Compton Scattering Total Cross Section at NLO

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PhysRevLett.126.211801; arXiv: 2102.06718

with Roman Lee and Matthew Schwartz

Compton Scattering: witnesses the development of QFT

$$e^-(p_1) + \gamma(k_1) \to e^-(p_2) + \gamma(k_2)$$

- One of the fundamental processes in both quantum mechanics and quantum electrodynamics (QED)

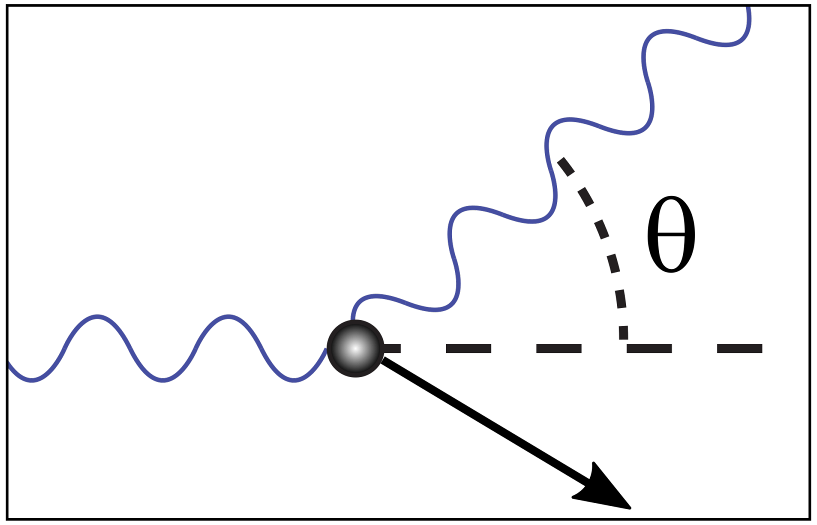
- Thomson scattering: elastic scattering in classical EM

$$\sigma = \frac{8\pi\alpha^2}{3m^2}$$

- Compton effect (1923): quantum effect

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

The photon always loses energy, unless $\theta = 0$



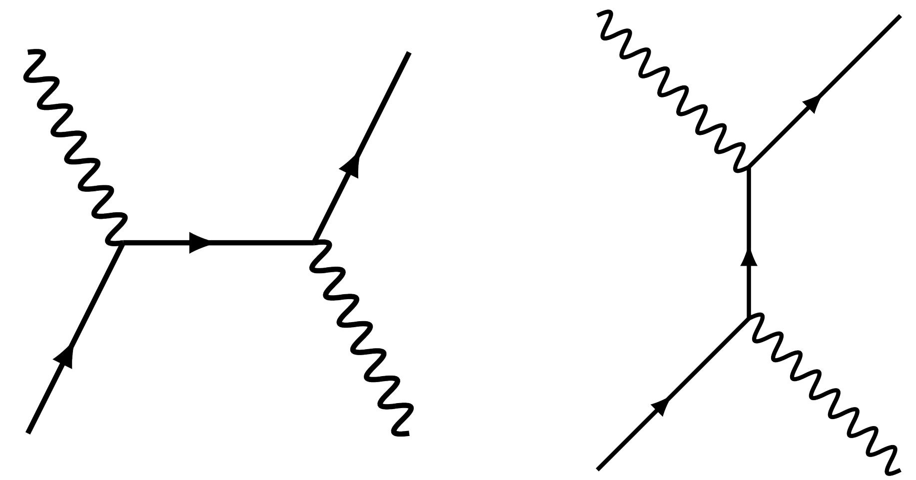
$e^-\gamma \to e^-\gamma$ [wiki]

What's the quantum version of cross section?

Compton Scattering: essential to prove Dirac equation

- Klein-Nishina formula (1929):

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right)$$

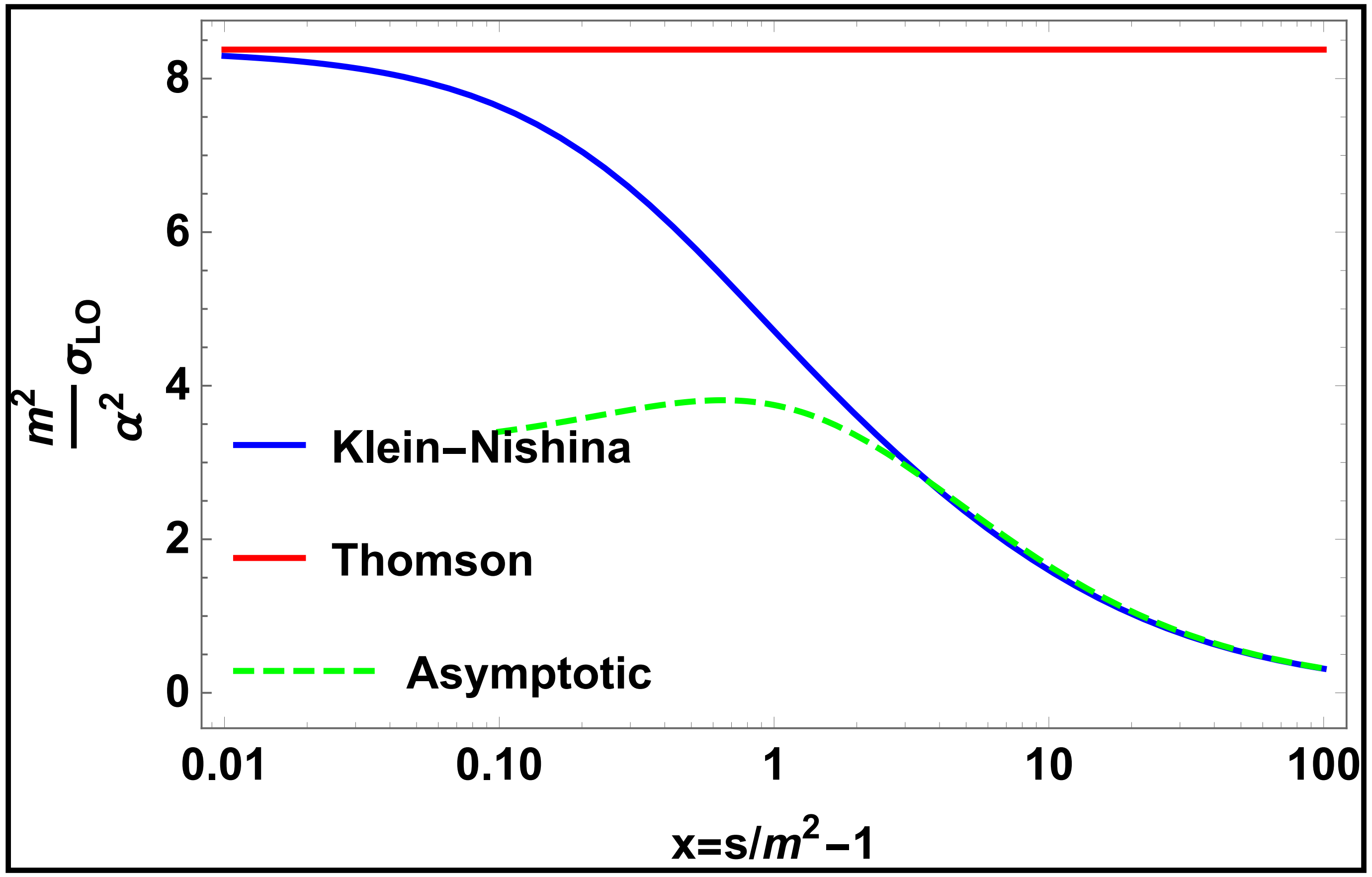


$$\sigma = \pi\alpha^2\left(\frac{2\left(3m^4 + 6m^2 s - s^2\right)\log\left(\frac{s}{m^2}\right)}{\left(m^2 - s\right)^3} + \frac{m^6 - m^4 s + 15m^2 s^2 + s^3}{s^2\left(m^2 - s\right)^2}\right) + \mathcal{O}(\alpha^3)$$

At high energies, $s \gg m^2$

$$\sigma \sim \frac{\pi\alpha^2}{s}\left[2\log\left(\frac{s}{m^2}\right) + 1\right]$$

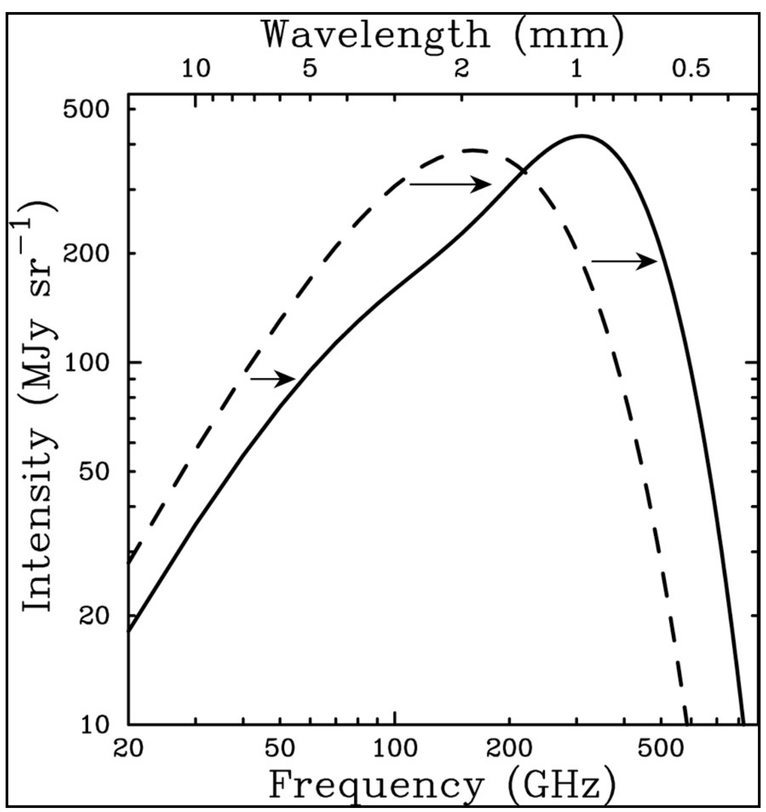
One of the first results in QED!



Motivation for studying Compton scattering

- Important in many aspects of physics: from X-ray crystallography to astrophysics
  - A luminosity monitor for the electron-photon collider
  - A clean process: to measure the coupling constant
  - In astrophysics: inverse Compton scattering

e.g. Sunyaev–Zeldovich effect



[Sunyaev, Zeldovich, 1980]

Theoretical side:

- the fundamental question: what is an electron?
- IR finite total cross sections in QED/QCD: forward scattering and resummation

Motivation for studying Compton scattering: Forward scattering

[1810.10022, Frye, Hannesdottir, Paul, Schwartz, Yan]

- There is already a single log at the tree-level.

$$\sigma \sim \frac{\pi\alpha^2}{s}\left[2\log\left(\frac{s}{m^2}\right)+1\right], \quad s \gg m^2$$

- Usually we expect a log to show up at 1-loop, and it can be resummed with RG equations. For example, we introduce the running coupling to improve the efficiency in QED.

- Use dim reg instead and set $m = 0$, we see explicit divergence in the t-channel: $d = 4 - 2\epsilon$

$$\sigma_t = \frac{16\pi\alpha^2}{Q^2}\Gamma_d\left(-\frac{1}{2\epsilon}+1\right), \quad \text{with} \quad \Gamma_d = \left(\frac{4\pi e^{-\gamma_E}\mu^4}{Q^2}\right)^{\epsilon}$$

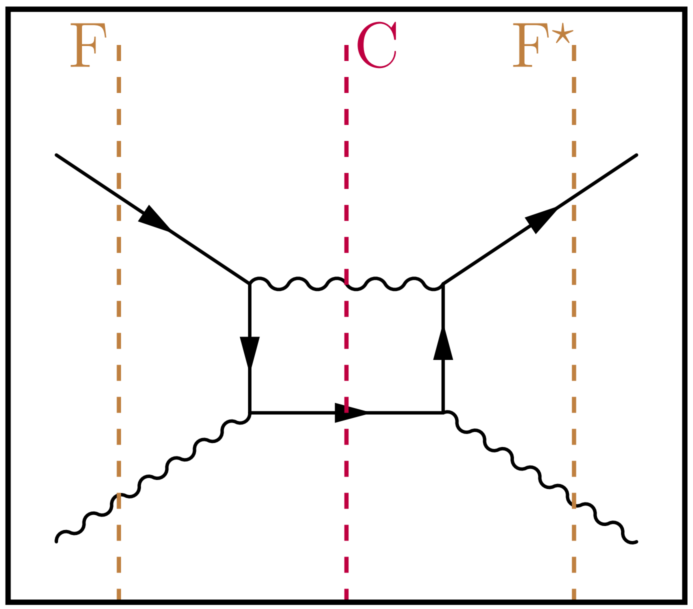
IR divergence comes from the outgoing γ collinear to the incoming e^-

Collinear Logarithms

Motivation for studying Compton scattering: Forward scattering

[1810.10022, Frye, Hannesdottir, Paul, Schwartz, Yan]

- IR finiteness requires the forward scattering included, where outgoing γ collinear to the incoming γ



$$\sigma_t = \frac{16\pi\alpha^2}{Q^2}\Gamma_d\left(-\frac{1}{2\epsilon}+1\right)$$

$$\sigma_F = \frac{16\pi\alpha^2}{Q^2}\Gamma_d\left(\frac{1}{2\epsilon}-1\right)$$

cancel!

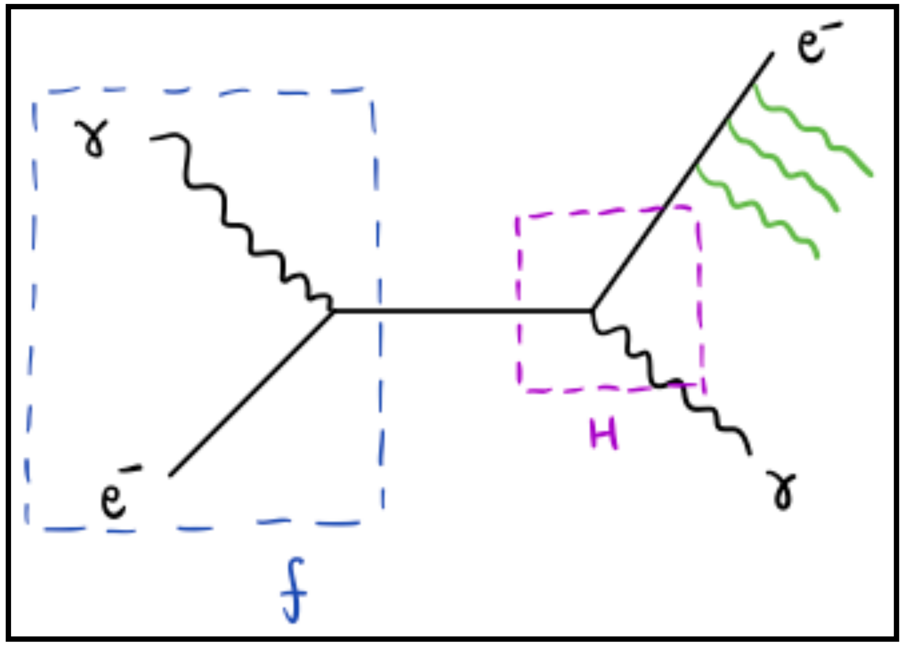
A hard photon and electron become effectively indistinguishable at high energies

- Kinoshita-Lee-Nauenberg (KLN) theorem: Unitarity guarantees the cancellation of infrared divergences when all final states and initial states are summed over.
- However, one only need to sum over initial or final states once the forward scattering is included.

Motivation for studying Compton scattering: Resummation

- If we include forward scattering and redefine the cross section, we would get something correct but not interesting.
- Alternatively, we can try to resume the logs using SCET. But it is hard to write down a factorization formula for total cross section

$$e^- + \gamma \to e^- + \gamma \; (+n\gamma)$$



- Do DGLAP equations reproduce all logarithms?
- To see the logs and resum them, we need to calculate NLO.

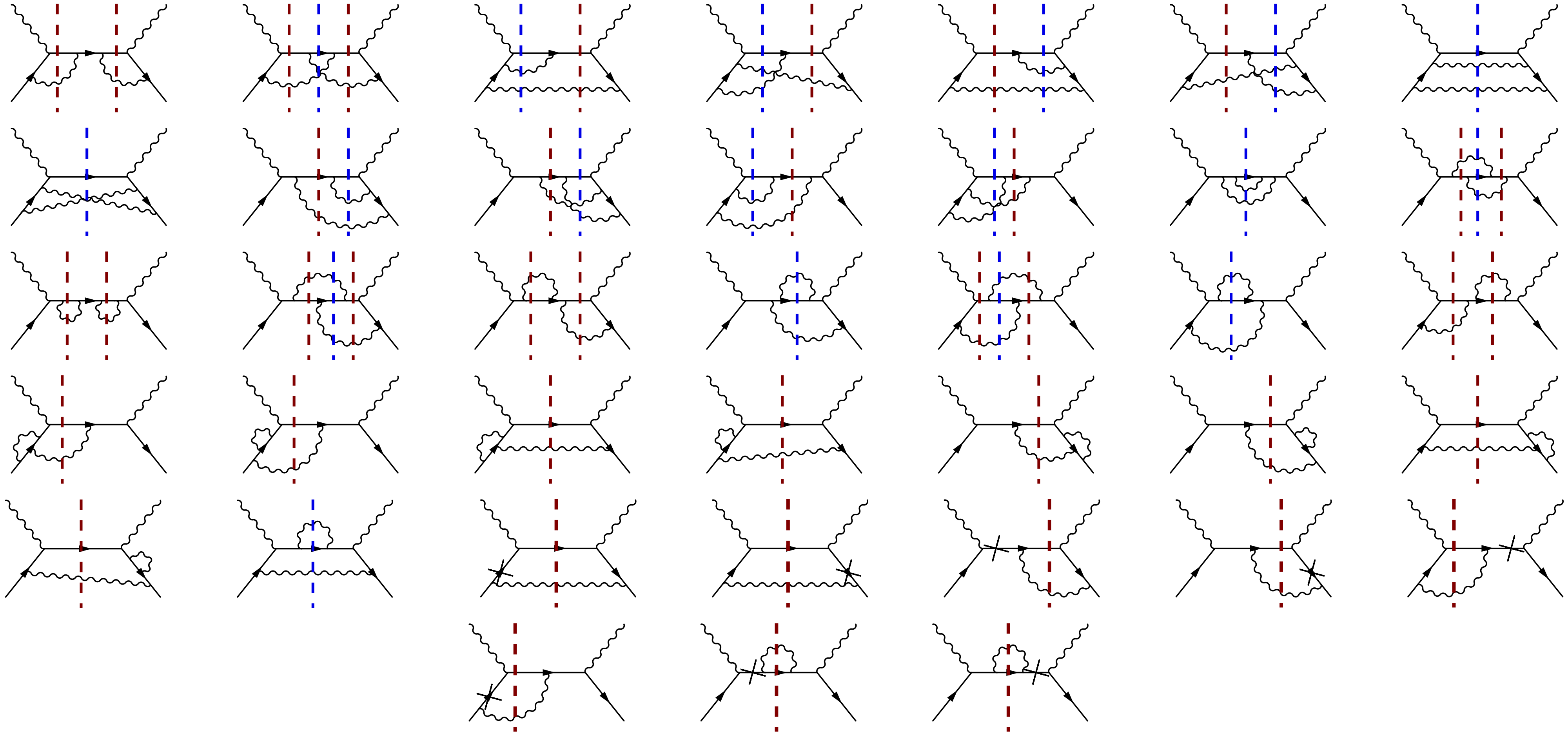
Compton scattering beyond leading order

- There are very few analytic results of the total cross section beyond LO in QED. For Compton,
  - real emissions: $e^-(p_1) + \gamma(k_1) \to e^-(p_2) + \gamma(k_2) + \gamma(k_3)$
  - virtual corrections: $e^-(p_1) + \gamma(k_1) \overset{1-loop}{\to} e^-(p_2) + \gamma(k_2)$
  - pair productions:
    $e^-(p_1) + \gamma(k_1) \to e^-(p_2) + e^+(k_2) + e^-(k_3), e^-(p_2) + \mu^+(k_2) + \mu^-(k_3) \ldots$
- Several developments:
  - Brown and Feynman (1951): the virtual differential cross section regulated by the photon mass
  - Mandl and Skyrme (1952): double Compton scattering: the amplitudes for the hard-photonic bremsstrahlung
  - Milton, Tsai, De Raad (1972), Gongora-T. and Stuart (1989), Veltman (1989), Swartz, hep-ph/9711447 — polarized scattering
  - Denner and Dittmaier, hep-ph/9805443 — numerical total cross section (Monte Carlo)
  - Lee, Lyubyakin, Stotsky, 2010.15430 — first analytic result for real emissions and pair productions!

Compton scattering beyond leading order

Failure: discontinuities of forward amplitude

- The main difficulties:
  - two-loop massive diagrams are hard to evaluate, even after region expansions
  - cannot separate different processes (difficult to separate elliptical sectors)



Compton scattering beyond leading order

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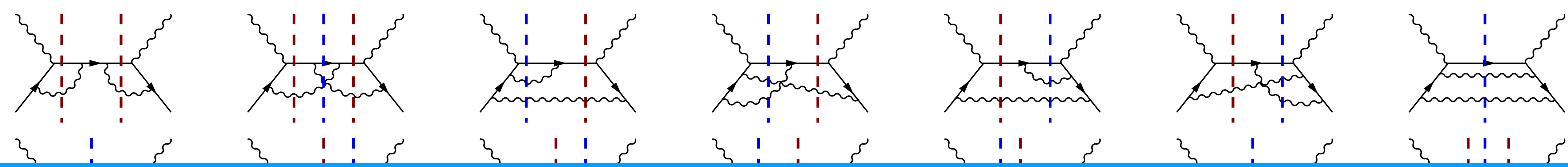
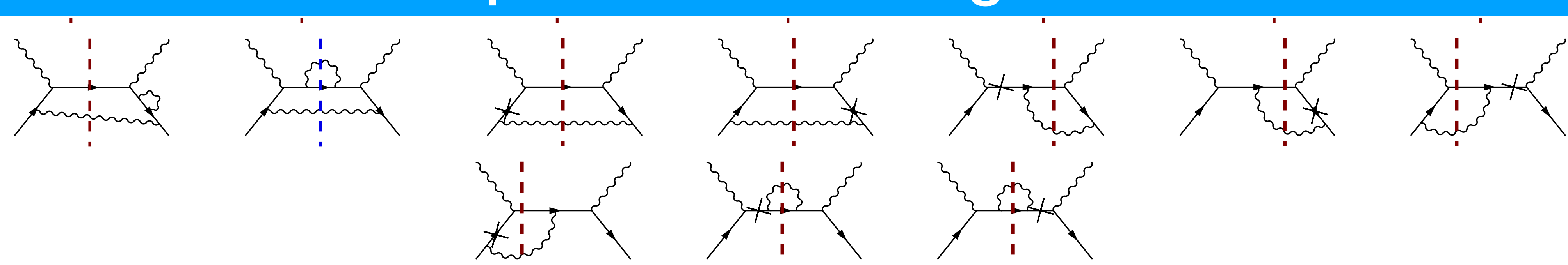


**Instead we use direct phase space integrations,
and
evaluate the forward diagrams in Fiesta to check our phase space master integrals.**

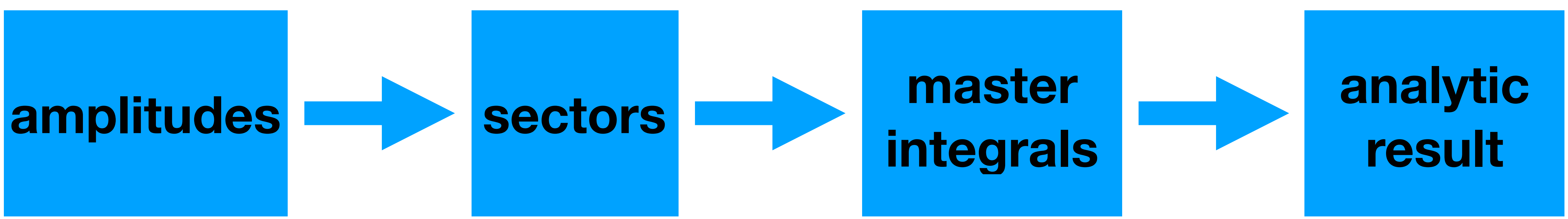


Introduction to multi-loop techniques

[Two loop integrals and QCD scattering, Anastasiou]

[Evaluating Feynman Integrals, Smirnov]

- As mentioned above, there are two kinds of integrals:
  - loop integrals: amplitudes
  - phase space integrals: cross sections, event shape observables



topology classification/ decomposition
[1111.0868, Pak]

Integration by parts (IBP)
[Chetyrkin, Tkachov, 1981]
[0102033, Laporta]

Differential equations (DE)
+boundary conditions

Next: leading order Compton as a warm-up example

Introduction to multi-loop techniques

amplitudes

Loop: $$\int \frac{d^d k_1}{i\pi^{d/2}} \cdots \int \frac{d^d k_m}{i\pi^{d/2}} \frac{f(1; k_1^\mu; k_2^\mu; k_1^\mu k_2^\nu; \cdots)}{A_1^{\nu_1} \cdots A_n^{\nu_n}}$$

[0207004, Anastasiou, Melnikov] [0306192,Anastasiou, Dixon, Melnikov, Petriello]

Phase space: $$2i\pi\delta(p^2 - m^2) \to \frac{1}{p^2 - m^2 + i0} - \frac{1}{p^2 - m^2 - i0}$$

Jet physics: Nontrivial constraints on the phase space; Nonlinear propagators; case by case

e.g. Energy correlators: Measurement functions as nonlinear cut propagators

[1801.03219, Dixon, Luo, Shtabovenko, Yang, Zhu] [2108.01674, Li, Schrijnder van Velzen, Waalewijn, Zhu]

Make it "look like" a loop integral (only) for IBP reductions

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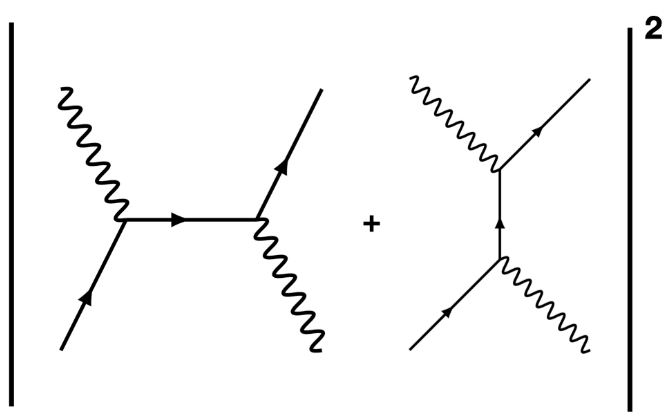

$e^-(p_1) + \gamma(k_1) \to e^-(p_2) + \gamma(k_2)$

$$\sigma = \frac{e^4}{2(s - m^2)} \times \frac{1}{4} \times \int \text{dLIPS}_2 \sum |\mathcal{M}|^2$$

$$\text{dLIPS}_2 = \frac{d^{d-1}k_2}{(2\pi)^{d-1}(2\omega_2)} \frac{d^{d-1}p_2}{(2\pi)^{d-1}(2E_2)} (2\pi)^d \delta^d(k_1 + p_1 - k_2 - p_2)$$

- Squared amplitudes for LO: generated by QGRAF or FeynCalc

Introduction to multi-loop techniques

sectors

[1111.0868, Pak]

- **topology classification**: different loop momentum shifts or external momentum renamings could give the same Feynman integrals.
- This classifies the amplitudes into different sectors.

$$\int \frac{d^d k_1}{i\pi^{d/2}} \cdots \int \frac{d^d k_m}{i\pi^{d/2}} \frac{f(1; k_1^\mu; k_2^\mu; k_1^\mu k_2^\nu; \cdots)}{A_1^{\nu_1} \cdots A_n^{\nu_n}}$$

- For Compton LO: decomposed into 1 sector

$$\frac{1}{[k_2^2]^{\nu_1}[(k_1+p_1-k_2)^2-m^2]^{\nu_2}[(k_2-p_1)^2-m^2]^{\nu_3}}$$

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master integrals

Integration by parts (IBP): a standard process that reduces the amplitude into a minimal set of integrals

IBP packages: LiteRed2

$$\int \text{dLIPS}_2 \sum |\mathcal{M}|^2 = -\frac{(d-2)\left(3ds^3-7ds^2+5ds-d-14s^3-2s^2-18s+2\right)}{(s-1)^2 s}\mathcal{J}_1 - \frac{2(d-2)\left(ds^2-2ds+d-2s^2-4s-10\right)}{s-1}\mathcal{J}_2$$

with $m=1$

- 2 master integrals: $\mathcal{J}_1 = \int \text{dLIPS}_2 \times 1, \qquad \mathcal{J}_2 = \int \text{dLIPS}_2 \times \frac{1}{(p_1-k_2)^2-m^2}$

Introduction to multi-loop techniques

[Evaluating Feynman Integrals, Smirnov]

master integrals → **Analytic results**

$$\mathcal{J}_1 = \int \mathrm{dLIPS}_2 \times 1 = \frac{4^{2-d}\pi^{\frac{3}{2}-\frac{d}{2}} s^{1-\frac{d}{2}} \left(s-m^2\right)^{d-3}}{\Gamma\left(\frac{d-1}{2}\right)}$$

$$\mathcal{J}_2 = \int \mathrm{dLIPS}_2 \times \frac{1}{(p_1-k_2)^2-m^2}$$

$$= -\frac{2^{3-2d}\pi^{\frac{3}{2}-\frac{d}{2}} s^{1-\frac{d}{2}} \left(m^2+s\right)\left(s-m^2\right)^{d-4}}{m^2\Gamma\left(\frac{d-1}{2}\right)} \left((d-2)\,{}_2F_1\left(-\frac{1}{2},1;\frac{d-1}{2};\frac{\left(m^2-s\right)^2}{\left(m^2+s\right)^2}\right)-d+3\right)$$

- **Differential Equations:** The basic idea is to form a close differential equation system satisfied by these master integrals.
- With boundary conditions, we obtain the analytic results instead of calculating the integrals directly.

[1412.2296, Henn]

$$\frac{d}{ds}\vec{J}(s;\cdots) = M(s;\cdots)\cdot\vec{J}(s;\cdots)$$

Differential equations (DE)

[1412.2296, Henn]

First, introduce $y = \sqrt{\frac{s-1}{s+3}}$

with $m = 1, d = 4 - 2\epsilon$

$$\frac{d}{dy}\begin{pmatrix}\mathcal{J}_1\\ \mathcal{J}_2\end{pmatrix} = \begin{pmatrix}\frac{2\left(2\epsilon y^2+2\epsilon+y^2-1\right)}{(y-1)y(y+1)\left(3y^2+1\right)} & 0\\ \frac{2\epsilon-1}{2y^3} & \frac{2}{(y-1)y(y+1)}\end{pmatrix}\begin{pmatrix}\mathcal{J}_1\\ \mathcal{J}_2\end{pmatrix}$$

Perform the transformation: $\vec{\mathcal{J}} = T\vec{\mathcal{G}}, \quad T = \frac{e^{-\gamma\epsilon}(2\pi)^{\epsilon-1}}{1-2\epsilon}\begin{pmatrix}\frac{y^2}{3y^2+1} & 0\\ 0 & \frac{(2\epsilon-1)(y-1)(y+1)}{\epsilon y^2}\end{pmatrix}$

ϵ-form / canonical form DE

$$\frac{d}{dy}\begin{pmatrix}\mathcal{G}_1\\ \mathcal{G}_2\end{pmatrix} = \epsilon\begin{pmatrix}-\frac{4\left(y^2+1\right)}{-3y^5+2y^3+y} & 0\\ \frac{y}{6y^4-4y^2-2} & 0\end{pmatrix}\begin{pmatrix}\mathcal{G}_1\\ \mathcal{G}_2\end{pmatrix}$$

Solutions: (UT)

$$\mathcal{G}_i(y,\epsilon) = \sum_{n=0} \epsilon^n g_i^{(n)}(y)$$

solve the equation order by order in ϵ

$$\mathcal{G}_1 = c_{0,1} + \epsilon\Big(c_{0,1}\log(1+y) - 4c_{0,1}\log(y) + c_{0,1}\log(1-y) + c_{0,1}\log\left(1-i\sqrt{3}y\right) + c_{0,1}\log\left(1+i\sqrt{3}y\right) + c_{1,1}\Big) + \mathcal{O}(\epsilon^2)$$

In forms of Polylogarithms!

Boundary conditions

- The most difficult piece to use DE method: usually vary case by case
- In general, we pick a kinematic point and calculate the master integrals (numerically or analytically), and then determine the unknown constants in the solution.
- For Compton LO, we evaluate the integral at $s \to m^2 = 1$ or $y \to 0$:

$$\mathcal{J}_1 = \int \text{dLIPS}_2 \times 1 \sim \int \text{dLIPS}_2^{(soft)}$$

$$\mathcal{J}_2 = \int \text{dLIPS}_2 \times \frac{1}{(p_1 - k_2)^2 - m^2} \sim \int \text{dLIPS}_2^{(soft)} \times \frac{1}{-4y^2}$$

The only integral we need to do:

The soft limit of 2-particle phase space:

$$\frac{(2\pi)^2}{2\pi^{d/2}} \int d^d k_2 d^d p_2 \delta^+(k_2^2) \delta^+(p_2^2 - 1) \delta^{(d)}(p_1 + k_1 - p_2 - k_2) \approx \frac{\pi^{3/2} 2^{1-2\epsilon}}{\Gamma(3/2 - \epsilon)} y^{2-4\epsilon} + \mathcal{O}(y^2)$$

Boundary conditions

- The asymptotic results of the master integrals help determine the unknown constants in the solution of DE:

$$\mathscr{G}_1 = \frac{\sqrt{\pi}2^{-\epsilon-1}e^{\gamma\epsilon}(1-2\epsilon)y^{-4\epsilon}}{\Gamma\left(\frac{3}{2}-\epsilon\right)} - \frac{\sqrt{\pi}2^{-\epsilon}e^{\gamma\epsilon}\epsilon(2\epsilon-1)y^{2-4\epsilon}}{\Gamma\left(\frac{3}{2}-\epsilon\right)} + \mathcal{O}(y^3)$$

$$\mathscr{G}_2 = -\frac{\sqrt{\pi}2^{-\epsilon-3}e^{\gamma\epsilon}\epsilon y^{2-4\epsilon}}{\Gamma\left(\frac{3}{2}-\epsilon\right)} + \mathcal{O}(y^3)$$

solution:

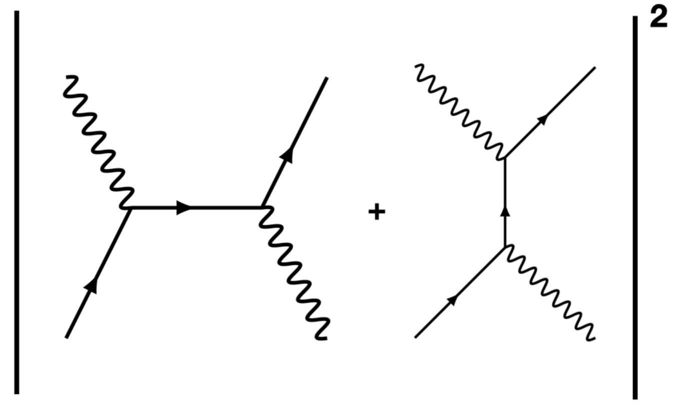
$$\mathscr{G}_1 = 1 + \epsilon\left(\log(y+1) - 4\log(y) + \log(1-y) + \log\left(3y^2+1\right) - \log(8)\right) + \mathcal{O}(\epsilon^2)$$

$$\mathscr{G}_2 = \frac{1}{16}\epsilon\left(\log(y+1) + \log(1-y) - \log\left(3y^2+1\right)\right) + \mathcal{O}(\epsilon^2)$$

LO calculation

Summary

amplitudes



$$\sigma = \frac{e^4}{2(s-m^2)} \times \frac{1}{4} \times \int \text{dLIPS}_2 \sum |\mathscr{M}|^2$$

sectors

$$\frac{1}{[k_2^2]^{\nu_1}[(k_1+p_1-k_2)^2-m^2]^{\nu_2}[(k_2-p_1)^2-m^2]^{\nu_3}}$$

master integrals

$$\int \text{dLIPS}_2 \sum |\mathscr{M}|^2 = -\frac{(d-2)\left(3ds^3-7ds^2+5ds-d-14s^3-2s^2-18s+2\right)}{(s-1)^2 s}\mathcal{J}_1 - \frac{2(d-2)\left(ds^2-2ds+d-2s^2-4s-10\right)}{s-1}\mathcal{J}_2$$

with $m = 1$

Analytic results

$$\mathcal{G}_1 = 1 + \epsilon\left(\log(y+1) - 4\log(y) + \log(1-y) + \log\left(3y^2+1\right) - \log(8)\right) + \mathcal{O}(\epsilon^2)$$

$$\mathcal{G}_2 = \frac{1}{16}\epsilon\left(\log(y+1) + \log(1-y) - \log\left(3y^2+1\right)\right) + \mathcal{O}(\epsilon^2)$$

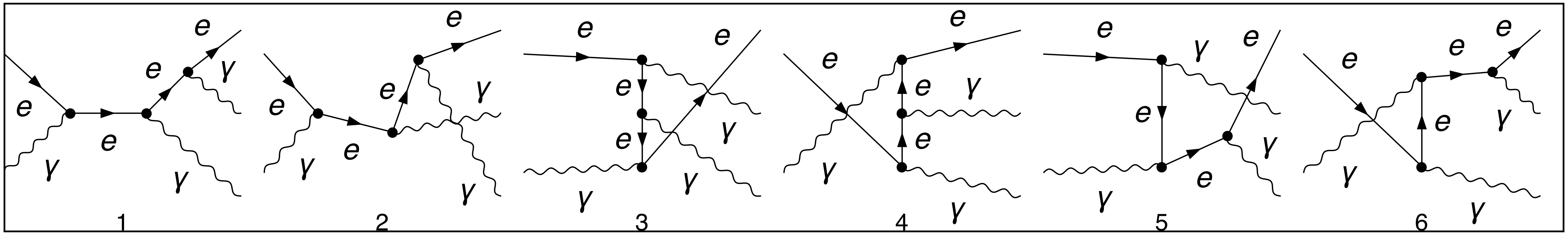
For NLO cross section, we have similar calculation setup, but with different boundary conditions

NLO total cross section

- Real emissions: $e^-(p_1) + \gamma(k_1) \to e^-(p_2) + \gamma(k_2) + \gamma(k_3)$

$$\sigma^{(R)} \sim \int \text{dLIPS}_3 \sum |\mathscr{M}^{(R)}|^2$$

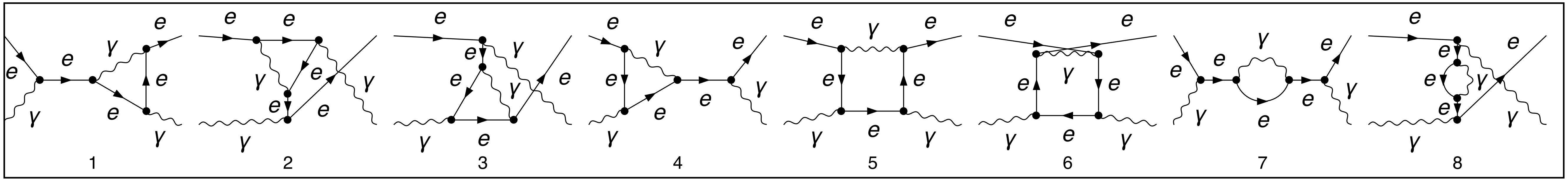
2 sectors:
14 master integrals



- Virtual corrections: $e^-(p_1) + \gamma(k_1) \xrightarrow{1-loop} e^-(p_2) + \gamma(k_2)$

$$\sigma^{(V)} \sim \int \text{dLIPS}_2 \int \frac{d^d k_3}{i\pi^{d/2}} \sum 2\text{Re}\left[\mathscr{M}^{(V)} \times \mathscr{M}^{(T)*}\right]$$

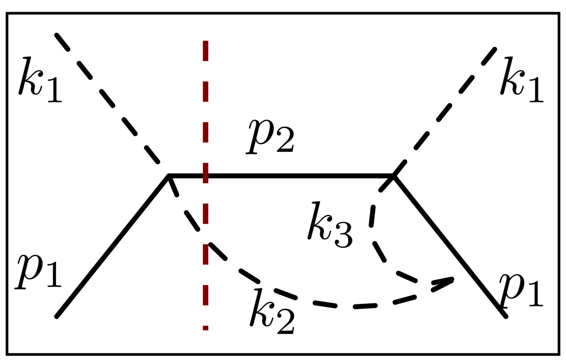
2 sectors:
24 master integrals



IBP: LiteRed2; Differential equations: Libra

Boundary conditions for NLO

- Real emissions: only 1 nonzero boundary; similar to tree-level
- Virtual corrections: 4 nonzero boundaries
  - Remarkably, the loop integral and phase space integral factorizes at threshold $s \to m^2 = 1$ or $y \to 0$



$$\mathcal{I} = \frac{(2\pi)^2}{2i\pi^d} \int d^d k_3 d^d p_2 d^d k_2 \delta(p_2^2 - m^2)\theta(p_2^0)\delta(k_2^2)\theta(k_2^0) \times \delta^d(p_1 + k_1 - p_2 - k_2) \frac{1}{k_3^2[(p_1 - k_2 - k_3)^2 - m^2]}$$

$$\int \frac{d^d k_3}{i\pi^{d/2}} \frac{1}{k_3^2[(p_1 - k_2 - k_3)^2 - m^2]} \overset{p_1 \cdot k_2 \to 2y^2}{\approx} \int dx_1 dx_2 \delta(1 - x_1 - x_2)\Gamma(\epsilon)(x_1 + x_2)^{2\epsilon - 2} x_2^{-\epsilon} \left(4x_1 y^2 + x_2\right)^{-\epsilon}$$

no long depends on any outgoing momentum!

$$\mathcal{I} \overset{\text{factorizes}}{\approx} \left(\int \text{dLIPS}_2^{(soft)} \right) \times (\text{feynman integral})$$

NLO total cross section

on-shell renormalization scheme $\sigma^{NLO} = \sigma^{NLO}_{bare} + (Z_\psi^2 Z_A^2 Z_\alpha^2 - 1)\sigma^{born} + \delta\sigma_m$

$$\sigma^{NLO} = \frac{\alpha^3}{m^2 x^3}\Bigg\{ -\frac{x\left(273x^3-982x^2-2960x-1744\right)}{24(x+1)^2}$$

$$+\frac{37x^4-54x^3-339x^2-428x-184}{4(x+1)^2}\ln(x+1)+\frac{x^2\left(14x^4+17x^3-17x^2-22x-8\right)}{2(1-x)(1+x)^3}\ln x$$

$$-\frac{4x^6+35x^5-31x^4-755x^3-1765x^2-1506x-440}{2(x+1)^2(x+4)}\ln^2(x+1)+\left(x^2-x+2\right)\left[\mathrm{Li}_2(1-x)-\frac{\pi^2}{6}\right]$$

$$-\frac{x^6+7x^5-28x^4-239x^3-449x^2-338x-88}{(x+1)^2(x+4)}\mathrm{Li}_2(-x)+\frac{x^4+7x^3+x^2-3x-2}{(x+1)^2}\ln(x+1)\ln x$$

$$-\frac{4\left(x^5+26x^4+146x^3+316x^2+288x+96\right)}{(x+1)^2(x+4)}G(-2,-1;x)+\frac{3x^4+18x^3+44x^2-8x-64}{x}yG(y,-1;x)$$

$$+T_3(x)\Bigg\} + \mathcal{O}(\epsilon)$$

→ transcendental weight 3

where $x = \dfrac{s-m^2}{m^2}$ $y = \sqrt{\dfrac{s-m^2}{s+3m^2}}$

$$G(a,a_1,\ldots,a_n;x) = \int_0^x dw_a(x')G(a_1,\ldots,a_n;x'), \quad dw_y(x) = \frac{y\,dx}{x}, \quad dw_a(x) = \frac{dx}{x-a} \quad (a=-4,-2,-1,0)$$

NLO total cross section

$$T_3(x) = \left(x^2+2x-6\right) g_1 - \frac{1}{3}\left(x^2-16x-23\right) g_2 + 8\left(x^2-4x-6\right) g_3 + 4(2x^2-x-6)g_4$$
$$+2\left(2x^2-7x-12\right) g_5 - (5x^2+32x-8)g_6 - 3(x-2)(x+4)yg_7 + 3\left(3x^2-8\right) g_{10}$$
$$-\frac{8y\left(x^4+3x^3-18x^2-68x-24\right)}{(x+4)x} g_8 + \frac{3y\left(5x^4+14x^3-96x^2-352x-128\right)}{(x+4)x} g_9$$
$$-\frac{16y\left(x^4+2x^3-24x^2-80x-48\right)}{(x+4)x} g_{11} - \frac{6y\left(x^3-12x-8\right)}{x} g_{12}$$

transcendental weight 3 bases

$$g_1 = \left[\mathrm{Li}_3(x^2) - \mathrm{Li}_2(x^2)\ln x\right], g_2 = \ln^3(x+1), g_3 = G(-1,-2,-1;x), g_4 = G(-1,-1,0;x),$$
$$g_5 = G(-1,0,-1;x), g_6 = G(0,-1,-1;x), g_7 = \left[G(0,y,-1;x) + 2G(y,-1,0;x)\right],$$
$$g_8 = G(y,0,-1;x), g_9 = G(y,-1,-1;x), g_{10} = G(y,y,-1;x), g_{11} = G(y,-2,-1;x),$$
$$g_{12} = G(-4,y,-1;x)$$

where $x = \frac{s-m^2}{m^2}$ $\quad y = \sqrt{\frac{s-m^2}{s+3m^2}}$

$$G(a,a_1,\ldots,a_n;x) = \int_0^x dw_a(x')G(a_1,\ldots,a_n;x'), \qquad dw_y(x) = \frac{ydx}{x}, \quad dw_a(x) = \frac{dx}{x-a} \quad (a=-4,-2,-1,0)$$

From GPLs to PolyLogs

[Dersy, Schwartz, Zhang, in progress]

- The alphabet from DE is

$$y = \sqrt{\frac{s-1}{s+3}}, \quad m = 1$$

$$l_0 = \frac{1}{s}, \quad l_1 = \frac{1}{s-1}, \quad l_2 = \frac{1}{s+1}, \quad l_3 = \frac{1}{s-2}, \quad l_4 = \frac{1}{s+3}, \quad l_5 = \frac{y}{s-1} = \frac{1}{\sqrt{(s-1)(s+3)}}$$

- The general question is:

  How to write down the simplest solution to the differential equation?

- For example, consider

$$I[l_5, l_0, s] = \int_1^s ds_1 \frac{1}{\sqrt{(s_1-1)(s_1+3)}} \int_1^{s_1} \frac{ds_2}{s_2} = \left\{ \frac{s+1+\sqrt{(s-1)(s+3)}}{2} \right\} \otimes s = r \otimes \frac{1-r+r^2}{r}$$

$$r = \frac{s+1+\sqrt{(s-1)(s+3)}}{2}$$

$$r \otimes (1-r+r^2) = r \otimes (1+r^3) - r \otimes (1+r) = \frac{1}{3} r^3 \otimes (1+r^3) - r \otimes (1+r)$$

$$I[l_5, l_0, s] = -\frac{1}{3} Li_2(-r^3) + Li_2(-r) - \frac{1}{2} \ln^2(r) + \frac{\pi^2}{18}$$

From GPLs to PolyLogs

[Dersy, Schwartz, Zhang, in progress]

$$I[l_5, l_0, s] = \int_1^s ds_1 \frac{1}{\sqrt{(s_1-1)(s_1+3)}} \int_1^{s_1} \frac{ds_2}{s_2} = \left\{ \frac{s+1+\sqrt{(s-1)(s+3)}}{2} \right\} \otimes s = r \otimes \frac{1-r+r^2}{r}$$

$$I[l_5, l_0, s] = -\frac{1}{3} Li_2(-r^3) + Li_2(-r) - \frac{1}{2} \ln^2(r) + \frac{\pi^2}{18}$$

- Direct integration gives rise to complex numbers $(-1)^{1/3}$, where we need an Li_2 identity to simplify the expression

$$Li_2(e^{\frac{2\pi i}{3}} r) + Li_2(e^{\frac{4\pi i}{3}} r) = \frac{1}{3} Li_2(r^3) - Li_2(r)$$

- What do we mean by the simplest?

  At least get rid of the complex numbers

- The main difficulties to automate this calculation:

  1. definition of the complexity; 2. change of the variables

- ML side: it is a tough game for reinforcement learning

- Alternatively, could we find a function version of PSLQ or Lattice reduction?

  Symbolic regression

[AI Feynman, 1905.11481, Udrescu, Tegmark]

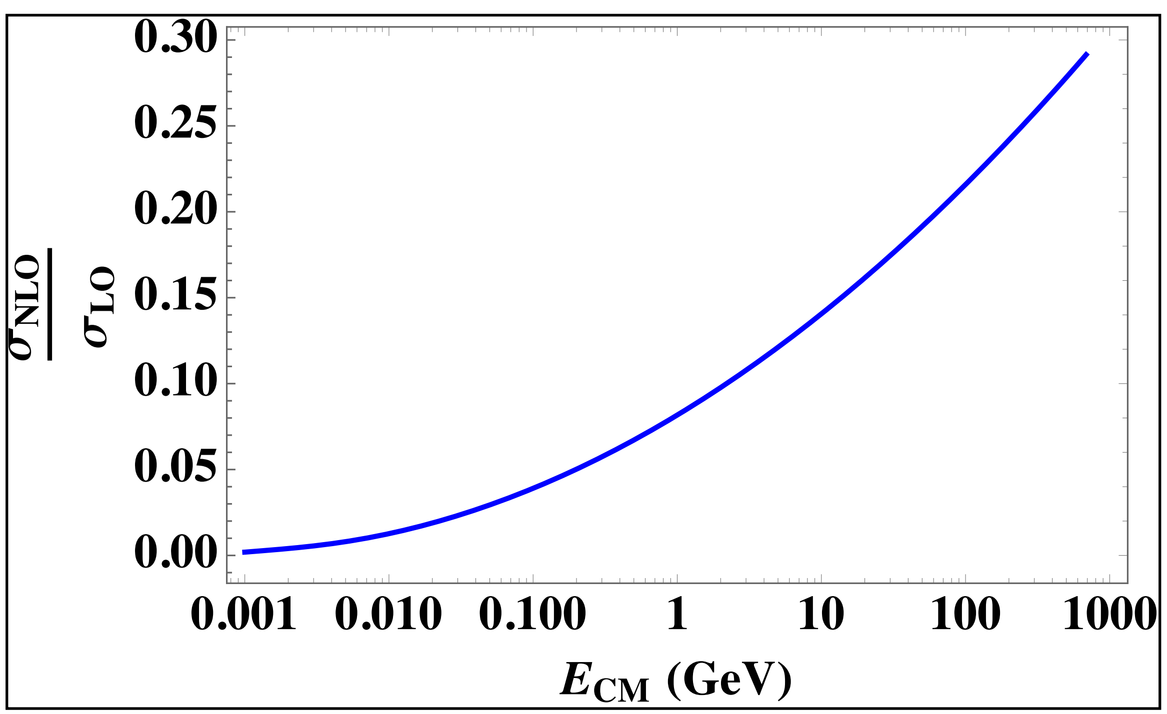
Asymptotic Behaviors

- Bloch-Nordsieck theorem: IR divergences cancel when both real and virtual contributions are summed over
- Thirring's theorem: near the threshold $s \to m^2$, NLO cross section vanishes

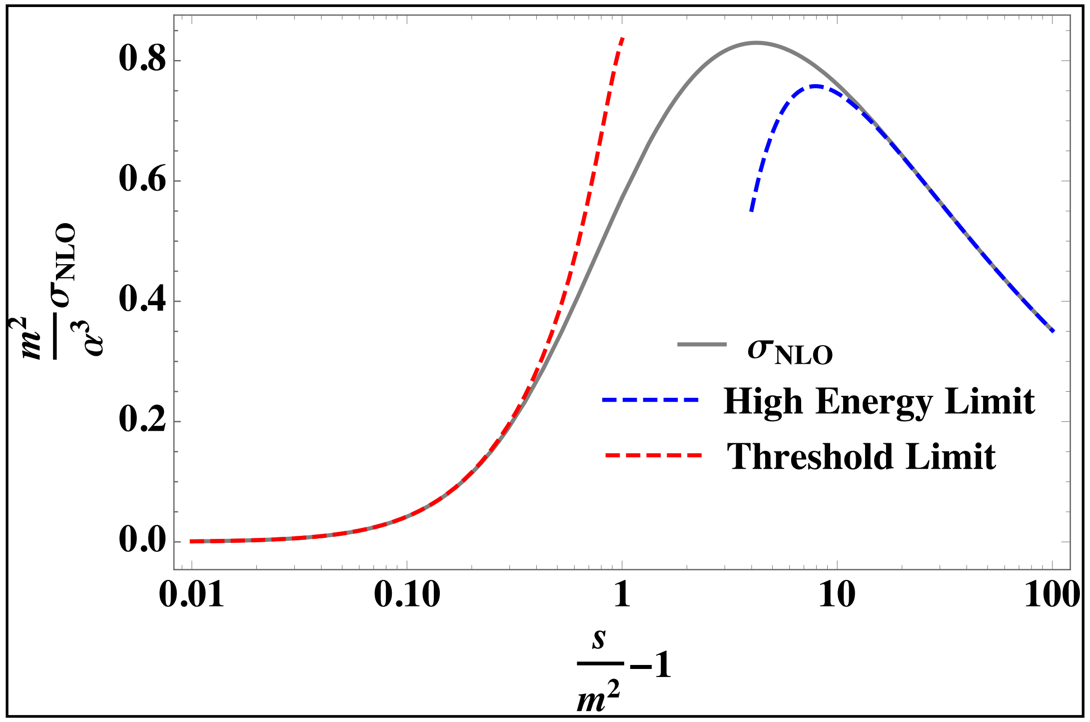
[9704368, Dittmaier]

Threshold: $$\sigma = \frac{\pi\alpha^2}{m^2}\left[\frac{8}{3} - \frac{8}{3}x + \cdots\right] + \frac{\alpha^3}{m^2}x^2\left[-\frac{16}{9}\ln x + \frac{7}{15} + \cdots\right], \quad x = \frac{s-m^2}{m^2}$$

High energy: $$\sigma = \frac{\pi\alpha^2}{s}\left[2\ln\frac{s}{m^2} + 1 + \cdots\right] + \frac{\alpha^3}{s}\left[\frac{1}{3}\ln^3\frac{s}{m^2} - \frac{1}{2}\ln^2\frac{s}{m^2} + \frac{17}{4}\ln\frac{s}{m^2} - \frac{75}{8} - \frac{\pi^2}{2} + 4\zeta_3 + \cdots\right]$$



NLO corrections to total cross section



Asymptotic expansions

Total Cross Sections in QED

- Compton scattering: $e^-\gamma \to e^-\gamma$ $$\sigma = \frac{2\pi\alpha^2}{s}\ln\frac{s}{m^2}\left(1+\frac{\alpha}{6\pi}\ln^2\frac{s}{m^2}+\cdots\right)$$
- Pair production: $\gamma\gamma \to e^+e^-$ $$\sigma = \frac{4\pi\alpha^2}{s}\ln\frac{s}{m^2}\left(1+\frac{\alpha}{12\pi}\ln^2\frac{s}{m^2}+\cdots\right)$$
- Photon production: $e^+e^- \to \gamma\gamma$ $$\sigma = \frac{2\pi\alpha^2}{s}\ln\frac{s}{m^2}\left(1+\frac{\alpha}{6\pi}\ln^2\frac{s}{m^2}+\cdots\right)$$

- DGLAP equations cannot reproduce all logarithms:

  PDFs predict $\frac{\alpha^3}{s}\ln^2\frac{s}{m^2}$ at NLO (collinear logarithms)

- Conceptually, naive factorization doesn't work for the total cross section:

  σ ~ (PDFs/EDFs) $\otimes$ (soft) $\otimes$ (collinear) $\otimes$ (Hard)

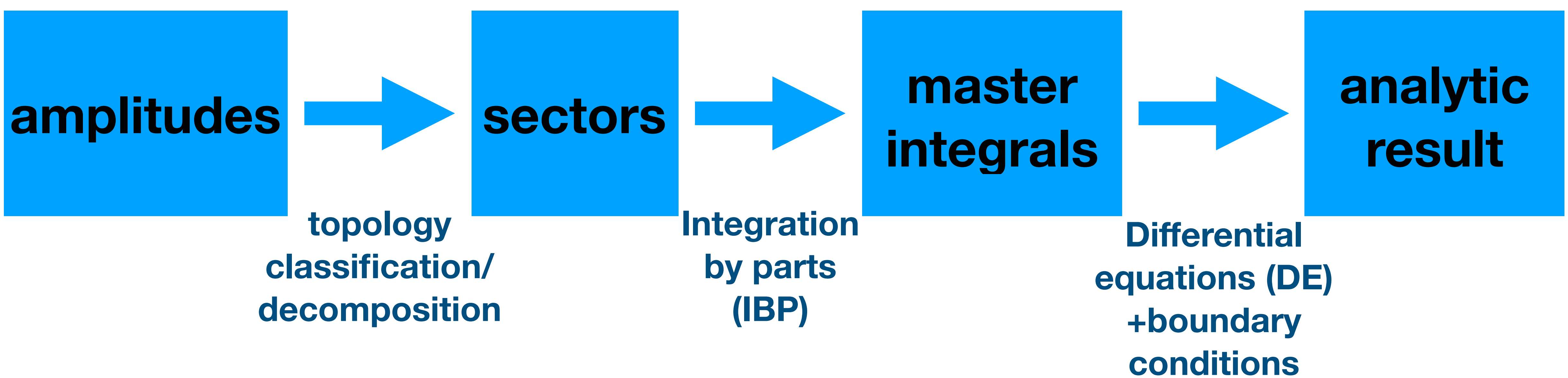
  since (off-shell) Glauber region is essential

- How to do resummation is not clear without a running scale

[2110.02978, Bhattacharya, Manohar, Schwartz]

Summary

- Compton scattering is one of the first results in QED. It plays an important role in all aspects of physics, in particular, essential to study the infrared structures and forward scattering.
- The multi-loop techniques have promoted the development of both amplitudes and precision QCD/collider physics.



- The total cross section of Compton scattering can be computed with multi-loop techniques. We present the complete calculation at LO and highlight the key points at NLO.