圈积分的解析与数值计算若干方法与程序简介

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2021-5-27

微分方程方法:

$$\frac{d}{dx}\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} A_{11} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{n1} & \dots & A_{n,n} \end{pmatrix}\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

x are Lorentz invariant kinematics

A. V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett.* **B254** (1991) 158–164.

A. V. Kotikov, *Differential equation method: The Calculation of N point Feynman diagrams*, *Phys. Lett.* **B267** (1991) 123–127. [Erratum: Phys. Lett.B295,409(1992)].

Multiloop integrals in dimensional regularization made simple

Johannes M. Henn (Princeton, Inst. Advanced Study) (Apr 5, 2013)

Published in: *Phys.Rev.Lett.* 110 (2013) 251601 • e-Print: 1304.1806 [hep-th]

Choosing canonical basis (basis with uniform transcendentality)

For random basis g， we may have:

$$\partial_x \vec{g}(x;\epsilon) = B(x,\epsilon)\,\vec{g}(x;\epsilon)$$

We can choose new basis f:

d=4-2∊

$$\vec{f} = T\vec{g},$$

$$d\,\vec{f}(x,\epsilon) = \epsilon\,\left(d\,\tilde{A}\right)\,\vec{f}(x;\epsilon)$$

$$\tilde{A} = \left[\sum_k A_k \log \alpha_k(x)\right]\,.$$

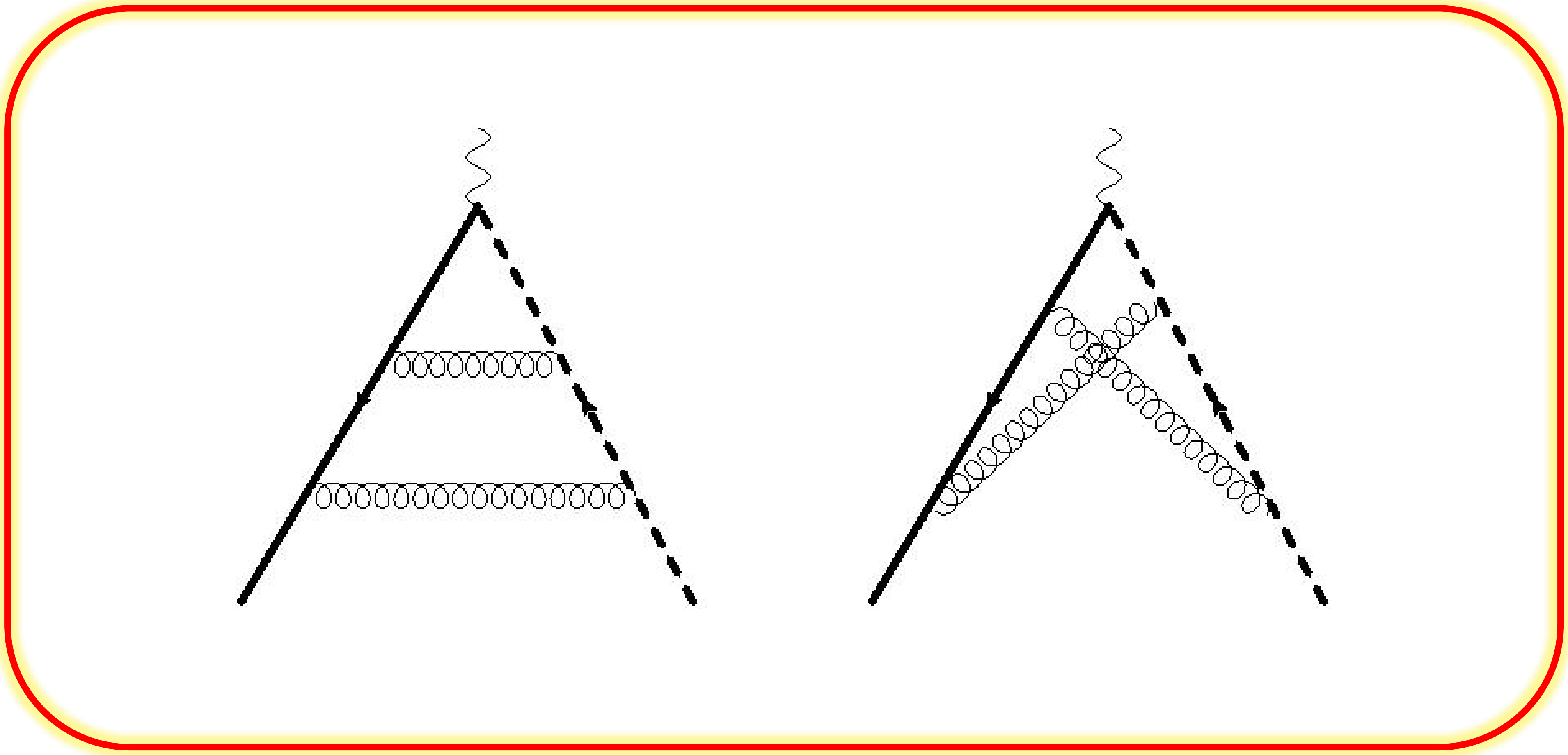
Master Integrals Calculation Example

Two-loop master integrals for heavy-to-light form factors of two different massive fermions

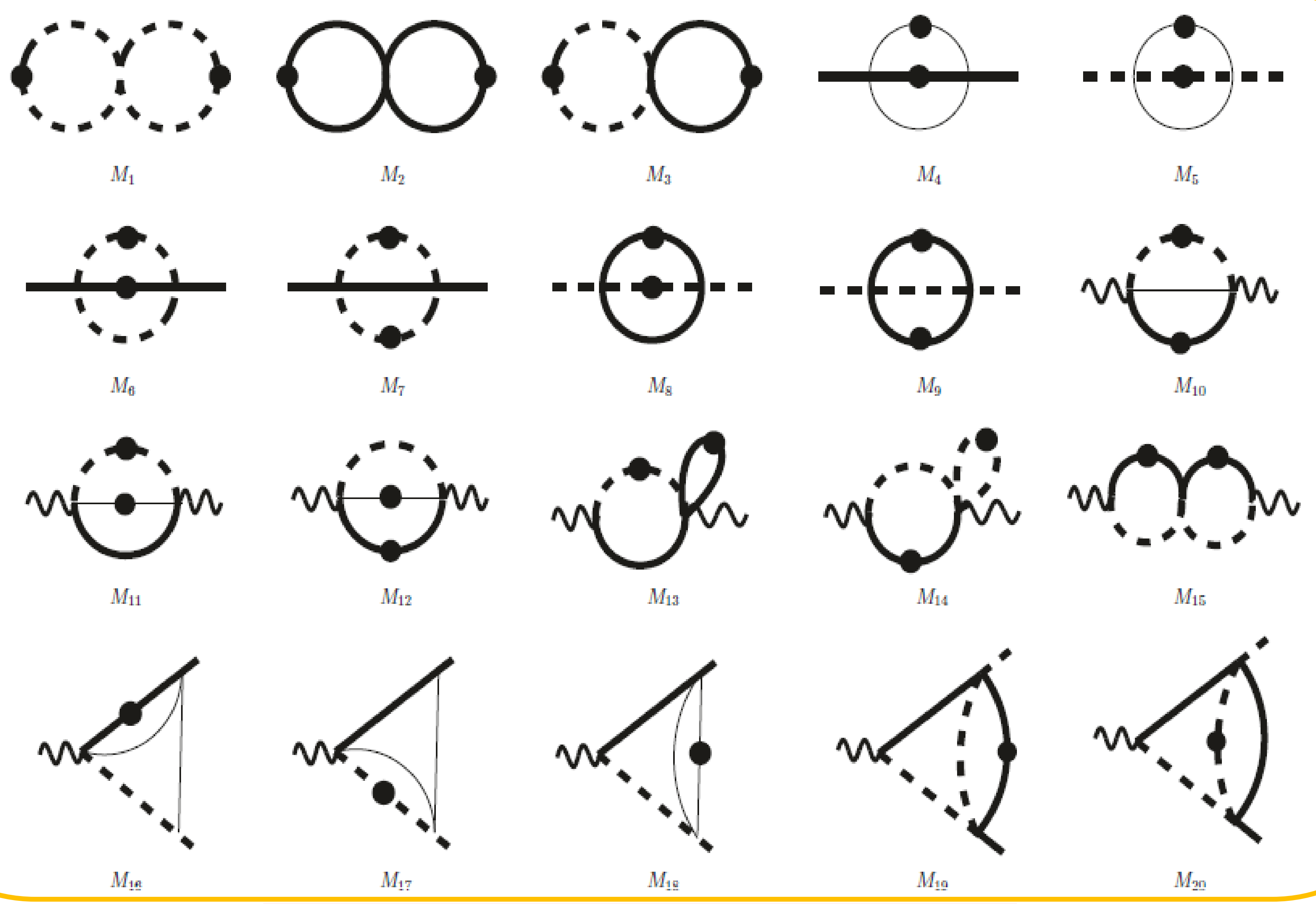
Massive quark decays to massive quark

$$(t \to b + W^+(l + \bar{\nu}), b \to c + l + \bar{\nu})$$

JHEP 02 (2018) 066



Sample of Two-loop Feynman diagrams

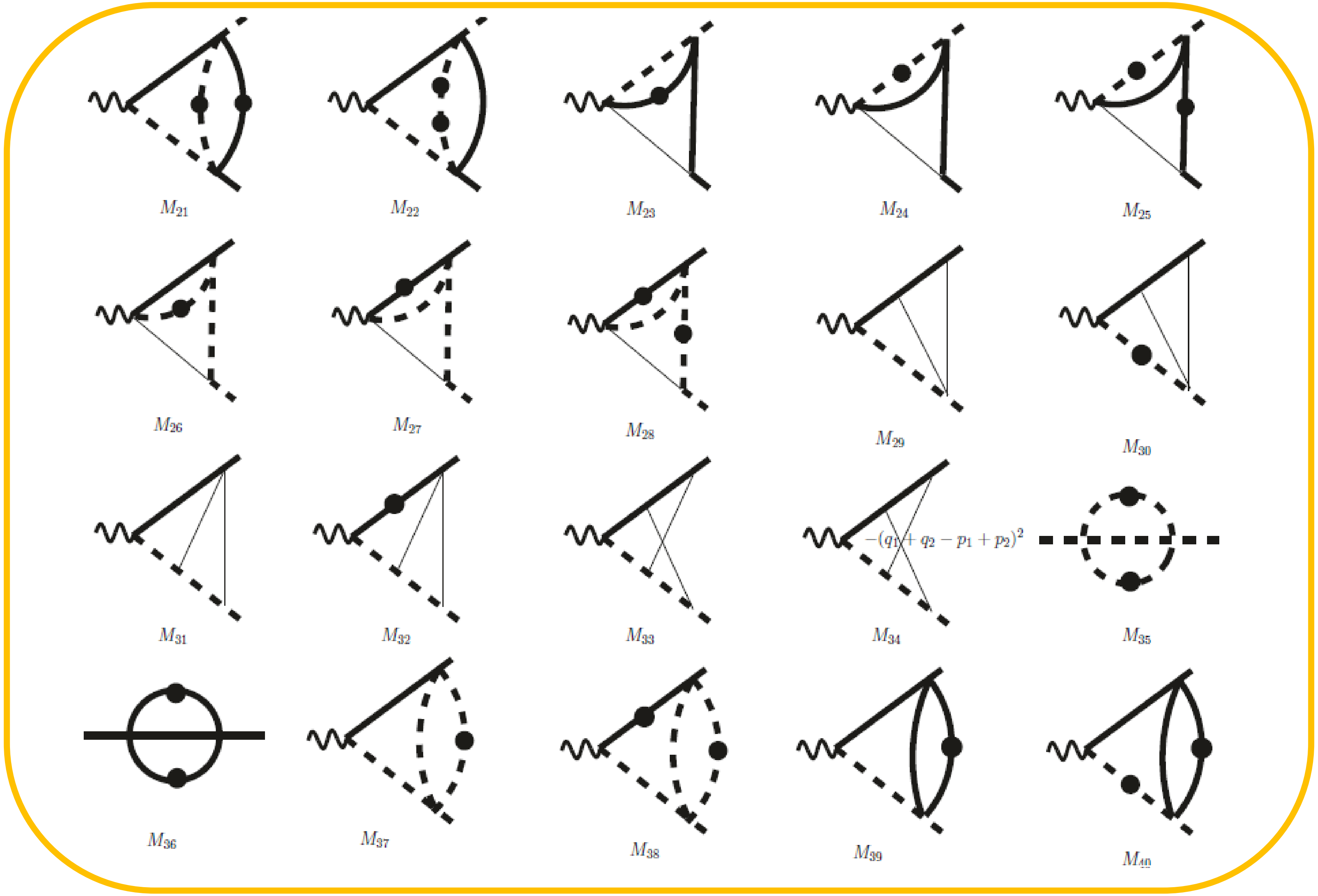


Tensor integrals are reduced by IBP (integrate-by-parts) method.

40 Master Integrals

Solid: Heavy; Dash solid: Light;

Thick: massless.



Three scales: s,m1,m2

$$
\begin{aligned}
F_{16} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{16}\,,\\
F_{17} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{17}\,,\\
F_{18} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{18}\,,\\
F_{19} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{19}\,,\\
F_{20} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{20}\,,\\
F_{21} &= \epsilon^2 \, \big(2s\,(\epsilon(M_{19}+2M_{20}) - m_2^2 M_{21} - 2m_1^2 M_{22})\\
&\quad +2(m_2^2-m_1^2)(\epsilon(M_{19}+2M_{20}) + m_2^2 M_{21} - 2m_1^2 M_{22})\big) + 2\frac{m_2}{m_1}F_9\,,\\
F_{22} &= \epsilon^2 \, \big(2(m_1^2-m_2^2)(2\epsilon(M_{19}+2M_{20}) + (m_1^2+m_2^2-s)M_{21} - 4m_1^2 M_{22})\big)\\
&\quad +2\frac{m_1}{m_2}F_7 - 2\frac{m_2}{m_1}F_9\,,\\
F_{23} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{23}\,,\\
F_{24} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{24}\,,\\
F_{25} &= \epsilon^2 \frac{s\,m_2^2(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}{(s-m_1^2+m_2^2)^2} M_{25}\\
&\quad +\epsilon^3 \frac{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}{2(s-m_1^2+m_2^2)} (M_{23}-M_{24})\\
&\quad -\frac{m_2\, s\,(s-m_1^2-m_2^2)}{m_1(s-m_1^2+m_2^2)^2}F_9 - \frac{\sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}}{4(s-m_1^2+m_2^2)}F_{11}\\
&\quad +\frac{s\,m_2^2}{(s-m_1^2+m_2^2)^2}(F_2 - F_3 - 6F_8 + 2F_{12})\,,
\end{aligned}
$$

Canonical Basis

Differential Equations In Canonical Form:

$$\mathrm{d}\mathbf{F}(x,y;\epsilon) = \epsilon\, \mathrm{d}\,\tilde{A}(x,y)\,\mathbf{F}(x,y;\epsilon)$$

$$\begin{aligned}\tilde{A}(x,y) = {} & A_1 \ln(x) + A_2 \ln(x+1) + A_3 \ln(x-1) + A_4 \ln(x+y) + A_5 \ln(x-y) \\ & + A_6 \ln(x\,y+1) + A_7 \ln(x\,y-1) + A_8 \ln(y) + A_9 \ln(y+1) + A_{10} \ln(y-1) \\ & + A_{11} \ln(x^2\,y - 2x + y) + A_{12} \ln(x^2 - 2y\,x + 1). \end{aligned} \tag{4.41}$$

A_i are rational matrices

$$s = m_1^2 \frac{(x-y)(1-x\,y)}{x}, \quad \text{and} \quad m_2 = m_1\,y.$$

Determination of Boundary Conditions

$$
\begin{aligned}
F_{10} &= \epsilon^2\, s\, M_{10}\,, \\
F_{11} &= \epsilon^2 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\,(M_{10}+M_{11}+M_{12})\,, \\
F_{12} &= \epsilon^2\, \left((m_1^2-m_2^2)\,(M_{10}+M_{11}+M_{12}) + s\,(M_{11}-M_{12})\right)\,,
\end{aligned}
$$

$$
\frac{\partial F_{12}}{\partial x} = \epsilon\left(-\frac{F_{11}+F_{12}}{x-y} + y\,\frac{F_{11}-F_{12}}{x\,y-1} + \frac{F_{12}}{x}\right)
$$

$$
\begin{aligned}
F_{11}|_{x=y} &= -F_{12}|_{x=y}\,. \\
F_{11}|_{x=\frac{1}{y}} &= F_{12}|_{x=\frac{1}{y}}\,.
\end{aligned}
$$

Goncharov Polylogarithms

$$G_{a_1,a_2,\ldots,a_n}(x) \equiv \int_0^x \frac{\mathrm{d}t}{t-a_1} G_{a_2,\ldots,a_n}(t) \,,$$

$$G_{\vec{0}_n}(x) \equiv \frac{1}{n!} \ln^n x \,.$$

$$G_{a_1,\ldots,a_m}(x) G_{b_1,\ldots,b_n}(x) = \sum_{c \in a \text{Ш} b} G_{c_1,c_2,\ldots,c_{m+n}}(x) \,.$$

A. B. Goncharov, *Multiple polylogarithms, cyclotomy and modular complexes*, Math. Res. Lett. **5**, (1998) 497–516, [arXiv:1105.2076].

Results:

$$
\begin{aligned}
F_{10} = {} & 2\epsilon^2\left[G_{0,0}(y) - G_{0,0}(x)\right] + \epsilon^3\Big[\frac{G_{0,0}(y)}{2}(G_{\frac{1}{y}}(x) + G_y(x) + 4G_y(1) - 4G_{\frac{1}{y}}(1) - G_{\frac{1}{y}}(y)) \\
& + 4G_0(y)(G_{0,y}(x) - G_{0,\frac{1}{y}}(x) + G_{0,0}(x) + G_{0,\frac{1}{y}}(y) - G_{\frac{1}{y},0}(1) - G_{y,0}(1) + \frac{\pi^2}{3}) \\
& + G_0(x)(\frac{G_{0,0}(y)}{2} - 4(G_{\frac{1}{y}}(1) - G_y(1))G_0(y) - 4G_{\frac{1}{y},0}(1) - 4G_{y,0}(1) + \pi^2) - 6G_{0,0,0}(x) \\
& + 12(G_{0,-1,0}(x) + G_{0,1,0}(x) - G_{0,-1,0}(y) - G_{0,1,0}(y)) - 2(2G_{0,\frac{1}{y},0}(x) + 2G_{0,y,0}(x) \\
& + G_{\frac{1}{y},0,0}(x) + G_{y,0,0}(x)) + 2G_{\frac{1}{y},0,0}(y) + 4G_{0,\frac{1}{y},0}(y) + 6\zeta(3)\Big] + \mathcal{O}(\epsilon^4), \\
F_{33} = {} & \epsilon^3\left[2G_{0,0,0}(x) - 2G_{0,1,0}(x) - 2G_{0,-1,0}(x) + \frac{1}{6}\pi^2 G_0(x) + \zeta(3)\right] + \mathcal{O}(\epsilon^4).
\end{aligned}
$$

Goncharov Polylogarithms

Check:

$$M_{33}^{\text{SecDec}}(-5.4, 1.0, 0.2) = \frac{-0.4466129 \pm 0.0000004}{\epsilon} - 0.507366 \pm 0.000006,$$

$$M_{33}^{\text{FIESTA}}(-5.4, 1.0, 0.2) = \frac{-0.446613 \pm 0.000005}{\epsilon} - 0.507387 \pm 0.000049,$$

$$M_{33}^{\text{Ours}}(-5.4, 1.0, 0.2) = \frac{-0.4466129967\ldots}{\epsilon} - 0.5073683817\ldots.$$

Numerical evaluation of multiple polylogarithms

Ginac

Jens Vollinga (Mainz U., Inst. Phys.), Stefan Weinzierl (Mainz U., Inst. Phys.) (Oct, 2004)

Published in: *Comput.Phys.Commun.* 167 (2005) 177 • e-Print: hep-ph/0410259 [hep-ph]

ginac.de

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Anti Software Patents Demonstration

Stop the nonsense, Stop the fraud!

For the last few years the European Patent Office (EPO) has, contrary to the letter and spirit of the existing law, granted more than 30000 patents on computer-implemented rules of organisation and calculation (programs for computers). Now Europe's patent movement is pressing to consolidate this practise by writing a new law.

Unlike copyright, patents can block independent creations. Software patents can render software copyright useless. **One copyrighted work can be covered by hundreds of patents of which the author doesn't even know but for whose infringement he and his users can be sued.** Some of these patents may be impossible to work around, because they are broad or because they are part of communication standards.

Evidence from economic studies shows that software patents have lead to a decrease in R&D spending.

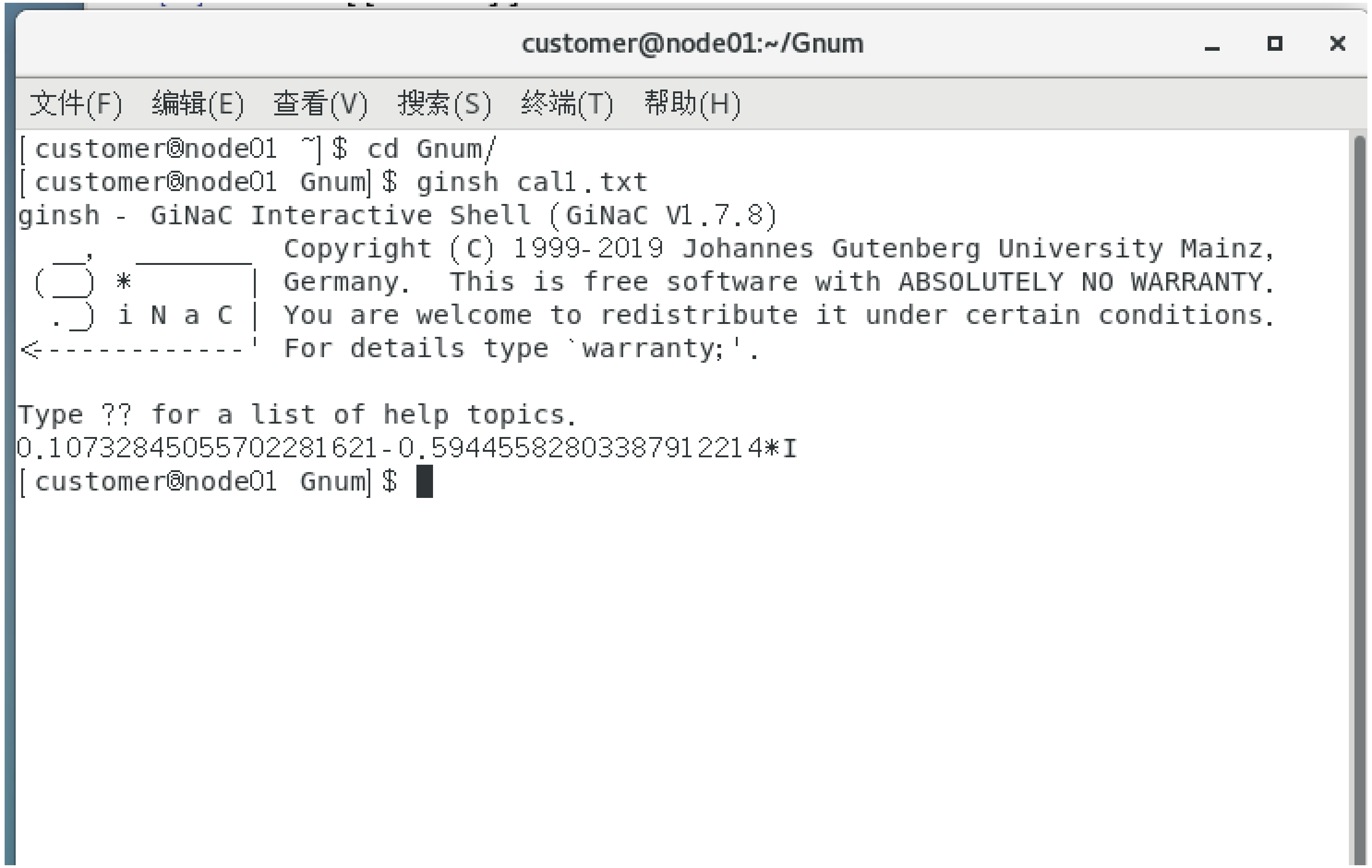
Advances in software are advances in abstraction. While traditional patents were for concrete and physical *inventions*, software patents cover *ideas*. Instead of patenting a specific mousetrap, you patent any "means of trapping mammals" or "means of trapping data in an emulated environment". The fact that the universal logic device called "computer" is used for this does not constitute a limitation.

When software is patentable, anything is patentable!

For more information about software patents in Europe, visit this website: swpat.ffii.org.

```
Digits = 15:
expression = 2*G({0.9393543905251667 + 7.589370620706021e-10*I,0.,1.,-4.696771952625833 + 4.246188272542417e-9*I},1) :
evalf(expression);
quit:
```

Save as txt file



1. arXiv:1904.07279 [pdf, other] hep-th hep-ph doi 10.1007/JHEP08(2019)135

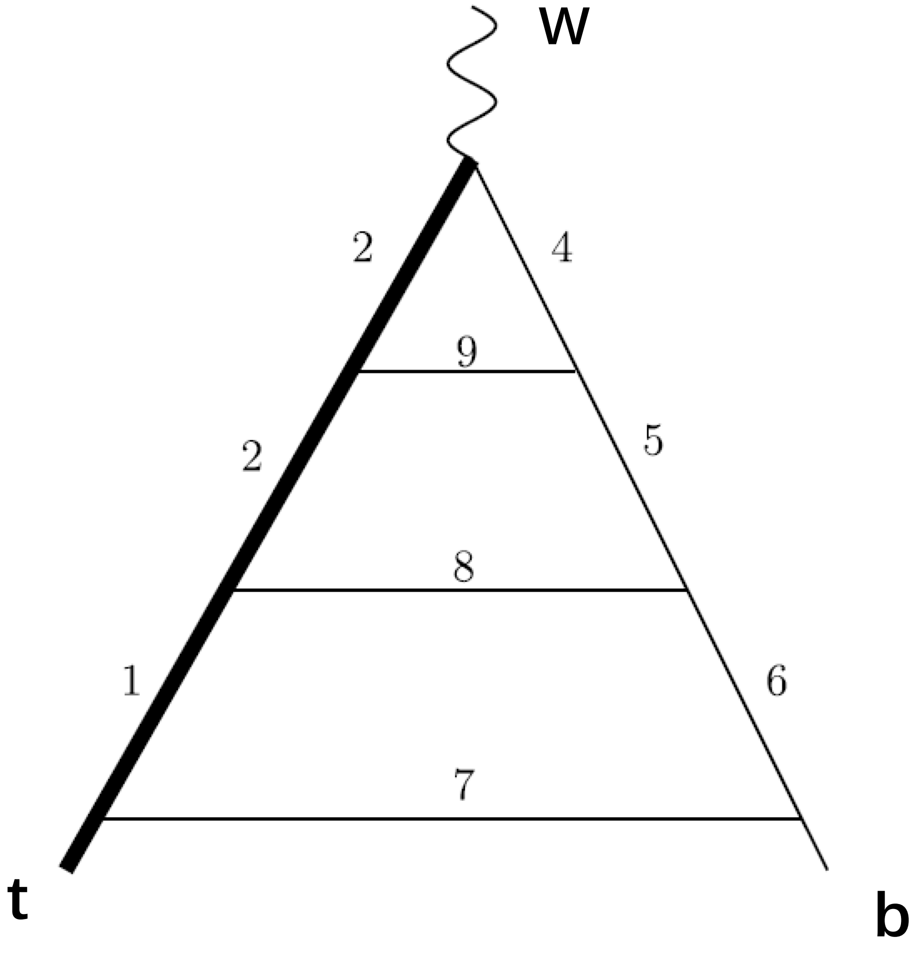
PolyLogTools - Polylogs for the masses

Authors: Claude Duhr, Falko Dulat

```
In[1]:=    T = 1/(1-x)*G[1/(1-x),0,0,1,1,1/(1+z)];
In[2]:=    Ginsh[ T, {x->0.3, z->0.45} ]

Out[2]:=   -0.029417947048466250336759719341660323827 9
```

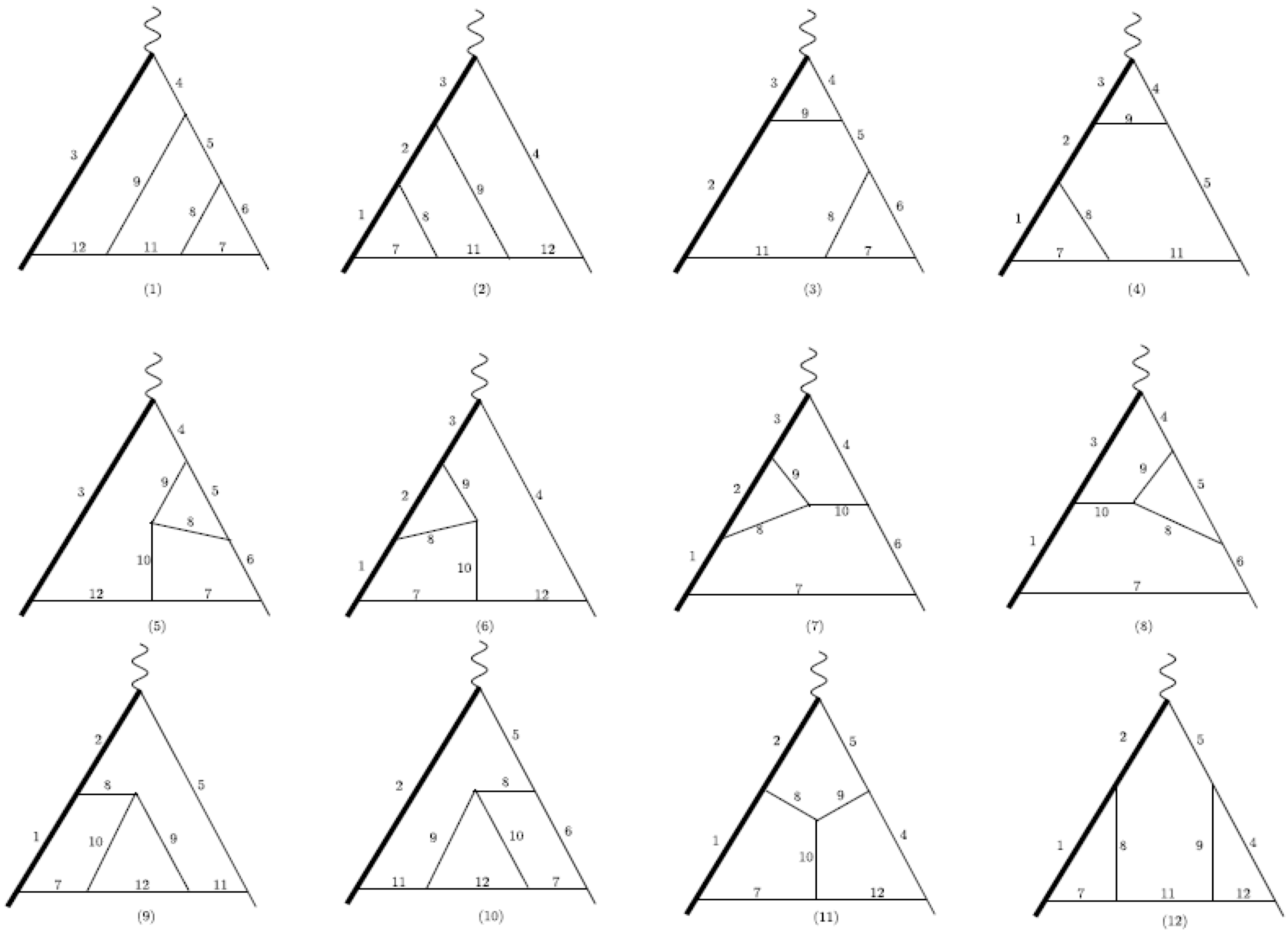
```
In[4]:=    Ginsh[ T, {x->0.3, z->0.45}, PrecisionGoal -> 10]
In[5]:=    Ginsh[ T, {x->0.3, z->0.45}, PrecisionGoal -> 100]

Out[4]:=   -0.02941794704846625
Out[5]:=   -0.02941794704846625033675971934166032382890897
           19017828790593790231936752156076091770442309841
           1162406117224765098927 7119
```

Three-Loop Heavy-to-light Form factors



Phys.Lett. B786, 453

Color-planar diagrams
(Leading Color Contribution)



Integrals can be parameterized by

d=4-2ϵ

$$I_{n_1,n_2,\ldots,n_{12}} = \int \frac{\mathcal{D}^d k_1 \, \mathcal{D}^d k_2 \, \mathcal{D}^d k_3}{D_1^{n_1} \, D_2^{n_2} \, D_3^{n_3} \, D_4^{n_4} \, D_5^{n_5} \, D_6^{n_6} \, D_7^{n_7} \, D_8^{n_8} \, D_9^{n_9} \, D_{10}^{n_{10}} \, D_{11}^{n_{11}} \, D_{12}^{n_{12}}}$$

$$D_1 = -(k_1+p_1)^2 + m^2, \, D_2 = -(k_2+p_1)^2 + m^2,$$
$$D_3 = -(k_3+p_1)^2 + m^2, \, D_4 = -(k_3+p_2)^2,$$
$$D_5 = -(k_2+p_2)^2, \, D_6 = -(k_1+p_2)^2, \, D_7 = -k_1^2,$$
$$D_8 = -(k_1-k_2)^2, \, D_9 = -(k_2-k_3)^2, \, D_{10} = -(k_1-k_3)^2,$$
$$D_{11} = -k_2^2, \, D_{12} = -k_3^2,$$

All integrals can be reduced to 71 Master Integrals

$$\mathcal{D}^d k_i \equiv \frac{m^{2\epsilon}}{\pi^{d/2}\Gamma(1+\epsilon)} d^d k_i, \quad d = 4-2\epsilon \, .$$

$$\frac{\partial}{\partial s} = \frac{1}{s - m^2} p_2 \cdot \frac{\partial}{\partial p_2}$$

Canonical Basis:

$$
\begin{aligned}
F_1 &= m^6 \, I_{3,3,3,0,0,0,0,0,0,0,0,0}\,, \\
F_2 &= \epsilon^2 m^4 \, I_{0,2,3,0,0,0,1,2,0,0,0,0}\,, \\
F_3 &= \epsilon^3 m^2 \, I_{0,0,2,0,0,0,2,2,1,0,0,0}\,, \\
F_4 &= (\epsilon - 1)(1 + 4\epsilon)\epsilon \, m^2 \, I_{2,0,2,0,0,0,0,2,1,0,0,0}\,, \\
F_5 &= \epsilon \, s \, m^4 \, I_{3,3,2,1,0,0,0,0,0,0,0,0}\,, \\
F_6 &= \epsilon^3 s \, I_{2,0,0,2,0,0,0,2,1,0,0,0}\,, \\
F_7 &= \epsilon^2 m^2 \left(2\epsilon \, I_{2,0,0,2,0,0,0,2,1,0,0,0} \right. \\
&\quad \left. + (s - m^2) \, I_{3,0,0,2,0,0,0,2,1,0,0,0}\right), \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

$$F_{67} = \epsilon^5 \, (s - m^2) \, I_{1,1,0,1,1,-1,1,1,0,2,0,0} \, ,$$

$$F_{68} = \epsilon^5 \, (s - m^2)^2 \, I_{1,1,0,1,1,0,1,1,0,2,0,0} \, ,$$

$$F_{69} = \epsilon^6 \, (s - m^2) \, I_{1,1,0,1,0,0,1,1,1,0,0,1} \, ,$$

$$F_{70} = \epsilon^6 \, (s - m^2)^2 \, I_{1,1,0,1,1,0,1,1,1,1,-1,1} \, ,$$

$$F_{71} = \epsilon^6 \, (s - m^2) \, I_{1,1,0,1,1,-1,1,1,1,1,-1,1} + \frac{1}{12(1-2\epsilon)} (12F_2 + 6F_3 + 3F_4 - 2F_7 + 6F_9 - 18F_{14} + 2F_{24} + 12F_{25}) \, .$$

$$\frac{\partial \mathbf{F}(x,\epsilon)}{\partial x} = \epsilon \left(\frac{\mathbf{P}}{x} + \frac{\mathbf{Q}}{x-1} \right) \mathbf{F}(x,\epsilon).$$

P and Q are 71*71
rational matrices

Boundary Conditions

Known

$$F_1 = \frac{1}{8},$$

$$F_2 = \frac{1}{8} + \epsilon^2 \frac{\pi^2}{12} + \epsilon^3 \zeta(3) + \epsilon^4 \frac{4\pi^4}{45} + 2\epsilon^5 \frac{27\zeta(5) + \pi^2\zeta(3)}{3} + \epsilon^6 \left(\frac{229\pi^6}{1890} + 4\zeta^2(3) \right) + \mathcal{O}(\epsilon^7),$$

$$x \equiv \frac{s}{m^2}.$$

$$\frac{\partial F_{38}}{\partial x} = \epsilon \left(\frac{-4F_2 - F_3 + 6F_{11} + 2F_{19} - 6(3F_{38} - 2F_{39})}{6x} + \frac{2F_{38}}{x-1} \right),$$

Regular at x=0

$$6(3F_{38} - 2F_{39})|_{x=0} = (-4F_2 - F_3 + 6F_{11} + 2F_{19})|_{x=0}$$

$$
\begin{aligned}
F_{38} &= \epsilon^5 \, (s - m^2) \, I_{0,1,1,1,0,0,1,2,1,0,0,0} \, , \\
F_{39} &= \epsilon^4 \, m^2 \, (s - m^2) \, I_{0,1,2,1,0,0,1,2,1,0,0,0} \, ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F_{38}}{\partial x} &= \epsilon \left(\frac{-4F_2 - F_3 + 6F_{11} + 2F_{19} - 6(3F_{38} - 2F_{39})}{6x} + \frac{2F_{38}}{x-1} \right) , \\
\frac{\partial F_{39}}{\partial x} &= \epsilon \left(\frac{-20F_2 + 4F_3 + 3F_{11} - 12F_{16} + 30F_{18} + 16F_{19} - 30(3F_{38} - 2F_{39})}{12x} \right. \\
&\quad \left. -2 \frac{3F_{39} - 4F_{38}}{x-1} \right) .
\end{aligned}
$$

$$
\begin{aligned}
6(3F_{38} - 2F_{39})|_{x=0} &= (-4F_2 - F_3 + 6F_{11} + 2F_{19})|_{x=0} \, , \\
30(3F_{38} - 2F_{39})|_{x=0} &= (-20F_2 + 4F_3 + 3F_{11} - 12F_{16} + 30F_{18} + 16F_{19})|_{x=0} \, .
\end{aligned}
$$

$$F_{37}|_{x=0} = -\frac{11}{240} - \epsilon^2 \frac{\pi^2}{180} + \epsilon^3 \frac{3\zeta(3)}{4} + \epsilon^4 \frac{379\pi^4}{5400} - \epsilon^5 \left(\frac{4\pi^2\zeta(3)}{3} - \frac{1107\zeta(5)}{20} \right) + \epsilon^6 \left(\frac{901\pi^6}{4725} + \frac{21\zeta^2(3)}{2} \right) + \mathcal{O}(\epsilon^7),$$

$$F_{39}|_{x=0} = -\epsilon^2 \frac{\pi^2}{36} + \epsilon^4 \frac{79\pi^4}{1080} + \epsilon^5 \left(\frac{143\pi^2\zeta(3)}{18} + \frac{5\zeta(5)}{2} \right) + \epsilon^6 \left(\frac{18737\pi^6}{22680} + 48\zeta^2(3) \right) + \mathcal{O}(\epsilon^7),$$

$$F_{41}|_{x=0} = \frac{7}{180} - \epsilon^2 \frac{7\pi^2}{270} - \epsilon^3 \frac{89\zeta(3)}{45} - \epsilon^4 \frac{139\pi^4}{900} - \epsilon^5 \frac{353\pi^2\zeta(3) + 8469\zeta(5)}{135} - \epsilon^6 \left(\frac{92077\pi^6}{170100} + \frac{2503\zeta^2(3)}{45} \right) + \mathcal{O}(\epsilon^7),$$

Results of Boundary conditions are expressed in terms of zeta functions.

$$
\begin{aligned}
F_{71} \;=\;& \epsilon^4 \left(H_{0,1,0,1}(x) - H_{0,0,1,1}(x) + \frac{\pi^2}{6} H_{0,1}(x) - \frac{\pi^4}{30} \right) \\
+\;& \epsilon^5 \Big(-2H_{0,0,0,0,1}(x) - 2H_{0,0,0,1,1}(x) - 2H_{0,0,1,0,1}(x) - 10H_{0,0,1,1,1}(x) \\
+\;& 2H_{0,1,0,1,1}(x) + 6H_{0,1,1,0,1}(x) - \frac{\pi^2}{6} H_{0,0,1}(x) + \pi^2 H_{0,1,1}(x) + 2\zeta(3) H_{0,1}(x) \\
-\;& \frac{7\pi^2\zeta(3)}{6} - \zeta(5) \Big) \\
+\;& \epsilon^6 \Big(-\left(2\zeta(5) + \frac{\pi^2\zeta(3)}{3}\right) H_1(x) + \frac{9\pi^4}{40} H_{0,1}(x) \\
+\;& \zeta(3)(-13H_{0,0,1}(x) + 9H_{0,1,1}(x) - 2H_{1,0,1}(x)) - \pi^2 \Big(- H_{0,0,0,1}(x) - \frac{5}{6} H_{0,0,1,1}(x) \\
+\;& H_{0,1,0,1}(x) + 6H_{0,1,1,1}(x) + \frac{1}{3} H_{1,0,0,1}(x) \Big) - 11H_{0,0,0,0,0,1}(x) - 11H_{0,0,0,0,1,1}(x) \\
-\;& 20H_{0,0,0,1,0,1}(x) - 20H_{0,0,0,1,1,1}(x) - 16H_{0,0,1,0,0,1}(x) - 26H_{0,0,1,0,1,1,}(x) \\
-\;& 29H_{0,0,1,1,0,1}(x) - 76H_{0,0,1,1,1,1}(x) - 14H_{0,1,0,0,0,1}(x) - 12H_{0,1,0,0,1,1}(x) \\
+\;& 2H_{0,1,0,1,0,1}(x) - 4H_{0,1,0,1,1,1}(x) + 3H_{0,1,1,0,0,1}(x) + 12H_{0,1,1,0,1,1}(x) \\
+\;& 36H_{0,1,1,1,0,1}(x) + 4H_{1,0,0,0,0,1}(x) + 2H_{1,0,0,1,0,1}(x) + 2H_{1,0,1,0,0,1}(x) \\
-\;& \frac{1219\pi^6}{15120} \Big) + \mathcal{O}(\epsilon^7), \qquad (14)
\end{aligned}
$$

H are Harmonic Polylogarithms

HPL-2.0

Check: (s=-1.3,m=1.0)

$$I^{\text{analytic}}_{1,1,0,1,1,0,1,1,1,1,-1,1} = \frac{0.00078765}{\epsilon^6} - \frac{0.00393624}{\epsilon^5} + \frac{0.0190587}{\epsilon^4} - \frac{0.0151068}{\epsilon^3} + \frac{0.290244}{\epsilon^2} + \frac{1.37654}{\epsilon} + 4.82542,$$

$$I^{\text{analytic}}_{1,1,0,1,1,-1,1,1,1,1,-1,1} = \frac{-6.69426}{\epsilon} - 63.1207.$$

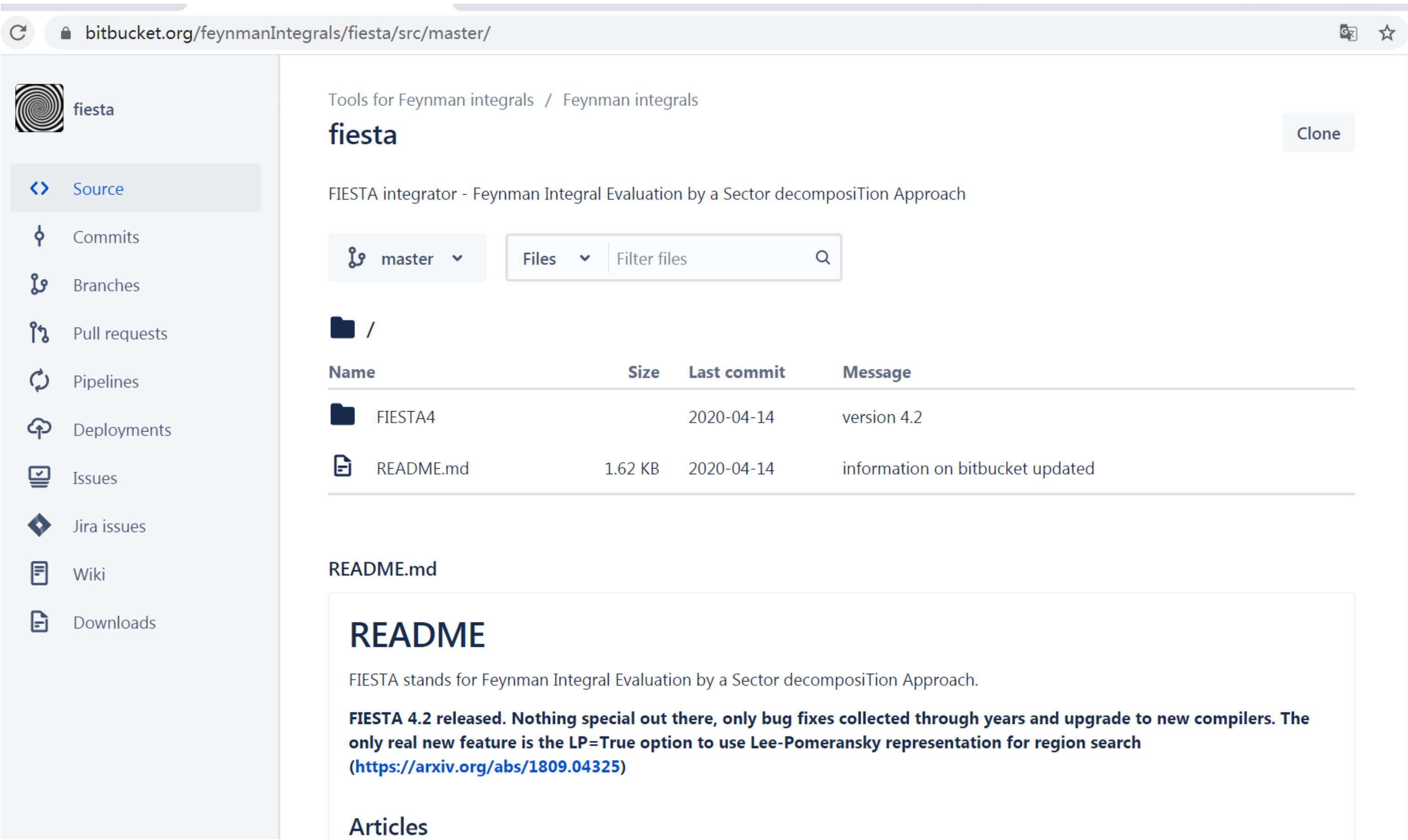
$$I^{\text{numeric}}_{1,1,0,1,1,0,1,1,1,1,-1,1} = \frac{0.000788}{\epsilon^6} - \frac{0.003936}{\epsilon^5} + \frac{0.019058 \pm 0.000002}{\epsilon^4} - \frac{0.015109 \pm 0.000035}{\epsilon^3} + \frac{0.290192 \pm 0.000756}{\epsilon^2} + \frac{1.37606 \pm 0.01581}{\epsilon} + 4.80886 \pm 0.31758,$$

$$I^{\text{numeric}}_{1,1,0,1,1,-1,1,1,1,1,-1,1} = \frac{-6.69429 \pm 0.00003}{\epsilon} - 63.1213 \pm 0.0004,$$

FIESTA packages

https://bitbucket.org/feynmanIntegrals/fiesta/src/master/

FIESTA



Install:
make dep
make

1511.03614

$$\mathcal{F}(a_1,\ldots,a_n) = \int \cdots \int \frac{\mathrm{d}^d k_1 \ldots \mathrm{d}^d k_l}{E_1^{a_1} \ldots E_n^{a_n}}$$

$$d = 4 - 2\epsilon$$

$$E_i \equiv E_i - i0$$

Prefactor

$$\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}$$

FIESTA

pySecDec

A GPU compatible quasi-Monte Carlo integrator interfaced to pySecDec

#1

S. Borowka (CERN), G. Heinrich (Munich, Max Planck Inst.), S. Jahn (Munich, Max Planck Inst.), S.P. Jones (CERN and Munich, Max Planck Inst.), M. Kerner (Munich, Max Planck Inst. and Zurich U.) et al. (Nov 28, 2018)

Published in: *Comput.Phys.Commun.* 240 (2019) 120-137 • e-Print: 1811.11720 [physics.comp-ph]

Before installing pySecDec, make sure that recent versions of numpy (`http://www.numpy.org/`) and sympy (`http://www.sympy.org/`) are installed. The pySecDec program (which includes the QMC integrator library) can be downloaded from `https://github.com/mppmu/secdec/releases`. To install pySecDec, perform the following steps

```
tar -xf pySecDec-<version>.tar.gz
cd pySecDec-<version>
make
<copy the highlighted output lines into your .bashrc>
```

DiffExp: 2006.05510

DiffExp, a Mathematica package for computing Feynman integrals in terms of one-dimensional series expansions

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Based on 1907.13234