Introduction to Sector Decomposition

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Why numerical approach

- Long time that numerical approach can check analytical results.
- Need for NNLO (even higher order) radiation effects @ LHC
- Multi-scale Feynman (master) integrals could become extremely difficult in the analytical approach, e.g. elliptic integral.
- Pheno analysis cares only about the numbers: production rates, distributions, asymmetries etc.
- Compromise to the numerical approach.

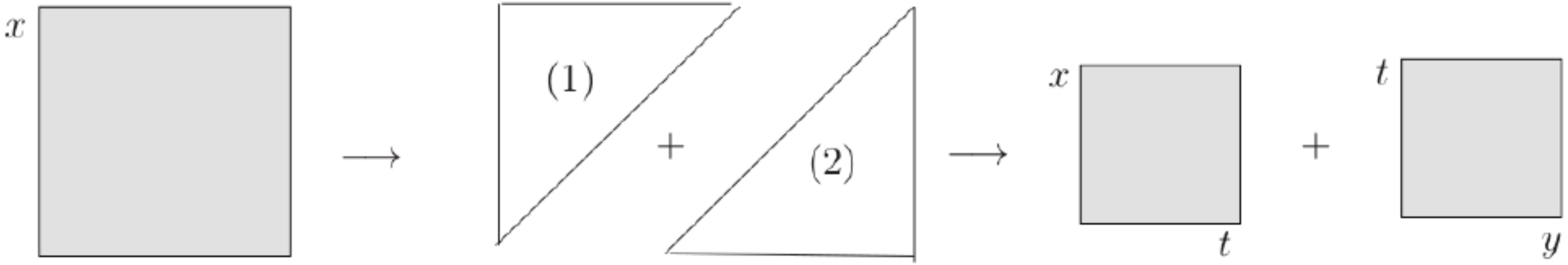
Popular numerical approaches

- Sector Decomposition
- Mellin-Barnes
- Differential equations
- …

Sector Decomposition

arXiv:0803.4177

$$I = \int_0^1 dx \int_0^1 dy\ x^{-1-\epsilon} y^{-\epsilon} (x + (1-x)y)^{-1}$$



$$I = \int_0^1 dx\ x^{-1-\epsilon} \int_0^1 dt\ t^{-\epsilon} (1 + (1-x)t)^{-1}$$
$$+ \int_0^1 dy\ y^{-1-2\epsilon} \int_0^1 dt\ t^{-1-\epsilon} (1 + (1-y)t)^{-1}$$

$$G^{\mu_1\cdots\mu_R}_{l_1\cdots l_R} = \int \prod_{l=1}^{L} d^D\kappa_l \, \frac{k_{l_1}^{\mu_1}\cdots k_{l_R}^{\mu_R}}{\prod_{j=1}^{N} P_j^{\nu_j}\left(\{k\},\{p\},m_j^2\right)} \, ,$$

$$d^D\kappa_l = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}}}\, d^Dk_l \, , \qquad P_j\left(\{k\},\{p\},m_j^2\right) = \left(q_j^2 - m_j^2 + i\delta\right) ,$$

Feynman parameterization

$$\frac{1}{\prod_{j=1}^{N} P_j^{\nu_j}} = \frac{\Gamma(N_\nu)}{\prod_{j=1}^{N}\Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^{N} dx_j \, x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^{N} x_i\right) \frac{1}{\left[\sum_{j=1}^{N} x_j P_j\right]^{N_\nu}} \, ,$$

where $N_\nu = \sum_{j=1}^{N}\nu_j$, leads to

$$G^{\mu_1\cdots\mu_R}_{l_1\cdots l_R} = \frac{\Gamma(N_\nu)}{\prod_{j=1}^{N}\Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^{N} dx_j \, x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^{N} x_i\right) \int d^D\kappa_1 \cdots d^D\kappa_L$$
$$\times\, k_{l_1}^{\mu_1}\cdots k_{l_R}^{\mu_R} \left[\sum_{i,j=1}^{L} k_i^{\mathrm{T}} M_{ij} k_j - 2\sum_{j=1}^{L} k_j^{\mathrm{T}}\cdot Q_j + J + i\delta\right]^{-N\nu} ,$$

Integrate out loop momenta

$$G^{\mu_1\cdots\mu_R}_{l_1\cdots l_R} = (-1)^{N_\nu} \frac{1}{\prod_{j=1}^{N}\Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^{N} dx_j\, x_j^{\nu_j-1}\delta\left(1-\sum_{l=1}^{N} x_l\right)$$

$$\times \sum_{m=0}^{[R/2]} \left(-\frac{1}{2}\right)^m \Gamma(N_\nu - m - LD/2)[(\tilde{M}^{-1}\otimes g)^{(m)}\tilde{l}^{(R-2m)}]^{\Gamma_1,\ldots,\Gamma_R}$$

$$\times \frac{\mathcal{U}^{N_\nu-(L+1)D/2-R}}{\mathcal{F}^{N_\nu-LD/2-m}}, \qquad (7)$$

where

$$v_l = \sum_{i=1}^{L} M^{-1}_{li} Q_i\,. \qquad \mathcal{F}(\mathbf{x}) = \det(M)\left[\sum_{j,l=1}^{L} Q_j M^{-1}_{jl} Q_l - J - i\delta\right], \qquad (8)$$

$$\mathcal{U}(\mathbf{x}) = \det(M)\,, \quad \tilde{M}^{-1} = \mathcal{U}M^{-1}\,, \quad \tilde{l} = \mathcal{U}v$$

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$[\tilde{l}] \Rightarrow [x]^L$

where

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$[\tilde{M}^{-1}] \Rightarrow [x]^{L-1}$

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$[J] = [x]$
$[\mathcal{U}] \Rightarrow [x]^L$
$[\mathcal{F}] = [x]^{L+1}$

$$\mathcal{U}(\mathbf{x}) = \det(M)\,, \quad \tilde{M}^{-1} = \mathcal{U}M^{-1}\,, \quad \tilde{l} = \mathcal{U}v$$

Example

$$
\begin{aligned}
G_{112}^{\mu_1\mu_2\mu_3} &= \int \mathrm{d}^D\kappa_1\,\mathrm{d}^D\kappa_2\,\frac{k_1^{\mu_1}k_1^{\mu_2}k_2^{\mu_3}}{\prod\limits_{j=1}^{N} P_j^{\nu_j}(\{k\},\{p\},m_j^2)} \\
&= \frac{(-1)^{N_\nu}}{\prod_{j=1}^{N}\Gamma(\nu_j)}\int\limits_0^\infty \prod_{j=1}^{N} dx_j\;x_j^{\nu_j-1}\,\delta(1-\sum_{l=1}^{N}x_l) \\
&\quad \Big\{\Gamma(N_\nu-D)\,\frac{\mathcal{U}^{N_\nu-3D/2-3}}{\mathcal{F}^{N_\nu-D}}\,\tilde{l}_1^{\mu_1}\tilde{l}_1^{\mu_2}\tilde{l}_2^{\mu_3} \\
&\quad -\frac{1}{2}\Gamma(N_\nu-1-D)\,\frac{\mathcal{U}^{N_\nu-3D/2-3}}{\mathcal{F}^{N_\nu-D-1}}\;\times \\
&\quad \left[(\tilde{M}^{-1}\otimes g)_{11}^{\mu_1\mu_2}\,\tilde{l}_2^{\mu_3}+(\tilde{M}^{-1}\otimes g)_{12}^{\mu_1\mu_3}\,\tilde{l}_1^{\mu_2}+(\tilde{M}^{-1}\otimes g)_{12}^{\mu_2\mu_3}\,\tilde{l}_1^{\mu_1}\right]\Big\}\,,
\end{aligned}
$$

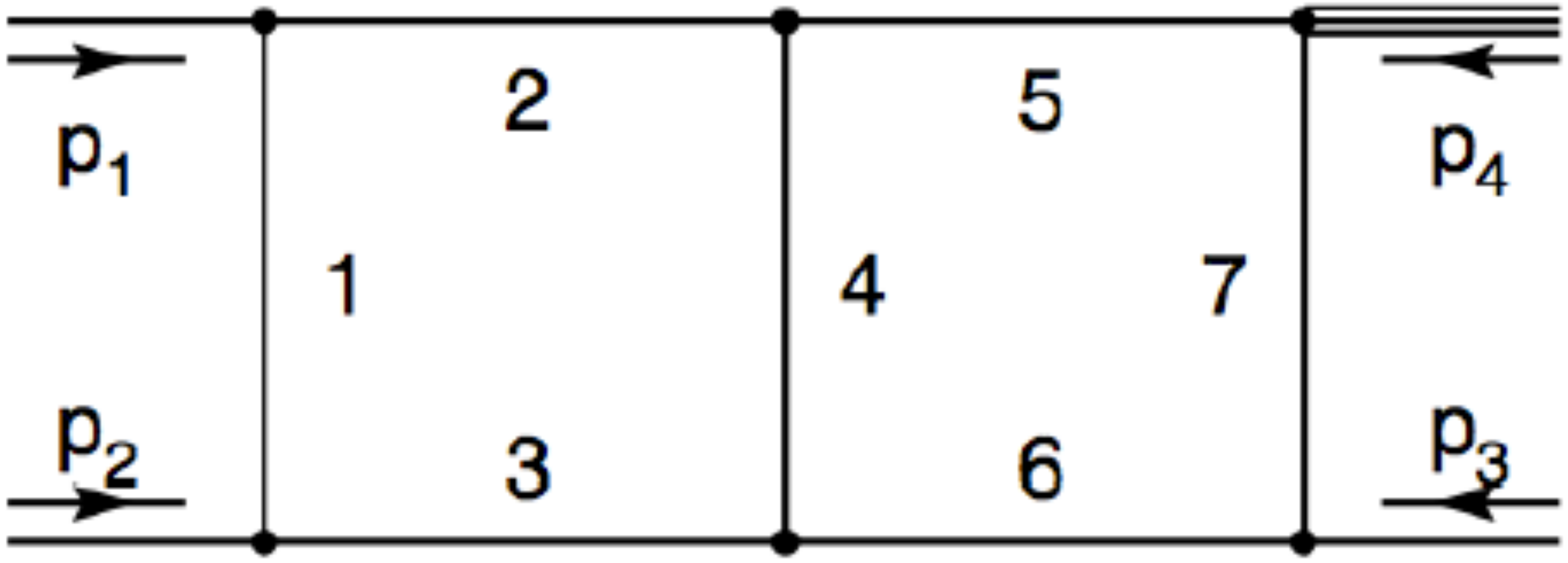
$$
(\tilde{M}^{-1}\otimes g)^{\mu\nu} = \begin{pmatrix} \tilde{M}_{11}^{-1}\,g^{\mu\nu} & \tilde{M}_{12}^{-1}\,g^{\mu\nu} \\ \tilde{M}_{21}^{-1}\,g^{\mu\nu} & \tilde{M}_{22}^{-1}\,g^{\mu\nu} \end{pmatrix}\,.
$$

U and F can be determined geometrically

$$\mathcal{U}(\mathbf{x}) = \sum_{T \in \mathcal{T}_1} \Big[\prod_{j \in \mathcal{C}(T)} x_j \Big] ,$$

$$\mathcal{F}_0(\mathbf{x}) = \sum_{\hat{T} \in \mathcal{T}_2} \Big[\prod_{j \in \mathcal{C}(\hat{T})} x_j \Big] (-s_{\hat{T}}) ,$$

$$\mathcal{F}(\mathbf{x}) = \mathcal{F}_0(\mathbf{x}) + \mathcal{U}(\mathbf{x}) \sum_{j=1}^{N} x_j m_j^2 .$$

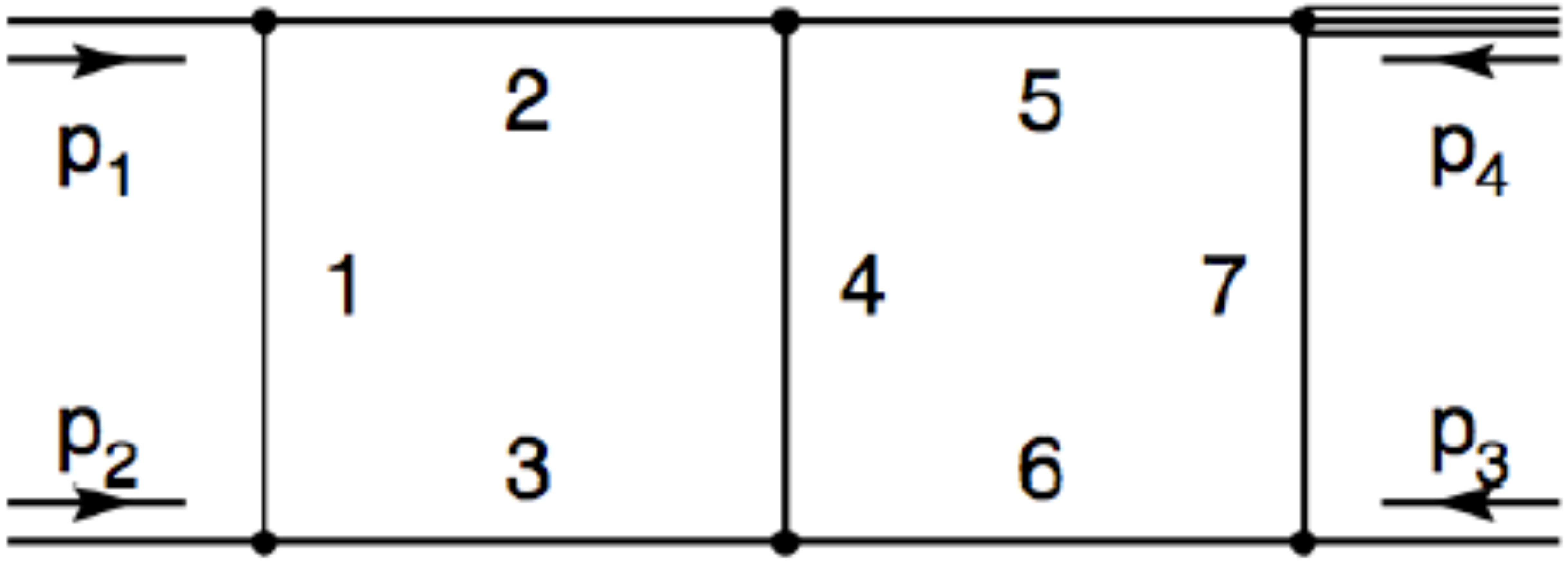


$$\mathcal{U} = x_{123} x_{567} + x_4 x_{123567} ,$$

$$\mathcal{F} = (-s_{12})(x_2 x_3 x_{4567} + x_5 x_6 x_{1234} + x_2 x_4 x_6 + x_3 x_4 x_5) + (-s_{23}) x_1 x_4 x_7 + (-p_4^2) x_7 (x_2 x_4 + x_5 x_{1234}) ,$$

where $x_{ijk\cdots} = x_i + x_j + x_k + \cdots$ and $s_{ij} = (p_i + p_j)^2$.

U and F can be determined geometrically



$$\mathcal{U}(\mathbf{x}) = \sum_{T\in\mathcal{T}_1} \Bigg[\prod_{j\in\mathcal{C}(T)} x_j \Bigg],$$

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$$\mathcal{F}(\mathbf{x}) = \mathcal{F}_0(\mathbf{x}) + \mathcal{U}(\mathbf{x}) \sum_{j=1}^{N} x_j m_j^2 .$$

$$[\mathcal{U}] \Rightarrow [x]^L$$
$$[\mathcal{F}] = [x]^{L+1}$$

$$\mathcal{U} = x_{123}x_{567} + x_4 x_{123567},$$

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where $x_{ijk\ldots} = x_i + x_j + x_k + \cdots$ and $s_{ij} = (p_i + p_j)^2$.

First generate primary sectors to eliminate Delta function

$$\int_0^\infty d^N x = \sum_{l=1}^{N} \int_0^\infty d^N x \prod_{\substack{j=1 \\ j\neq l}}^{N} \theta(x_l \geq x_j)\,.$$

$$x_j = \begin{cases} x_l t_j & \text{for } j < l\,, \\ x_l & \text{for } j = l\,, \\ x_l t_{j-1} & \text{for } j > l \end{cases}$$

$$G_l = \int_0^1 \prod_{j=1}^{N-1} dt_j \, \frac{\mathcal{U}_l^{N_\nu-(L+1)D/2}(\mathbf{t})}{\mathcal{F}_l^{N_\nu-LD/2}(\mathbf{t})}\,, \quad l = 1, \ldots, N\,.$$

$$[u] = [x]^L$$

$$u \Rightarrow x_\ell^L u_\ell(\vec{t})$$

$$u = x_{123} x_{567} + x_4 x_{12}$$

$$u = (x_1 + x_2 + x_3)(x_5 + x_6 + x_7) + x_4 (x_1 + x_2)$$

$$u_2 = (t_1 + 1 + t_2)(t_4 + t_5 + t_6) + t_3 (t_1 + 1)$$

$$x_l^{N_\nu - 1}$$

$$G^{\mu_1 \dots \mu_R}_{l_1 \dots l_R} = (-1)^{N_\nu} \frac{1}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^{N} dx_j \; x_j^{\nu_j - 1} \delta(1 - \sum_{l=1}^{N} x_l)$$

$$\sum_{m=0}^{[R/2]} (-\frac{1}{2})^m \Gamma(N_\nu - m - LD/2) \left[(\tilde{M}^{-1} \otimes g)^{(m)} \; \tilde{l}^{(R-2m)} \right]^{\Gamma_1, \dots, \Gamma_R}$$

$$\times \frac{\mathcal{U}^{N_\nu - (L+1)D/2 - R}}{\mathcal{F}^{N_\nu - LD/2 - m}}$$

$$(x_l^{L-1})^m \qquad (x_l^L)^{R-2m}$$

$$\frac{(x_l^L)^{N_\nu - (L+1)D/2 - R}}{(x_l^{L+1})^{N_\nu - LD/2 - m}}$$

$$\int dx_l \; \delta(1 - x_1 - \cdots - x_N)\, x_l^{-1}$$

$$= \int dx_l \; \delta(1 - x_l t_1 - x_l t_2 - \cdots - x_l - \cdots - x_l t_{N-1})\, x_l^{-1}$$

$$= \int dx_l \; \delta[1 - (1 + t_1 + \cdots + t_{N-1}) x_l]\, x_l^{-1}$$

$$= 1$$

$$G_l = \int_0^1 \prod_{j=1}^{N-1} dt_j \, \frac{\mathcal{U}_l^{N_\nu - (L+1)D/2}(\mathbf{t})}{\mathcal{F}_l^{N_\nu - LD/2}(\mathbf{t})} \,, \quad l = 1, \ldots, N \,.$$

Determine a sub-set of parameters ti

$$\mathcal{S} = \{t_{\alpha_1}, \ldots, t_{\alpha_r}\}$$

Then divide into r sub-sectors

$$\prod_{j=1}^{r} \theta(1 \geq t_{\alpha_j} \geq 0) = \sum_{k=1}^{r} \prod_{\substack{j=1 \\ j \neq k}}^{r} \theta(t_{\alpha_k} \geq t_{\alpha_j} \geq 0) \,.$$

$$t_{\alpha_j} \to \begin{cases} t_{\alpha_k} t_{\alpha_j} & \text{for } j \neq k \,, \\ t_{\alpha_k} & \text{for } j = k \,. \end{cases}$$

$$G_{lk} = \int_0^1 \left(\prod_{j=1}^{N-1} dt_j \, t_j^{a_j - b_j \epsilon} \right) \frac{\mathcal{U}_{lk}^{N_\nu - (L+1)D/2}}{\mathcal{F}_{lk}^{N_\nu - LD/2}} , \quad k = 1, \ldots, r \, .$$

$$\mathcal{U}_{lk_1 k_2 \ldots} = 1 + u(\mathbf{t}) \, , \quad \mathcal{F}_{lk_1 k_2 \ldots} = -s_0 + \sum_\beta (-s_\beta) f_\beta(\mathbf{t}) \, ,$$

All the coefficients of divergences are finite (complicated).

$$I_j = \int_0^1 dx_j \, x_j^{a_j + b_j \epsilon} \, \mathcal{I}(x_j, \{x_{i \neq j}\}, \epsilon) \, .$$

$$x^{-1+b\,\epsilon} = \frac{1}{b\,\epsilon}\,\delta(x) + \sum_{n=0}^{\infty} \frac{(b\,\epsilon)^n}{n!} \left[\frac{\ln^n(x)}{x}\right]_+ ,$$

NLO: eps^-3, eps^-2, eps^-1, eps^0, eps^1

NNLO: eps^-5, …

integrals >> # sectors

Strategy for sector decomposition

- Iterative decomposition strategy:

C. Bogner and S. Weinzierl, Comput. Phys. Commun., 178: 596–610 (2008)

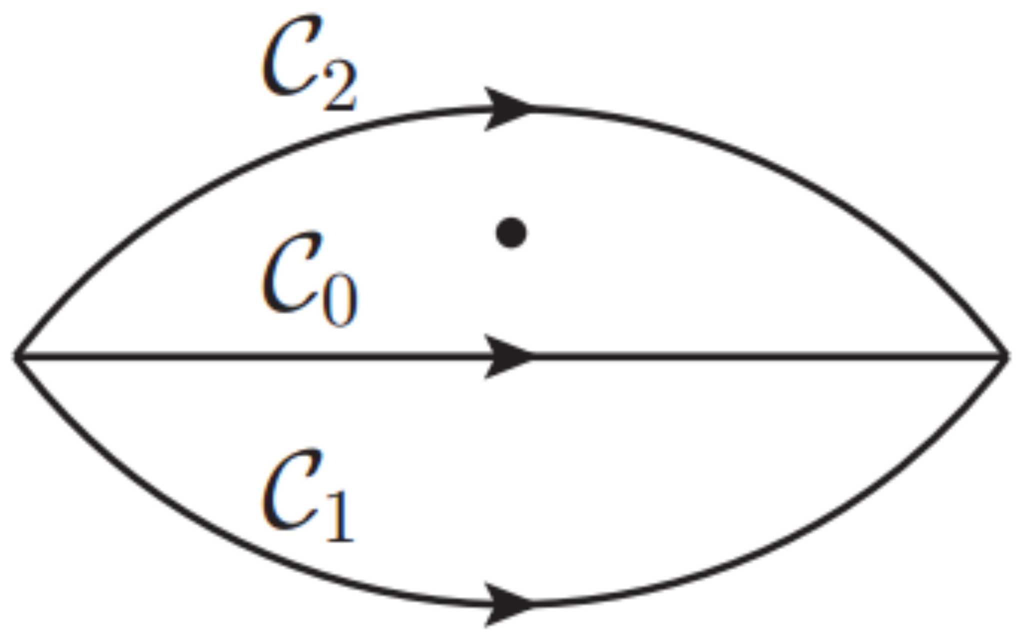
- Geometric strategy:

T. Kaneko and T. Ueda, Comput. Phys. Commun., 181: 1352–1361 (2010)

- Heuristic strategy etc.

Diagram	A	B	C	S	X	H	This method	Exponential S.D.
Bubble	2	2	2	2*	2		2	2
Triangle	3	3	3	3*	3		3	3
Box	12	12	12	12	12		12	8
Tbubble	58	48	48	48*	48		48	36
Double box, $p_i^2 = 0$	775	586	586	362	293	282	266	106
Double box, $p_4^2 \neq 0$	543*	245*	245*	230*	192*	197	186	100
Double box, $p_i^2 = 0$ nonplanar	1138	698	698	441*	395		360	120
D420	8898	564	564	180	F		168	100
3 loop vertex (A8)	4617*	1196*	1196*	871*	750*	684	684	240
Triple box	M	114256	114256	22657	10155		6568	856

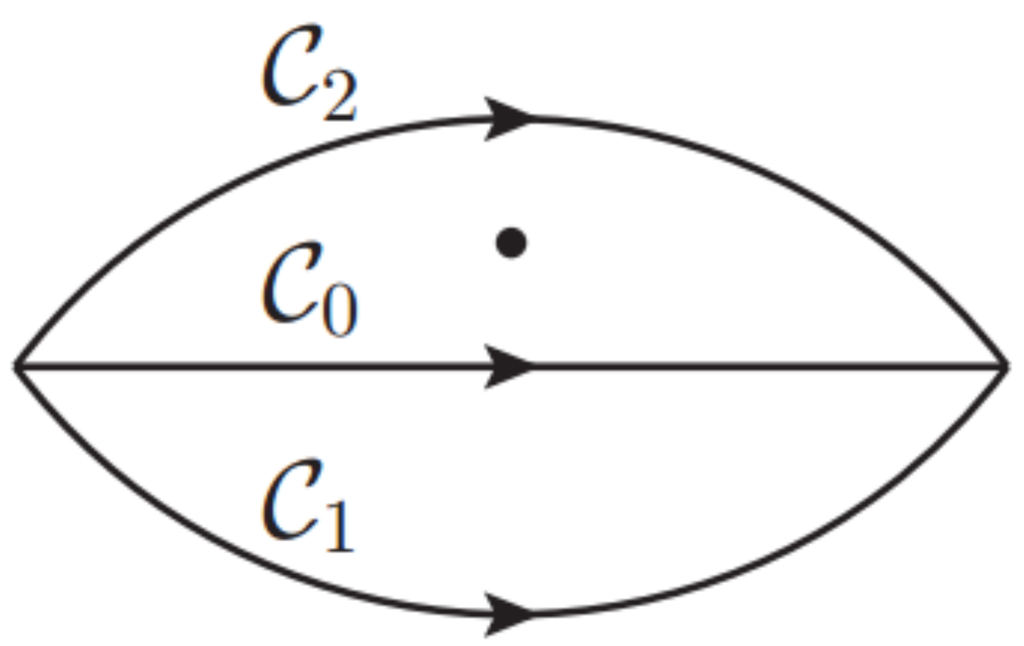
$$I_s = C(\epsilon) \lim_{\delta \to 0} \int_0^1 \frac{\mathcal{D}(\vec{x}, \epsilon)\, \mathcal{H}_s(\vec{x}, \epsilon)}{[\mathcal{F}_s(\vec{x}, m_i^2, s_{jk}) - i\delta]^{a+b\epsilon}}$$



$$z_i = x_i - i\lambda x_i^{\alpha} (1 - x_i)^{\beta} \frac{\partial \mathcal{F}_s}{\partial x_i}$$

$$\lim_{\delta \to 0} \int_0^1 \frac{\mathcal{D}(\vec{x}, \epsilon)\, \mathcal{H}_s(\vec{x}, \epsilon)}{[\mathcal{F}_s(\vec{x}, m_i^2, s_{jk}) - i\delta]^{a+b\epsilon}} = \int_{\mathcal{C}} \frac{\mathcal{D}(\vec{z}, \epsilon)\, \mathcal{H}_s(\vec{z}, \epsilon)}{[\mathcal{F}_s(\vec{z}, m_i^2, s_{jk})]^{a+b\epsilon}}$$

$$I_s = C(\epsilon) \lim_{\delta \to 0} \int_0^1 \frac{\mathcal{D}(\vec{x}, \epsilon)\, \mathcal{H}_s(\vec{x}, \epsilon)}{[\mathcal{F}_s(\vec{x}, m_i^2, s_{jk}) - i\delta]^{a+b\epsilon}}$$



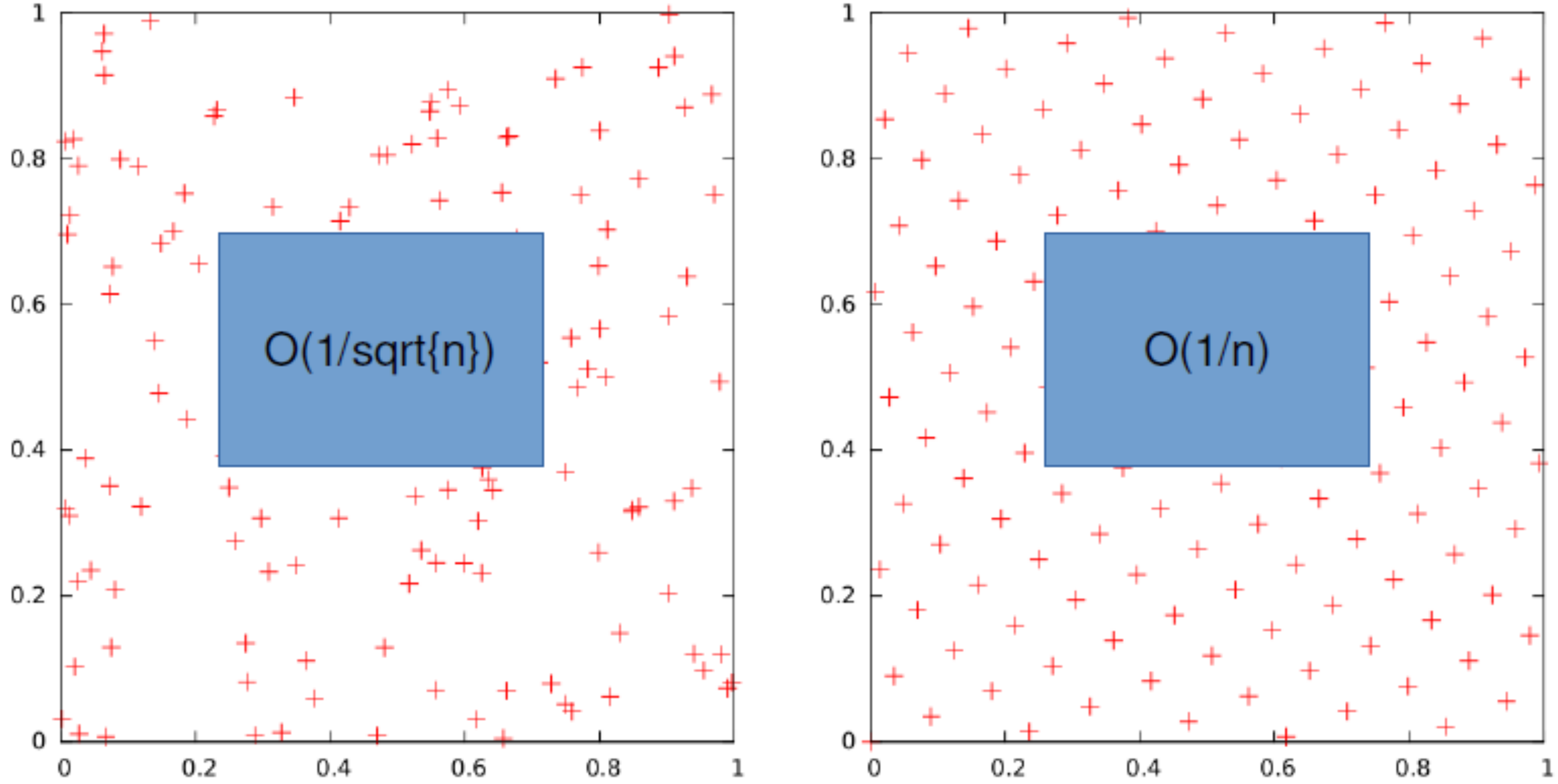
$$z_i \;=\; x_i - i\lambda x_i^{\alpha}(1 - x_i)^{\beta} \frac{\partial \mathcal{F}_s}{\partial x_i}$$

$$\lim_{\delta \to 0} \int_0^1 \frac{\;\cdots\; \mathcal{H}_s(\vec{z}, \epsilon)}{[\mathcal{F} \;\cdots\;, s_{jk})]^{a+b\epsilon}}$$

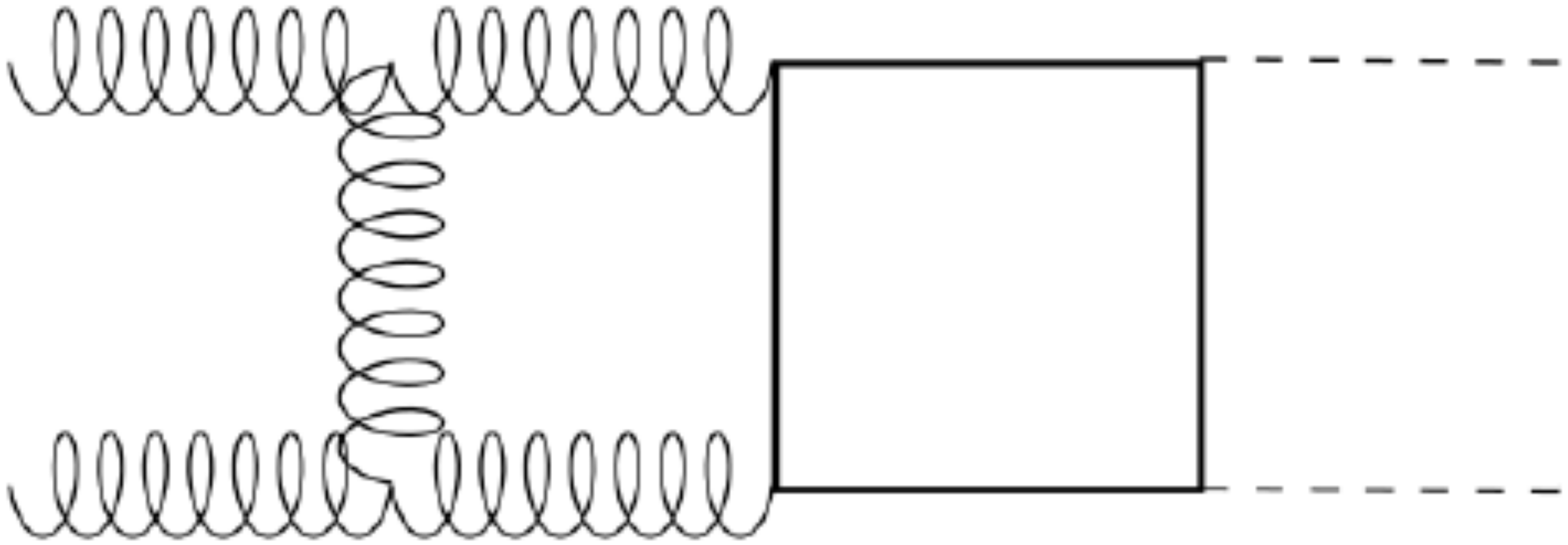
1. λ is small: the convergence could be bad
2. λ is big: Jacobi could generate trouble

Implement Quasi-Monte-Carlo

$$I(f) = \int_0^1 d^s x f(\vec{x}) \qquad I_{estimate}(f) = \sum_{i=0}^{n-1} f(\vec{x}_i)$$



arxiv:1508.02512



$$I_C = e^{-2\epsilon\gamma_E} s^{-3-2\epsilon} \sum_{i=0}^{i=2} \frac{P_i}{\epsilon^i}.$$

	Vegas/CPU	QMC/GPU
P_2	$-7.959 \pm 0.009 - 10.586i \pm 0.009i$	$-7.949 \pm 0.003 - 10.585i \pm 0.005i$
P_1	$3.9 \pm 0.1 - 28.1i \pm 0.1i$	$3.831 \pm 0.005 - 28.022i \pm 0.005i$
P_0	$-3.9 \pm 0.8 + 92.3i \pm 0.8i$	$-4.63 \pm 0.07 + 92.13i \pm 0.07i$
Integration Time	45540s	19s

Tools for sector decomposition

- FIESTA
  https://github.com/fiestaIntegrator/fiesta
- pySecDec
  https://github.com/gudrunhe/secdec
- SeDe.jl
  In preparation

Czakon's subtraction at NNLO

arXiv:1005.0274

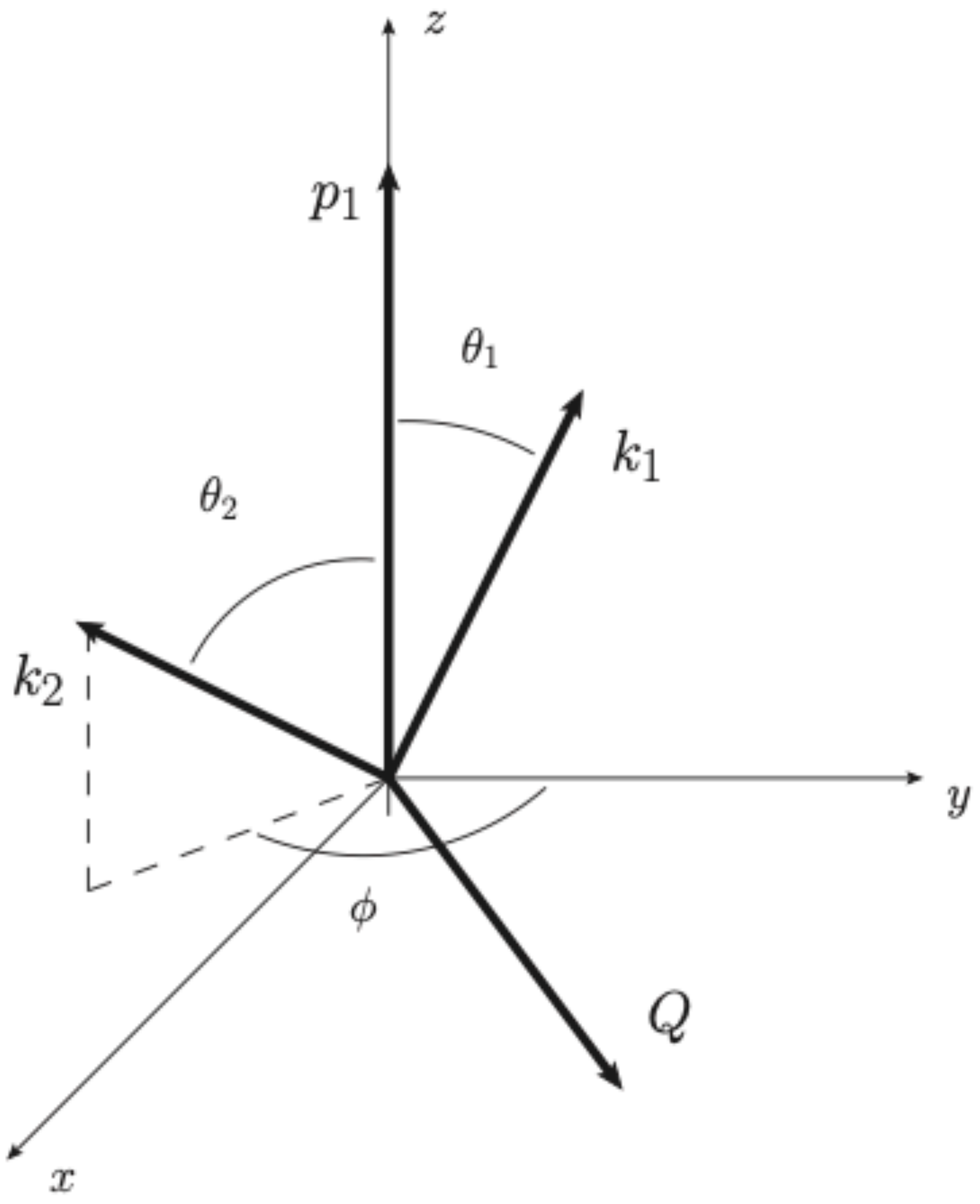
$$
\begin{aligned}
\int d\Phi_{n+2} &= \int \frac{d^{d-1}k_1}{(2\pi)^{d-1}2k_1^0}\frac{d^{d-1}k_2}{(2\pi)^{d-1}2k_2^0} \\
&\times \int \prod_{i=1}^{n} \frac{d^{d-1}q_i}{(2\pi)^{d-1}2q_i^0}(2\pi)^d\delta^{(d)}(q_1+\cdots+q_n-Q) \\
&\equiv \int d\Phi_3 \int d\Phi_n(Q)\,,
\end{aligned}
$$

$$Q = p_1 + p_2 - k_1 - k_2\,.$$

$$
\begin{aligned}
1 \;=\; & \\
&\left.\begin{array}{l} +\theta_1(k_1)\theta_1(k_2) \\ +\theta_2(k_1)\theta_2(k_2) \end{array}\right\}\text{triple-collinear sector} \\
&\left.\begin{array}{l} +\theta_1(k_1)\theta_2(k_2)(1-\theta_3(k_1,k_2)) \\ +\theta_2(k_1)\theta_1(k_2)(1-\theta_3(k_1,k_2)) \end{array}\right\}\text{double-collinear sector} \\
&+(\theta_1(k_1)\theta_2(k_2)+\theta_2(k_1)\theta_1(k_2))\theta_3(k_1,k_2)\;\Big\}\,\text{single-collinear sector}\,.
\end{aligned}
$$

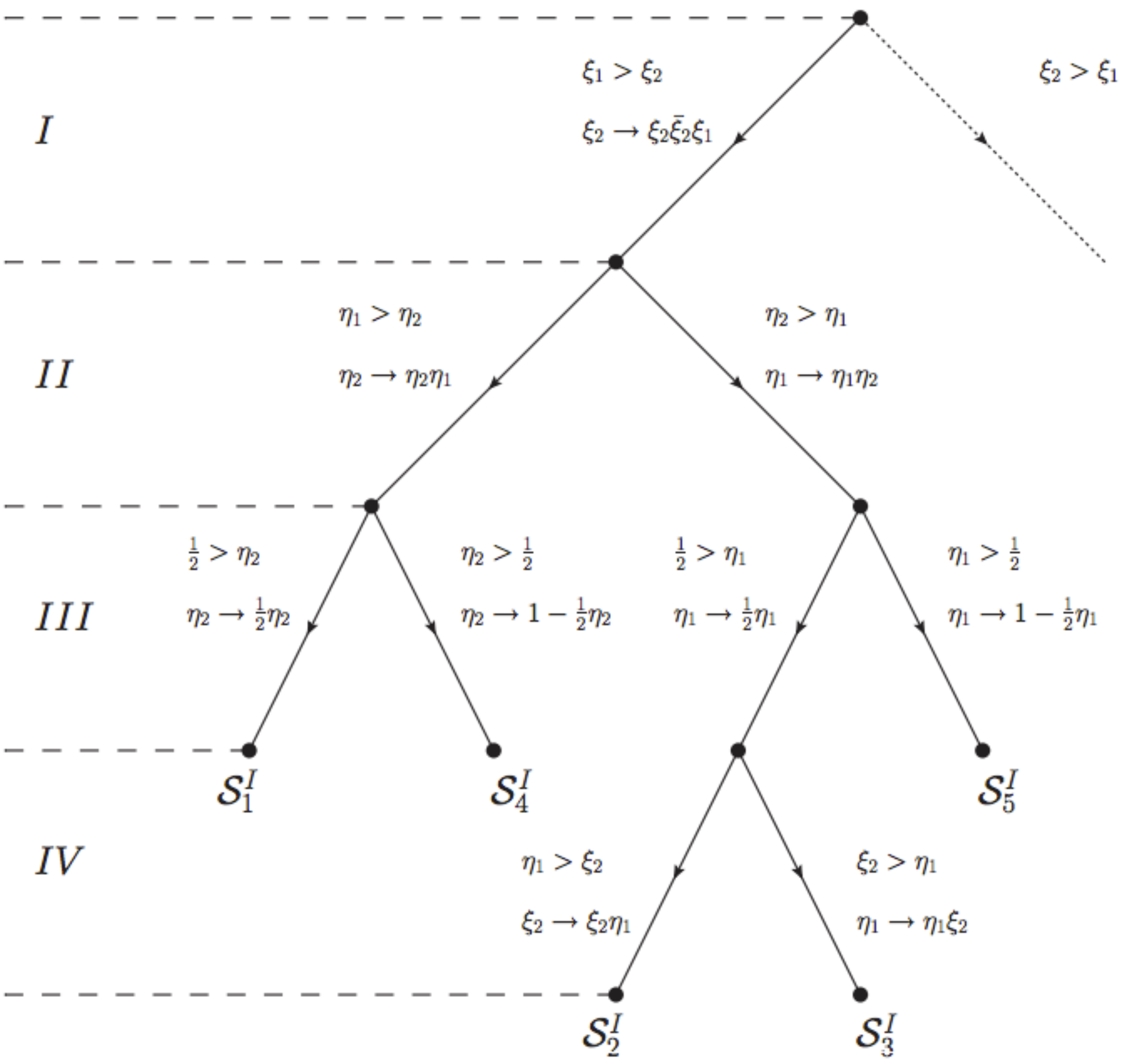
$$\theta_1(k) = \theta(\vec{k}\cdot\vec{p}_1)$$

$$
\begin{aligned}
p_1^\mu &= \frac{\sqrt{s}}{2}(1, \vec{0}^{(d-2)}, 1)\ , \\
k_1^\mu &= \frac{\sqrt{s}}{2}\beta^2\xi_1(1, \vec{0}^{(d-3)}, \sin\theta_1, \cos\theta_1)\ , \\
k_2^\mu &= \frac{\sqrt{s}}{2}\beta^2\xi_2(1, \vec{0}^{(d-4)}, \sin\phi\sin\theta_2, \cos\phi\sin\theta_2, \cos\theta_2)\ .
\end{aligned}
$$



$$
\begin{aligned}
\int d\Phi_3 &= \int \frac{d^{d-1}k_1}{(2\pi)^{d-1}2k_1^0}\frac{d^{d-1}k_2}{(2\pi)^{d-1}2k_2^0} \\
&= \frac{1}{8(2\pi)^{5-2\epsilon}\Gamma(1-2\epsilon)} s^{2-2\epsilon}\beta^{8-8\epsilon} \\
&\times \int_0^1 d\eta_1\ (\eta_1(1-\eta_1))^{-\epsilon}\int_0^1 d\eta_2\ (\eta_2(1-\eta_2))^{-\epsilon}\int_{-1}^1 d\cos\phi\ (1-\cos^2\phi)^{-\frac{1}{2}-\epsilon} \\
&\quad \iint d\xi_1 d\xi_2\ \xi_1^{1-2\epsilon}\xi_2^{1-2\epsilon}\ ,
\end{aligned}
$$

$$
\eta_{1,2} = \frac{1}{2}(1-\cos\theta_{1,2})\ .
$$



I
II
III
IV
$\xi_1 > \xi_2$
$\xi_2 \to \xi_2\bar{\xi}_2\xi_1$
$\xi_2 > \xi_1$
$\eta_1 > \eta_2$
$\eta_2 \to \eta_2\eta_1$
$\eta_2 > \eta_1$
$\eta_1 \to \eta_1\eta_2$
$\frac{1}{2} > \eta_2$
$\eta_2 \to \frac{1}{2}\eta_2$
$\eta_2 > \frac{1}{2}$
$\eta_2 \to 1 - \frac{1}{2}\eta_2$
$\frac{1}{2} > \eta_1$
$\eta_1 \to \frac{1}{2}\eta_1$
$\eta_1 > \frac{1}{2}$
$\eta_1 \to 1 - \frac{1}{2}\eta_1$
$\mathcal{S}_1^I$
$\mathcal{S}_4^I$
$\mathcal{S}_5^I$
$\eta_1 > \xi_2$
$\xi_2 \to \xi_2\eta_1$
$\xi_2 > \eta_1$
$\eta_1 \to \eta_1\xi_2$
$\mathcal{S}_2^I$
$\mathcal{S}_3^I$

$$\begin{aligned}\sigma_{\mathcal{O}}^{(\mathcal{S})} &= \frac{1}{2s}\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{3\epsilon}\int d\mu_\zeta\, d\eta_1\, d\eta_2\, d\xi_1\, d\xi_2\, d\Phi_2\, \mu_{\mathcal{S}}^{\rm reg}\, \theta_{\mathcal{S}}\, F_J\, \frac{1}{\eta_1^{1-b_1\epsilon}}\frac{1}{\eta_2^{1-b_2\epsilon}}\frac{1}{\xi_1^{1-b_3\epsilon}}\frac{1}{\xi_2^{1-b_4\epsilon}}\,\mathfrak{M}_{\mathcal{S}} \\ &= \int d\zeta\, d\eta_1\, d\eta_2\, d\xi_1\, d\xi_2\, d\cos\theta_Q\, d\phi_Q\, d\cos\rho_Q\, \Sigma_{\mathcal{O}}^{(\mathcal{S})}\,, \end{aligned} \tag{59}$$

$$\frac{1}{\lambda^{1-b\epsilon}} = \frac{1}{b}\frac{\delta(\lambda)}{\epsilon} + \sum_{n=0}^{\infty}\frac{(b\epsilon)^n}{n!}\left[\frac{\ln^n(\lambda)}{\lambda}\right]_+ ,$$

$$\lim_{\mathbf{X}\to\mathbf{0}}\mathfrak{M}_{\mathcal{S}} = g^2\,\langle\mathcal{M}_3|\mathbf{V}|\mathcal{M}_3\rangle \quad \text{or} \quad \lim_{\mathbf{X}\to\mathbf{0}}\mathfrak{M}_{\mathcal{S}} = g^4\,\langle\mathcal{M}_2|\mathbf{V}|\mathcal{M}_2\rangle\,,$$

$$\mathbf{V}_{a_1a_5a_6}^{ss'} = \lim_{\mathbf{X}\to\mathbf{0}}\mathfrak{R}_{\mathcal{S}}\frac{4\hat{P}_{a_1a_5a_6}^{ss'}}{s_{156}^2}\,.$$

Thank you!