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Counting Master Integral: A Fancinating Problem

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Lee-Pomeransky Representation, a Brief Review

In this section we give a brief review of Lee-Pomeransky representation of Master integrals in different sector. We denote

$$\mathcal{I}(d;\nu_1,\cdots,\nu_n)=\int\prod_{i=1}^{L}\frac{\mathrm{d}^d l_i}{i\pi^{d/2}}\frac{1}{\prod_{i=1}^{n}D_i^{\nu_i}} \tag{1}$$

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to obtain the α representation:

$$\mathcal{I}(d;\nu_1,\cdots,\nu_n)=\int\prod_{i=1}^{N}\frac{\alpha_i^{\nu_i-1}\mathrm{d}\alpha_i}{\Gamma(\nu_i)}\frac{1}{U^{d/2}}\exp\left(-\frac{F}{U}\right) \tag{3}$$

where F and U are symanzik polynomials with $\deg(F)-\deg(U)=1$.

$$\mathcal{I}(d;\nu_1,\cdots,\nu_n) = \int \Gamma(|\nu| - \frac{dL}{2}) \prod_{i=1}^{N} \frac{x_i^{\nu_i - 1} \mathrm{d}x_i}{\Gamma(\nu_i)} \delta(\sum_{i=1}^{N} x_i - 1) \frac{F^{dL/2-|\nu|}}{U^{d(L+1)/2-|\nu|}} \tag{4}$$

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Then we use the definition of Beta function, to derive that

$$\frac{F^{dL/2-|\nu|}}{U^{d(L+1)/2-|\nu|}} = \int_0^\infty \frac{s^{|\nu|-dL/2-1}\mathrm{d}s}{(F+sU)^{d/2}} \frac{1}{B(|\nu| - \frac{dL}{2}, \frac{d(L+1)}{2} - |\nu|)} \tag{5}$$

$$\mathcal{I}(d;\nu_1,\cdots,\nu_n)=\int\Gamma(|\nu|-\frac{dL}{2})\prod_{i=1}^{N}\frac{x_i^{\nu_i-1}\mathrm{d}x_i}{\Gamma(\nu_i)}\delta(\sum_{i=1}^{N}x_i-1)\frac{F^{dL/2-|\nu|}}{U^{d(L+1)/2-|\nu|}} \tag{4}$$

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Then we find

$$\mathcal{I}(d;\nu_1,\cdots,\nu_n)=\int\frac{\Gamma(|\nu|-\frac{dL}{2})}{B(|\nu|-dL/2,\frac{d(L+1)}{2}-|\nu|)}\prod_{i=1}^{N}\frac{x_i^{\nu_i-1}\mathrm{d}x_i}{\Gamma(\nu_i)}$$

$$\delta(\sum_{i=1}^{N}x_i-1)\int_0^\infty\frac{s^{|\nu|-dL/2-1}\mathrm{d}s}{(F+sU)^{d/2}}$$

$$=\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d(L+1)}{2}-|\nu|)}\int_0^\infty\prod_{i=1}^{N}\frac{x_i^{\nu_i-1}\mathrm{d}x_i}{\Gamma(\nu_i)}\frac{1}{(F+U)^{d/2}}$$

where in the last step, we redefined $x_i \to x_i/s$, and integrate the delta function with s, and find the final answer. The result ten is called Lee-Pomeransky representation of the master integral, where the dependdence of the topology of the diagram is now characterized by $G := F + U$.

Remark

In Lee-Pomeransky representation, the dimensional regularization scheme of the divergence in Feynman integral turns out to be analytic continuation in d and $\nu = (\nu_1, \cdots, \nu_N)$. It is then necessary to study the property of $\mathcal{I}$ in terms of those variables. Notice in general there will be a small $i\epsilon$ term which specifies the pole distribution in the Feynman integral, and it is vital that one keeps those ϵ terms in G. Here we have neglected it since in this note we are maily focusing on the algebraic structure of $\mathcal{I}$.

Definition (Twisted Mellin Transformation)

Let $f : \mathbb{R}_+^N \to \mathbb{C}$, the following function on $\mathbb{C}^N$ is called the twisted Mellin transformation of f:

$$\mathcal{M}(f)(\nu_1, \cdots, \nu_N) := \Big(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i - 1} \mathrm{d}x_i}{\Gamma(\nu_i)}\Big) f(x_1, \cdots, x_N) \qquad (6)$$

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whenever such integral exists.

It is obvious that the Feynman integral can be expressed in terms of :

$$\mathcal{I}(d, \nu) = \frac{\Gamma(d/2)}{\Gamma(d/2 - \omega)} \mathcal{M}(G^{-d/2})(\nu_1, \cdots, \nu_N) \tag{7}$$

where $\omega = dL/2 - \sum_{i=1}^{N} \nu_i$.

Lemma

Let $\alpha, \beta \in \mathbb{C}, \nu \in \mathbb{C}^N, 1 \leq i \leq N$, *and* $f, g : \mathbb{R}_+^N \to \mathbb{C}$. *Writing* e_i *for* $i-th$ *unit vector in* $\mathbb{C}^N$. *The twisted Mellin transfomation has the following property:*

- *Linearity :* $\mathcal{M}(\alpha f + \beta g)(\nu) = \alpha \mathcal{M}\{f\}(\nu) + \beta \mathcal{M}\{g\}(\nu)$.
- *Multiplication:* $\mathcal{M}\{x_i f\}(\nu) = \mathcal{M}(f)(\nu + 1)$
- *Differentiation:* $\mathcal{M}\{\partial_i f\}(\nu) = -\mathcal{M}\{f\}(\nu - e_i)$

These properties are the building blocks to the application of D-module theory later.

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As a remainder, we note that $\mathcal{M}$ is actually inversible in the space of functions in $\mathbb{R}_+^N$:

Theorem (Inverse Mellin Transformation)

Suppose the Mellin transformation $f^*(\nu) := \mathcal{M}(f)(\nu)$ converges in the domain of the form $a_i \leq \mathrm{Re}(\nu_i) \leq b_i$, for $1 \leq i \leq N$, where $a, b \in \mathbb{R}^N$. Then its inverse is given by:

$$f(x) = \mathcal{M}^{-1}(f^*)(x) := (\prod_{i=1}^{N} \int_{\sigma_i + i\mathbb{R}} \mathrm{d}\nu_i \frac{\Gamma(\nu_i)}{x^{\nu_i} 2\pi i}) f^*(\nu), \text{ where } x \in \mathbb{R}_+^N \tag{8}$$

This multiple integral along lines parallel to the imaginary axis converges for $a_i \leq \sigma_i \leq b_i$, which does not depend on the choice of σ_i.

Proof.

We only consider one variable case by direct computation:

$$\begin{aligned}
\int_{\sigma+i\mathbb{R}} \mathrm{d}\nu \frac{\Gamma(\nu_1)}{x^{\nu_1} 2\pi i} f^*(\nu) &= \int_{\sigma+i\mathbb{R}} \mathrm{d}\nu \frac{\Gamma(\nu)}{x^{\nu} 2\pi i} \int \frac{\mathrm{d}y y^{\nu-1}}{\Gamma(\nu)} f(y) \\
&= \int_{\mathbb{R}} \mathrm{d}y \int_{\sigma+i\mathbb{R}} \frac{\mathrm{d}\nu}{2\pi i} \frac{y^{\nu-1}}{x^{\nu}} f(y) \\
&= \int_{\mathbb{R}} \mathrm{d}y \int_{\sigma+i\mathbb{R}} \frac{\mathrm{d}\nu}{2\pi i} \frac{1}{x} e^{(\nu-1)t} f(y) \\
&= \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}} \frac{\mathrm{d}u}{2\pi} \frac{1}{x} e^{(\sigma-1)t+iut} f(y) \\
&= \int_{\mathbb{R}} \mathrm{d}y \frac{1}{x} \delta(\log(y/x)) f(y) \\
&= \int_0^{\infty} \mathrm{d}t \delta(t) e^t f(xe^t) \\
&= f(x)
\end{aligned}$$

□

Remark

This means we can formulate the theory of Feynmna integrals in the Mellin transformed space. Now let's formulate the IBP relation in this space.

Definition

The momentum space IBP relations of Feynman integral $\mathcal{I}(d;\nu_1,\cdots,\nu_N)$ are those relations between scalar Feynman integrals that are obtained from Stokes' theorem:

$$(\int\prod_{n=1}^{L}d^d l_n)o_j^i f=0 \tag{9}$$

where

$$o_j^i:=\frac{\partial}{\partial q_i^\mu}q_j^\mu,\forall i\in\{1,\cdots,L\},j\in\{1,\cdots,M\} \tag{10}$$

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We then search for the representation of o_j^i in the Mellin transformation space:

Given a set of $N = |\Theta|$ denominators D such that the matrix $\mathcal{A}$ deifned by $D_a = \sum_{\{i,j\}\in\Theta} \mathcal{A}_a^{\{i,j\}} s_{\{i,j\}} + \lambda_a$, where $\Theta = \{\{i,j\} | 1 \leq i \leq L, 1 \leq j \leq L+E\}, |\Theta| = \frac{L(L+1)}{2} + LE$, is invertible. Then every momentum-space IBP can be written explicitly as

$$\mathbf{O}_j^i \mathcal{I}(\nu) = 0 \tag{11}$$

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where for $1 \le i \le L, 1 \le j \le L+E$, it can be expressed as:

$$\mathbf{O}_j^i = \begin{cases} d\delta_{ij} - \sum_{a,b=1}^{N} C_{aj}^{bi} a^+(b^- - \lambda_b) & j \le \\ -\sum_{a,b=1}^{N} C_{aj}^{bi} a^+(b^- - \lambda_b) - \sum_{a=1}^{N} \sum_{m=L+1}^{L+E} A_a^{\{i,m\}} q_j q_m a^+ & j > \end{cases} \tag{12}$$

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where the coefficient matrix C_{aj}^{bi} is defined as:

$$C_{aj}^{bi} := \begin{cases} \sum_{m=1}^{L+E} \mathcal{A}_a^{\{i,m\}} \mathcal{A}_{\{m,j\}}^b (1 + \delta_{mi}) & j \leq L \\ \sum_{m=1}^{L} \mathcal{A}_a^{\{i,m\}} \mathcal{A}_{\{m,j\}}^b (1 + \delta_{mi}) & j > L \end{cases} \tag{13}$$

Proof:

The action of o^i_j on the integrad is:

$$o^i_j f = d\delta^i_j f + f \sum_{a=1}^{N} \frac{-\nu_a}{D_a} q_j \frac{\partial D_a}{\partial q_i} \tag{14}$$

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By definition, we know

$$\begin{aligned} q_j \frac{\partial D_a}{\partial q_i} &= q_j \frac{\partial}{\partial q_i} \sum_{\{k,m\}\in\Theta} \mathcal{A}_a^{\{k,m\}} q_k q_m \\ &= \sum_{m=1}^{L+E} \mathcal{A}_a^{\{i,m\}} q_j q_m (1 + \delta^i_m) \end{aligned}$$

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Then we express the scarlar products $q_j q_m$ in terms of denominators using:

$$q_i q_j = \sum_{a=1}^{N} \mathcal{A}^a_{\{i,j\}} (D_a - \lambda_a) \tag{15}$$

Then we find

$$o^i_j f = d\delta^i_j f - f \sum_{a,b=1}^{N} C^{bi}_{aj} \frac{\nu_a}{D_a}(D_b - \lambda_b) \text{ for } 1 \leq j \leq L \text{ and}$$

$$o^i_j f = -f \sum_{a,b=1}^{N} C^{bi}_{aj} \frac{\nu_a}{D_a}(D_b - \lambda_b) - f \sum_{a=1}^{N} \sum_{m=L+1}^{L+E} \mathcal{A}^{\{i,m\}}_a q_j q_m \frac{\nu_a}{D_a}$$

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Then by noticing that multiplying f with ν_a / D_a is equivalent to the action of a^+, and $\frac{1}{D_b}$ is equivalent to b^-, we are able to conclude the proposition.

Introduction of D-module

To go head and formulate the problem, we introduce the concept of D-module in this section. Let's start with the integral

$$F(t) = \frac{1}{2\pi}\int_{D(t)} \exp\left(-\frac{1}{2}(x^2+y^2)\right)\mathrm{d}x\mathrm{d}y, \quad D(t) = \{(x,y)\in\mathbb{R}^2 | xy \leq t \tag{16}$$

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The non-triviality of the integral lies in the integration region. To go ahead, we introduce the Heaviside function

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$$F(t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\left(-\frac{1}{2}(x^2 + y^2)\right) \theta(t - xy) \mathrm{d}x\mathrm{d}y \tag{18}$$

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Differentiation with the integral sign yields

$$v(t) = F'(t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\left(-\frac{1}{2}(x^2 + y^2)\right) \delta(t - xy) \mathrm{d}x \mathrm{d}y \tag{19}$$

The integrand $u(x, y, t) = \exp\bigl(-\frac{1}{2}(x^2 + y^2)\bigr)\delta(t - xy)$ then satisfies a holonomic system:

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We have an equality that:

$$y\partial_t(\partial_y + x\partial_t + y) - y(\partial_x + y\partial_t + x) + (\partial_t^2 - 1)(t - xy) = -\partial_x y + \partial_y y\partial_t + t\partial_t^2 + \partial_t - t$$

The integrand $u(x, y, t) = \exp\left(-\frac{1}{2}(x^2 + y^2)\right)\delta(t - xy)$ then satisfies a holonomic system:

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$$= -\partial_x y + \partial_y y\partial_t + t\partial_t^2 + \partial_t - t$$

Notice in the right hand side, the first two terms are total derivatives in x or y, therefore we gain

$$(t\partial_t^2 + \partial_t - t)v(t) = 0 \qquad (21)$$

by setting $t = -iz, u(z) = v(-iz)$, it is easy to verify that

$$(z^2 \frac{\mathrm{d}}{\mathrm{d}z^2} + z \frac{\mathrm{d}}{\mathrm{d}z} + z^2)u = 0 \tag{22}$$

u satisfies the bessel equation.

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In the above example, we find that the analytic structure of the integral on t can be obtained from the integrand, with combinatorics of differential operators acting on it. To study such combinatorics, we should consider the space of all such operators, which forms a ring structure, together with the representation space of such a ring, which is a module over it. We introduce the following concepts.

Definition

A Weyl algebra A^N in N variables $x_1, \cdots, x_N$ is the non-commutative algebra of polynomial differential operators:

$$A^N := \mathbb{C}[x_1, \cdots, x_N, \partial_1, \cdots, \partial_N] / \sim \quad (24)$$

where $\sim$ denotes the ideal $\langle \partial_i x_j - x_j \partial_i - \delta_{ij} \rangle |_{i,j \in \{1, \cdots, N\}}$. Note that with the multiple index notatoin

$$x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \partial^\beta = \partial_1^{\beta_1} \partial_2^{\beta_2} \cdots \partial_N^{\beta_N} \quad (25)$$

every $P \in A^N$ can be written uniquely in the form

$$P = \sum_{\alpha, \beta} c_{\alpha\beta} x^\alpha \partial^\beta \quad (26)$$

Later, we shall extend the field to $\mathbb{C}(s)$ which can be viewed as a localization of s in $\mathbb{C}[s]$. We denote $k = \mathbb{C}(s)$, and $A_k^N = A^N \otimes_{\mathbb{C}} k$. Now, we start to construct the space where all the annilators in **??** lives in

Definition

Given a polynomial $f \in \mathbb{C}[x]$, the $A^N[s]$-module $\mathbb{C}[s, x, \frac{1}{f}]f^s$ consists of elements of the form $p/f^k \cdot f^s$, where $p \in \mathbb{C}[s, x], k \in \mathbb{N}_0$, with the $A^N[s]$ action that:

$$q(\frac{p}{f^k}f^s) = \frac{qp}{f^k} \cdot f^s, \partial_i(\frac{p}{f^k}f^s) = \frac{f\partial_i p + (s-k)\partial_i f}{f^{k+1}} \cdot f^s \qquad (27)$$

We denote the submodule generated by f^s under the action of $A^N[s]$ as $A^N[s]f^s$.

Counting Master Integrals

We are now prepared in formulating our proposal in the language of D-modules. We firstly construct the space of all Feynman integrals (in the ν space), and then assign it with a D-module structure.

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Given $G \in \mathbb{C}[x_1, \cdots, x_N]$, the (rescaled-)Feynman integral is defined as $\tilde{\mathcal{I}}_G = \mathcal{M}\{G^s\}$. The vector space of all Feynman integrals associated to G is then defined as

$$V_G := \sum_{n \in \mathbb{Z}^N} \mathbb{C}(s, \nu) \cdot \tilde{\mathcal{I}}(s, \nu + n) = \mathbb{C}(s, \nu) \otimes_{\mathbb{C}(s)} (\mathcal{M}\{A_k^N\} \cdot \tilde{\mathcal{I}}_G) \quad (28)$$

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where $\mathcal{M}\{A_k^N\}$ is the Mellin transformation of the operators in the Weyl algebra.

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Notice then $\mathcal{M}$ is inversible and therefore we can rephrase the above space in terms of parametric intergands, which is:

$$\mathfrak{C}(G) = \dim_{\mathbb{C}(s,\theta)}(\mathbb{C}(s,\theta) \otimes_{\mathbb{C}[s,\theta]} A^N[s] \cdot G^s) \tag{30}$$

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where $\theta = (\theta_1, \theta_2, \cdots, \theta_N)$, and can be understood of localization of $x_i \partial_i$ in the fractional field $\mathbb{C}(s,\theta)$. For later convenience, we introduce $k = \mathbb{C}(s), R = \mathbb{C}[s,\theta], F = \mathbb{C}(s,\theta)$, and the module $\mathbb{M} = A^N[s]G^s$, then

$$\mathfrak{C}(G) = \dim_F(F \otimes_R \mathbb{M}) \tag{31}$$

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In the following, we shall study the module in a slightly different context, where we do localization of elements in A_k^N in the hypersurface $x_i = 0$:

$$D_k^N = A_k^N[x^{-1}] := k[x^{\pm 1}] \otimes_k A_k^N \tag{32}$$

We can view this space (which is the space of derivatives with coefficients being rational functions of xs) as the pull back $D_k^N = \iota^* A_k^N$ with the open inclusion map:

$$\iota^* \mathbb{G}_{m,k}^N \to \mathbb{A}_k^N \tag{33}$$

Then we state our main result in this section:

Theorem

Let $\mathbb{M}$ denote a holonomic A_k^N-module, then $F \otimes_R \mathbb{M}$ is a finite-dimensional vector space over F. Moreover, its dimension is given by the Euler characteristic $\dim_F(F \otimes_R \mathbb{M}) = \chi(\iota^* \mathbb{M})$.

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Remark

Here the Euler characteristic is assigned to the de Rham cohomology with $\iota^* \mathbb{M}$ *as a module on* R*, with connection derivative* $\mathrm{d} = \mathrm{d}x^i \partial_i$ *. We shall explain in detail about this cohomology in the process of proof.*

Lemma

Let $\mathbb{M}$ denote a holonomic D_k^N-module, then for any $1 \le i \le N$, and $\mathbb{M}(\theta_i, \cdots, \theta_N) := k(\theta_i, \cdots, \theta_N) \otimes_{k[\theta_i, \cdots, \theta_N]} \mathbb{M}$ denotes the D_k^N-module where $\theta_i = x_i \partial_i$ is localized in $\theta_i = 0$, and is called algebraic Mellin transform. Then $\mathbb{M}(\theta_i, \cdots, \theta_N)$ is a holomorphic $D^{i-1}_{k(\theta_i, \cdots, \theta_N)}$ module.

Proof :
Since $\mathbb{M}(\theta_i, \theta_{i+1}, \cdots, \theta_N) = \mathbb{M}(\theta_{i+1}, \cdots, \theta_N)(\theta_i)$, it suffices to show the case $i = N$. To declare the holonomicity of the algebraic Mellin transformation, we formulate this module as a quotient module. Let's consider

$$\mathbb{M}[\nu] := k[\nu] \otimes_k \mathbb{M} \tag{34}$$

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$$\mathbb{M}[\nu] := k[\nu] \otimes_k \mathbb{M} \tag{34}$$

This is a D_k^N-module with coefficients being ν polynomial. Then we consider the sub-module $(\partial_N + \nu/x_N)\mathbb{M}$, and it's quotient $\mathfrak{M} := \mathbb{M}[\nu]/(\partial_N + \nu/x_N)\mathbb{M}$. Since then $k(\nu)$ is flat in $k[\nu]$, $(\partial_N + \nu/x_N)\mathbb{M}$ remains to be a sub-module after tensoring with $k(\nu)$ in $k[\nu]$.

Therefore, we conclude that

$$\frac{\mathbb{M} \otimes_k k(\nu)}{(\partial_N + \nu_N / x_N)(\mathbb{M} \otimes_k k(\nu))} \simeq \mathfrak{M} \otimes_{k[\nu]} k(\nu) \simeq \mathbb{M}(\nu) \tag{35}$$

Therefore, we conclude that

$$\frac{\mathbb{M} \otimes_k k(\nu)}{(\partial_N + \nu_N/x_N)(\mathbb{M} \otimes_k k(\nu))} \simeq \mathfrak{M} \otimes_{k[\nu]} k(\nu) \simeq \mathbb{M}(\nu) \qquad (35)$$

are isomorphic as $D^{N-1}_{k(\nu)}$-modules. The LHS is the quotient of $\mathbb{M}x_N^{\nu}/\partial_N \mathbb{M}x_N^{\nu}$ of the $D^N_{k(\nu)}$-module $\mathbb{M}x_N^{\nu} := \mathbb{M} \otimes_k k(\nu)$ defoned by the original action of D^{N-1}_k and $x_N^{\pm}$ on $\mathbb{M}$, but twisting the connection ∂_N to $\partial_N + \nu/x_N$.

Therefore, we conclude that

$$\frac{\mathbb{M} \otimes_k k(\nu)}{(\partial_N + \nu_N/x_N)(\mathbb{M} \otimes_k k(\nu))} \simeq \mathfrak{M} \otimes_{k[\nu]} k(\nu) \simeq \mathbb{M}(\nu) \tag{35}$$

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Corollary

The full Mellin transformation $\mathbb{M}(\theta_1, \cdots, \theta_N)$ *is a finite-dimensional vector space over the field* $k(\theta_1, \cdots, \theta_N)$.

Theorem

If $\mathbb{M}$ denotes the holonomic D_k^1-module, then

$$\dim_{k(\theta_1)}\mathbb{M}(\theta_1) = \chi(\mathbb{M}) = \dim_k\left(\frac{\mathbb{M}}{\partial_1\mathbb{M}}\right) - \dim_k \ker(\partial_1) \tag{36}$$

Proof Due to holonomicity, we know that $\mathbb{M}$ is cyclic under the action of D_k^1. Therefore, we can pick a generator of $\mathbb{M}$ and extend it as a basis of $\mathbb{M}(\theta_1)$ over the field $k(\theta_1)$.

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$$b(\theta \pm j)\mathcal{N}_{j+1} = b(\theta \pm j)x^{\mp j}\mathcal{N}_1 = x^{\mp j} b(\theta)\mathcal{N}_1 \subset x^{\mp j}\mathcal{N} \subset \mathcal{N}_j \quad (37)$$

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which shows that the polynomial $b_{j+1}(\theta) = b(\theta + j)b(\theta - j) \in k[\theta]$ have the proeprty that $b_{j+1}(\theta)\mathcal{N}_{j+1} \subset \mathcal{N}_j$. Let $Z := b^{-1}(0)$ denotes the set of zeroes of b.

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Then we find that $b_{j+1}^{-1}(0) \subset (Z + j) \cup (Z - j)$. In other words, the zeroes of b_{j+1} is shifted by j. Thus $\exists j_0 \in \mathbb{N}$, such that $\forall j > j_0$, we have $b_{j+1}(0) \neq 0$. Then by Bezout theorem, $\exists u_{j+1}(\theta), v_{j+1}(\theta)$, such that

$$1 = u_{j+1}(\theta)b_{j+1}(\theta) + v_{j+1}(\theta)\theta \quad (39)$$

Now suppose $m \in \mathcal{N}_{j+1}$, we know

$$m = 1 \cdot m = b_{j+1}(\theta) u_{j+1}(\theta) m + v_{j+1}(\theta) \theta m \subset b_{j+1}(\theta) \mathcal{N}_j + v_{j+1}(\theta) \theta m \tag{40}$$

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Thus we know that $\ker(\theta) \cap \mathcal{N}_{j+1} \subset \mathcal{N}_j$. Therefore, we find

$$\ker(\theta) \cap \mathcal{N}_{j+1} \subset \ker(\theta) \cap \mathcal{N}_j \subset \cdots \subset \ker(\theta) \cap \mathcal{N}_{j_0} \tag{41}$$

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In addition, let $x \in \mathcal{N}_{j+1}, j \geq j_0$, and $\theta x \in \mathcal{N}_{j_0}$, then

$$x \subset \mathcal{N}_j + \mathcal{N}_{j_0} \tag{42}$$

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In consequence, we find

$$\frac{\mathbb{M}}{\partial_1(\mathbb{M})} \simeq \frac{\mathbb{M}}{\theta_1(\mathbb{M})} \simeq \frac{\mathcal{N}_{j_0}}{\theta_1(\mathcal{N}_{j_0})} \tag{43}$$

which yields a quasi-isomorphism between $\mathrm{DR}(\mathbb{M})$ and $\mathrm{DR}(\mathcal{N}_{j_0})$. The statement of the theorem thus reduces to

$$\dim_{k(\theta)}\mathbb{M}(\theta) = \dim_{k(\theta)}(\mathcal{N}_{j_0}) = \chi(\mathcal{N}_{j_0}) \tag{44}$$

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for j_0 large enough. Now since both sides are additive under short exact sequences, the claim reduces to the case of free rank one $k[\theta]$-module. Now of course

$$\dim_{k(\theta)} k[\theta] = 1$$

$$\dim_{k(\theta)} \frac{k[\theta]}{\theta k[\theta]} - \dim_{k(\theta)}\ker(\theta) = 1 - 0 = 1$$

holds. Thus proving the theorem.

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Let $\mathbb{M}$ denote a holomorphic D_k^N-module and suppose we have proven the theorem for variables less than N. In particular, we make invoke the claim for D_k^{N-1}-modules $\ker\partial_N$ and $\mathbb{M}/\partial_N\mathbb{M}$. Let $\theta' = (\theta_1, \cdots, \theta_{N-1})$, and $\mathbb{M}' := \mathbb{M}(\theta_1, \cdots, \theta_{N-1})$. From the last theorem, we know

$$\chi(\mathbb{M}/\partial_N\mathbb{M}) = \dim_{k(\theta')} \mathbb{M}'/\partial_N\mathbb{M}' \tag{45}$$

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where the LHS is the image of $\mathbb{M}/\partial_N\mathbb{M}$ under the push forward $\pi : \mathbb{G}^N_{m,k} :\to \mathbb{G}^{N-1}_{m,k}$. and

$$\chi(\ker(\partial_N)) = \dim_{k(\theta')} \ker(\partial_N)(\theta_1 \cdots, \theta_{N-1}) = \dim_{k(\theta')}\mathbb{M}' \tag{47}$$

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Then we find

$$\begin{aligned}\chi(\mathbb{M}/\partial_N\mathbb{M}) - \chi(\ker(\partial_N)) &= \mathsf{dim}_{k(\theta')}(\frac{\mathbb{M}'}{\partial_N\mathbb{M}'}) - \dim_{k(\theta')}(\ker(\partial'_N)) \\ &= \chi(\mathbb{M}') \\ &= \dim_{k'(\theta_N)}\mathbb{M}'(\theta_N) \\ &= \dim_{k(\theta_1,\cdots,\theta_N)}\mathbb{M}(\theta_1,\cdots,\theta_N)\end{aligned}$$

Therefore, we only need to show that the LHS is equal to $\chi(\mathbb{M})$. THe identity $\chi(\mathbb{M}/\partial_N\mathbb{M}) - \chi(\mathrm{ker}\partial_N) = \chi(\mathbb{M})$ follows from the long exact sequence:

$$\cdots \to H^{i+1}(\mathrm{DR}(\mathrm{ker}\partial_N)) \to H^i(\mathrm{DR}(\mathbb{M})) \to H^i(\mathrm{DR}(\mathbb{M}/\partial_N\mathbb{M})) \to H^{i+2}(\mathrm{I} \tag{48}$$

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We shall give a concrete construction of this long-exact sequence in the appendix.

Remark

The space $\mathbb{M}(\theta)$ can be well understood as doing localization to $x_i\partial_i$s. Notice in this case, the mutiplication of x becomes inversible since

$$(1 + x\partial)^{-1}\partial x = (1 + x\partial)^{-1}(1 + x\partial) = 1 \tag{49}$$

Remark

In the above setting, we have not taken the discrete symmetry such as the symmetry between different propagators or permutation of external momentums into account. Also, in actual use, the concept of master integrals vary in different articles. Here, we formulate the notion as the dimension of the Mellin transformed space, which is a D_k^N *module and serves as a linear space over* $k(\theta)$*, where* $\theta = x\partial$ *is localized, and take values in* $\mathbb{C}$*.*

Calculation of The Dimension

In actual use, we do not need to construct the cohomology and calculate $\chi(\mathbb{M})$ by definition. We can transform it to a problem of linear reduction of Symanzik polynomials. We state without proof here the theorem which connects $\chi(\mathbb{M})$ with the Euler characteristics of an affine space:

In actual use, we do not need to construct the cohomology and calculate $\chi(\mathbb{M})$ by definition. We can transform it to a problem of linear reduction of Symanzik polynomials. We state without proof here the theorem which connects $\chi(\mathbb{M})$ with the Euler characteristics of an affine space:

Theorem

Let $G \in \mathbb{C}[x_1, \cdots, x_N]$ be a polynomial and set $k = \mathbb{C}(s)$. Then the Euler characteristics of the algebraic de Rham complexes of the holonoic A_k^N-module $\iota^* A_k^N G^s$ and the holonomic $A_{\mathbb{C}}^N$-module $\mathbb{C}[x^{\pm 1}, G^{-1}] = \mathcal{O}(\mathbb{G}_{m,k}^N \backslash \mathbb{V}(G))$ coincide:

$$\chi(\iota^* A_k^N G) = \chi(\mathbb{C}[x^{\pm}, G]) \tag{50}$$

In particular, we can dispose of the parameter s completely and compute with the algebraic de Rham complex of $\mathbb{C}[x^{\pm 1}, G^{-1}]$, which is the ring of regular functions of the complement of the hypersurface $\mathbb{V}(G) := \{x : G(x) = 0\}$in the torus G_m^N.

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$$\mathfrak{C}(G) = \chi(\mathbb{C}[x^{\pm 1}, G^{-1}]) \tag{51}$$

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$$\mathfrak{C}(G) = \chi(\mathbb{C}[x^{\pm 1}, G^{-1}]) \tag{51}$$

Via Grothendieck's comparison isomorphism, this is the same as the topological Euler characteristics, up to a sign:

$$\mathfrak{C}(G) = (-1)^N \chi(\mathbb{C}^N \setminus \{x_1 x_2 \cdots x_N G = 0\}) = (-1)^N \chi(G_m^N \setminus \{G = 0\}) \tag{52}$$

Since now we are interested mainly in the calculation of Euler characteristic, we can simplify calculations by abstracting from a concrete variety $\mathbb{V}(G)$ to its class $[G]$ in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$. This ring is the free Abelian group generated by isomorphism classes $[X]$ of varieties over $\mathbb{C}$. Where the equivalent relation is introduced with

$$[X] = [X \backslash Z] + [Z] \tag{53}$$

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$$[X] = [X \backslash Z] + [Z] \tag{53}$$

for closed subvarieties $Z \subset X$. It is a unital ring for the product

$$[X] \cdot [Y] := [X \times_k Y] \tag{54}$$

with unit $1 = [\mathbb{A}^0]$ given by the class of the point. The crucial fact of Euler characteristic χ is the factorization proerty, namely we have

$$\chi(X \backslash Z) = \chi(X) - \chi(Z), \chi(X \times Z) = \chi(X)\chi(Y) \qquad (55)$$

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The class $\mathbb{L} = [\mathbb{A}^1]$ of the affine line is called Lefschetz motive and fullfills $\chi(\mathbb{L}) = 1$. For several polynomials $P_1, \cdots, P_n$, we write $\mathbb{V}(P_1, \ldots, P_n) := \{P_1 \cdots, P_n = 0\}$. Now we are ready for the following :

Theorem

Let $A, B \in \mathbb{C}[x_1, \cdots, x_{N-1}]$ *, then*

$$[\mathbb{G}_m^N \backslash \mathbb{V}(A + x_N B)] = \mathbb{L}[\mathbb{G}_m^{N-1} \backslash \mathbb{V}(A, B)] - [\mathbb{G}_m^{N-1} \backslash \mathbb{V}(A)] - [\mathbb{G}_m^{N-1} \backslash \mathbb{V}(B)] \tag{56}$$

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Proof.

From definition, we know that

$$[\mathbb{G}_m^N \backslash \mathbb{V}(A + x_N B)] = [\mathbb{G}_m]([\mathbb{G}_m^N \backslash \mathbb{V}(A)] + [\mathbb{G}_m^N \backslash \mathbb{V}(B)]) - \mathbb{L}[\mathbb{G}_m^{N-1} \backslash \mathbb{V}(A \cdot B)] \tag{57}$$

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Then by $[\mathbb{G}_m] = \mathbb{L} - 1$, and $\mathbb{V}(A \cdot B) = \mathbb{V}(A) + \mathbb{V}(B) - \mathbb{V}(A, B)$, we obtains the original theorem.

□

A two-loop Example:

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We consider the master integral of the following family:

$$I(\nu_1, \cdots, \nu_5) = \int \frac{\mathrm{d}^d l_1}{i\pi^{d/2}} \int \frac{\mathrm{d}^d l_2}{i\pi^{d/2}} \frac{1}{(-l_1^2)^{\nu_1}(-l_2^2)^{\nu_2}(-(l_2-p)^2)^{\nu_3}(-(l_1-p)} \quad (58)$$

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The symanzik polynomials are:

$$\mathcal{U} = (x_1 + x_4)(x_2 + x_3) + x_5(x_1 + x_2 + x_3 + x_4)$$
$$\mathcal{F} = x_1 x_2(x_3 + x_4) + x_3 x_4(x_1 + x_2) + x_5(x_1 + x_2)(x_3 + x_4)$$

where we have normalized $-p^2 = 1$. Calculation of the Grothendieck variety, we find

$$[\mathbb{G}_m^5 \backslash \mathbb{V}(\mathcal{U} + \mathcal{F})] = -\mathbb{L}^4 + 5\mathbb{L}^3 - 13\mathbb{L}^2 + 21\mathbb{L} - 15 \quad (59)$$

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thus

$$\chi[G_m^5 \backslash \mathbb{V}(\mathcal{U} + \mathcal{F})] = 3 = \mathfrak{C}(\mathcal{U} + \mathcal{F}) \qquad (60)$$