A short introduction to method of regions and SCET

圈积分及相空间积分计算系列讲座

王健 2021年6月10日

Motivations

- Multi-scale Feynman integrals have been the bottleneck in the application of perturbative QFT.
- They are difficult because of the large space of functions and combinatorics in multiple variables in the final integrated result. (Disaster of dimensions!)
- In the method of differential equations, boundary conditions are still demanding. How to choose a wise boundary?

Method of regions

- One way to simplify the multi-scale integrals is to use the method of regions.
- This kind of analysis has led to (and is the base of) effective field theories, such as SCET.
- The idea is to expand a complicated integral within dim-reg to a sum of simpler integrals, at the integrand level.
- Simpler means fewer scales involved.
- The ultimate version: Each term depends only on a single scale.
- The accuracy can be improved systematically.

4 steps

1. Determine the large and small scales in the problem
2. Divide the loop integration domain into regions in which each loop momentum is of the order of one of the scales in the problem
3. Perform a Taylor expansion in the parameters, which are small in the given region.
4. Integrate over the entire loop integration domain in every region.

An example

$$I = \int_0^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{\ln \frac{M}{m}}{M^2 - m^2}$$

- First, we assume m<M. The full result can be expanded as

$$I = \frac{\ln \frac{M}{m}}{M^2} \left(1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots \right)$$

- This is called an asymptotic expansion since the result is not analytic in m/M.
- We may naively Taylor expand the integrand in the small m limit.

$$I \neq \int_0^\infty dk \, \frac{k}{k^2(k^2+M^2)} \left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots \right)$$

Introduce a scale: $m \ll \Lambda \ll M$

$$I = \underbrace{\int_0^\Lambda dk \frac{k}{(k^2+m^2)(k^2+M^2)}}_{I_{(I)}} + \underbrace{\int_\Lambda^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)}}_{I_{(II)}}$$

In the low-energy scale region:

$$I_{(I)} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)M^2}\left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \cdots\right)$$

In the high-energy scale region:

$$I_{(II)} = \int_\Lambda^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_\Lambda^\infty dk \frac{k}{k^2(k^2+M^2)}\left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right)$$

$$I_{(1)} = -\frac{1}{M^2}\ln\left(\frac{m}{\Lambda}\right) - \frac{\Lambda^2}{2M^4} + \cdots \qquad I_{(II)} = -\frac{1}{M^2}\ln\left(\frac{\Lambda}{M}\right) + \frac{\Lambda^2}{2M^4} + \cdots$$

$$I_{(I)} + I_{(II)} = -\frac{1}{M^2}\ln\left(\frac{m}{M}\right) + \cdots$$

Introducing a scale makes the calculation complicated. Try working in dim-reg.

In the low-energy scale region:

$$I_{(I)} = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)M^2} \left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \cdots\right)$$

In the high-energy scale region:

$$I_{(II)} = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{k^2(k^2+M^2)} \left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right)$$

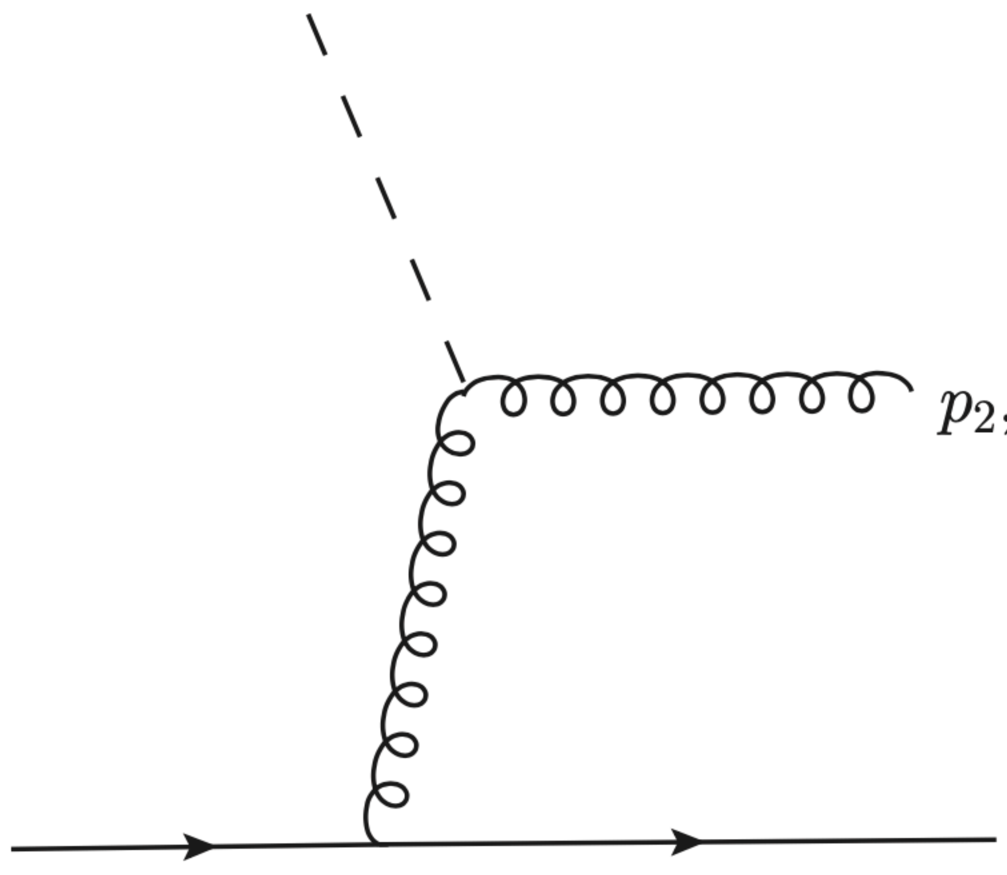
$$I_{(I)} = \frac{m^{-\varepsilon}}{2M^2}\Gamma\left(1-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{1}{M^2}\left(\frac{1}{\varepsilon} - \ln m + \mathcal{O}(\varepsilon)\right)$$

$$I_{(II)} = -\frac{M^{-\varepsilon}}{2M^2}\Gamma\left(1-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{1}{M^2}\left(-\frac{1}{\varepsilon} + \ln M + \mathcal{O}(\varepsilon)\right)$$

Overlapping region:

$$R = \int_0^\infty dk k^{-\varepsilon} \frac{k}{k^2 M^2} \left(1 - \frac{m^2}{k^2} - \frac{k^2}{M^2} + \cdots\right)$$

Another example



$$\int_k \frac{1}{k^2(k-p_2)^2(k+p_1-p_3)^2} = \frac{i}{16\pi^2}\frac{1}{\epsilon^2}\frac{1}{-2p_2\cdot p_3}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}(1-z^{-\epsilon})$$

$$p_1 \sim Q(z,\lambda^2,\lambda\sqrt{z}), \quad p_2 \sim Q(1,\lambda^2,\lambda), \quad p_3 \sim Q(\lambda^2,1,\lambda) \qquad p \sim (n_+p, n_-p, p_\perp)$$

In the hard region: $k \sim Q(1,1,1)$

$$\begin{aligned} I_1^{\text{IR}} &= \int_k \frac{1}{k^2(k-p_2)^2(k+p_1-p_3)^2}\Bigg|_{\text{hard}}^{\text{IR poles}} \\ &= \int_k \frac{1}{[k^2+i0][k^2-n_-kn_+p_2+i0][k^2-n_+kn_-p_3+i0]}\Bigg|^{\text{IR poles}} \\ &= \frac{-i}{16\pi^2}\frac{1}{\epsilon^2}\frac{1}{Q^2}\left(1+\epsilon\ln\frac{\mu^2}{Q^2}\right)+O(\epsilon^0) \end{aligned}$$

In the z-anti-hard-collinear region: $k \sim Q(z, 1, \sqrt{z})$

$$I_2^{\rm IR} = \int_k \frac{1}{k^2(k-p_2)^2(k+p_1-p_3)^2}\Bigg|_{\rm z-anti-hard-collinear}^{\rm IR\ poles}$$
$$= \int_k \frac{1}{[k^2+i0][-n_-kn_+p_2+i0][k^2-n_+kn_-p_3+n_-kn_+p_1-n_-p_3n_+p_1+i0]}\Bigg|^{\rm IR\ poles}$$
$$= \frac{i}{16\pi^2}\frac{1}{\epsilon^2}\frac{1}{Q^2}\left(1+\epsilon\ln\frac{\mu^2}{zQ^2}\right)+O(\epsilon^0)$$

The sum of the above two regions reproduce the full result.
The poles in the full result arise in the IR regions, so one may look at these regions.

In the p2-collinear region:

$k \sim Q(1, \lambda^2, \lambda)$

$$I_1^{\rm UV} = \int_k \frac{1}{k^2(k-p_2)^2(k+p_1-p_3)^2}\Bigg|_{\rm coll\ k||p_2}^{\rm UV\ poles}$$
$$= \int_k \frac{1}{[k^2+i0][(k-p_2)^2+i0][-n_+kn_-p_3+i0]}\Bigg|^{\rm UV\ poles}$$
$$= \frac{-i}{16\pi^2}\mu^{2\epsilon}\int_0^{n_+p_2} dn_+k\int_0^\infty dk_T^2(k_T^2)^{-\epsilon}(k_T^2-x\bar{x}p_2^2)^{-1}\frac{1}{-n_+p_2}\frac{1}{[-n_+kn_-p_3+i0]}$$
$$= \frac{i}{16\pi^2}\frac{1}{\epsilon}\mu^{2\epsilon}\int_0^1 dx(-x\bar{x}p_2^2)^{-\epsilon}\frac{1}{[-xn_+p_2n_-p_3]}$$
$$= \frac{i}{16\pi^2}\frac{1}{\epsilon^2}\frac{1}{Q^2}\left(1+\epsilon\ln\frac{\mu^2}{-p_2^2}\right)+O(\epsilon^0)$$

In the p1-collinear region:

$$k \sim Q(z, \lambda^2, \lambda\sqrt{z})$$

$$\begin{aligned}
I_2^{\text{UV}} &= \int_k \frac{1}{k^2(k-p_2)^2(k+p_1-p_3)^2}\Bigg|_{\text{p}_1\ \text{coll.}}^{\text{UV poles}} \\
&= \int_k \frac{1}{[k^2+i0][\underbrace{-n_+p_2 n_-k + p_2^2}_{\lambda^2}+i0][\underbrace{-(n_+k+n_+p_1)\cdot n_-p_3}_{zQ^2}+i0]}\Bigg|^{\text{UV poles}} \\
&= \frac{-i}{16\pi^2}\mu^{2\epsilon}\int_0^\infty dn_+k \int_0^\infty dk_T^2 (k_T^2)^{-\epsilon}(k_T^2 - xp_2^2)^{-1}\frac{1}{-n_+p_2}\frac{1}{[-(n_+k+n_+p_1)\cdot n_-p_3+i0]} \\
&= \frac{i}{16\pi^2}\frac{1}{\epsilon}\mu^{2\epsilon}\int_0^\infty dx(-xp_2^2)^{-\epsilon}\frac{1}{[-(xn_+p_2+n_+p_1)\cdot n_-p_3+i0]} \\
&= \frac{-i}{16\pi^2}\frac{1}{\epsilon^2}\frac{1}{Q^2}\left(1+\epsilon\ln\frac{\mu^2}{-p_2^2}\right)z^{-\epsilon}
\end{aligned}$$

The sum of the above two regions is

$$I_1^{\text{UV}} + I_2^{\text{UV}} = \frac{i}{16\pi^2}\frac{1}{\epsilon^2}\frac{1}{Q^2}\left(1-z^{-\epsilon}\right) = \frac{i}{16\pi^2}\frac{1}{Q^2}\frac{\ln z}{\epsilon}$$

Useful references and website

https://www.ttp.kit.edu/~asmirnov/Tools-Regions.htm

Beneke, Smirnov, hep-ph/9711391
Pak, Smirnov, 1011.4863
Ananthanarayan et al, 1810.06270

Bauer,Fleming,Pirjol,Rothstein,Stewart, PRD63,114020, PRD65,054022,PRD66,014017
Beneke,Chapovsky,Diehl,Feldmann, NPB643,431,PLB553,267
Becher,Broggio,Ferroglia, 1410.1892

Introduction to SCET

- The method of regions focuses mainly on the momentum space. The Feynman diagrams are still generated by traditional methods.
- Effective field theories set up power counting from the Lagrangian, for both momenta and fields. Symmetries are also more explicitly implemented.
- Soft collinear effective theory (SCET) is very suitable for QCD (loop) integrals since it describes the regions where divergences appear.
- Tremendous improvement has been achieved during the last two decades, e.g., the general structures of divergences of amplitudes, the relations between (external) massless amplitudes and massive ones, the anomalous dimensions of TMD PDFs, the global subtraction of IR divergences in differential (N)NNLO QCD corrections, the subleading power structures of large logarithms in cross sections, non-global logs, super-leading logs.

Fermion fields

In QCD, the fermion fields are described by: $\mathcal{L}_q = \bar{\psi} i D_\mu \gamma^\mu \psi$

In SCET, the fermion fields split to different modes: $\psi = \psi_c + \psi_s$

To extract the large component, we define
(see Peskin's book, sec.3.3)

$$\psi_c(x) \equiv \xi(x) + \eta(x)$$

$$\xi = P_+ \psi_c \equiv \frac{\not{n}\not{\bar{n}}}{4}\psi_c\,, \qquad \eta = P_- \psi_c \equiv \frac{\not{\bar{n}}\not{n}}{4}\psi_c$$

$$\not{n}\,\xi(x) = 0\,, \qquad \not{\bar{n}}\,\eta(x) = 0$$

Scaling property of fields:

$$\langle 0|T\left\{\xi(x)\bar{\xi}(0)\right\}|0\rangle = \frac{\not{n}\not{\bar{n}}}{4}\langle 0|T\left\{\psi_c(x)\bar{\psi}_c(0)\right\}|0\rangle\frac{\not{\bar{n}}\not{n}}{4} = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i0}e^{-ip\cdot x}\frac{\not{n}\not{\bar{n}}}{4}\not{p}\frac{\not{\bar{n}}\not{n}}{4} \sim \lambda^4\frac{1}{\lambda^2} = \lambda^2$$

$$\xi(x) \sim \lambda, \quad \eta(x) \sim \lambda^2, \quad \psi_s(x) \sim \lambda^3$$

Gauge fields and E.o.M

In SCET, the gauge fields split to different modes: $A^\mu(x) \to A_c^\mu(x) + A_s^\mu(x)$

$$\langle 0| T\left\{A^\mu(x) A^\nu(0)\right\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2+i0} e^{-ip\cdot x}\left[-g^{\mu\nu} + \xi \frac{p^\mu p^\nu}{p^2}\right]$$

Scaling property of fields: $A_c^\mu(x) \sim p_c^\mu, \quad A_s^\mu(x) \sim p_s^\mu$

$$\mathcal{L}_c = \left(\bar{\xi} + \bar{\eta}\right)\left[\frac{\not{n}}{2} i\bar{n}\cdot D + \frac{\bar{\not{n}}}{2} in\cdot D + i\not{D}_\perp\right](\xi + \eta) = \bar{\xi}\frac{\bar{\not{n}}}{2} in\cdot D\xi + \bar{\xi} i\not{D}_\perp \eta + \bar{\eta} i\not{D}_\perp \xi + \bar{\eta}\frac{\not{n}}{2} i\bar{n}\cdot D\eta$$

E.o.M of fields:

$$\not{D}_\perp \xi = -\frac{\not{n}}{2}\bar{n}\cdot D\eta \qquad\qquad \eta = -\frac{\bar{\not{n}}}{2\bar{n}\cdot D}\not{D}_\perp \xi$$

Lagrangian of collinear quark fields:

$$\mathcal{L}_c = \bar{\xi}\frac{\bar{\not{n}}}{2} in\cdot D\xi + \bar{\xi} i\not{D}_\perp \frac{1}{i\bar{n}\cdot D} i\not{D}_\perp \frac{\bar{\not{n}}}{2}\xi \qquad\qquad in\cdot D = in\cdot\partial + g\, n\cdot A_c(x) + g\, n\cdot A_s(x_-)$$

Lagrangian of SCET

$$\mathcal{L}_{\rm SCET} = \bar{\psi}_s i \not{D}_s \psi_s + \bar{\xi} \frac{\not{\bar{n}}}{2} \left[i n \cdot D + i \not{D}_{c\perp} \frac{1}{i\bar{n} \cdot D_c} i \not{D}_{c\perp} \right] \xi - \frac{1}{4} \left(F^{s,a}_{\mu\nu} \right)^2 - \frac{1}{4} \left(F^{c,a}_{\mu\nu} \right)^2$$

Collinear gauge field strengths: $igF^c_{\mu\nu} = [iD_\mu, iD_\nu]\,, \quad D^\mu = n \cdot D \frac{\bar{n}^\mu}{2} + \bar{n} \cdot D_c \frac{n^\mu}{2} + D^\mu_{c\perp}$

The interaction between collinear and soft modes can be removed by field redefinitions.

$$\xi(x) \to S_n(x_-) \xi^{(0)}(x)\,, \qquad A^\mu_c(x) \to S_n(x_-) A^{(0)\mu}_c(x) S^\dagger_n(x_-) \qquad S_n(x) = \mathbf{P} \exp \left[ig \int_{-\infty}^0 ds\, n \cdot A_s(x + sn) \right]$$

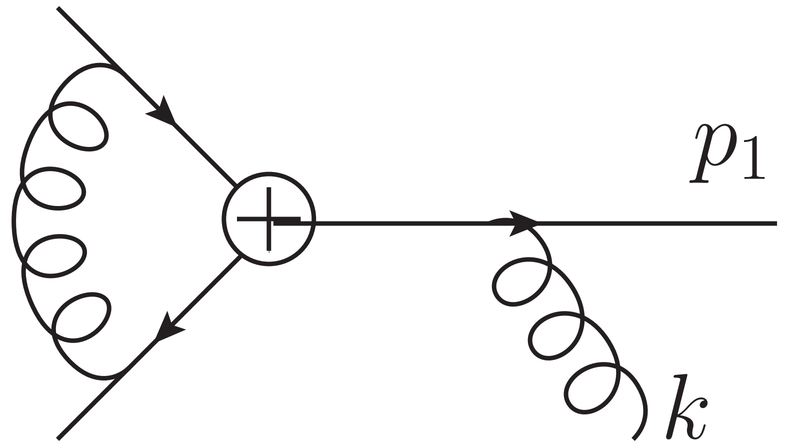
After using the properties of Wilson lines, we see

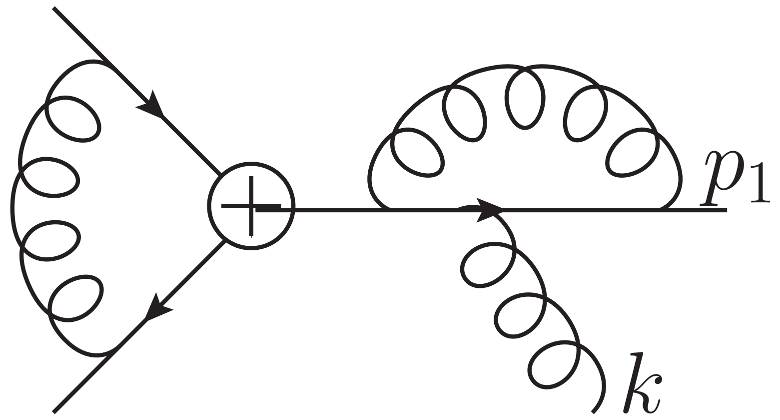
$$in \cdot D\xi(x) = S_n(x_-) \left(in \cdot \partial + gn \cdot A^{(0)}_c(x) \right) \xi^{(0)}(x) \equiv S_n(x_-) in \cdot D^{(0)}_c \xi^{(0)}(x)$$

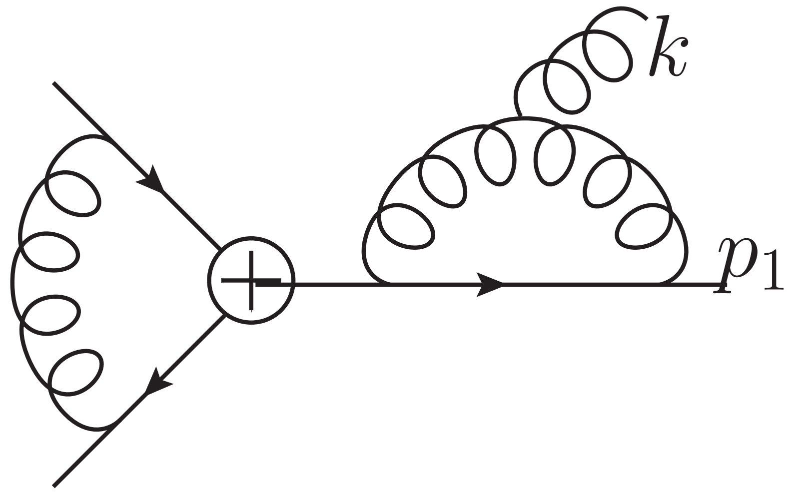
This conclusion holds to all orders, though it is not so clear from the diagrams.

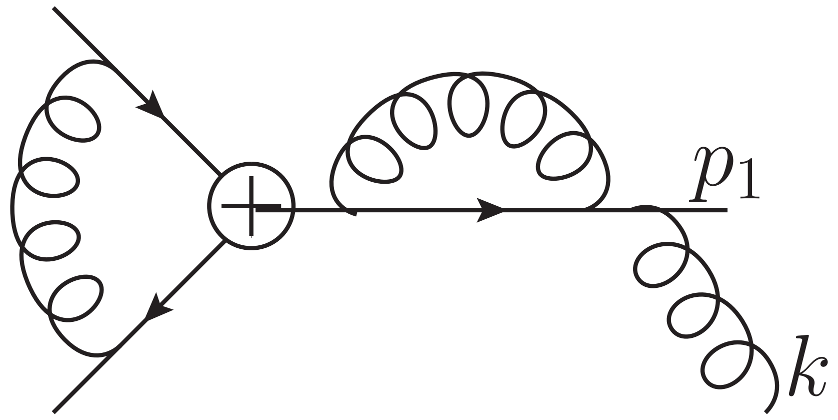
Decoupling of soft gauge fields

Collinear momenta: p_1, p_2 , soft momentum: k

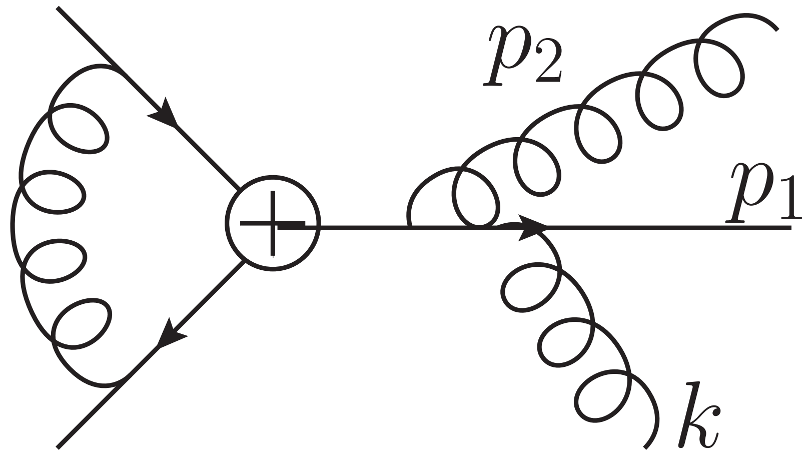


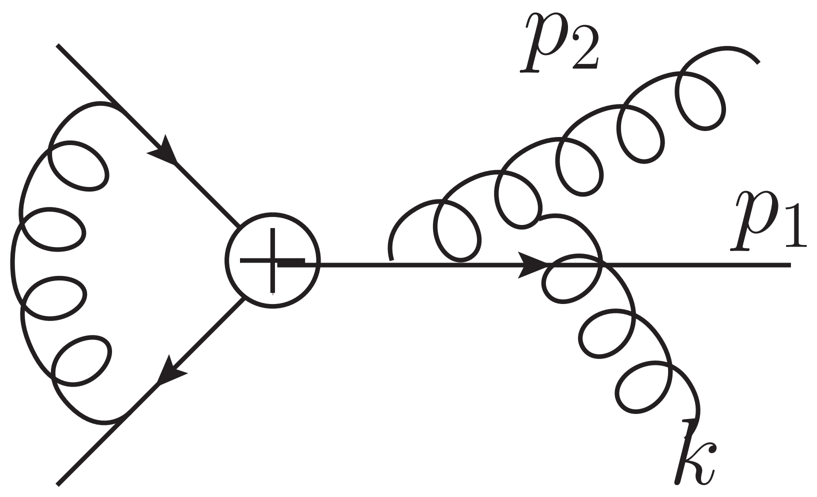


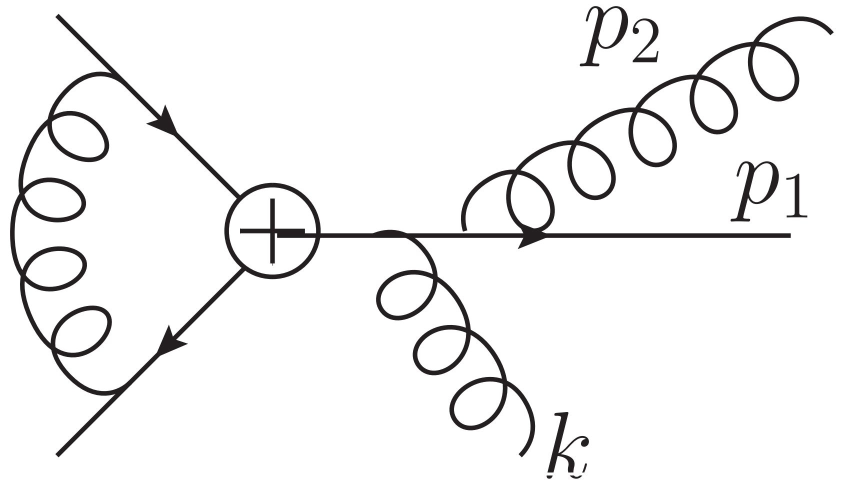




$$\frac{1}{(p_1 + l + k)^2}$$







$$\frac{1}{(p_1 + p_2 + k)^2}$$

Operators of SCET

The SCET Lagrangian describes the interaction in the single collinear or soft modes, as it should be.

The interaction between different collinear directions involves hard scales that live beyond the control of SCET, and are governed by (hard) operators and associating Wilson coefficients.

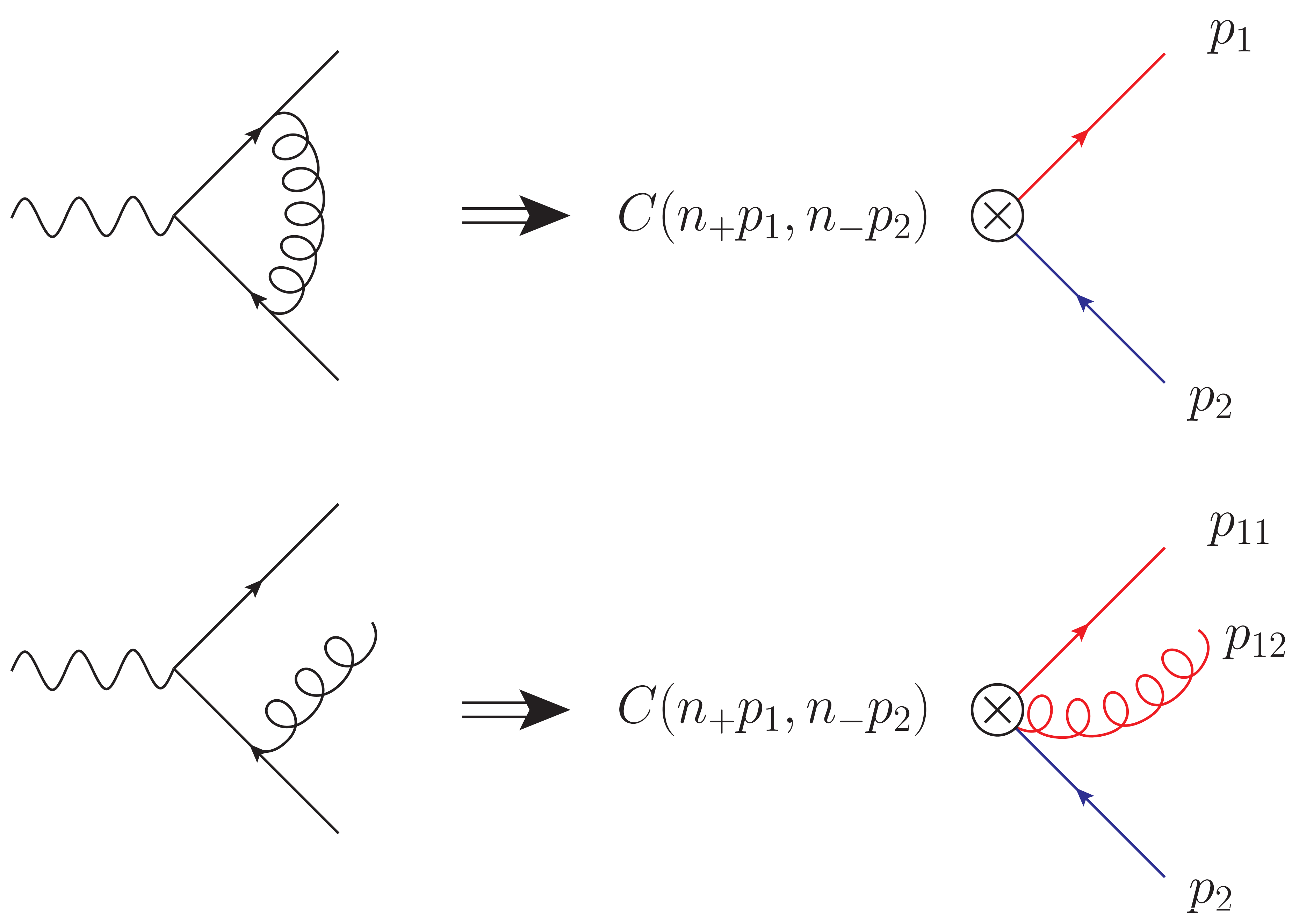
Consider the simple current operator in QCD: $J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$. Since there are only two quark fields, we can choose a frame in which they move back-to-back. Matching this operator into that in SCET.

$$J^\mu(x) \to \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dt\, C_V(s,t)\bar{\chi}_c\,(x+s\bar{n})\gamma_\perp^\mu \chi_{\bar{c}}(x+tn)$$

The new collinear fields are defined by $\chi_c = W_c^\dagger \xi_c \quad W_c(x) = \mathbf{P}\exp\left[ig\int_{-\infty}^{0} ds\,\bar{n}\cdot A_c(x+s\bar{n})\right]$

Matching & Collinear Wilson lines

We perform matching by calculating the matrix elements of the operators in both the full and effective theories. The matching coefficient should not depend on the external states one chooses.



Coll. gauge invariant building block: $\chi_c, \bar{\chi}_c$

Decoupling transformation:

$$\chi_c(x) \to S_n\,(x_-)\chi_c^{(0)}(x)\,, \quad \chi_{\bar{c}}(x) \to S_{\bar{n}}\,(x_+)\chi_{\bar{c}}^{(0)}(x)$$

$$J^\mu(x) = \int ds \int dt\, C_V(s,t)\bar{\chi}_c^{(0)}\,(x+s\bar{n})S_n^\dagger\,(x_-)\,S_{\bar{n}}\,(x_+)\gamma_\perp^\mu \chi_{\bar{c}}^{(0)}(x+tn)$$

Or $$\sum_{q,k} \bar{\chi}_{c,q}\,(0)\,\tilde{C}_V(\bar{\mathcal{P}}^\dagger, \mathcal{P})\,\gamma_\perp^\mu\,\chi_{\bar{c},k}(0)$$

Matching Wilson Coefficient

Oneloop calculation shows

$$\tilde{C}_V^{\text{bare}}(\varepsilon, Q^2) = 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{\pi^2}{6} + \mathcal{O}(\varepsilon) \right) \left(\frac{Q^2}{\mu^2} \right)^{-\varepsilon} + \mathcal{O}\left(\alpha_s^2\right)$$

which contains UV poles. Use MSbar renormalization scheme,

$$\tilde{C}_V(Q^2, \mu) = 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left(-\ln^2 \frac{Q^2}{\mu^2} + 3 \ln \frac{Q^2}{\mu^2} + \frac{\pi^2}{6} - 8 \right) + \mathcal{O}(\alpha_s^2)$$

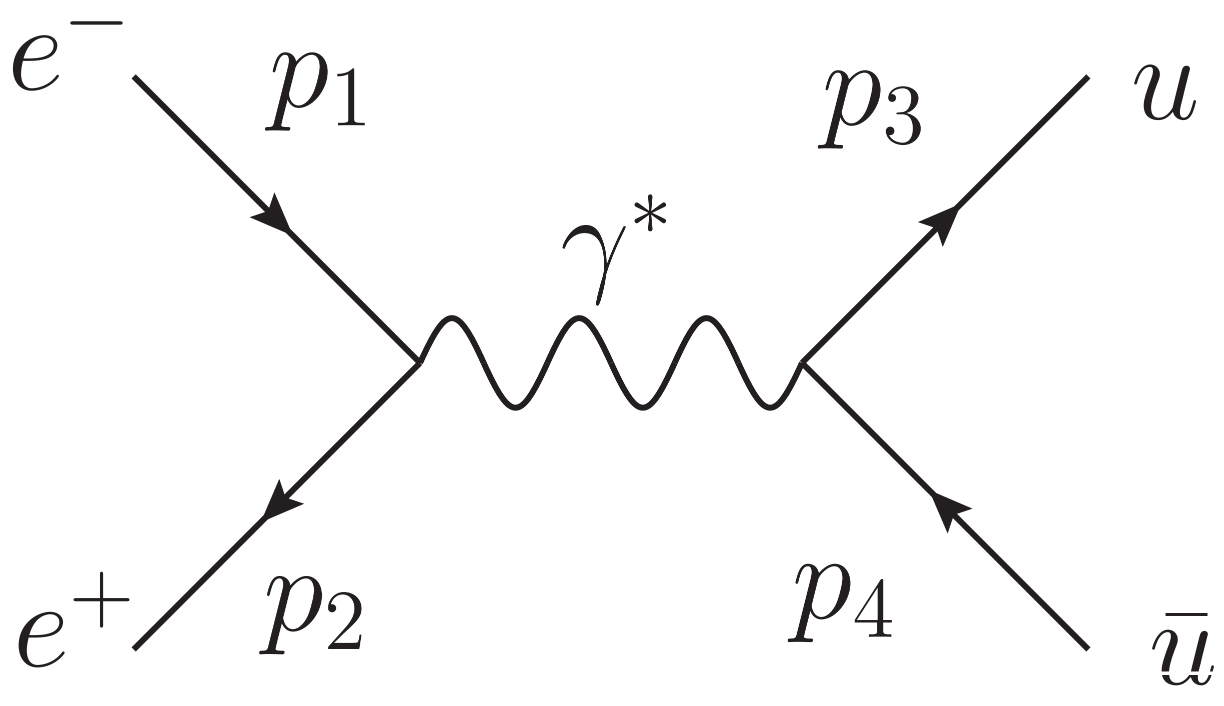
$$Z\left(\varepsilon, Q^2, \mu\right) = 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left(-\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon} \ln \frac{Q^2}{\mu^2} - \frac{3}{\varepsilon} \right) + \mathcal{O}\left(\alpha_s^2\right)$$

Solve the renormalization group equation to obtain the all-order result

$$\frac{d}{d \ln \mu} \tilde{C}_V(Q^2, \mu) = \left[C_F \gamma_{\text{cusp}}(\alpha_s) \ln \frac{Q^2}{\mu^2} + \gamma_V(\alpha_s) \right] \tilde{C}_V(Q^2, \mu)$$

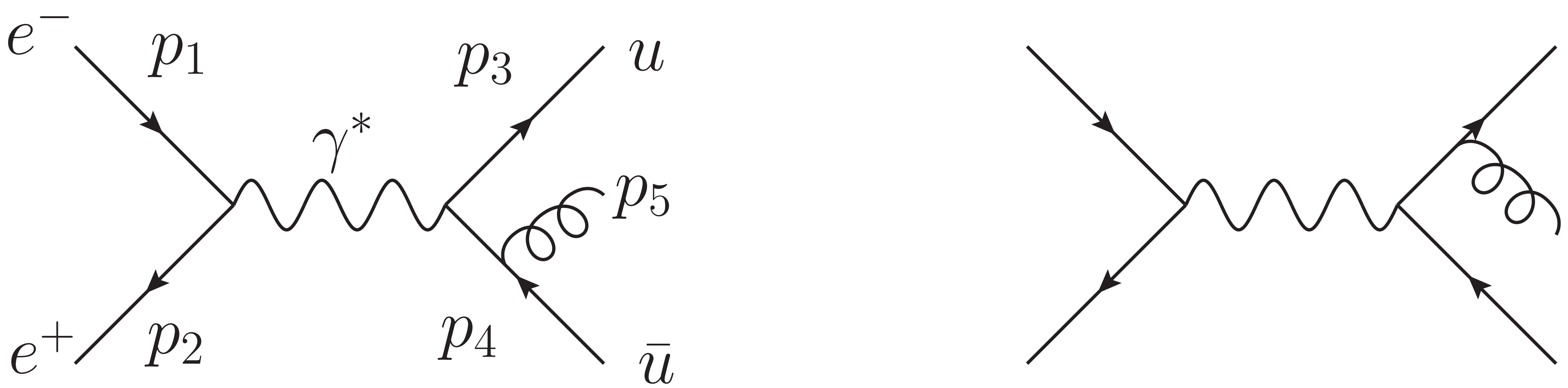
The cusp part controls the structure of double logs and has been studied in the renormalization of closed Wilson lines with cusps in 1980's.

Factorization of x-sec. for thrust

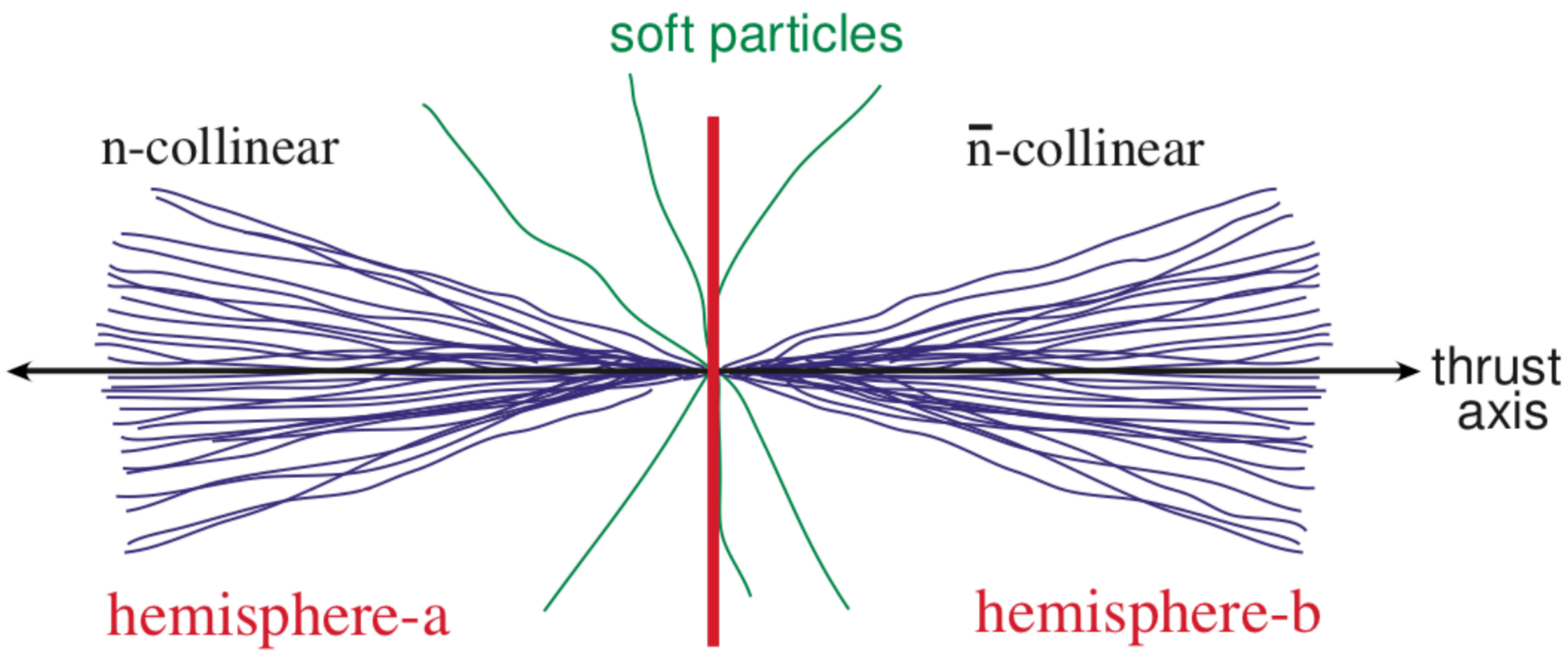


An observable: $T \equiv \max_{\vec{n}} T_{\vec{n}} = \max_{\vec{n}} \frac{\sum_i |\vec{n} \cdot \vec{p}_i|}{\sum_i |\vec{p}_i|}$

In the limit $T \to 1$, $T \approx 1 - \frac{m_L^2 + m_R^2}{Q^2}$



(1) (2)



Factorization of x-sec. for thrust

The fixed-order result is

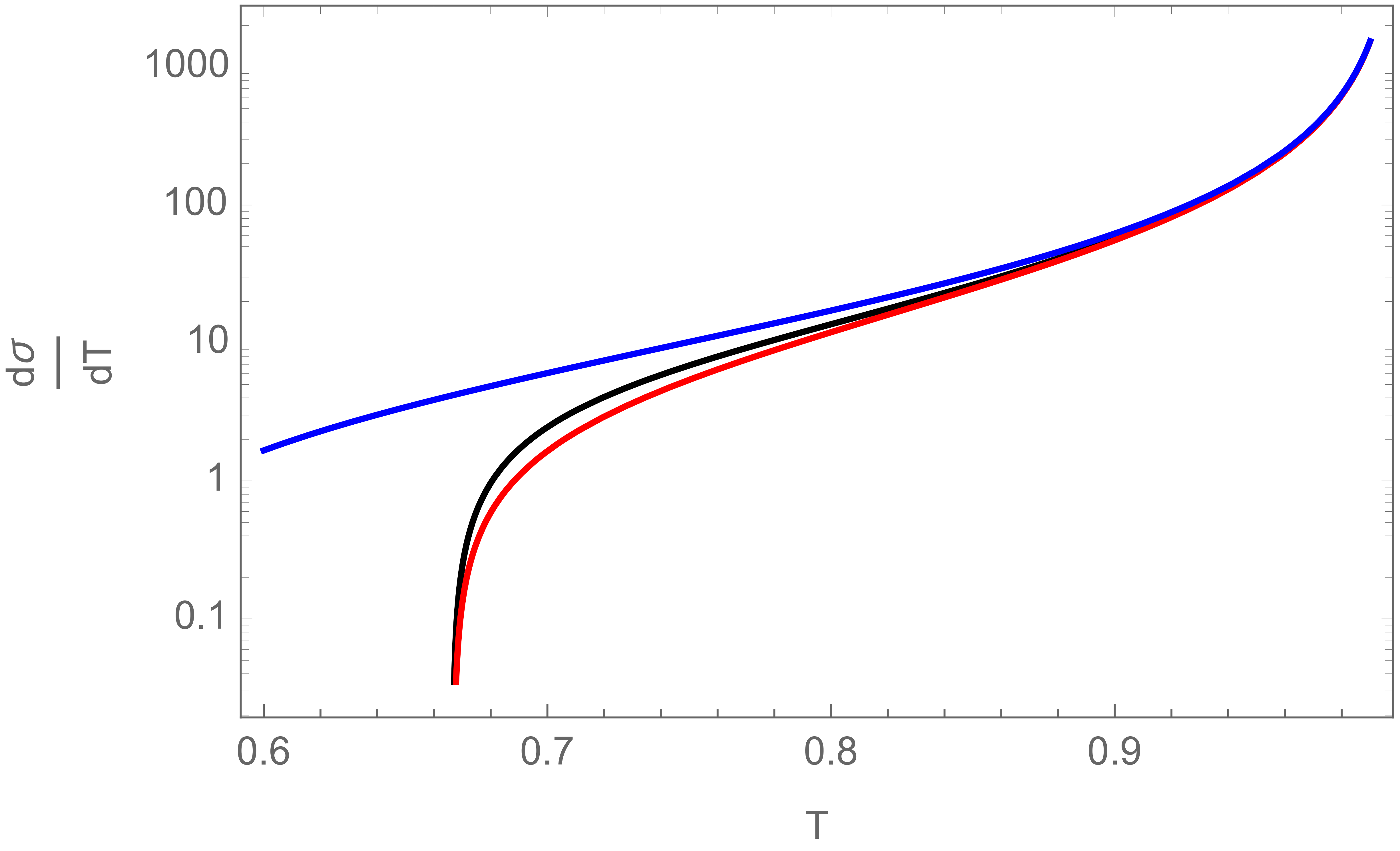
$$\frac{d\sigma}{\sigma_B dT} = \frac{\alpha_s C_F}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\ln\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{(1-T)}\right]$$

The leading power logarithmic result is

$$\frac{d\sigma}{\sigma_B dT} = \frac{\alpha_s C_F}{2\pi}\left[\frac{-4\ln(1-T)-3}{1-T}\right]$$

The next-to-leading power is

$$\frac{d\sigma}{\sigma_B dT} = \frac{\alpha_s C_F}{2\pi}\left[2(\ln(1-T)-1)\right]$$



Factorization of x-sec. for thrust

In the limit of $\tau \equiv 1 - T \to 0$, the cross section becomes

$$\frac{1}{\sigma_B}\frac{d\sigma}{d\tau} = H(Q^2, \mu^2)\int dm_L^2 dm_R^2 J(m_L^2, \mu) J(m_R^2, \mu) S_T(k, \mu)\delta(\tau - \frac{m_L^2 + m_R^2}{Q^2} - \frac{k}{Q})$$

with

$$H(Q^2, \mu^2) = |C(Q^2, \mu^2)|^2$$

$$J(p^2, \mu) = \frac{1}{\bar{n} \cdot p}\text{Disc}\left[i\int d^4x e^{-ipx}\langle 0 | T\left\{\bar{\chi}(x)\frac{\bar{n}^\mu \gamma_\mu}{2}\chi(0)\right\}\right]$$

$$S_T(k, \mu) = \sum_X |\langle X | S_n^\dagger S_{\bar{n}} | 0\rangle|^2 \delta(k - n \cdot p_{X_n} - \bar{n} \cdot p_{X_{\bar{n}}})$$

Factorization of x-sec. for thrust

The x-section contains contribution from the scales:

$$Q^2, \quad m^2 \sim Q^2\lambda^2 \sim Q^2\tau, \quad k^2 \sim Q^2\lambda^4 \sim Q^2\tau^2$$

The summation of large log comes from the running from one scale to another:

$$\ln\tau = \ln\frac{m^2}{Q^2} = \ln\frac{\mu_h^2}{Q^2} + \ln\frac{\mu^2}{\mu_h^2} - \ln\frac{\mu^2}{m^2}$$

where the second term is a result of RG solution.

Ansatz: $\quad H(Q^2, \mu^2) = e^{U_1(\mu)+U_2(\mu)} \left(\dfrac{Q^2}{\mu_h^2}\right)^{U_3(\mu)} H(Q^2, \mu_h^2)$

Factorization of x-sec. for thrust

The U's satisfy

$$\frac{dU_1}{d\ln\mu} = 2\gamma_H(\alpha_s),\ \frac{dU_3}{d\ln\mu} = 2\Gamma_{\text{cusp}}(\alpha_s),\ \frac{dU_2}{d\ln\mu} = -4\Gamma_{\text{cusp}}(\alpha_s)\ln\frac{\mu}{\mu_h}$$

The solutions are

$$U_1(\mu,\mu_h) = 2\int_{\alpha_s(\mu_h)}^{\alpha_s(\mu)} d\alpha\frac{\gamma_H(\alpha)}{\beta(\alpha)},\ U_2(\mu,\mu_h) = -4\int_{\alpha_s(\mu_h)}^{\alpha_s(\mu)} d\alpha\frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}\int_{\alpha_s(\mu_h)}^{\alpha} d\alpha'\frac{1}{\beta(\alpha')}$$

They have perturbative expansions. Since they appear in the exponent, the large log of $\ln\frac{\mu}{\mu_h}$ has been resummed.

Factorization of x-sec. for thrust

The jet function's RGE is non-local.

$$\frac{dJ(p^2,\mu)}{d\ln\mu} = \left[-2\Gamma_{\text{cusp}}\ln\frac{p^2}{\mu^2} - 2\gamma_J\right] J(p^2,\mu) + 2\Gamma_{\text{cusp}}\int_0^{p^2} dq^2 \frac{J(p^2,\mu) - J(q^2,\mu)}{p^2 - q^2}$$

An elegant solution is obtained by using the Laplace transform.

$$J(p^2,\mu) = e^{U_4(\mu_j,\mu)}\tilde{j}(\partial_{\eta_j},\mu_j)\frac{1}{p^2}\left(\frac{p^2}{\mu_j^2}\right)^{\eta_j}\frac{e^{-\gamma_E\eta_j}}{\Gamma(\eta_j)}$$

A similar solution can be found for the soft function.

And then the convolution between two jet and one soft function can be done analytically.

Summary

- In this talk, I give a short introduction to the method of regions, which can be used to obtain the integrals in certain limits. This can also be used to determine the boundary constants of differential equations.
- In another part of the talk, I review the basic ideas of SCET. The development in this field deepens our understanding of the IR structure of the QCD amplitudes and many results have been used in pQCD.
- Many more contents have been skipped.

Thanks a lot !