Bootstrapping a two-loop four-point form factor

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Based on the work with Yuanhong Guo (郭圆宏)、Lei Wang (王磊)

圈积分-相空间积分学习群系列报告, 2021.06.03

Generic strategy of loop computation

Feynman integrals → **Expansion in a basis of integrals** → **Final result in functional form**

Feynman diagrams, On-shell unitarity method, …

Integral reductions: PV reduction, IBP reduction, …

Solving integrals, functional identities to simplify the result, …

$\sum$ (integrand) $\quad$ $\sum c_i M_i$ $\quad$ $\sum$ functions

Generic strategy of loop computation

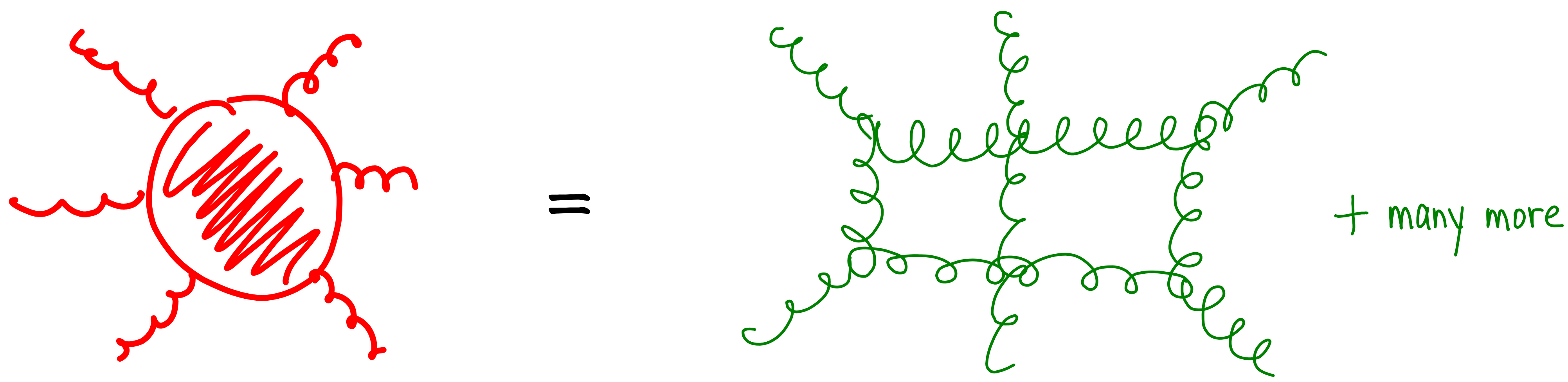
Feynman integrals → Expansion in a basis of integrals → Final result in functional form

Feynman integrals	Expansion in a basis of integrals	Final result in functional form
Feynman diagrams, On-shell unitarity method, …	Integral reductions: PV reduction, IBP reduction, …	Solving integrals, functional identities to simplify the result, …
$\sum$ (integrand)	$\sum c_i M_i$	$\sum$ functions

Complicated intermediate expressions

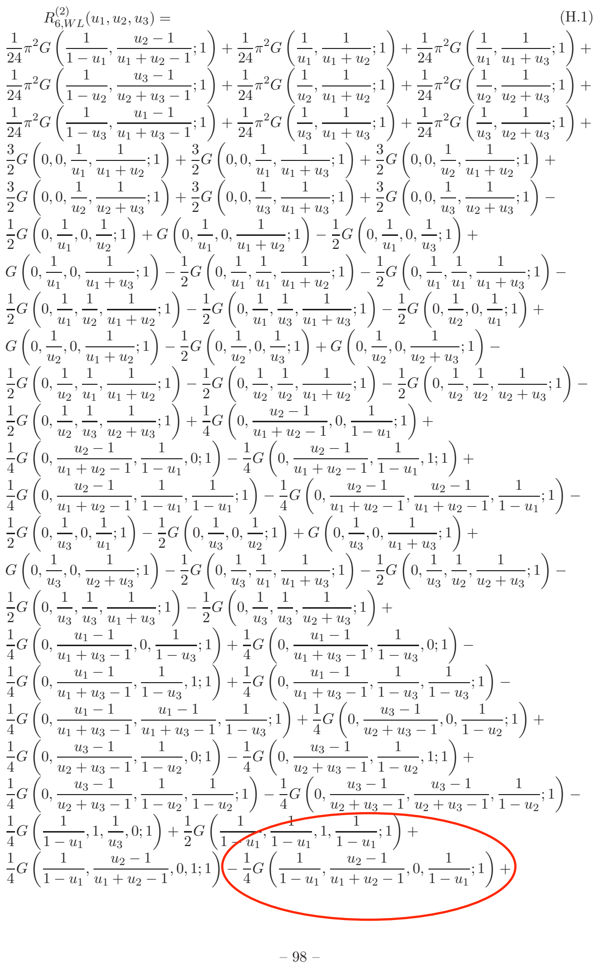
Compact analytic form

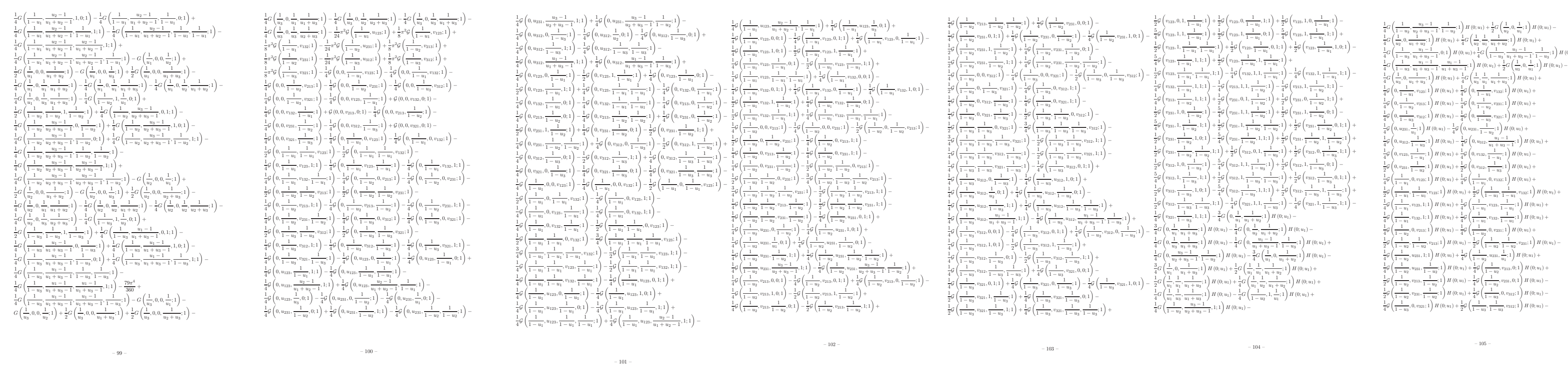
Two-loop six-gluon amplitudes in N=4



[Del Duca, Duhr, Smirnov 2010]

17 page complicated functions



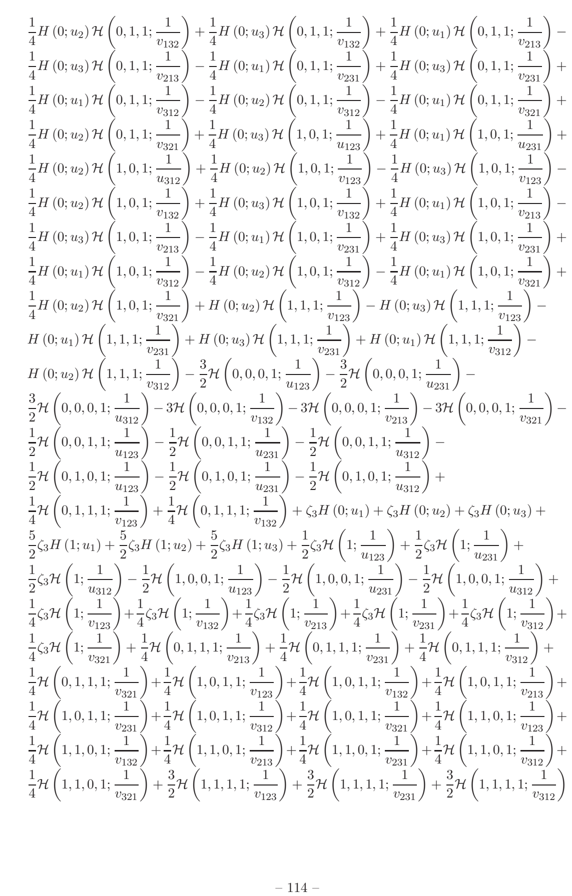


[Del Duca, Duhr, Smirnov 2010]

"multiple(Goncharov)-polylogrithm function"

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)$$

$$G(a_k,a_{k-1},\ldots,a_1;z)=\int_0^z G(a_{k-1},\ldots,a_1;t)\frac{dt}{t-a_k},\quad G(z)=1$$



Result can be remarkably simple

17 pages =

[Goncharov, Spradlin, Vergu, Volovich 2010]

$$\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right)-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2+\frac{1}{24}J^4+\frac{\pi^2}{12}J^2+\frac{\pi^4}{72}$$

$$L_4(x^+,x^-)=\frac{1}{8!!}\log(x^+x^-)^4+\sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-)) \qquad \ell_n(x)=\frac{1}{2}\left(\mathrm{Li}_n(x)-(-1)^n\,\mathrm{Li}_n(1/x)\right) \qquad J=\sum_{i=1}^{3}(\ell_1(x_i^+)-\ell_1(x_i^-)).$$

a line result in terms of classical polylogarithms!

Such simplicity is totally unexpected using traditional Feynman diagrams!

Mathematical tool: “symbol”

From function to "Symbol"

Recursion definition of "Symbol":

$$\mathrm{d} f_k = \sum_i f^i_{k-1}\, \mathrm{dLog}(R_i), \qquad \mathrm{Symbol}(f_k) = \sum_i \mathrm{Symbol}(f^i_{k-1}) \otimes R_i$$

Function	Differential	**symbol**
R	d R	0
log(R)	d log(R)	R
log(R1)log(R2)	logR1 dlogR2+logR2 dlogR1	R1 $\otimes$ R2 + R2 $\otimes$ R1
Li_2(R)	Li_1(R) dlogR	-(1-R) $\otimes$ R

Symbol

Algebraic relations:

$$R_1 \otimes \ldots \otimes (c\, R_i) \otimes \ldots \otimes R_n = R_1 \otimes \ldots \otimes R_i \otimes \ldots \otimes R_n$$ *c* is const

$$R_1 \otimes \ldots \otimes (R_i\, R_j) \otimes \ldots \otimes R_n = R_1 \otimes \ldots \otimes R_i \otimes \ldots \otimes R_n + R_1 \otimes \ldots \otimes R_j \otimes \ldots \otimes R_n$$

Make it easy to prove non-trivial identities, e.g.:

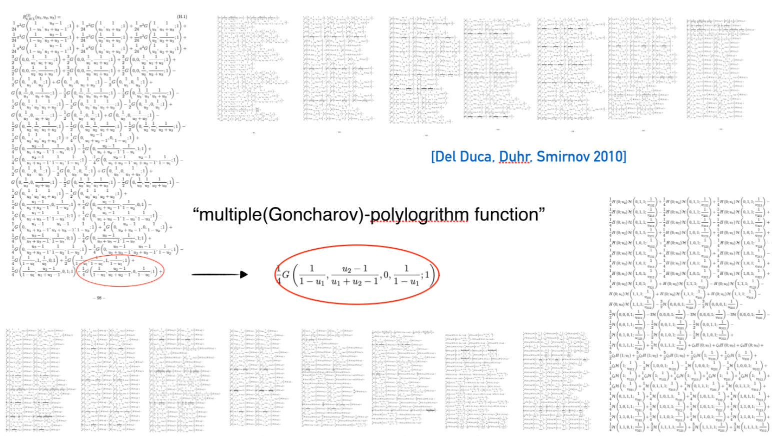
$$\mathrm{Li}_2(z) = -\mathrm{Li}_2(1-z) - \log(1-z)\log(z) + \frac{\pi^2}{6}$$

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{1}{z}\right) - \frac{1}{2}\log^2(-z) - \frac{\pi^2}{6} \;/;\; z \notin (0,\, 1)$$

$$\mathrm{Li}_2\left(\frac{x}{1-y}\right) + \mathrm{Li}_2\left(\frac{y}{1-x}\right) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \mathrm{Li}_2\left(\frac{xy}{(1-x)(1-y)}\right) = \mathrm{Log}(1-x)\mathrm{Log}(1-y)$$

Applications

Complicated expression → symbol → Simple expression



$$\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right)-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2+\frac{1}{24}J^4+\frac{\pi^2}{12}J^2+\frac{\pi^4}{72}$$

Applications

Complicated expression ⟶ symbol ⟶ Simple expression

A better strategy:

Derive symbol directly without knowing function in advance.

Bootstrap strategy   Dixon, Drummond, Henn 2011, ...

We will apply a different strategy based on master integrand expansion.

Outline

Background and Motivation

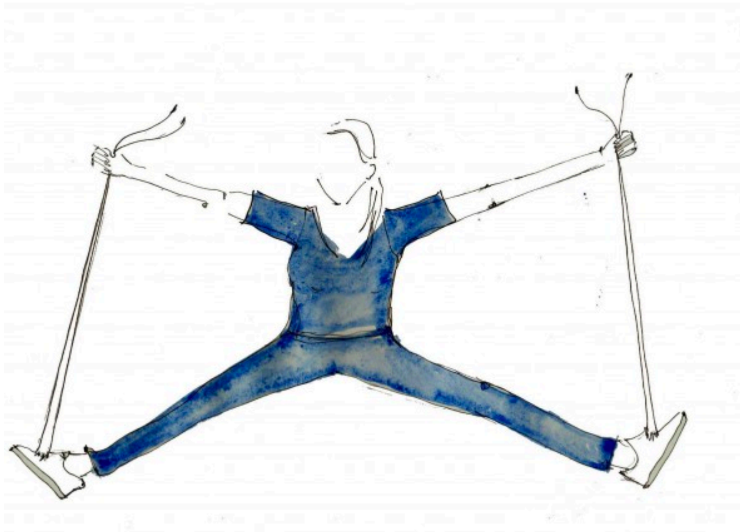
New bootstrap strategy

Two-loop four-point form factor

Summary and outlook

Bootstrap

Bootstrap



Bootstrap

Top-down

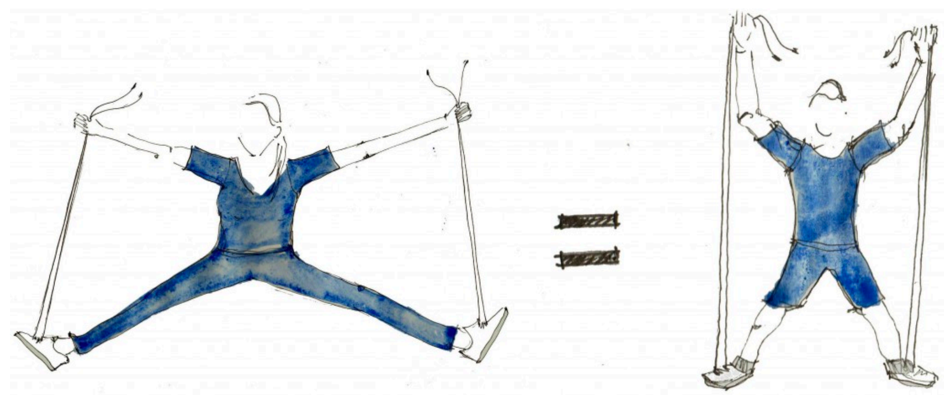
Bottom-up

S-matrix program

"One should try to calculate S-matrix elements directly, without the use of field quantities, by requiring them to have some general properties that ought to be valid, …."

— Eden et.al, "The Analytic S-matrix", 1966

Conformal bootstrap



Compute anomalous dimensions and correlation functions

Alexander M. Polyakov

Vyacheslav S. Rychkov

2-dim → D-dim

Bootstrap of amplitudes

Symbol bootstrap

Computing the finite remainder functions using symbol techniques.

Ansatz in symbols → **Physical constraints** → Solution

$$S_{\text{ansatz}}(R) = \sum_i c_i \left[\otimes_a W_{i,a} \right]$$

$$S(R) = \sum_i c_i \left(\otimes_a W_{i,a} \right)$$

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The new strategy we will use

Ansatz in master integrals → **Physical constraints** → Solution of coefficients

$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

$$\mathcal{F}^{(l)} = \sum_i C_i I_i^{(l)}$$

“master bootstrap”

Ansatz in master integral expansion → **Physical constraints** → Solution of coefficients

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

- Symmetry property
- IR divergences
- Collinear factorization
- Unitarity cut

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i {\color{red}C_i} I_i^{(l)}$$

“master bootstrap”

Ansatz in master integral expansion → **Physical constraints** → Solution of coefficients

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Symmetry property

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$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

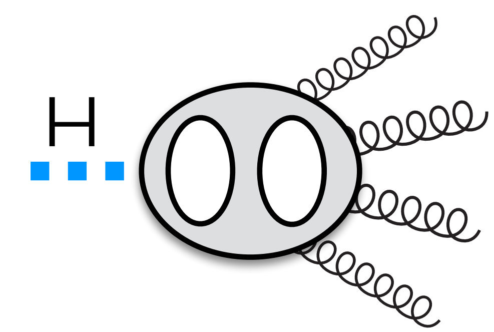
Application:
two-loop four-point form factor

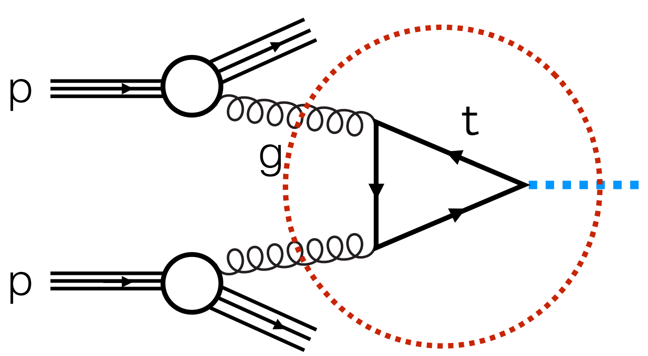
Form factors

We consider two-loop four-point form factor in N=4 SYM:

$$\mathscr{F}_{\mathcal{O},4} = \int d^4x\, e^{-iq\cdot x}\langle 1_\phi, 2_\phi, 3_\phi 4^+ \,|\, \mathrm{tr}(\phi^3)(x) \,|\, 0\rangle$$

It is a N=4 version of Higgs+4-parton amplitudes in QCD:





$m_t \to \infty$

$$\mathcal{L}_{\text{eff}} = \hat{C}_0 H \,\mathrm{tr}(F^2) + \mathcal{O}\left(\frac{1}{m_{\text{t}}^2}\right)$$

Form factors

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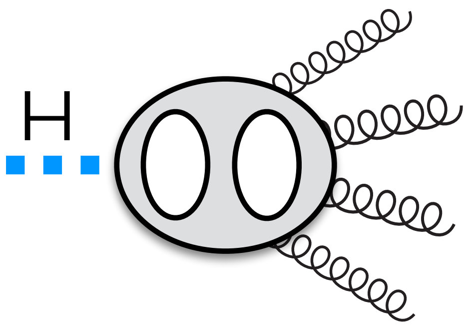


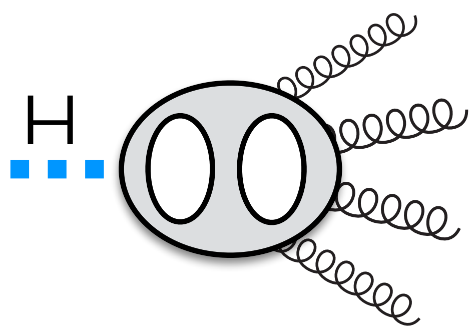
Five-point two-loop amplitudes are at frontier and under intense study:

There have been many massless five-point two-loop amplitudes obtained in analytic form. See e.g. Abreu, Dormans, Cordero, Ita, Page 2019 and many others….

For five-point two-loop amplitudes with one massive leg, so far only one result is available: $u\bar{d} \to W^+ b\bar{b}$ Badger, Hartanto, Zoia 2021

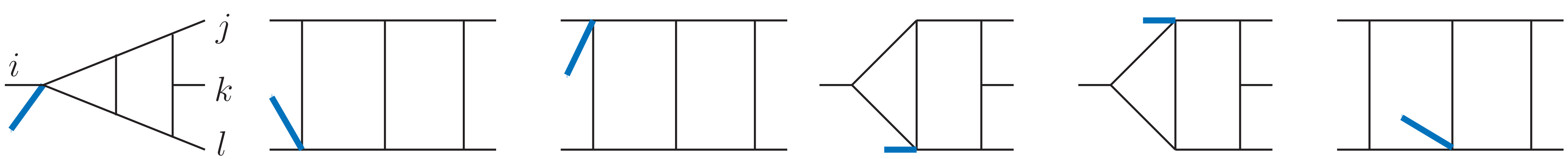
Form factors

Our result provides a first two-loop five-point example with a color-singlet off-shell leg.



$$\mathscr{F}_{\mathcal{O},4} = \int d^4x\, e^{-iq\cdot x} \langle 1_\phi, 2_\phi, 3_\phi 4^+ \,|\, \mathrm{tr}(\phi^3)(x) \,|\, 0 \rangle$$

$$\{s_{12}, s_{23}, s_{34}, s_{14}, s_{13}, s_{24}, \mathrm{tr}_5\}; \qquad \mathrm{tr}_5 = 4i\varepsilon_{p_1 p_2 p_3 p_4}$$



Planar master integrals have been evaluated recently.

Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020

Canko, Papadopoulos, Syrrakos 2020

Ansatz

$$\mathscr{F}_{\mathcal{O},4} = \int d^4x\, e^{-iq\cdot x} \langle 1_\phi, 2_\phi, 3_\phi 4^+ \,|\, \mathrm{tr}(\phi^3)(x) \,|\, 0 \rangle$$

Tree-level:

$$\mathcal{F}_4^{(0)} = \mathcal{F}^{(0)}_{\mathrm{tr}(\phi_{12}^3)}(1^\phi, 2^\phi, 3^\phi, 4^+) = \frac{\langle 31\rangle}{\langle 34\rangle\langle 41\rangle}\,.$$

One-loop:

$$\mathcal{F}_4^{(1)} = \mathcal{F}_4^{(0)} \mathcal{I}_4^{(1)} = \mathcal{F}_4^{(0)} \Big(B_1\, \mathcal{G}_1^{(1)} + B_2\, \mathcal{G}_2^{(1)} \Big)$$

$$B_1 = \frac{\langle 12\rangle\,\langle 34\rangle}{\langle 13\rangle\,\langle 24\rangle}\,, \quad B_2 = \frac{\langle 14\rangle\,\langle 23\rangle}{\langle 13\rangle\,\langle 24\rangle}\,, \quad B_1 + B_2 = 1\,,$$

$$\mathcal{G}_1^{(1)} = -\frac{1}{2} I^{(1),\mathrm{UT}}_{\mathrm{Box}}(4,1,2) - \frac{1}{2} I^{(1),\mathrm{UT}}_{\mathrm{Box}}(3,4,1) - I^{(1),\mathrm{UT}}_{\mathrm{Bubble}}(4,1,2) - I^{(1),\mathrm{UT}}_{\mathrm{Bubble}}(3,4,1) + I^{(1),\mathrm{UT}}_{\mathrm{Bubble}}(4,1) - I^{(1),\mathrm{UT}}_{\mathrm{Bubble}}(2,3)\,.$$

Ansatz

$$\mathscr{F}_{\mathscr{O},4} = \int d^4x\, e^{-iq\cdot x} \langle 1_\phi, 2_\phi, 3_\phi 4^+ \,|\, \mathrm{tr}(\phi^3)(x) \,|\, 0 \rangle$$

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Two-loop ansatz: $\boxed{\mathcal{F}_4^{(2)} = \mathcal{F}_4^{(0)} \Big(B_1\, \mathcal{G}_1^{(2)} + B_2\, \mathcal{G}_2^{(2)} \Big)}$

$$\mathcal{G}_a^{(2)} = \sum_{i=1}^{221} c_{a,i} I_i^{(2),\mathrm{UT}}\,, \qquad \mathcal{G}_2^{(2)} = \mathcal{G}_1^{(2)}|_{(p_1 \leftrightarrow p_3)}$$

Ansatz

$$\mathscr{F}_{\mathscr{O},4} = \int d^4x\, e^{-iq\cdot x} \langle 1_\phi, 2_\phi, 3_\phi 4^+ \,|\, \mathrm{tr}(\phi^3)(x) \,|\, 0 \rangle$$

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Constraints

IR divergences

Collinear factorization

Spurious pole

Unitarity cut

BDS ansatz

$$\mathcal{I}^{(2),\mathrm{BDS}} = \frac{1}{2}\left(\mathcal{I}^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\mathcal{I}^{(1)}(2\epsilon)$$

$$\mathcal{R}_n^{(2)} = \left[\mathcal{I}^{(2)} - \mathcal{I}^{(2),\mathrm{BDS}}\right]_{\mathrm{fin}} \xrightarrow{p_i \parallel p_{i+1}} \mathcal{R}_{n-1}^{(2)}$$

We introduce:

$$\mathcal{I}_{4,\mathrm{BDS}}^{(2)} = \sum_{a=1}^{2} B_a \left[\frac{1}{2}\left(\mathcal{G}_a^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\mathcal{G}_a^{(1)}(2\epsilon)\right]$$

which captures all IR and collinear singularities.

$$\mathcal{R}_{4\text{-pt}}^{(2)} := \left(\mathcal{I}_4^{(2)} - \mathcal{I}_{4,\mathrm{BDS}}^{(2)}\right)\Big|_{\mathcal{O}(\epsilon^0)} \xrightarrow[\text{or } p_4 \parallel p_1]{p_4 \parallel p_3} \mathcal{R}_{3\text{-pt}}^{(2)}$$

Constraints

IR divergences

Collinear factorization

Spurious pole

Unitarity cut

Constraints	Parameters left
Symmetry of $(p_1 \leftrightarrow p_3)$	221
IR (Symbol)	82
Collinear limit (Symbol)	38
Spurious pole (Symbol)	22
IR (Function)	17
Collinear limit (Funcion)	10

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Spurious pole gives no new constraint.

Constraints

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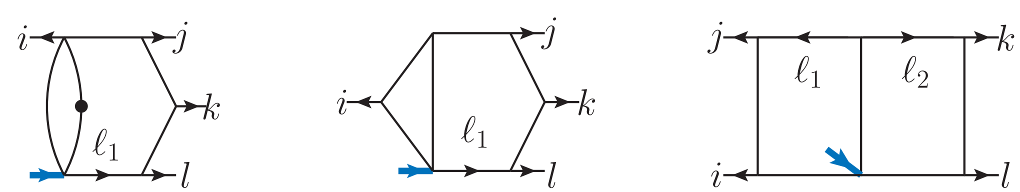
Collinear factorization

Spurious pole

Unitarity cut

Remaining 10 parameters are related to following master integrals:



(a) BPb (b) TP (c) dBox2c

$\sum_{i=1}^{10} x_i \tilde{G}_i,$

$$\tilde{G}_1 = I_{\mathrm{TP}}^{\mathrm{UT}}(1,2,3,4) + I_{\mathrm{TP}}^{\mathrm{UT}}(3,2,1,4)\,,$$
$$\tilde{G}_2 = I_{\mathrm{BPb}}^{\mathrm{UT}}(1,2,3,4) - I_{\mathrm{BPb}}^{\mathrm{UT}}(4,3,2,1) + (p_1 \leftrightarrow p_3)$$
$$\tilde{G}_3 = B_1 I_{\mathrm{dBox2c}}^{\mathrm{UT}}(1,2,3,4) + B_2 I_{\mathrm{dBox2c}}^{\mathrm{UT}}(3,2,1,4)\,,$$

Free of above constraints.

Constraints

IR divergences

Collinear factorization

Spurious pole

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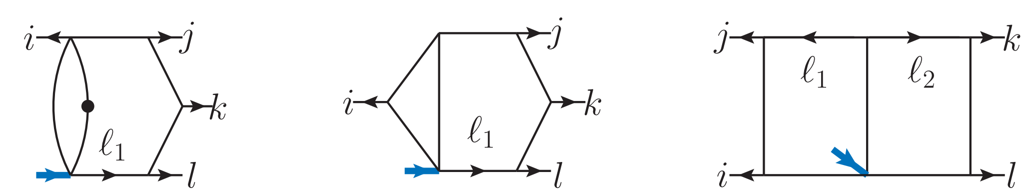

(a) BPb (b) TP (c) dBox2c

$\mathrm{tr}_5 \times \mu^2$ $\mathcal{O}(\epsilon^1)$

$$\mathrm{tr}_5 = 4i\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$$

$$\mu_{ij} = \ell_i^{-2\epsilon} \cdot \ell_j^{-2\epsilon}$$

Constraints

IR divergences

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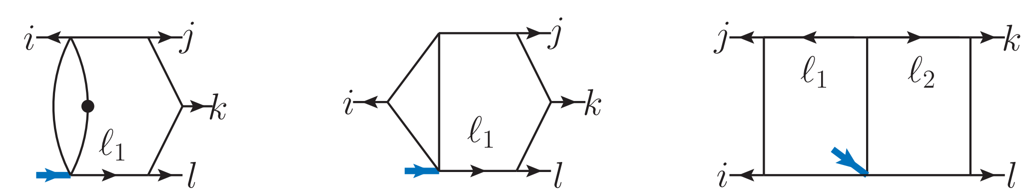
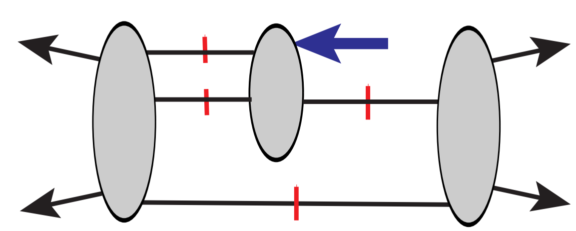


(a) BPb (b) TP (c) dBox2c

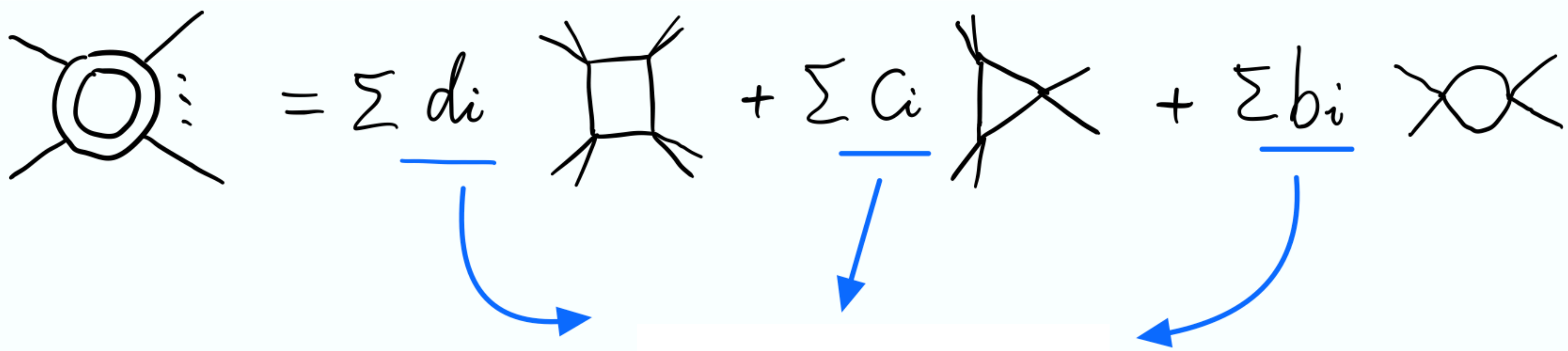
Can be fixed via **simple two-double cuts**:



$\mathcal{F}_3^{(0)}\,\mathcal{A}_4^{(0),\mathrm{MHV}}\,\mathcal{A}_5^{(0),\mathrm{MHV}}$

Unitarity cuts

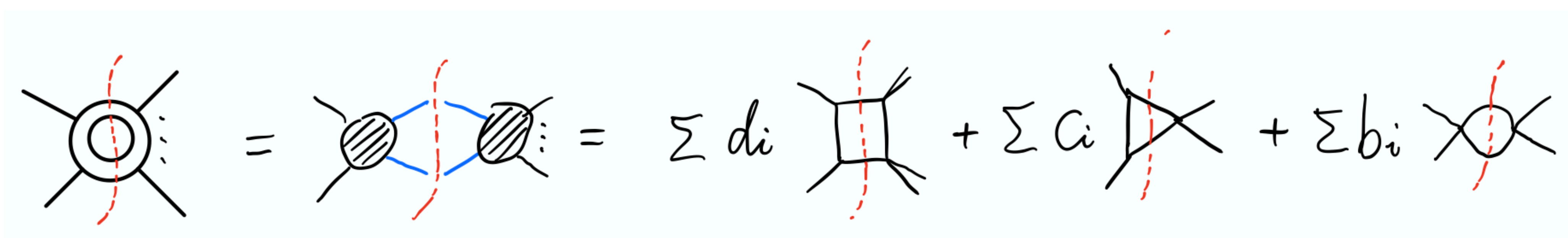
Consider one-loop amplitudes:



What we really want

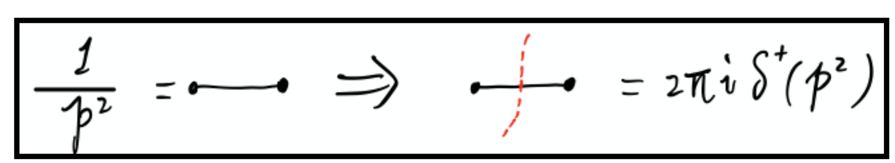
Unitarity cuts

We can perform unitarity cuts:



and from tree products, we derive the coefficients more directly.

Cutkosky cutting rule:



Constraints

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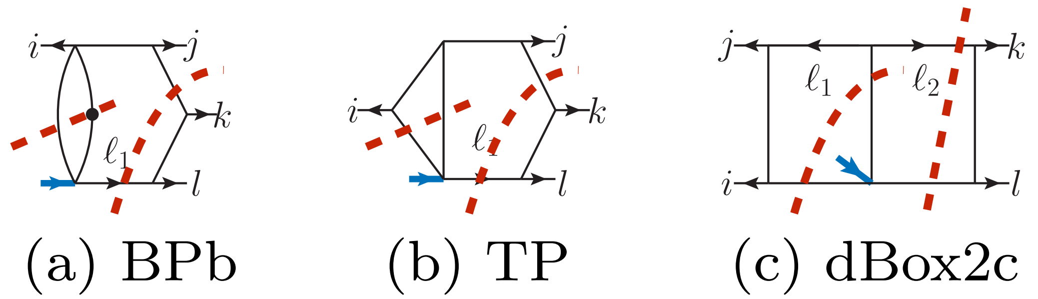

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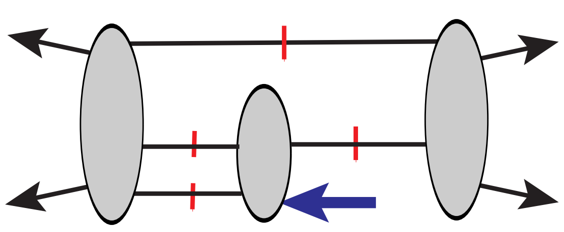

$\mathcal{F}_3^{(0)} \mathcal{A}_4^{(0),\mathrm{MHV}} \mathcal{A}_5^{(0),\mathrm{MHV}}$

A summary

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IR (Symbol)	82
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If keeping only to ϵ^0 order	6
Simple unitarity cuts	0

Numerics

Substituting in the master integral results, we have the full analytic form in GPLs, and they can be evaluated with GiNaC to 'arbitrary' high precision:

	$\mathcal{F}^{(2)}/\mathcal{F}^{(0)}$
ϵ^{-4}	8
ϵ^{-3}	$-10.888626564448543787 + 25.132741228718345908i$
ϵ^{-2}	$-31.872672672370517258 - 16.558017711981028644i$
ϵ^{-1}	$-24.702889082481070673 - 2.9923229294749490751i$
ϵ^{0}	$-86.211269185142415564 - 128.27562636360640808i$
$\mathcal{R}_4^{(2)}$	$8.3794306422137831973 - 14.941297169128279600i$

$\{s_{12} = 241/25,\ s_{23} = -377/100,\ s_{34} = 13/50,\ s_{14} = -161/100,\ s_{13} = s_{24} = -89/100,\ \mathrm{tr}_5 = i\sqrt{1635802}/2500\}$

Numerics: collinear limit

Collinear limit: $\mathcal{R}^{(2)}_{\text{4-pt}} := \left(\mathcal{I}^{(2)}_4 - \mathcal{I}^{(2)}_{4,\text{BDS}}\right)\big|_{\mathcal{O}(\epsilon^0)} \xrightarrow[\text{or } p_4 \parallel p_1]{p_4 \parallel p_3} \mathcal{R}^{(2)}_{\text{3-pt}}$

$$\{s_{12} = 24/5,\ s_{23} = 1037/1000,\ s_{34} = 3111/(16 \times 10^{43}),\ s_{14} = 351/1000,\ s_{13} = 549/1000,\ s_{24} = 663/1000,\ \text{tr}_5 = i9333\sqrt{156 \times 10^{38} - 1/10^{44}}\}.$$

	$\mathcal{F}^{(2)}/\mathcal{F}^{(0)}$
ϵ^{-4}	8
ϵ^{-3}	$372.73227729764577740 + 50.265482457436691815i$
ϵ^{-2}	$22299.426450303417729 + 2341.9459709432377859i$
ϵ^{-1}	$989445.74441873599952 + 140772.89586692467156i$
ϵ^{0}	$36885962.819916639458 + 6247689.7372657501908i$
$\mathcal{R}^{(2)}_{\text{4-pt}}$	$-13.79946362217945 + 9.616825584877344 \times 10^{-18} i$

$$\mathcal{R}^{(2)}_{\text{3-pt}}(\hat{s}_{12}, \hat{s}_{23}, \hat{s}_{13}) = \mathcal{R}^{(2)}_{\text{3-pt}}\left(\tfrac{24}{5}, \tfrac{17}{10}, \tfrac{9}{10}\right)$$

$$\mathcal{R}^{(2)}_{\text{3-pt}} - \mathcal{R}^{(2)}_{\text{4-pt}} = (1.9834 \times 10^{-37} + 9.6168 \times 10^{-18} i)$$

Numerics: spurious pole

Spurious pole cancellation:

$$\mathcal{I}_4^{(2)} = \frac{1}{2}\left(\mathcal{G}_1^{(2)} + \mathcal{G}_2^{(2)}\right) + \frac{B_1 - B_2}{2}\left(\mathcal{G}_1^{(2)} - \mathcal{G}_2^{(2)}\right) \qquad B_1 - B_2 = \frac{s_{12}s_{34} - s_{14}s_{23} - \mathrm{tr}_5}{s_{13}s_{24}} \sim \frac{1}{\hat{\delta}}$$

	$(\mathcal{G}_1^{(2)} - \mathcal{G}_2^{(2)})/s_{24}$
ϵ^{-4}	0
ϵ^{-3}	0
ϵ^{-2}	$-2.906457694101063804 - 2.221328138901874007 0i$
ϵ^{-1}	$7.976373135985054846 8 - 9.569684774251949437 9i$
ϵ^{0}	$24.83191732321506906 9 + 36.10209824140692533 8i$

Kinematics: $\{s_{12} = -11/5,\ s_{23} = -57/20,\ s_{34} = 18/5,\ s_{14} = 5/4,\ s_{13} = 3,\ s_{24} = 10^{-20},\ \mathrm{tr}_5 > 0\}$

Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020

Technical details: symbol letters

$$\mathrm{Sym}(\mathcal{R}_4^{(2)}) = \sum_i c_i W_{i_1} \otimes W_{i_2} \otimes W_{i_3} \otimes W_{i_4}$$

$$u_{ij} = \frac{s_{ij}}{s_{1234}}, \qquad u_{ijk} = \frac{s_{ijk}}{s_{1234}}.$$

Building blocks:

$$x^{\pm}_{ijkl} = \frac{1 + u_{ij} - u_{kl} \pm \sqrt{\Delta_{3,ijkl}}/s_{1234}}{2u_{ij}},$$

$$y^{\pm}_{ijkl} = \frac{u_{ij}u_{kl} - u_{ik}u_{jl} + u_{il}u_{jk} \pm P(ijkl)\mathrm{tr}_5/(s_{1234})^2}{2u_{ij}u_{il}},$$

$$z^{\pm\pm}_{ijkl} = 1 + y^{\pm}_{ijkl} - x^{\pm}_{lijk},$$

$$\Delta_{3,ijkl} = \mathrm{Gram}(p_i + p_j, p_k + p_l),$$

$$\mathrm{tr}_5 = 4i\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$$

Complicated letters:

$$U(p_i + p_j, p_k + p_l) = u_{ikl}u_{jkl} - u_{kl},$$

$$X_1(p_i + p_j, p_k, p_l) = \frac{u_{ij}x^{+}_{ijkl} - u_{ijl}}{u_{ij}x^{-}_{ijkl} - u_{ijl}},$$

$$X_2(p_i + p_j, p_k + p_l) = \frac{x^{+}_{ijkl}}{x^{-}_{ijkl}},$$

$$Y_1(p_i, p_j, p_k, p_l) = \frac{y^{+}_{ijkl}}{y^{-}_{ijkl}},$$

$$Y_2(p_i, p_j, p_k, p_l) = \frac{y^{+}_{ijkl} + 1}{y^{-}_{ijkl} + 1},$$

$$Z(p_i, p_j, p_k, p_l) = \frac{z^{++}_{ijkl}z^{--}_{ijkl}}{z^{+-}_{ijkl}z^{-+}_{ijkl}}.$$

Technical details: symbol letters

All 42 letters in remainder:

$$
\begin{gathered}
u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{34}, \\
u_{123}, u_{124}, u_{134}, u_{234}, \\
u_{123} - u_{12}, u_{123} - u_{23}, u_{124} - u_{12}, u_{124} - u_{14}, \\
u_{134} - u_{14}, u_{134} - u_{34}, u_{234} - u_{23}, u_{234} - u_{34}, \\
1 - u_{123}, 1 - u_{124}, 1 - u_{134}, 1 - u_{234}\,.
\end{gathered}
$$

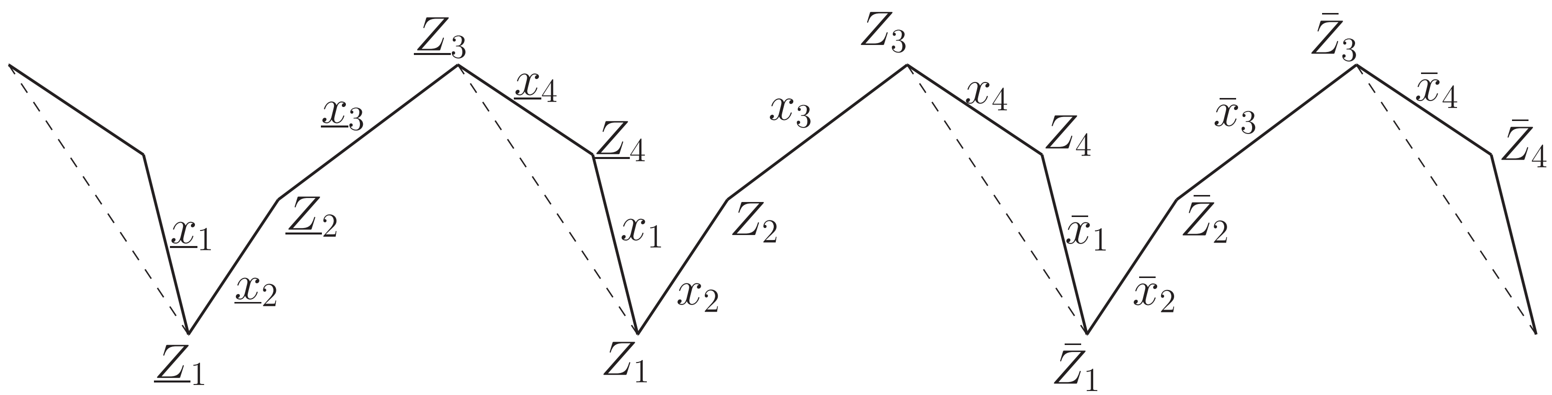
$$
\begin{gathered}
X_1(p_1+p_2,p_3,p_4), X_1(p_2+p_3,p_4,p_1), \\
X_1(p_1+p_4,p_2,p_3), X_1(p_3+p_4,p_1,p_2), \\
X_2(p_1+p_2,p_3+p_4), X_2(p_2+p_3,p_1+p_4), \\
X_2(p_1+p_4,p_2+p_3), X_2(p_3+p_4,p_1+p_2), \\
U(p_1+p_2,p_3+p_4), U(p_2+p_3,p_1+p_4), \\
U(p_1+p_4,p_2+p_3), U(p_3+p_4,p_1+p_2), \\
Y_1(p_1,p_2,p_3,p_4), Y_1(p_1,p_3,p_2,p_4) \\
Y_2(p_1,p_3,p_2,p_4), Y_2(p_3,p_1,p_2,p_4), \\
Y_2(p_1,p_3,p_4,p_2), Y_2(p_3,p_1,p_4,p_2), \\
Z(p_1,p_2,p_3,p_4), Z(p_3,p_2,p_1,p_4).
\end{gathered}
$$

Extra 4 letters that appear in master:

$$
q^2\,,\ \sqrt{\Delta_{3,1234}}\,,\ \sqrt{\Delta_{3,1423}}\,,\ \mathrm{tr}_5
$$

Technical details:
collinear limit of form factors

Dual momentum space



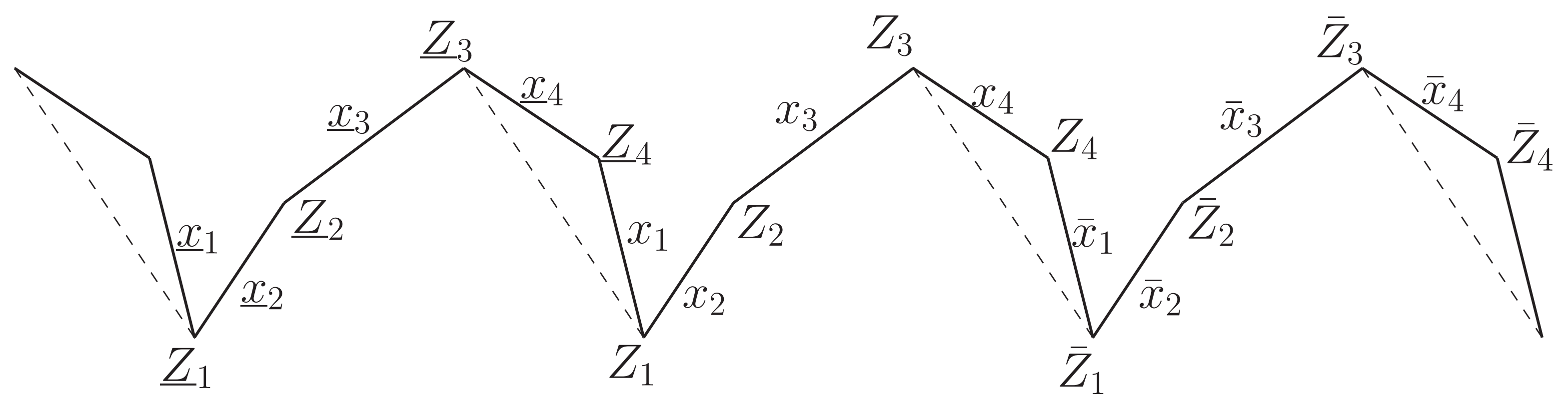
$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = p_i^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\widetilde{\lambda}_i^{\dot{\alpha}}\,, \quad \underline{x}_i - x_i = x_i - \bar{x}_i = q$$

$$Z_i^A = (\lambda_i^{\alpha}, \mu_i^{\dot{\alpha}})\,, \qquad \mu_i^{\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} \cdot \lambda_{i\alpha} = x_{i+1}^{\alpha\dot{\alpha}} \cdot \lambda_{i\alpha}$$

$$x_{ij}^2 = (x_i - x_j)^2 = \frac{\langle i-1, i, j-1, j\rangle}{\langle i-1, i\rangle\langle j-1, j\rangle} \qquad \langle Z_i Z_j Z_k Z_l\rangle = \langle ijkl\rangle$$

Technical details: collinear limit of form factors

Dual momentum space



Collinear limit parametrization:

$$Z_4 = Z_3 + \delta \frac{\langle \bar{1}\bar{2}13\rangle}{\langle \bar{1}\bar{2}12\rangle} Z_2 + \tau\delta \frac{\langle \bar{2}123\rangle}{\langle \bar{1}\bar{2}12\rangle} \bar{Z}_1 + \eta \frac{\langle \bar{1}123\rangle}{\langle \bar{1}\bar{2}12\rangle} \bar{Z}_2$$

$$\lambda_4 = \lambda_3 + \delta \frac{\langle \bar{1}\bar{2}13\rangle}{\langle \bar{1}\bar{2}12\rangle} \lambda_2 + \tau\delta \frac{\langle \bar{2}123\rangle}{\langle \bar{1}\bar{2}12\rangle} \bar{\lambda}_1 + \eta \frac{\langle \bar{1}123\rangle}{\langle \bar{1}\bar{2}12\rangle} \bar{\lambda}_2$$

taking first $\eta \to 0$, followed by $\delta \to 0$.

$$y^+_{1234} \to \frac{(1-t)\delta}{t} \frac{(\hat{u}_{12}+\hat{u}_{13})\hat{u}_{23}}{\hat{u}_{12}}, \quad y^-_{1234} \to -\frac{\eta}{\delta} \frac{\hat{u}_{23}}{\hat{u}_{12}+\hat{u}_{13}},$$

$$y^+_{1324} \to \frac{\hat{u}_{23}}{\hat{u}_{13}}, \qquad y^-_{1324} \to \frac{\hat{u}_{23}}{\hat{u}_{13}},$$

$$y^+_{3124} \to -\frac{t}{(1-t)\delta} \frac{\hat{u}_{12}}{\hat{u}_{13}(\hat{u}_{12}+\hat{u}_{13})}, \quad y^-_{3124} \to \frac{\delta}{\eta} \frac{\hat{u}_{12}+\hat{u}_{13}}{\hat{u}_{13}},$$

$$y^+_{1342} \to \frac{t\eta}{(1-t)\delta} \frac{\hat{u}_{23}}{\hat{u}_{12}+\hat{u}_{13}}, \quad y^-_{1342} \to -\delta \frac{(\hat{u}_{12}+\hat{u}_{13})\hat{u}_{23}}{\hat{u}_{12}},$$

$$y^+_{3142} \to \frac{t}{1-t}, \qquad y^-_{3142} \to \frac{t}{1-t}.$$

$$\tau = \frac{t-1}{t} \frac{s_{12}+s_{13}}{s_{12}+s_{23}}$$

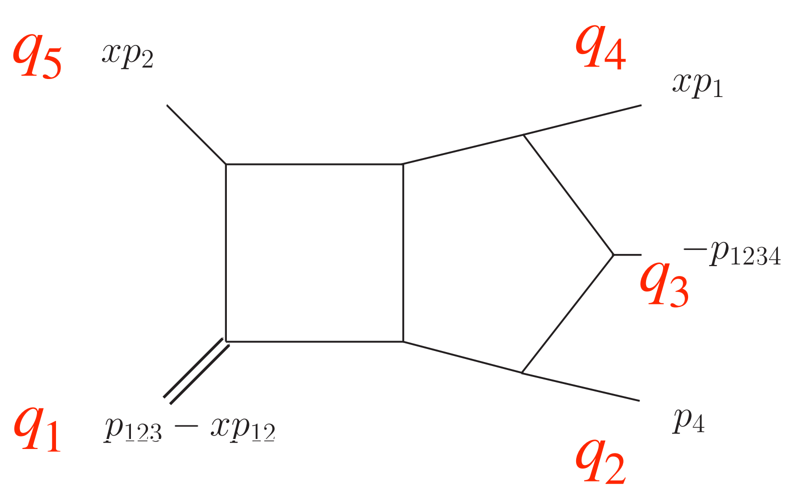
Technical details: numerical computation

Master integrals are evaluated in multiple polylogarithm.

Canko, Papadopoulos, Syrrakos 2020

A different set of kinematics are chosen.

$\{q_1, q_2, q_3, q_4, q_5\}$ with q_1 massive $\{x, S_{12}, S_{23}, S_{34}, S_{45}, S_{51}\}$.



$$\tilde{s}_{15} = (1-x)S_{45} + S_{23}x\,,$$
$$q_1^2 = (1-x)(S_{45} - S_{12}x)\,,$$
$$\tilde{s}_{12} = (S_{34} - S_{12}(1-x))x\,,$$
$$\tilde{s}_{23} = S_{45}\,,\ \tilde{s}_{34} = S_{51}x\,,\ \tilde{s}_{34} = S_{51}x$$

$$\tilde{s}_{ij} = (q_i + q_j)^2$$

Summary and outlook

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We present a first analytic computation of a two-loop five-point scattering with one color-singlet off-shell leg.

We develop a new bootstrap strategy based on master integral expansion, which applies efficiently for this case.

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Outlook:

Apply to more general observables.

Study the new constraints beyond collinear limit, such as OPE limit, Regge limit.

Thank you!

Extra slides

Symbol bootstrap

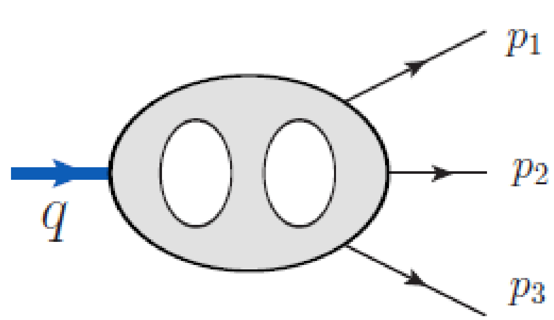
Computing the finite remainder functions using symbol techniques.

Ansatz in symbols → **Physical constraints** → Solution

$$S_{\text{ansatz}}(R) = \sum_i c_i \Big[\otimes_a W_{i,a} \Big]$$

$$S(R) = \sum_i c_i \Big(\otimes_a W_{i,a} \Big)$$

Two-loop 3-point example: Brandhuber, Travaglini, GY 2012



$$u = \frac{s_{12}}{q^2}, \quad v = \frac{s_{23}}{q^2}, \quad w = \frac{s_{31}}{q^2}$$

$$q^2 = s_{12} + s_{23} + s_{31}$$

$$u + v + w = 1$$

Symbol bootstrap: 2-loop 3-point form factor

Consider two-loop three-point form factor:

$$\mathcal{R}_3^{(2)} := \mathcal{G}_3^{(2)}(\epsilon) - \frac{1}{2}(\mathcal{G}_3^{(1)}(\epsilon))^2 - f^{(2)}(\epsilon)\,\mathcal{G}_3^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)$$

Compute its symbol directly, without knowing the result first.

Constraints:

- Variables in symbol : $\{u, v, w; 1-u, 1-v, 1-w\}$
- Entry conditions: restriction on the position of variables
- Collinear limit : $\text{Symbol} \to 0$
- Totally symmetric in kinematics
- Integrability condition $\sum dw_i \wedge dw_{i+1} (w_1 \otimes \cdots \otimes w_{i-1} \otimes w_{i+2} \otimes \cdots \otimes w_n) = 0$

Symbol bootstrap:
2-loop 3-point form factor

A unique solution of the remainder symbol:

$$
\begin{aligned}
\mathcal{S}^{(2)} \;=\; & -2u\otimes(1-u)\otimes(1-u)\otimes\frac{1-u}{u}+u\otimes(1-u)\otimes u\otimes\frac{1-u}{u} \\
& -u\otimes(1-u)\otimes v\otimes\frac{1-v}{v}-u\otimes(1-u)\otimes w\otimes\frac{1-w}{w} \\
& -u\otimes v\otimes(1-u)\otimes\frac{1-v}{v}-u\otimes v\otimes(1-v)\otimes\frac{1-u}{u} \\
& +u\otimes v\otimes w\otimes\frac{1-u}{u}+u\otimes v\otimes w\otimes\frac{1-v}{v} \\
& +u\otimes v\otimes w\otimes\frac{1-w}{w}-u\otimes w\otimes(1-u)\otimes\frac{1-w}{w} \\
& +u\otimes w\otimes v\otimes\frac{1-u}{u}+u\otimes w\otimes v\otimes\frac{1-v}{v} \\
& +u\otimes w\otimes v\otimes\frac{1-w}{w}-u\otimes w\otimes(1-w)\otimes\frac{1-u}{u} \\
& +\text{cyclic permutations}\,.
\end{aligned}
$$

It satisfies $\mathcal{S}^{(2)}_{abcd}-\mathcal{S}^{(2)}_{bacd}-\mathcal{S}^{(2)}_{abdc}+\mathcal{S}^{(2)}_{badc}-(a\leftrightarrow c\,,\,b\leftrightarrow d)\;=\;0$

therefore can be obtained from a function involving only classical polylog functions:

$\log x_1\log x_2\log x_3\log x_4\;,\;\mathrm{Li}_2(x_1)\log x_2\log x_3\;,\;\mathrm{Li}_2(x_1)\mathrm{Li}_2(x_2)\;,\;\mathrm{Li}_3(x_1)\log x_2$ and $\mathrm{Li}_4(x_i)$

Symbol bootstrap:
2-loop 3-point form factor

Reconstruct the function (plus collinear constraint) :

$$\mathcal{R}_3^{(2)} = -2\left[\mathrm{J}_4\left(-\frac{uv}{w}\right)+\mathrm{J}_4\left(-\frac{vw}{u}\right)+\mathrm{J}_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]$$
$$-2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i)+\frac{\log^2 u_i}{2!}\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4$$

$$\mathrm{J}_4(z) := \mathrm{Li}_4(z)-\log(-z)\mathrm{Li}_3(z)+\frac{\log^2(-z)}{2!}\mathrm{Li}_2(z)-\frac{\log^3(-z)}{3!}\mathrm{Li}_1(z)-\frac{\log^4(-z)}{48}\,.$$

Simple combination of classical polylog functions !

Symbol bootstrap: 2-loop 3-point form factor

N=4 result is identical to the maximally transcendental part in QCD!

$$
\begin{aligned}
&-2G(0,0,1,0,u)+G(0,0,1-v,1-v,u)+2G(0,0,-v,1-v,u)-G(0,1,0,1-v,u)+4G(0,1,1,0,u)-G(0,1,1-v,0,u)+G(0,1-v,0,1-v,u)\\
&+G(0,1-v,1-v,0,u)-G(0,1-v,-v,1-v,u)+2G(0,-v,0,1-v,u)+2G(0,-v,1-v,0,u)-2G(0,-v,1-v,1-v,u)-2G(1,0,0,1-v,u)\\
&-2G(1,0,1-v,0,u)+4G(1,1,0,0,u)-4G(1,1,1,0,u)-2G(1,1-v,0,0,u)+G(1-v,0,0,1-v,u)-G(1-v,0,1,0,u)-2G(-v,1-v,1-v,u)H(0,v)\\
&-2G(1-v,1,0,0,u)+2G(1-v,1,0,1-v,u)+2G(1-v,1,1-v,0,u)+G(1-v,1-v,0,0,u)+2G(1-v,1-v,1,0,u)-2G(1-v,1-v,-v,1-v,u)\\
&-G(1-v,-v,1-v,0,u)+4G(1-v,-v,-v,1-v,u)-2G(-v,0,1-v,1-v,u)-2G(-v,1-v,0,1-v,u)-2G(-v,1-v,1-v,0,u)+4G(1,0,1,0,u)\\
&+4G(-v,-v,1-v,1-v,u)-4G(-v,-v,-v,1-v,u)-G(0,0,1-v,u)H(0,v)-G(0,1,0,u)H(0,v)-G(0,1-v,0,u)H(0,v)+G(0,1-v,1-v,u)H(0,v)\\
&-G(0,-v,1-v,u)H(0,v)-2G(1,0,0,u)H(0,v)+G(1,0,1-v,u)H(0,v)+G(1,1-v,0,u)H(0,v)+G(1-v,0,0,u)H(0,v)-G(1-v,0,1-v,u)H(0,v)\\
&-G(1-v,1,0,u)H(0,v)-G(1-v,1-v,0,u)H(0,v)-G(1-v,-v,1-v,u)H(0,v)+G(-v,0,1-v,u)H(0,v)+G(-v,1-v,0,u)H(0,v)+H(1,0,0,1,v)\\
&-G(0,0,1-v,u)H(1,v)-G(0,0,-v,u)H(1,v)+G(0,1,0,u)H(1,v)-G(0,1-v,0,u)H(1,v)+G(0,1-v,-v,u)H(1,v)-2G(0,-v,0,u)H(1,v)\\
&+2G(0,-v,1-v,u)H(1,v)+2G(1,0,0,u)H(1,v)-G(1-v,0,0,u)H(1,v)+G(1-v,0,-v,u)H(1,v)-2G(1-v,1,0,u)H(1,v)-G(1-v,0,-v,1-v,u)\\
&+G(1-v,-v,0,u)H(1,v)-4G(1-v,-v,-v,u)H(1,v)+2G(-v,0,1-v,u)H(1,v)+2G(-v,1-v,0,u)H(1,v)-4G(-v,1-v,-v,u)H(1,v)\\
&-4G(-v,-v,1-v,u)H(1,v)+4G(-v,-v,-v,u)H(1,v)+G(0,0,u)H(0,0,v)+G(0,1-v,u)H(0,0,v)+G(1-v,0,u)H(0,0,v)+H(1,0,1,0,v)\\
&-G(0,0,u)H(0,1,v)+G(0,-v,u)H(0,1,v)-G(1,0,u)H(0,1,v)+2G(1-v,0,u)H(0,1,v)+2G(1-v,1-v,u)H(0,1,v)-3G(1-v,-v,u)H(0,1,v)\\
&-G(-v,0,u)H(0,1,v)-2G(-v,1-v,u)H(0,1,v)+4G(-v,-v,u)H(0,1,v)-G(0,0,u)H(1,0,v)+G(0,-v,u)H(1,0,v)-G(1,0,u)H(1,0,v)\\
&+2G(1-v,0,u)H(1,0,v)-2G(1-v,1-v,u)H(1,0,v)+G(1-v,-v,u)H(1,0,v)-G(-v,0,u)H(1,0,v)+2G(-v,1-v,u)H(1,0,v)+G(0,0,u)H(1,1,v)\\
&-2G(0,-v,u)H(1,1,v)-2G(-v,0,u)H(1,1,v)+4G(-v,-v,u)H(1,1,v)+G(0,u)H(0,0,1,v)-3G(1-v,u)H(0,0,1,v)+4G(-v,u)H(0,0,1,v)\\
&+G(0,u)H(0,1,0,v)+G(1-v,u)H(0,1,0,v)-G(0,u)H(0,1,1,v)+2G(-v,u)H(0,1,1,v)+G(0,u)H(1,0,0,v)+G(1-v,u)H(1,0,0,v)+H(1,1,0,0,v)\\
&-G(0,u)H(1,0,1,v)+2G(-v,u)H(1,0,1,v)-G(0,u)H(1,1,0,v)+4G(1-v,u)H(1,1,0,v)-2G(-v,u)H(1,1,0,v)+H(0,0,1,1,v)+H(0,1,0,1,v)\\
&+G(1-v,1-v,u)H(0,0,v)+2G(1-v,1-v,-v,u)H(1,v)-G(1-v,-v,0,1-v,u)+H(0,1,1,0,v)+G(1-v,0,1-v,0,u)-G(0,1-v,1,0,u)\\
&+4G(-v,1-v,-v,1-v,u)
\end{aligned}
$$

QCD

Gehrmann, Jaquier, Glover, Koukoutsakis 2011

$$\updownarrow$$

$$
\begin{aligned}
\mathcal{R}_3^{(2)} = & -2\left[\mathrm{J}_4\left(-\frac{uv}{w}\right)+\mathrm{J}_4\left(-\frac{vw}{u}\right)+\mathrm{J}_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]\\
& -2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i)+\frac{\log^2 u_i}{2!}\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4
\end{aligned}
$$

N=4 SYM

Brandhuber, Travaglini, GY 2012