IBP for gravitational wave physics

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Motivation

GW discovery in 2015 by LIGO / VIRGO. Future ground-based and space-based detectors offer much higher sensitivity.

Theoretical predictions for waveforms need orders of magnitude improvement in precision!
New approach: scattering amplitudes. Especially suited for post-Minkowskian expansion.

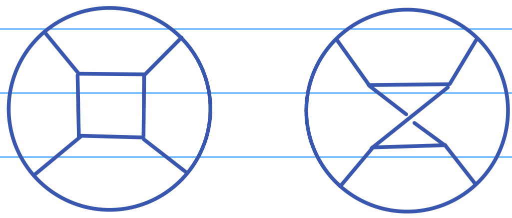
(see extra slides for definition.)

IBP is an essential technique.

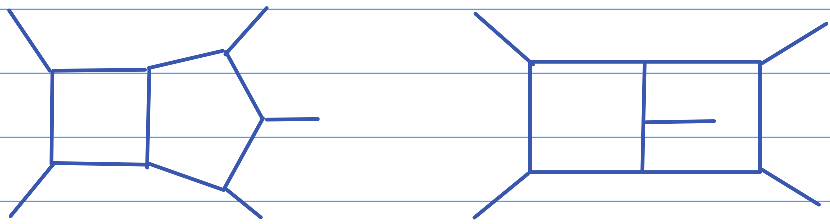
Other gravitational amplitudes in which IBP played an important role:

- 5-loop UV behavior of N=8 supergravity. [Bern, Carrasco, Chen, Edison, Johansson, Parra-Martinez, Roiban, MZ, '18]

(IBP for vacuum integrals)



- N=8 amplitudes for 3-loop 4-point [Henn, Mistlberger], 2-loop 5-point [Abreu, Dixon, Page, Herrmann, MZ '19; Chicherin, Gehrmann, Henn, Wasser, Zhang, '19]



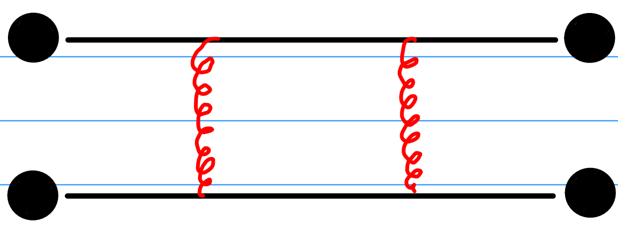
(Same type of integrals as in current frontier in QCD)

- post-Newtonian expansion of binary dynamics. Need multi-loop proagator integrals in $(3-2\epsilon)$ dimensions.

What amplitudes do we need for post-Minkowskian gravity?

Conservative dynamics (non-spinning black holes):

scalar (m1) + scalar (m2) --> scalar(m1) + scalar(m2), via graviton exchange.



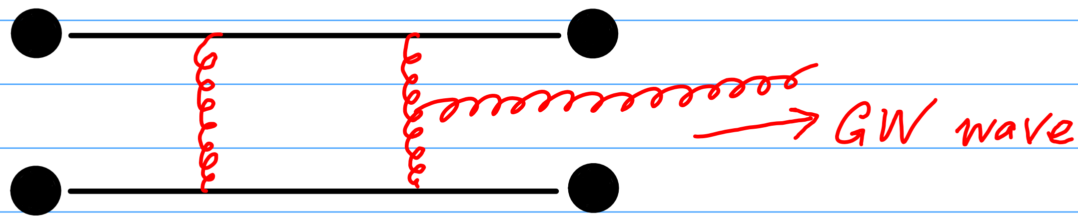
Black hole approximated as point particle when the distance is large.

Systematized by EFT expansion.

BH with spin : fermion / vector / higher-spin particles.

Radiative / dissipative dynamics:

scalar (m1) + scalar (m2) --> scalar(m1) + scalar(m2) + n gravitons



What's the current frontier?

Conservative dynamics: scalar+scalar --> scalar+scalar, at 3 loops (only potential region).

[Bern, Roiban, Ruf, Shen, Solon, MZ, '21 (PRL)]

i.e. $O(G^4)$, 4th-post Minkowskian order

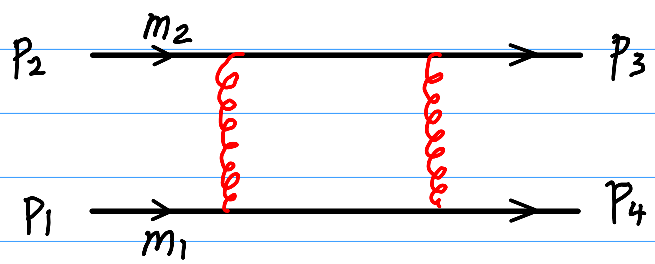
Radiative energy loss: scalar+scalar --> scalar+scalar+graviton, at 2 loops

[Herrmann, Parra-Martinez, Ruf, MZ, '21 (PRL)]

leading $O(G^3)$ energy loss, exact velocity dependence in v.

Vast field: radiation reaction, spin effects, finite-size (tidal) effects...

How challenging are the Feynman integrals?



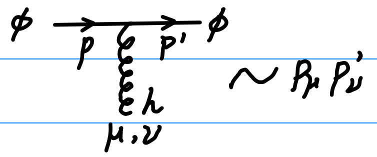
$$\left.\begin{aligned} s &= (p_1+p_2)^2, \\ t &= (p_2-p_3)^2, \\ m_1^2 &= p_1^2 = p_4^2, \\ m_2^2 &= p_2^2 = p_3^2. \end{aligned}\right\}$$
4 kinematic scales,
3 nontrivial parameters

Number of scales comparable to e.g. $q+\bar{q}\to W+Z$

But the tensor rank is very high.

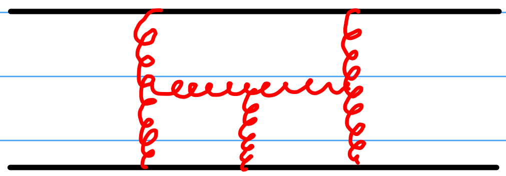
$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $g^{\mu\nu} \sim g^{-1}{}_{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2)\ldots$

$$L = \int d^4x \sqrt{g} \left[R + \frac{1}{2} \underbrace{g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi} - \frac{1}{2} m^2 \phi^2 \right]$$



$\sim p_\mu p'_\nu$

2 powers of loop momenta per vertex (for QCD, only 1 power).



Example 3-loop diagram: 8 vertices.
Numertor has degree / rank 16!

Twice as much as QCD.

A simpler problem: scattering of black holes in N=8 supergravity.

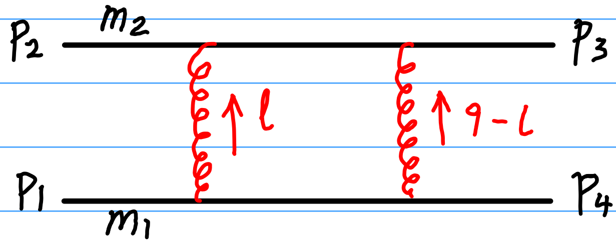
[Caron-Huot, Zahraee, '18; Parra-Martinez, Ruf, MZ, '20]

Degree=0, i.e. only scalar integrals for 2 loops. Degree=2 for 3 loops.

Suitable for testing new ideas about integration etc, like N=4 SYM for QCD calculations.

Simplification from asymptotic expansion

$$\lim_{|q| \ll |p_i|}$$



Exchanged momenta in t channel $\sim \hbar / R \ll m_i, |p_i|$.

We only need the amplitude as an expansion in small $\hbar$.

Need to carefully set up the expansion to eliminate as many scales as possible.

[Beneke, Smirnov, '98]
Method of regions: the full integral as a series in small $|q|$ to all orders, is a sum over two contributions.

(1) soft region $|q|,|l| \ll |p|$. Contains non-analytic behavior, e.g. $1/q^2$, $\log(-q^2)$.

Taylor expansion in small $|q|/|p|, |l|/|p|$, then integrate over ALL l.

e.g. $1/[(l+p_1)^2 - m_1^2] = 1/[2p_1 \cdot l + l^2] = 1/(2p_1 \cdot l) + \cdots$

$|q| \ll |l| \sim |p|$. *(will fine-tune the expansion strategy later)*

(2) hard region Gives Taylor series in q^2. Contact interaction in position space.

Taylor expansion in small $|q|/|p|$, then integrate over ALL l.

e.g. $1/[(l+p_1)^2 - m_1^2]$ is unexpanded,

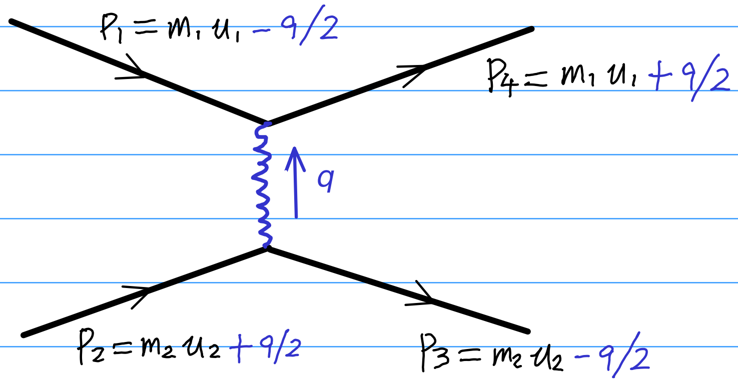
while $1/(q-l)^2 = 1/l^2 + \cdots$

small

In each region, the integrand is integrated over the entire domain.
Overlap betweeen regions vanishes in dimensional regularization.

Symmetric parametrization for soft region

[Glauber; Polkinghorne; Neill & Rothstein]



$$u_1 \cdot q = u_2 \cdot q = 0, u_1 \cdot u_1 = u_2 \cdot u_2 = 1,$$

$$u_1 \cdot u_2 = y, \quad q^2 = -t$$

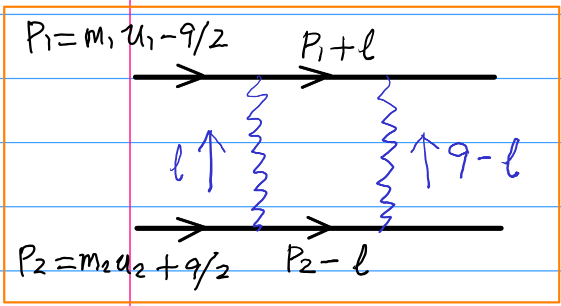
dependence fixed by mass dimension

The only nontrivial parameter which the master integrals depend on.

Used to be s/t, m_1^2/t, m_2^2/t.

Function of 3 variables ⟶ Function of 1 variable. Enormous reduction in complexity.

Example for soft expansion at one loop

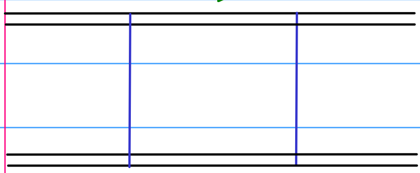


$$\frac{1}{2p_1\cdot l + l^2} = \frac{1}{2m_1 u_1\cdot l + (l^2 - q\cdot l)}$$
$$= \frac{1}{m_1}\frac{1}{2u_1\cdot l} - \frac{1}{m_1^2}\frac{l^2 - q\cdot l}{(2u_1\cdot l)^2}$$

Mass dependence factored out! No need for m_1, m_2 in IBP

$$\frac{1}{l^2}\frac{1}{(q-l)^2}\frac{1}{(p_1+l)^2 - m_1^2}\frac{1}{(p_2-l)^2 - m_2^2}$$

$$\longrightarrow \frac{1}{l^2}\frac{1}{(q-l)^2}\frac{1}{(2u_1\cdot l + i0)}\frac{1}{(-2u_2\cdot l + i0)} + \dots$$

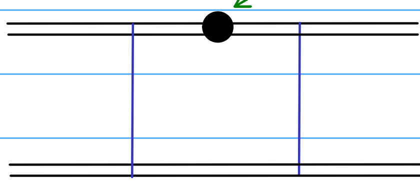


a master integral

Double line = linear propagator

squared linear propagator

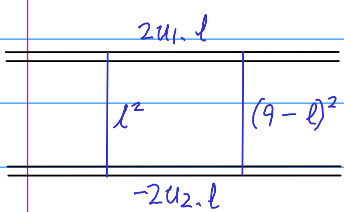
Higher orders in the expansion: will have e.g.



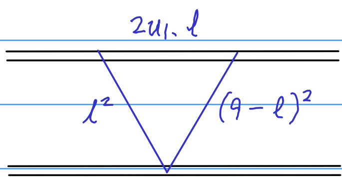
- *Numerators*

Recall that the more complicated integrals evarporate after IBP reduction.

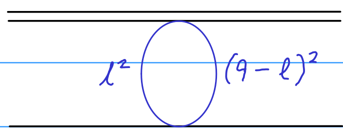
All masters at one loop

$I_{\text{box}} =$ 

linearized box

$I_{\text{tri}} =$ 

linearized triangle

$I_{\text{bub}} =$ 

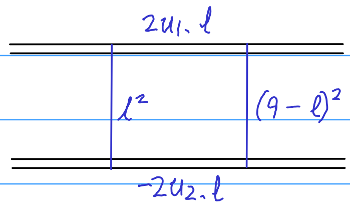
bubble

Therefore, we need to do IBP for integrals with linearized propagators, with only one scaleless kinematic variable $u_1 \cdot u_2 \equiv y$.

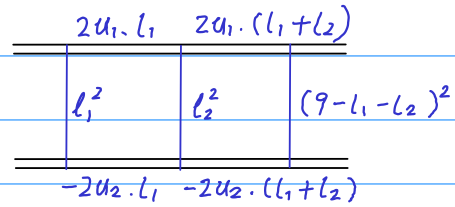
(dependence on q^2 from trivial dimensional analysis)

$p_1 = m_1 u_1 - q/2$

$p_2 = m_2 u_2 + q/2$



linearized box



IBP considerations

1. Choice of irreducible scalar products (ISPs)

This is not any different from usual Feynman integrals with quadratic propagators. Look at 2 loop example.

9 independent scalar products are

$u_1 \cdot l_1,\ u_2 \cdot l_1,\ q \cdot l_1,$

$u_1 \cdot l_2,\ u_2 \cdot l_2,\ q \cdot l_2,$

$l_1^2,\ l_2^2,\ l_1 \cdot l_2.$

They can be expressed as linear combinations of propagators (quadratic & linear ones) and ISPs - must have at least one quadratic propagator depending on l_1, l_2, and $l_1 + l_2$, but other propagators can be linear.

2. Decoupling of integrals by q-parity

Under soft expansion $|l_i| \sim |q|$, quadratc propagators, e.g. $1/(q-l)^2$ scale as $1/|q|^2$, linear propagators, e.g. $1/(2u_1 \cdot l)$ scale as $1/|q|$, integration measure $\left(\int d^4 l_i \right)$ scales as $|q|^4$.

But IBP reduction coefficients are analytic in $q^2 \implies$ q-even and q-odd integrals decouple. The L-loop classical potential between two black holes behave like

$$\left(\frac{GM}{R} \right)^{L+1} \sim (GM|q|)^{L-2}$$

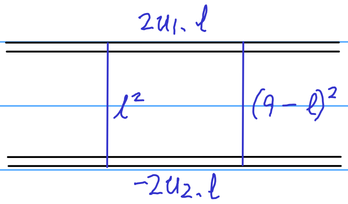
after Fourier transform from position space to momentum space.

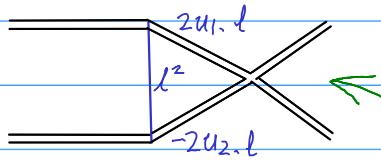
Only need to IBP-reduce q-even / q-odd terms at even / odd loop orders.

Differential equations also decouple into two separate systems. For example, only attempt to find the canonical form for one system.

3. Scaleless sectors and spanning cuts

Back to one-loop example. If we cancel the propagator $(q-l)^2$,

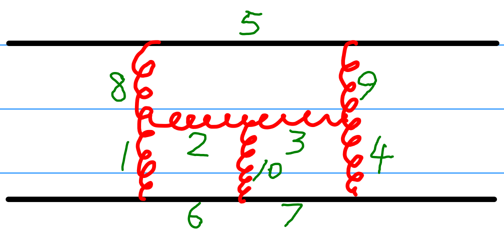




Then the remaining propagators have homogeneous scaling weights under $l^\mu \to \lambda\, l^\mu \implies$ scaleless integrals, vanish in dimensional regularization.

This is ultimately a consequence of the soft $\hbar$ expansion. Inituitively, contact interactions are irrelevant for long-range classical physics.

3-loop example

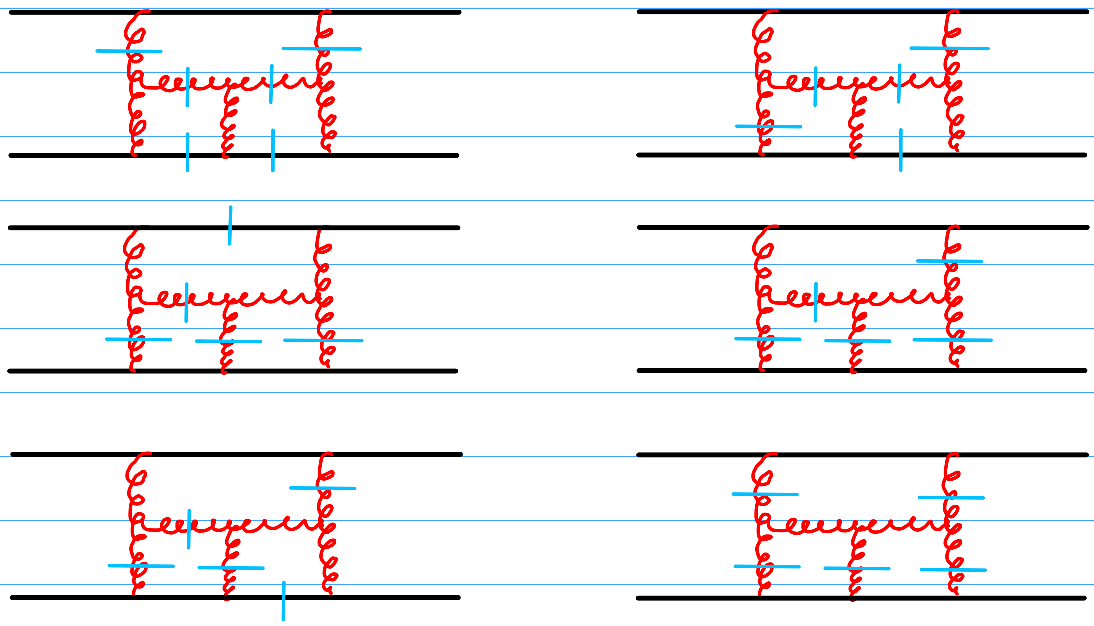


10 propagators +
5 ISPs

Cannot cancel propagators to make m1 line and m2 line touch each other. For example, (1, 8), (4, 9), (10, 2, 8).

All non-scaleless integrals have support on a set of spanning cuts. Can perform IBP on cuts and then perform cut merging. [Larsen, Zhang, '15]

3-loop IBP cut merging (work in progress):



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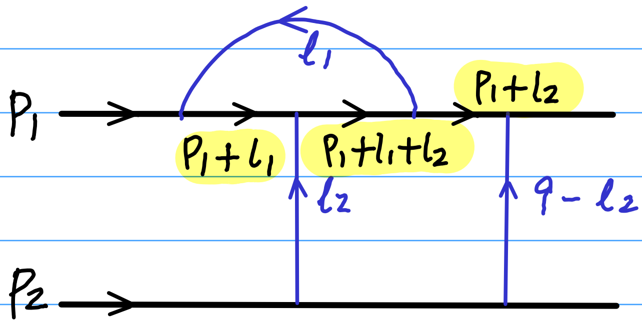
For the potential region, in which graviton momenta are off-shell and dominated by spatial components, also need one matter cut per loop, further simplifying calculation.

4. partial fractioning (algebraic reduction) after soft expansion



This diagram contributes to radiative dynamics, e.g. energy loss, in binary dynamics at $\mathcal{O}(G^3)$

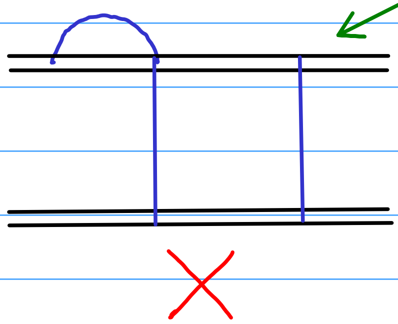
Recall that $p_1 = \underset{\mathcal{O}(1)}{m_1 u_1} - \underset{\mathcal{O}(|q|)}{q}/2,$

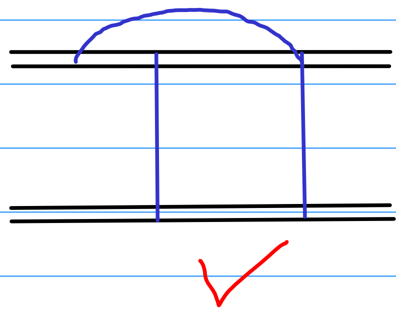
$$\underbrace{(p_1 + l_1)^2 \approx 2 m_1 u_1 \cdot l_1}_{\rho_1}, \quad \underbrace{(p_1 + l_1 + l_2)^2 \approx 2 m_1 (u_1 \cdot l_1 + u_1 \cdot l_2)}_{\rho_2}, \quad \underbrace{(p_1 + l_2)^2 \approx 2 m_1 u_1 \cdot l_2}_{\rho_3}$$

Linearly dependent, because of soft expansion

$$\rho_1 = \rho_2 - \rho_3 \implies \frac{1}{\rho_1 \rho_2 \rho_3} = \frac{1}{\rho_1^2 \rho_3} - \frac{1}{\rho_1^2 \rho_2}.$$

scaleless

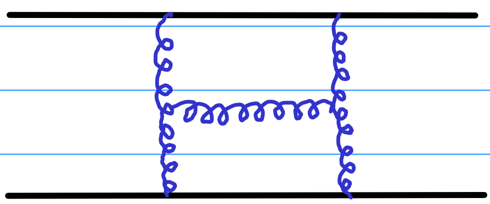




Phase space integrals and reverse unitarity

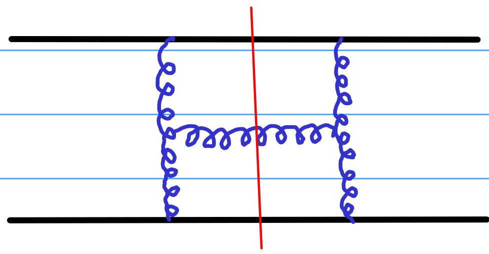
Setup: classical limit of observables from S-matrix. [Kosower, Maybee, O'Connell '18]

Virtual diagram $\mathcal{M}^{(2)}$



contributes to impulse on scattered black hole (deflection angle)

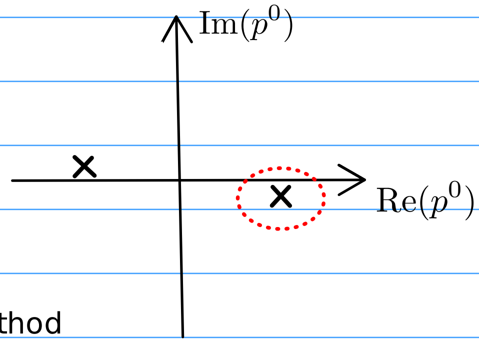
Real emission diagram $\mathcal{M}^{(1)} \mathcal{M}^{*(1)}$



contributes to impluse & energy loss

(uncut) Feynman propagator $1/(p^2 - m^2 + i0)$

cut propagtor for phase space
$2\pi\,\theta(p^0)\delta(p^2 - m^2)$
from picking up only the +ve energy residue
in Feynman propagator



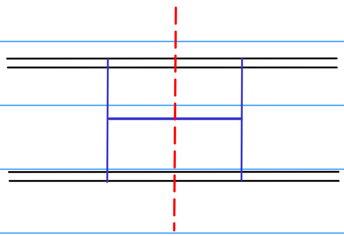
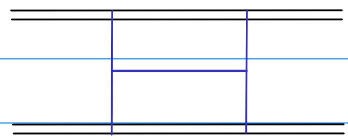
IBP & Differential equations unchanged!
Only change boundary conditions for DEs, known as method
of Reverse Unitarity.

Important in perturbative QCD for Higgs cross sections at NNLO and N3LO, and
energy correlations in electron-positron collider event shapes.

First application of reverse unitarity to gravitational physics in [Herrmann, Parra Martinez,
Ruf, MZ, 2101.07255 (PRL), 2104.03957].

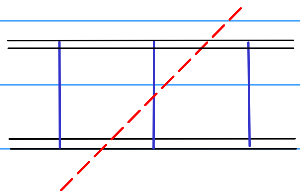
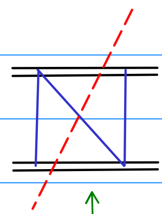
We re-used DEs in canonical basis for virtual integrals in [Parra-Martinez, Ruf, MZ, '20].

Example use of reverse unitarity

 $= 2\,\mathrm{Im}\Big($  $\Big)$

(only one Cutkosky cut, optical
theorem enough)

(Virtual integrals computed via
differential equations)

$\frac{\partial}{\partial v}\Big[v^2$  $\Big]$ = simpler integrals  etc.

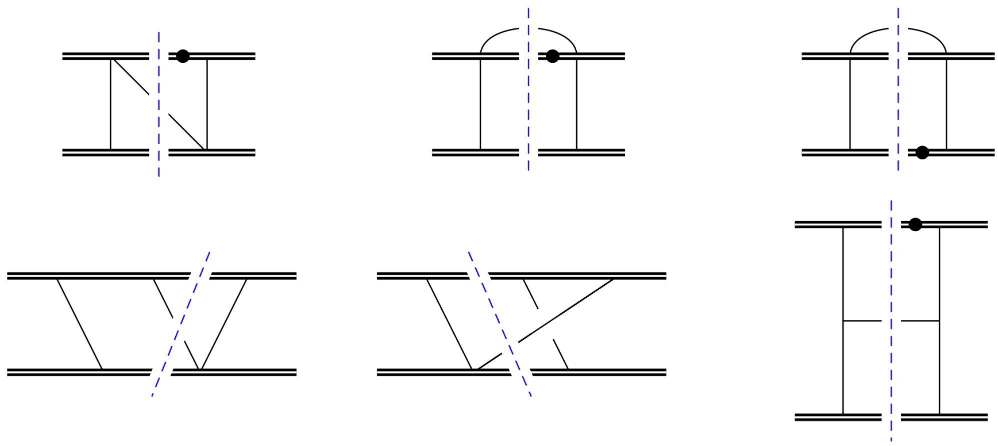
More than one Cutkosky cut. Need serious
use of reverse unitarity, including DEs
on cut.

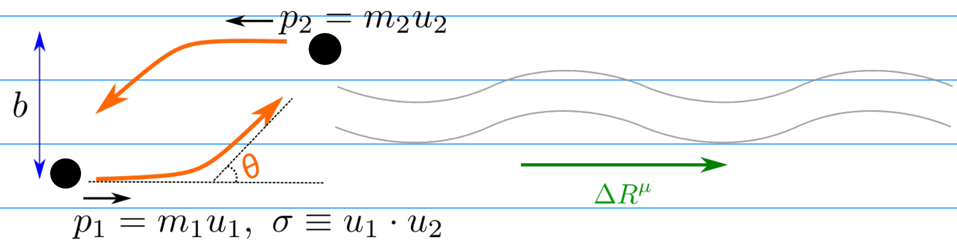
Known!

Result for radaiated energy at 3rd-post-Minkowskian order

[Herrmann, Parra-Martinez, Ruf, MZ, '21 (PRL)

The only needed master integrals:





$$\Delta R^{\mu} = \frac{G^3 m_1^2 m_2^2}{|b|^3} \frac{u_1^{\mu} + u_2^{\mu}}{\sigma + 1} \mathcal{E}(\sigma) + \mathcal{O}(G^4)\,.$$

$$\mathcal{E}(\sigma) = f_1 + f_2 \log\left(\frac{\sigma+1}{2}\right) + f_3 \frac{\sigma \operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\,,$$

$$f_1 = \frac{210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151}{48\left(\sigma^2-1\right)^{3/2}}\,,$$

$$f_2 = -\frac{35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5}{8\sqrt{\sigma^2-1}}\,, \quad f_3 = \frac{\left(2\sigma^2-3\right)\left(35\sigma^4 - 30\sigma^2 + 11\right)}{8(\sigma^2-1)^{3/2}}$$

simple weight-1 result at 2 loops!

Future challenges

Precision at GW detectors motivate studying the post-Minkowskian expansion at one loop higher - i.e. 4 loops, involving one-parameter IBP problem. Challenging but within reach.

Explore other IBP methods, e.g. using syzygy equations. [Gluza, Kajda, Kosower; Ita; Larsen, Zhang]

Function space: 2 loops - weight 1 (poly)log. 3 loops - weight 2 polylog + elliptic integrals. 4 loops? Do we need to investigate numerical techniques?

See next pages for extra material

POST-NEWTONIAN (PN) EXPANSION

Joint expansion in $GM/r \sim v^2$, locked by Virial theorem.

Conservative Hamiltonian in c.o.m. frame:

$$m = m_A + m_B, \qquad \nu = \mu/m$$
$$\mu = m_A m_B / m$$

$\mathcal{O}(v^2)$ $\mathcal{O}(G)$

$$\frac{H}{\mu} = \boxed{\frac{P^2}{2} - \frac{Gm}{R}} + H_{1\mathrm{PN}} + H_{2\mathrm{PN}} + H_{3\mathrm{PN}} + H_{4\mathrm{PN}} \ldots$$

(1980) (2000) (2014)

0PN, Newton

$$\mathcal{O}(v^4) + \mathcal{O}(Gv^2) + \mathcal{O}(G^2)$$

$$\boxed{+\frac{1}{c^2}\left\{-\frac{P^4}{8} + \frac{3\nu P^4}{8} + \frac{Gm}{R}\left(-\frac{P_R^2\nu}{2} - \frac{3P^2}{2} - \frac{\nu P^2}{2}\right) + \frac{G^2m^2}{2R^2}\right\}}$$

1PN, Einstein, Infeld, Hoffman, 1938

POST-MINKOWSKIAN (PM) EXPANSION

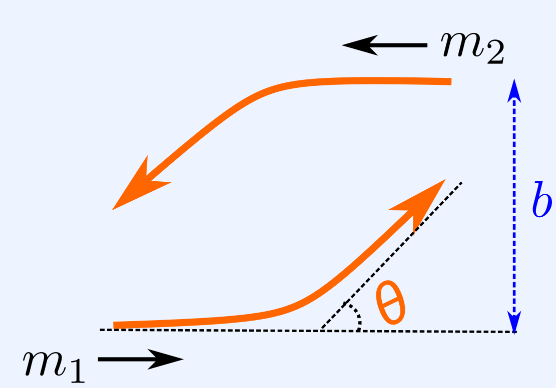
Expansion in coupling GM/r, **exact velocity dependence**

[Bertotti, Kerr, Plebanski, Portilla, Westpfahl, Gollder, Bel, Damour, Derulle, Ibanez, Martin, Ledvinka, Scaefer, Bicak...]

Most accurate PM scattering angle until ~ 2019 [Westpfahl, '85]

Scattering angle of two black holes, as function of $m_1, m_2, b, E_{\mathrm{cm}}$.

$$2\sin\left(\frac{\theta}{2}\right) = \frac{4G(m_1+m_2)}{b}\left(\frac{\hat{E}^4 - 2m_1^2m_2^2}{\hat{E}^4 - 4m_1^2m_2^2} + \frac{3\pi}{16}\frac{G(m_1+m_2)}{b}\frac{5\hat{E}^4 - 4m_1^2m_2^2}{\hat{E}^4 - 4m_1^2m_2^2}\right)$$

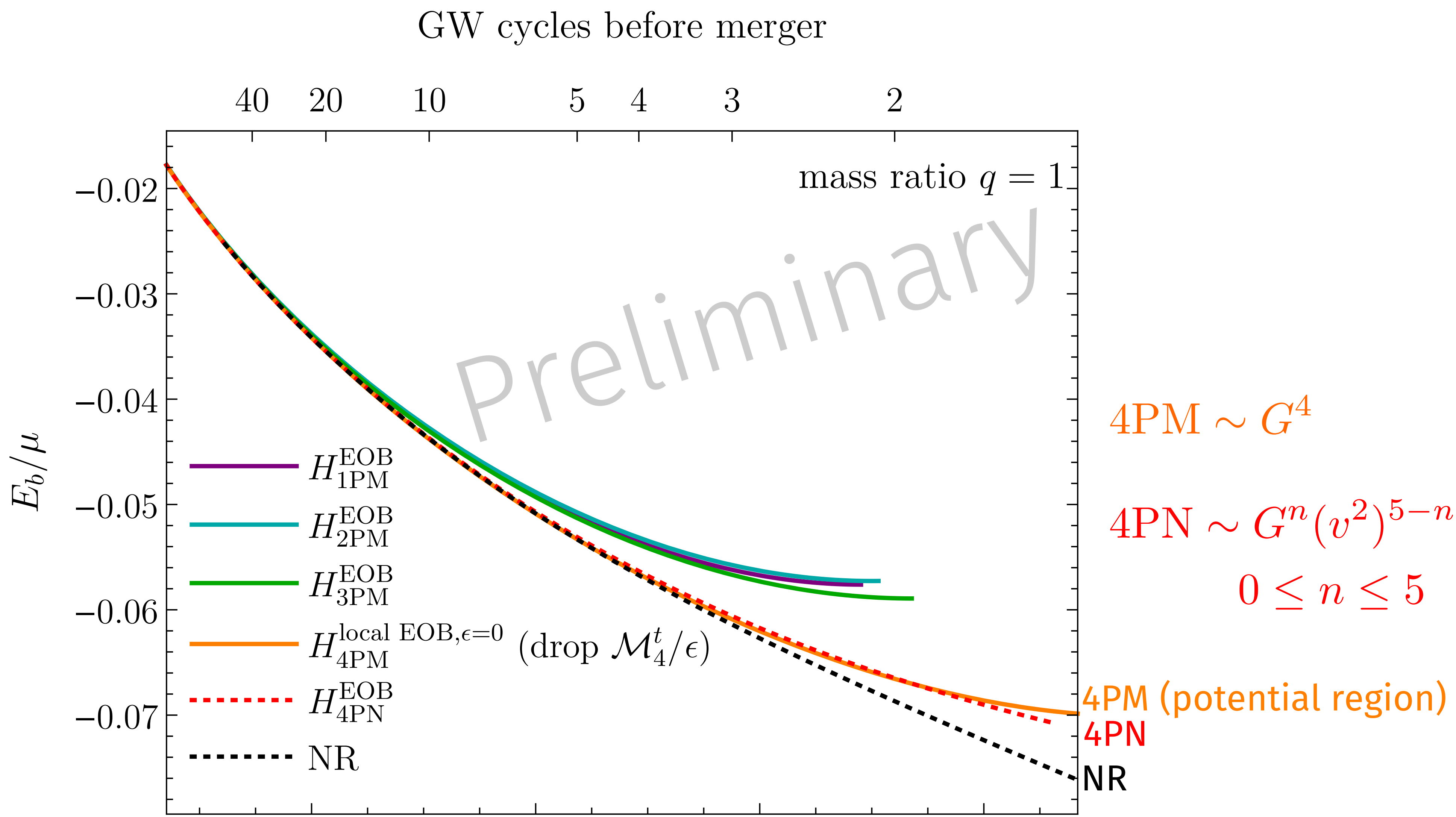


where $\hat{E}^2 \equiv E_{\mathrm{cm}}^2 - (m_1^2 + m_2^2),\ c = 1$

Similar to expansion in **relativistic QFT** - can QFT help push it further?

4PM BINDING ENERGY VS. NUMERICAL RELATIVITY

[Khalil, Buonanno, Steinhoff, Vines, preliminary]



$4\mathrm{PM} \sim G^4$

$4\mathrm{PN} \sim G^n (v^2)^{5-n},$
$0 \leq n \leq 5$