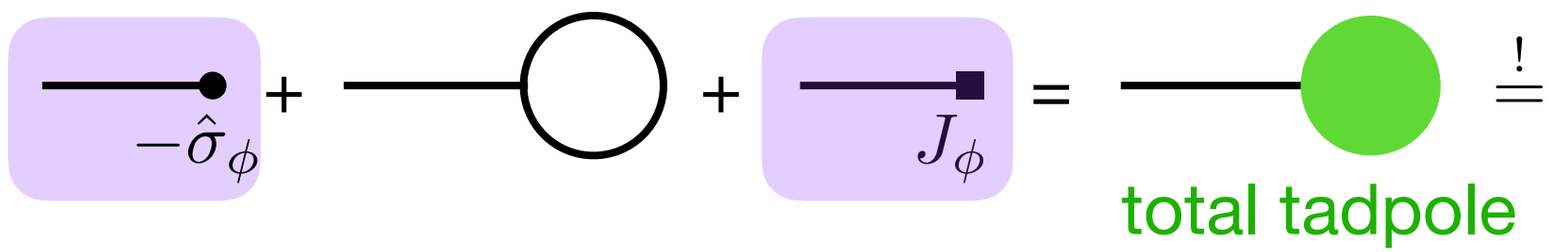
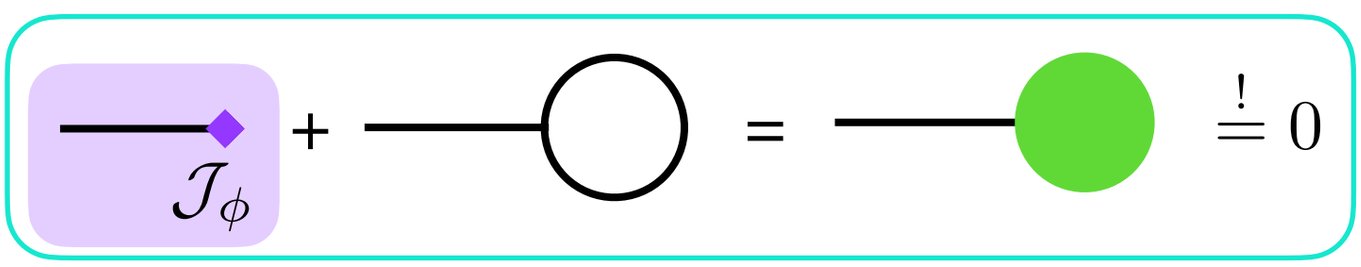
$\langle\phi_q\rangle =$ 

At tree level, set $\quad J_\phi = \hat{\sigma}_\phi$

At 1-loop, set $\quad J_\phi = \hat{\sigma}_\phi + \mathcal{J}_\phi$

formally 1-loop order



$\Rightarrow$ The exponent in the path integral can be re-organized

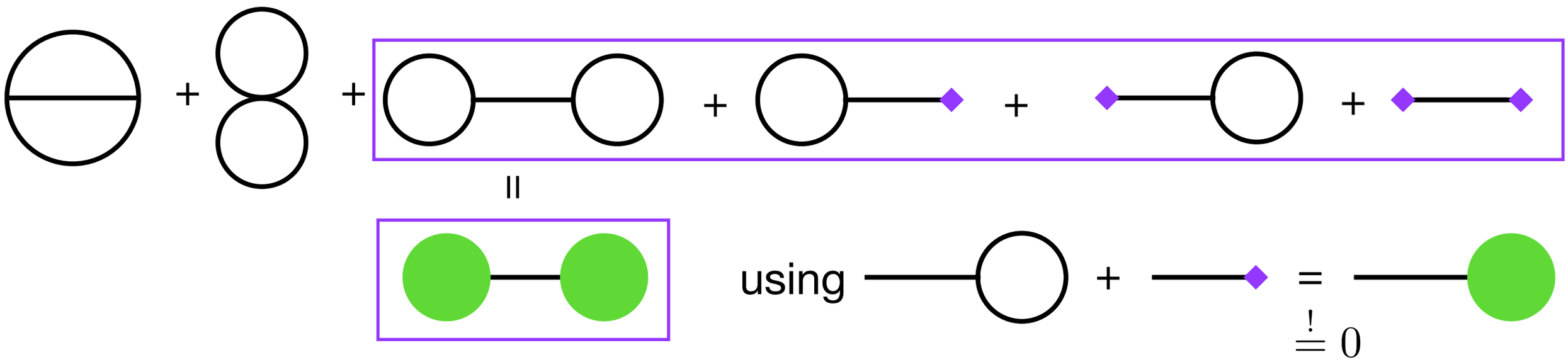
$$e^{\frac{i}{\hbar}\Gamma[\hat{\phi},\hat{\chi}]} = \int D\chi_q D\phi_q \ \exp\frac{i}{\hbar}\int d^4x\ \widehat{\mathcal{L}}$$

$$\widehat{\mathcal{L}} = -\hat{\Lambda} + \mathcal{J}_\phi\phi_q + \mathcal{J}_\chi\chi_q \quad + \text{(quadratic)} + \text{(cubic)} + \text{(quartic)}$$

On Tadpoles

The 1PI condition is equivalent to the vanishing tadpole condition.

- E.g. the 2 loop vacuum graphs in terms of $\mathcal{J}_\phi$ are:



Shift $\phi \to \phi_q + \hat{\phi}$, drop the linear terms in ϕ_q, compute 1PI vacuum bubbles

$\Leftrightarrow$

Include explicit $\mathcal{J}_\phi$ source terms, and impose the vanishing

$$\langle \phi_q \rangle = 0 \quad \langle \chi_q \rangle = 0$$

can be implemented in the EFT!

When to shift?

Conceptual Subtlety 2:
To compute, we shifted with respect to $\hat{\phi}, \hat{\chi}$, with $\langle \phi \rangle = \hat{\phi}$.

After we integrate out χ_q, $\langle \phi \rangle$ in the EFT $\neq \langle \phi \rangle$ at high energies, due to matching corrections.

Shift *then* integrate out $\neq$ integrate out *then* shift

We need to do the shift at high energies so that the expectation value $\langle \phi \rangle = \hat{\phi}$ retains its original meaning.

Method:

$\mathcal{L}(\phi, \chi; \mu_H) \quad \lambda_i(\mu_H) \qquad \mu_H \sim 1$

1) ↓ Shift and replace $\hat{\sigma} \to \mathcal{J}$

$\widehat{\mathcal{L}}(\phi_q, \chi_q; \mu_H) \quad \hat{\lambda}_i(\hat{\phi}, \hat{\chi}; \mu_H)$

2) ↓ Integrate out χ_q, match onto EFT

$\mathcal{L}_{\mathrm{EFT}}(\phi_q; \mu_H) \quad \widetilde{\lambda}_i(\hat{\phi}, \hat{\chi}; \mu_H)$

3) ↓ Run using EFT β-functions

$\mathcal{L}_{\mathrm{EFT}}(\phi_q; \mu_L) \quad \widetilde{\lambda}_i(\hat{\phi}, \hat{\chi}; \mu_L) \qquad \mu_L \sim z$

4) ↓ Do the path integral over ϕ_q

$\langle \chi_q \rangle \stackrel{!}{=} 0 \xrightarrow{\textbf{5)}} V_{\mathrm{eff}}(\hat{\phi}, \hat{\chi}; \mu_L)$

Compute RG-improved $\mathcal{J}_\chi$ via vanishing tadpole constraint

Step 1: Shift and replace linear term

$$\mathcal{L}(\phi, \chi; \mu_H) \quad \lambda_i(\mu_H)$$

1) ↓

$$\widehat{\mathcal{L}}(\phi_q, \chi_q; \mu_H) \quad \hat{\lambda}_i(\hat{\phi}, \hat{\chi}; \mu_H)$$

- Shift $\phi = \phi_q + \hat{\phi}\,, \;\; \chi = \chi_q + \hat{\chi}$

  and replace $J_\phi = \hat{\sigma}_\phi + \mathcal{J}_\phi\,, \;\; J_\chi = \hat{\sigma}_\chi + \mathcal{J}_\chi$

$$\lambda_i = \{\lambda_\chi, \lambda_\phi, \lambda_3\}$$
$$\mu_H \sim 1$$

- The exponent leftover in the path integral is then

$$\widehat{\mathcal{L}} = -\hat{\Lambda} + \mathcal{J}_\phi \phi_q + \mathcal{J}_\chi \chi_q \quad \text{+ (quadratic) + (cubic) + (quartic)}$$

As we already noted, the constant piece is:

$$\begin{aligned}\hat{\Lambda} &= V_{\text{tree}}(\hat{\phi}, \hat{\chi}) + \Lambda \\ &= \frac{\lambda_\phi}{24}(\hat{\phi}^2 - v_\phi^2)^2 + \frac{\lambda_\chi}{24}(\hat{\chi}^2 - v_\chi^2)^2 + \frac{\lambda_3}{4}(\hat{\phi}^2 - v_\phi^2)(\hat{\chi}^2 - v_\chi^2) + \Lambda\end{aligned}$$

Step 2: Match onto EFT

$$\widehat{\mathcal{L}}(\phi_q, \chi_q; \mu_H) \qquad \hat{\lambda}_i(\hat{\phi}, \hat{\chi}; \mu_H)$$

2) $\downarrow$

$$\mathcal{L}_{\rm EFT}(\phi_q; \mu_H) \qquad \widetilde{\lambda}_i(\hat{\phi}, \hat{\chi}; \mu_H)$$

- Integrate out χ_q and match at $\mu = \mu_H \sim 1$ onto

$$\mathcal{L}_{\rm EFT} = \frac{1}{2}(\partial_\mu \phi_q)^2 - \widetilde{\sigma}\phi_q - \frac{\widetilde{m}^2}{2}\phi_q^2 - \frac{\widetilde{\rho}}{6}\phi_q^3 - \frac{\widetilde{\lambda}}{24}\phi_q^4 - \widetilde{\Lambda} + \mathcal{O}(z^6)$$

- To get the LL series, we need **tree level matching** + 1 loop running.
  - $\hat{\Lambda}$ goes into the EFT cosmological constant, $\widetilde{\Lambda}(\mu_H)$

Step 2: Match onto EFT

$$\widehat{\mathcal{L}}(\phi_q, \chi_q; \mu_H) \qquad \hat{\lambda}_i(\hat{\phi}, \hat{\chi}; \mu_H)$$

$$\xrightarrow{\;2)\;}$$

$$\mathcal{L}_{\rm EFT}(\phi_q; \mu_H) \qquad \widetilde{\lambda}_i(\hat{\phi}, \hat{\chi}; \mu_H)$$

- Integrate out χ_q and match at $\mu = \mu_H \sim 1$ onto

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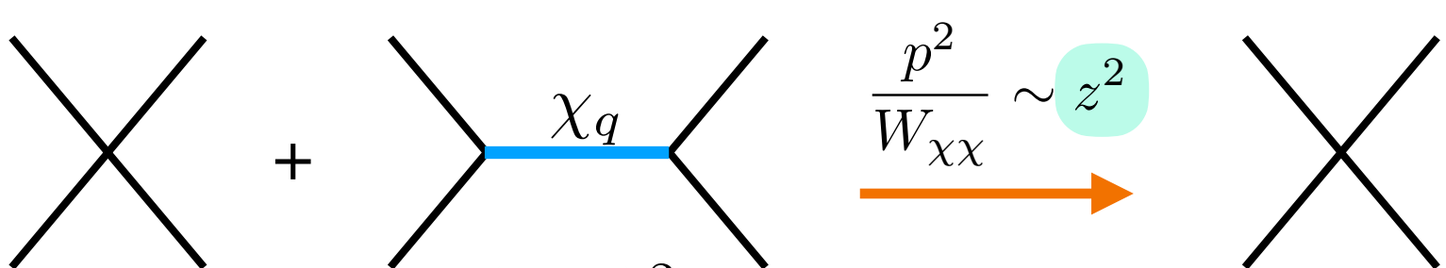
  ✳ Matching $\widetilde{\lambda}(\mu_H)$:



$\hat{\lambda}$ $\qquad \dfrac{(\lambda_3 \hat{\chi})^2}{(p_1+p_2)^2 - W_{\chi\chi}}$ + perm. $\qquad \widetilde{\lambda} \;\Rightarrow\; \widetilde{\lambda} \overset{\mu=\mu_H}{=} \lambda_\phi - \dfrac{9\lambda_3^2 \hat{\chi}^2}{\lambda_\chi v_\chi^2} + (z^2)$

$$\frac{1}{3}\lambda_\chi v_\chi^2 + \mathcal{O}(z^2)$$

Revisiting a subtlety

- At this point, it is clear that we must shift in the high-energy theory before integrating out χ_q.
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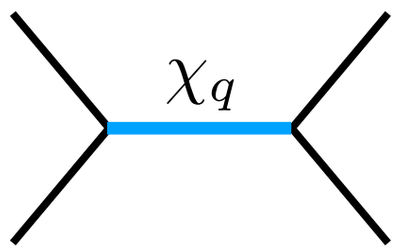
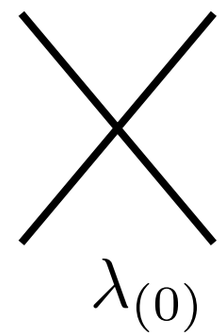


contributes to

match



contributes to

shift

$$\widetilde{m}^2(\mu_H) \supset \frac{\lambda_{(0)}\hat{\phi}^2}{2}$$

There are 6 ways of replacing the 4 ϕ-legs with 2 ϕ_q , 2 $\hat{\phi}$.

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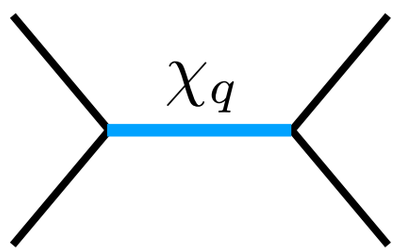
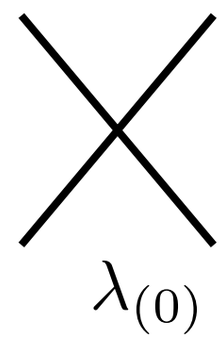


contributes to match

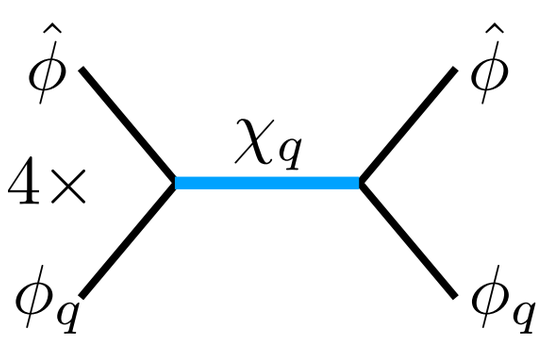


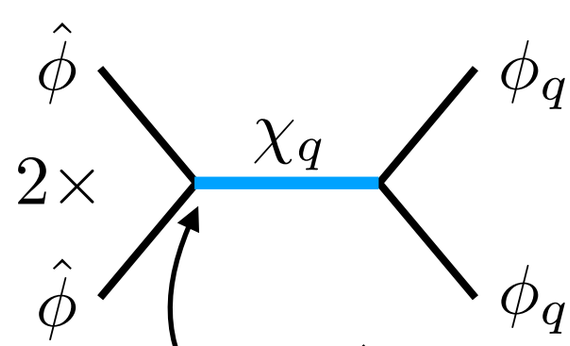
contributes to shift

$$\widetilde{m}^2(\mu_H) \supset \frac{\lambda_{(0)}\hat{\phi}^2}{2}$$

There are 6 ways of replacing the 4 ϕ-legs with 2 ϕ_q, 2 $\hat{\phi}$.

These arise from 2 different topologies at high-energies:





$\Rightarrow$Only 4/6 contribute to the EFT.

No $\hat{\phi}^2\chi_q$ vertex in the full theory!

- The correct answer has $\widetilde{m}^2(\mu_H) \supset -\dfrac{3\lambda_3^2\hat{\chi}^2\hat{\phi}^2}{\lambda_\chi v_\chi^2}$ **not** 3/2 times this.

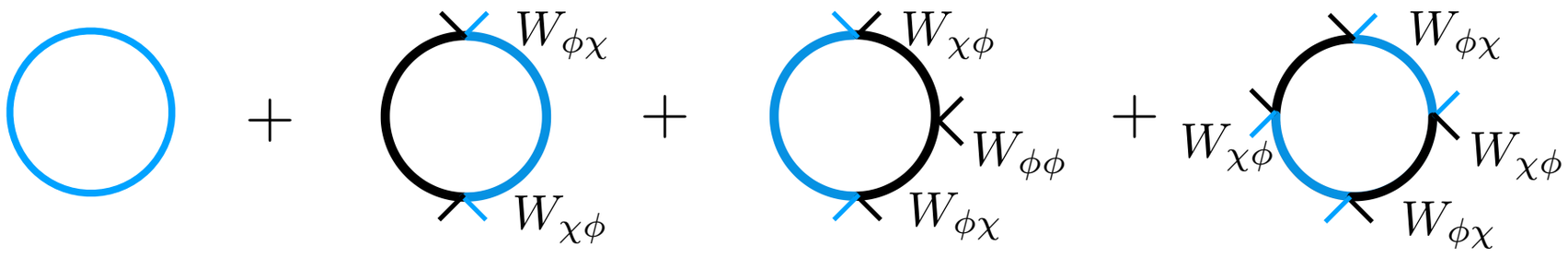
Step 2b: One loop matching

- Since we compute the full 1-loop contribution to $V_{\rm eff}$, we also require 1-loop matching.

$$V_{\rm eff} = \begin{bmatrix} & 1 & & & & & \\ +\lambda & (L & + & 1) & & & \\ +\lambda^2 & (L^2 & + & L & + & 1) & \\ +\lambda^3 & (L^3 & + & L^2 & + & L & + & 1) \end{bmatrix}$$

- This comes from the χ_q vacuum bubble expanded in *z.*

$V_{\rm match}(\mu_H) =$



$+\mathcal{O}(z^6)$

$$= \frac{1}{p^2 - W_{\phi\phi}} = \frac{1}{p^2} + \mathcal{O}(z^2) \quad \Rightarrow$$ All the logs are of the form $\log \frac{W_{\chi\chi}}{\mu_H^2}$

$$= \frac{1}{p^2 - W_{\chi\chi}}$$

Step 2b: One loop matching

$$V_{\rm match}(\mu_H) = \frac{1}{64\pi^2}\left\{ \left[\ln\frac{W_{\chi\chi}}{\mu_H^2} - \frac{3}{2}\right] \overbrace{W_{\chi\chi}^2}^{\sim 1} + \left[\ln\frac{W_{\chi\chi}}{\mu_H^2} - 1\right] 2\overbrace{W_{\chi\phi}W_{\phi\chi}}^{\sim z^2} \right.$$
$$\left. + \left[\ln\frac{W_{\chi\chi}}{\mu_H^2} - 1\right] \frac{2W_{\chi\phi}W_{\phi\chi}W_{\phi\phi}}{\underbrace{W_{\chi\chi}}_{\sim z^4}} - \left[\ln\frac{W_{\chi\chi}}{\mu_H^2} - 2\right] \frac{W_{\chi\phi}^2 W_{\phi\chi}^2}{\underbrace{W_{\chi\chi}^2}_{\sim z^4}} \right\} + \mathcal{O}(z^6)$$

- Intuition: this matches the expansion of the large eigenvalue contribution
- Independent of $\phi_q \Rightarrow$ contributes to the EFT cosmological constant,

$$\widetilde{\Lambda}(\mu_H) = V_{\rm tree}(\mu_H) + V_{\rm match}(\mu_H) - \frac{3\mathcal{J}_\chi^2}{2\lambda_\chi v_\chi^2}$$

- Note: this and the other EFT parameters depend on the source $\mathcal{J}_\chi$, which still needs to be determined! (to discuss in the last step)

Step 3: Run in the EFT

$\mathcal{L}_{\rm EFT}(\phi_q;\mu_H) \quad \widetilde{\lambda}_i(\hat{\phi},\hat{\chi};\mu_H)$

3) ↓

$\mathcal{L}_{\rm EFT}(\phi_q;\mu_L) \quad \widetilde{\lambda}_i(\hat{\phi},\hat{\chi};\mu_L)$

Now, run using the EFT β-functions to a scale $\mu_L \sim z$. We'll need:

$$\widetilde{m}^2(\mu_L) = \widetilde{m}^2(\mu_H)\left\{\eta^{-1/3}\left[1-\frac{1}{2}\xi\right]+\frac{1}{2}\xi\eta^{-1}\right\},$$

$$\widetilde{\Lambda}(\mu_L) = \widetilde{\Lambda}(\mu_H) + \frac{\widetilde{m}^4(\mu_H)}{2\widetilde{\lambda}(\mu_H)}\left\{\frac{1}{3}\left(3-6\xi+2\xi^2\right)-\frac{1}{2}\xi\left(\xi-2\right)\eta^{-1/3} - \frac{1}{4}\left(\xi-2\right)^2\eta^{1/3}+\frac{1}{12}\xi^2\eta^{-1}\right\}$$

$$t = \frac{1}{16\pi^2}\ln\frac{\mu_L}{\mu_H}, \qquad \eta = 1-3\widetilde{\lambda}(\mu_H)t, \qquad \xi = \frac{\widetilde{\rho}^2(\mu_H)}{\widetilde{\lambda}(\mu_H)\widetilde{m}^2(\mu_H)} = \frac{\hat{\phi}^2\widetilde{\lambda}(\mu_H)}{\widetilde{m}^2(\mu_H)}$$

resummed!

Aside: A nice path to the β-functions

$$\left(\beta_i \frac{\partial}{\partial \lambda_i} - \gamma_i \phi_i \frac{\partial}{\partial \phi_i}\right) V_{\text{tree}} + \mu \frac{d}{d\mu} V_{\text{1-loop}} = 0$$

$$\Downarrow$$

$$\boxed{\left(\beta_i \frac{\partial}{\partial \lambda_i} - \gamma_i \hat{\phi}_i \frac{\hat{\partial}}{\partial \hat{\phi}_i}\right) V_{\text{tree}} = \frac{1}{32\pi^2}\left(\text{Tr}W^2 - 2\text{Tr}(M_f^\dagger M_f)^2 + 3\text{Tr}M_v^4\right)}$$

For example, take the two-scalar theory we started with.

- Compute the LHS, collecting powers of $\hat{\phi}, \hat{\chi}$.
- The RHS computes to $\frac{1}{32\pi^2}\left(W_{\phi\phi}^2 + 2W_{\phi\chi}W_{\chi\phi} + W_{\chi\chi}^2\right)$
- Equate powers of $\hat{\phi}$ and $\hat{\chi}$. The result:

$$\beta_{\lambda_\chi} = \frac{3(\lambda_\chi^2 + \lambda_3^2)}{16\pi^2} \qquad \beta_{\lambda_\phi} = \frac{3(\lambda_\phi^2 + \lambda_3^2)}{16\pi^2} \qquad \beta_{\lambda_3} = \frac{\lambda_3(4\lambda_3 + \lambda_\phi + \lambda_\chi)}{16\pi^2}$$

as well as for $\beta_{v_\phi}, \beta_{v_\chi}, \beta_\Lambda \ldots$

Step 4: Finish the path integral

$$\mathcal{L}_{\rm EFT}(\phi_q;\mu_L) \quad \widetilde{\lambda}_i(\hat{\phi},\hat{\chi};\mu_L)$$

4) $\downarrow$

$$V_{\rm eff}(\hat{\phi},\hat{\chi};\mu_L)$$

Perform the functional integral over ϕ_q to quadratic order.

$$V_{\rm eff} = \widetilde{\Lambda}(\mu_L) + \frac{\widetilde{m}^4(\mu_L)}{64\pi^2}\left[\log\frac{\widetilde{m}^2(\mu_L)}{\mu_L^2} - \frac{3}{2}\right] + \mathcal{O}(z^6)$$

- The running required all the EFT interactions, not just the quadratic ones.

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- The running required all the EFT interactions, not just the quadratic ones.

Conceptual subtlety 3: We did NOT compute $V_{\rm eff}$ of the EFT. There is no sense in which we shift ϕ in the EFT to compute its 1-loop effective potential.

Step 5: Tadpole running

$$\langle \chi_q \rangle \stackrel{!}{=} 0 \xrightarrow{\text{5)}} V_{\text{eff}}(\hat{\phi}, \hat{\chi}; \mu_L)$$

We started with $e^{\frac{i}{\hbar}\Gamma[\hat{\phi},\hat{\chi}]} = \int D\chi_q D\phi_q e^{\frac{i}{\hbar}\int d^4x \, (\mathcal{L}|_{\text{shift}-\text{linear}} + \underbrace{\mathcal{J}_\phi \phi_q + \mathcal{J}_\chi \chi_q}_{\widehat{\mathcal{L}}})}$

$$\int D\phi_q e^{\frac{i}{\hbar}\int d^4x \, \mathcal{L}_{\text{EFT}}}$$

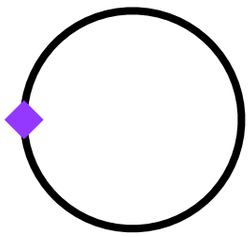
- Even though χ_q has been integrated out, we can determine $\mathcal{J}_\chi$ via

$$\frac{\delta}{\delta \mathcal{J}_\chi} \int D\phi_q e^{i\int d^4x \mathcal{L}_{\text{EFT}}} = 0 \quad \Rightarrow \quad \langle \chi_q \rangle = 0$$

(the same for $\mathcal{J}_\phi$ gives $\mathcal{J}_\phi = 0$)

This results in $\mathcal{J}_\chi(\mu_H) = \frac{1}{2}\lambda_3(\mu_H)\hat{\chi}\langle[\phi_q^2]\rangle_{\mu_H} + \lambda_3(\mu_H)\hat{\chi}\hat{\phi}\cancel{\langle\phi_q\rangle}_{\mu_H}$

Step 5: Tadpole running

- At low energies, we compute the matrix element 

$$\langle [\phi_q^2] \rangle_{\mu_L} = \frac{\widetilde{m}^2}{16\pi^2} \left(\log \frac{\widetilde{m}^2}{\mu_L^2} - 1 \right)$$

- To RG improve, integrate the operator anomalous dimension

$$\mu \frac{d}{d\mu} [\phi_q^2] = \frac{1}{16\pi^2} \left(-\tilde{\lambda} [\phi_q^2] - 2\widetilde{\rho}\phi_q - 2\widetilde{m}^2 \right)$$

to obtain $\langle [\phi_q^2] \rangle_{\mu_H}$, and plug into $\mathcal{J}_\chi$.

Checking the answer

$$V_{\rm eff} = \widetilde{\Lambda}(\mu_L) + \frac{\widetilde{m}^4(\mu_L)}{64\pi^2}\left[\log\frac{\widetilde{m}^2(\mu_L)}{\mu_L^2} - \frac{3}{2}\right]$$

- If we neglect the running and compute at a single scale $\mu_L = \mu_H$, this result reduces to the 1-loop fixed order potential (expanded in *z*).

$$\widetilde{\Lambda}(\mu_L) \to \widetilde{\Lambda}(\mu_H) = V_{\rm tree}(\mu_H) + V_{\rm match}(\mu_H)$$

$$\widetilde{m}^2(\mu_L) \to m_-^2(\mu_H)$$

and $V_{\rm eff}$ reduces to the tree level potential + 1-loop contributions

$$\sim m_\pm^4 \log\frac{m_\pm^2}{\mu_H^2}$$

Checking the answer

$$V_{\text{eff}} = \widetilde{\Lambda}(\mu_L) + \frac{\widetilde{m}^4(\mu_L)}{64\pi^2}\left[\log\frac{\widetilde{m}^2(\mu_L)}{\mu_L^2} - \frac{3}{2}\right]$$

- The 2-loop LL term is the first nontrivial check that integrating 1-loop RGE correctly reproduces a 2-loop term. (Comes entirely from $\widetilde{\Lambda}(\mu_L)$)
- Expand in $t = \frac{1}{16\pi^2}\log\frac{\mu_L}{\mu_H}$, and compare the $\mathcal{O}(t^2)$ term to existing 2-loop fixed order results [Ford, Jones '92] expanded in *z.*

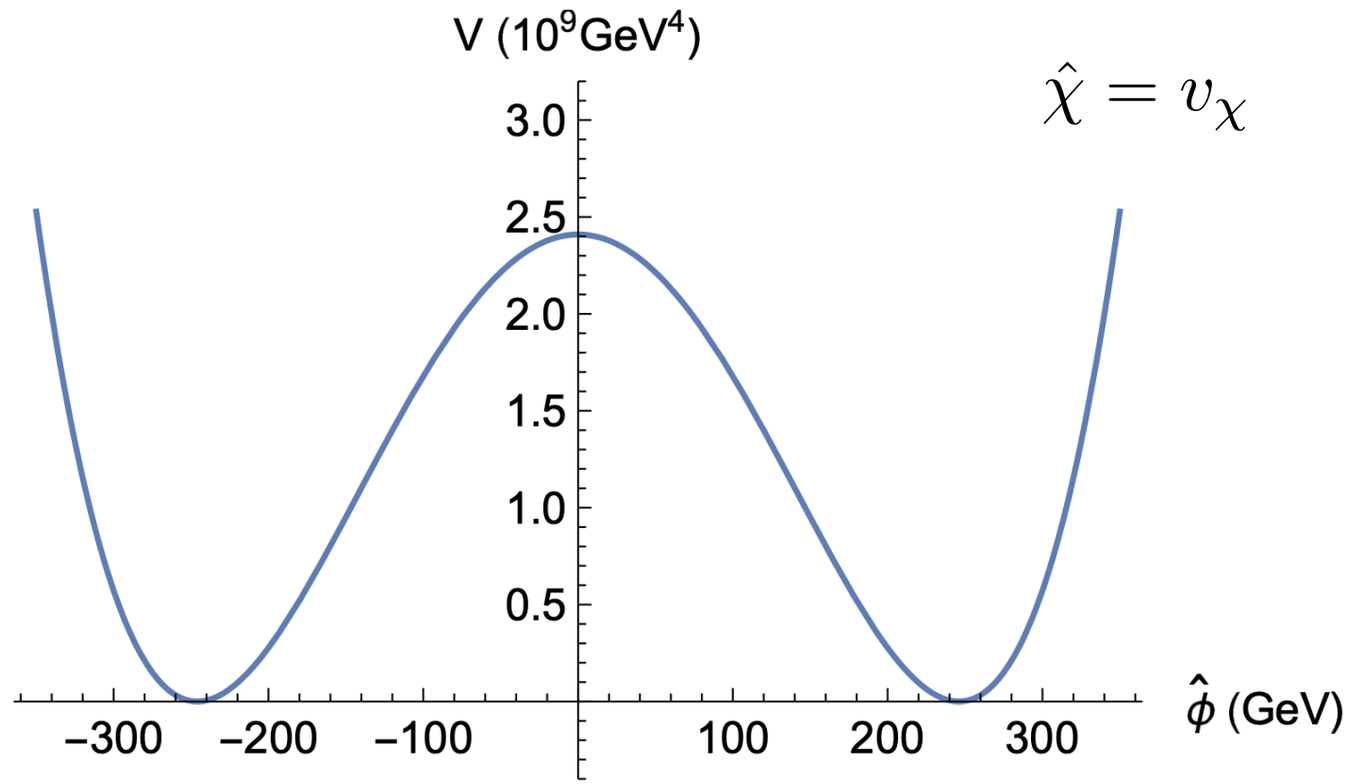
  ✳ Result: Perfect agreement.
  In particular, **all** the subtleties are crucial to getting this right.

Fun with plots

Choose starting values: $\frac{\lambda_\phi(\mu_H)}{16\pi^2} = 0.1$ $\frac{\lambda_\chi(\mu_H)}{16\pi^2} = 0.25$ $\frac{\lambda_3(\mu_H)}{16\pi^2} = 0.04$

$$v_\phi(\mu_H) = 246 \text{ GeV} \qquad v_\chi(\mu_H) = 500 \text{ TeV}$$

Then, the tree level effective potential is:



A numerical instability

With our input parameters satisfying $v_\phi \ll v_\chi$, the 1-loop $\mathcal{O}(z^2)$ terms in $V_{\rm match}(\mu_H)$ overwhelm the tree level potential and $\mathcal{O}(z^4)$ terms.

$$
\begin{aligned}
V_{\rm match}(\mu_H) = \frac{1}{64\pi^2}\Bigg\{ & \\
&\left[\ln\frac{W^{(0)}_{\chi\chi}}{\mu_H^2} - \frac{3}{2}\right]\left[W^{(0)}_{\chi\chi}\right]^2 + \left[\ln\frac{W^{(0)}_{\chi\chi}}{\mu_H^2} - 1\right] 2W^{(0)}_{\chi\chi}W^{(2)}_{\chi\chi} && + \left[\ln\frac{W^{(0)}_{\chi\chi}}{\mu_H^2}\right]\left[W^{(2)}_{\chi\chi}\right]^2 \\
&+0 && + \left[\ln\frac{W^{(0)}_{\chi\chi}}{\mu_H^2} - 1\right] 2W_{\phi\chi}W_{\chi\phi} && + \frac{2W_{\phi\chi}W_{\chi\phi}W^{(2)}_{\chi\chi}}{W^{(0)}_{\chi\chi}} \\
&+0 && +0 && + \left[\ln\frac{W^{(0)}_{\chi\chi}}{\mu_H^2} - 1\right]\frac{2W_{\phi\chi}W_{\chi\phi}W_{\phi\phi}}{W^{(0)}_{\chi\chi}} \\
&+0 && +0 && - \left[\ln\frac{W^{(0)}_{\chi\chi}}{\mu_H^2} - 2\right]\left(\frac{W_{\phi\chi}W_{\chi\phi}}{W^{(0)}_{\chi\chi}}\right)^2 \Bigg\} \\
&\sim 1 && \sim z^2 && \sim z^4
\end{aligned}
$$

A numerical instability

A simple fix is to make a judicious choice of matching scale:

$$\mu_H^2 = \frac{1}{e} W_{\chi\chi}^{(0)} = \frac{1}{3e} \lambda_\chi v_\chi^2$$

$$V_{\text{match}}(\mu_H) = \frac{1}{64\pi^2} \Bigg\{ \left[\ln \frac{W_{\chi\chi}^{(0)}}{\mu_H^2} - \frac{3}{2}\right] \left[W_{\chi\chi}^{(0)}\right]^2 + \left[\ln \frac{W_{\chi\chi}^{(0)}}{\mu_H^2} - \cancel{1}\right] 2W_{\chi\chi}^{(0)} W_{\chi\chi}^{(2)} + \left[\ln \frac{W_{\chi\chi}^{(0)}}{\mu_H^2}\right] \left[W_{\chi\chi}^{(2)}\right]^2$$

$$+0 + \left[\ln \frac{W_{\chi\chi}^{(0)}}{\mu_H^2} - \cancel{1}\right] 2W_{\phi\chi} W_{\chi\phi} + \frac{2W_{\phi\chi} W_{\chi\phi} W_{\chi\chi}^{(2)}}{W_{\chi\chi}^{(0)}}$$

$$+0 \quad +0 \quad + \left[\ln \frac{W_{\chi\chi}^{(0)}}{\mu_H^2} - \cancel{1}\right] \frac{2W_{\phi\chi} W_{\chi\phi} W_{\phi\phi}}{W_{\chi\chi}^{(0)}}$$

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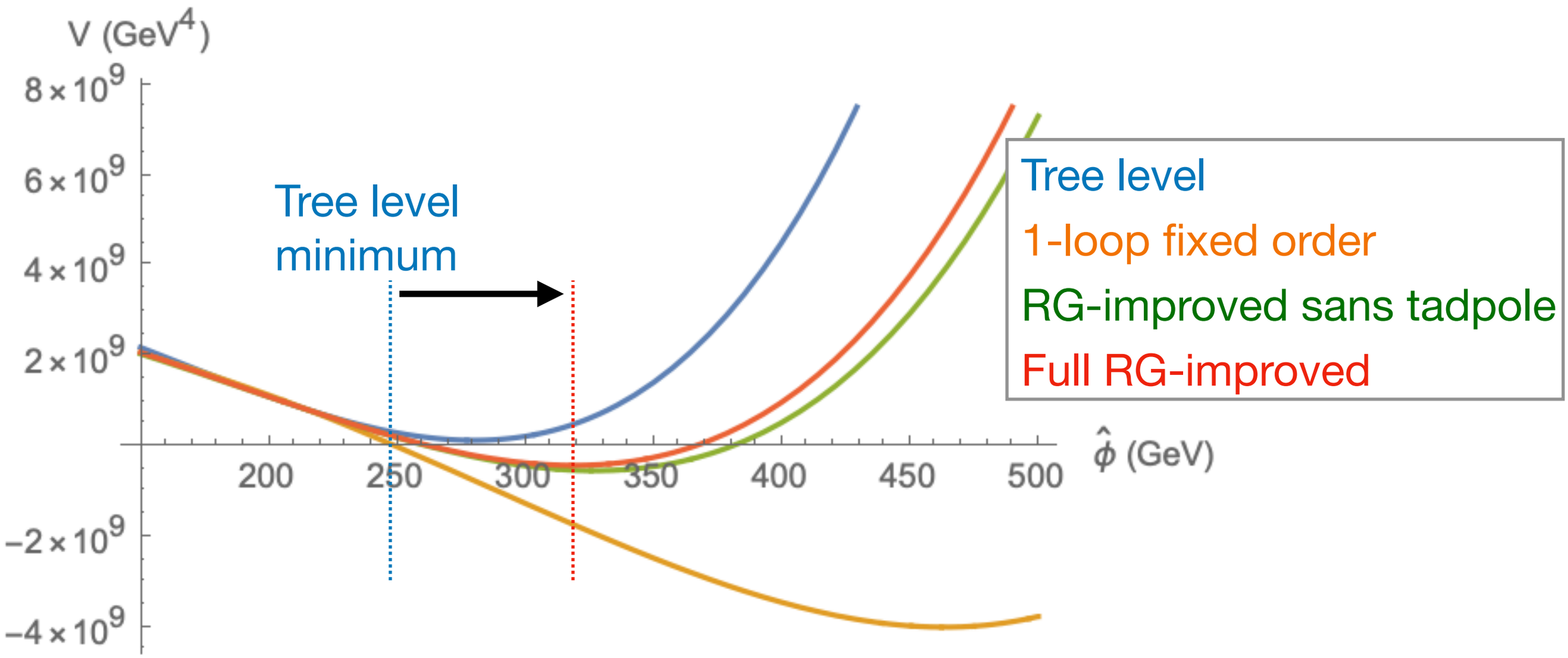
$$\sim 1 \qquad \sim z^2 \qquad \sim z^4$$

Fun with plots



Plot for $\hat{\chi}$ = its value at full RG improved minimum.

More results

We computed the RG improved 1-loop potentials in several examples, and in all cases found agreement with 2-loop fixed order. [Ford, Jones '92] [Martin '02]

- 2-scalar theory in the unbroken phase.

$$W_{\chi\chi} \sim 1 \qquad W_{\phi\phi} \sim z^2 \qquad \hat{\chi} \sim \hat{\phi} \sim W_{\phi\chi} \sim z$$

- O(*N*) model in the broken phase.

  N-1 light Goldstone modes $\vec{\phi}$, heavy radial mode with vev $\hat{\chi}$

$$\hat{\chi} \sim W_{\chi\chi} \sim 1 \qquad (\hat{\chi}^2 - v^2) \sim W_{\phi\phi} \sim z^2$$

- Higgs-Yukawa model. 2 cases:
  - ✳ Fermion much heavier than boson: $W \sim \lambda \sim z^2 \qquad \hat{\phi} \sim m_F(\hat{\phi}) \sim 1$
  - ✳ Boson much heavier than fermion: $W \sim 1 \qquad \hat{\phi} \sim m_F(\hat{\phi}) \sim z$

- O(2) coupled to a fermion: we show the result has no Goldstone boson catastrophe. [Martin '13, '14][Elias-Miro, Espinosa, Konstandin '14]

Thank you!

Additional Slides

Matching in the 2-scalar theory

$$\widetilde{\lambda} \overset{\mu=\mu_H}{=} \lambda_\phi - \frac{9\lambda_3^2\hat{\chi}^2}{\lambda_\chi v_\chi^2} + \mathcal{O}(z^2)$$

$$\widetilde{\rho} \overset{\mu=\mu_H}{=} \hat{\phi}\left[\lambda_\phi - \frac{9\lambda_3^2\hat{\chi}^2}{\lambda_\chi v_\chi^2}\right] + \mathcal{O}(z^3)$$

$$\widetilde{m}^2 \overset{\mu=\mu_H}{=} \frac{1}{6}\lambda_\phi\left(3\hat{\phi}^2 - v_\phi^2\right) + \frac{1}{2}\lambda_3\left(\hat{\chi}^2 - v_\chi^2\right) - \frac{3\lambda_3^2\hat{\chi}^2\hat{\phi}^2}{\lambda_\chi v_\chi^2} + \frac{3\lambda_3\hat{\chi}\mathcal{J}_\chi}{\lambda_\chi v_\chi^2} + \mathcal{O}(z^4)$$

$$\widetilde{\sigma} \overset{\mu=\mu_H}{=} -\mathcal{J}_\phi + \frac{3\lambda_3\hat{\chi}\hat{\phi}\mathcal{J}_\chi}{\lambda_\chi v_\chi^2} + \mathcal{O}(z^5)$$

$$\widetilde{\Lambda} \overset{\mu=\mu_H}{=} \hat{\Lambda} + V_{\text{match}} - \frac{3\left(\mathcal{J}_\chi\right)^2}{2\lambda_\chi v_\chi^2} + \mathcal{O}(z^6)$$