圈积分与相空间积分系列

FIRE6不完全指南

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2020年12月10日

Quick Review

❖ Consider L loops $(E+1)$ points diagram

- Feynman Integral

$$F[\alpha_1, \alpha_2, \ldots, \alpha_n] = \int \prod_j^L \frac{d^d l_j}{i\pi^{d/2}} \frac{1}{D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}}, \quad \alpha_i \in \mathbb{Z}, \quad D_i = q_i^2 - m_i^2 [+i\eta]$$

- Sector

$$\mathcal{S}(F[\alpha_1, \alpha_2, \ldots, \alpha_n]) = (s_1, s_2, \ldots, s_n) \qquad s_i = \begin{cases} +1 & \alpha_i > 0 \\ -1 & \alpha_i \leq 0 \quad [0 \text{ in Kira}] \end{cases}$$

- Complexity

$$N_+ = \sum_{\alpha_i > 0} \alpha_i - 1 \quad [\alpha_i \text{ in Kira}], \quad N_- = \sum_{\alpha_i < 0} -\alpha_i$$

- Ordering

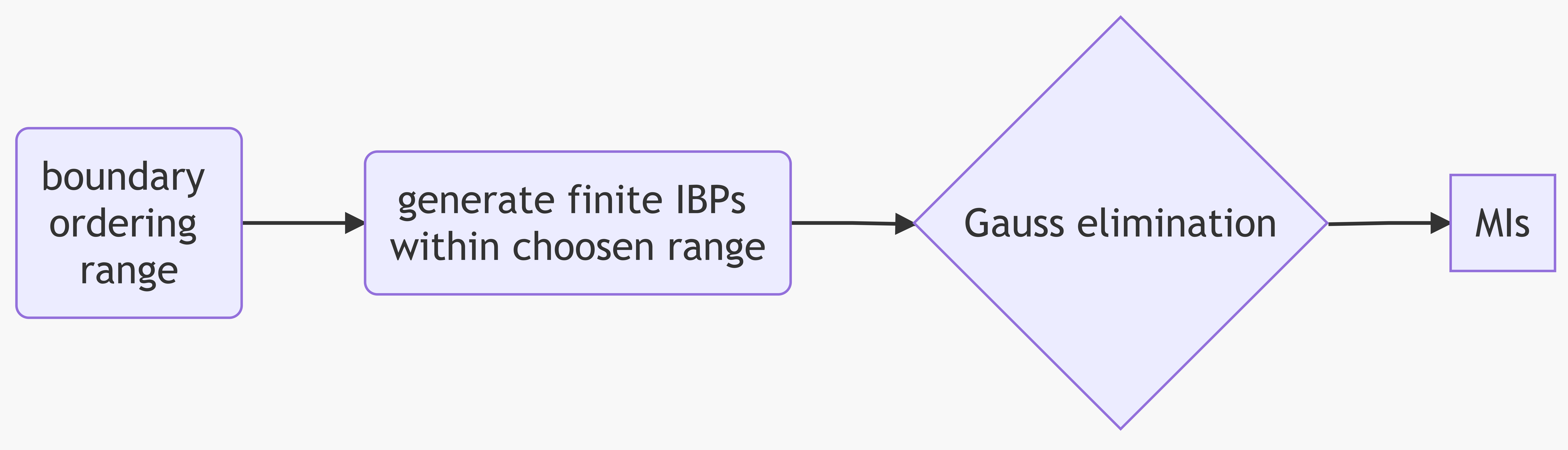
$$F[\alpha_1, \alpha_2, \ldots, \alpha_n]) \succ F[\beta_1, \beta_2, \ldots, \beta_n] \quad \Leftrightarrow \quad \mathcal{S}^{(\alpha)} \succ \mathcal{S}^{(\beta)} \qquad \text{e.g. Laporta Ordering}$$

- "Primary" $L(L+E)$ IBPs

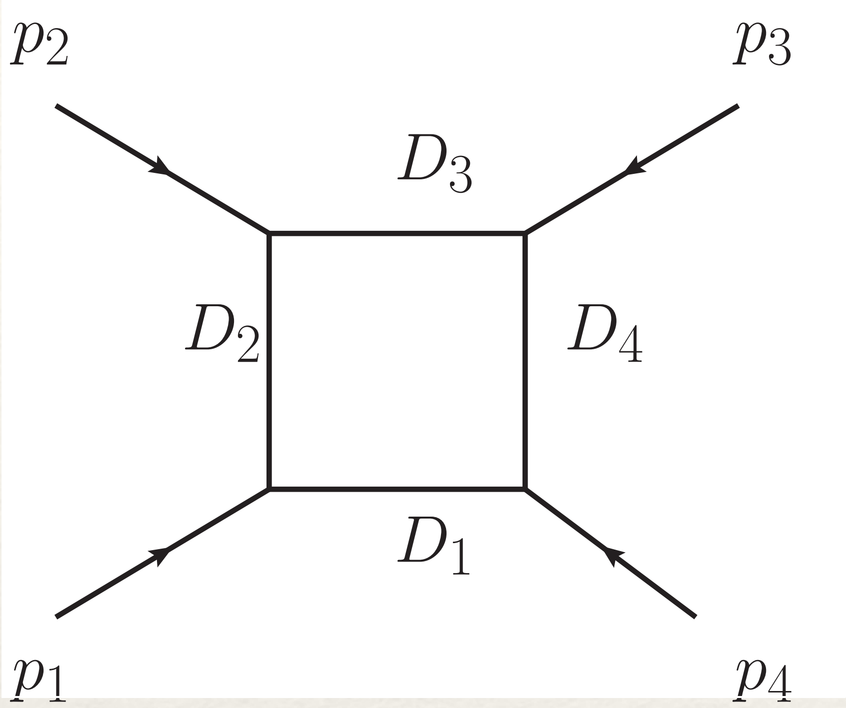
$$0 = \int \prod_j^L \frac{d^d l_j}{i\pi^{d/2}} \sum_{r=1}^{L} \sum_{k=1}^{L+E} \frac{\partial}{\partial l_r^\mu} \frac{v_k^\mu}{D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}}, \quad v \in \{l_1, \ldots, l_L, p_1, \ldots, p_E\}$$

❖ Consider L loops $(E+1)$ points diagram

- Laporta Algorithm(VERY brief)



❖ Example: One loop massless box



$$D_1 = l^2$$
$$D_2 = (l+p_1)^2$$
$$D_3 = (l+p_{12})^2 = (l+p_1+p_2)^2$$
$$D_4 = (l+p_{123})^2 = (l+p_1+p_2+p_3)^2$$
$$p_i^2 = 0,\ p_{12}^2 = s,\ \ p_{23}^2 = t$$

$$B_1 = l^\mu \frac{\partial D_1}{\partial l^\mu} = 2l^2 = 2D_1$$
$$B_2 = l^\mu \frac{\partial D_2}{\partial l^\mu} = 2l^2 + 2lp_1 = D_2 + D_1$$
$$B_3 = l^\mu \frac{\partial D_3}{\partial l^\mu} = 2l^2 + 2lp_{12} = D_3 + D_1 - s$$
$$B_4 = l^\mu \frac{\partial D_4}{\partial l^\mu} = 2l^2 + 2lp_{123} = D_4 + D_1$$

$$\frac{\partial}{\partial l^\mu}\frac{l^\mu}{D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}D_4^{\alpha_4}} = \frac{d}{D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}D_4^{\alpha_4}} - \frac{\alpha_1 B_1}{D_1^{\alpha_1+1}D_2^{\alpha_2}D_3^{\alpha_3}D_4^{\alpha_4}} - \frac{\alpha_2 B_2}{D_1^{\alpha_1}D_2^{\alpha_2+1}D_3^{\alpha_3}D_4^{\alpha_4}} - \frac{\alpha_3 B_3}{D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3+1}D_4^{\alpha_4}} - \frac{\alpha_4 B_4}{D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}D_4^{\alpha_4+1}}$$

$$\frac{\partial}{\partial l^\mu}\frac{l^\mu}{D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}D_4^{\alpha_4}} = \frac{d-2\alpha_1-\alpha_2-\alpha_3-\alpha_4}{D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}D_4^{\alpha_4}} - \frac{\alpha_2}{D_1^{\alpha_1-1}D_2^{\alpha_2+1}D_3^{\alpha_3}D_4^{\alpha_4}} - \frac{\alpha_3}{D_1^{\alpha_1-1}D_2^{\alpha_2}D_3^{\alpha_3+1}D_4^{\alpha_4}} - \frac{\alpha_4}{D_1^{\alpha_1-1}D_2^{\alpha_2}D_3^{\alpha_3}D_4^{\alpha_4+1}} + \frac{\alpha_3 s}{D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3+1}D_4^{\alpha_4}}$$
$$= [(d-2\alpha_1-\alpha_2-\alpha_3-\alpha_4) - \alpha_2 \mathbf{1}^-\mathbf{2}^+ - \alpha_3\mathbf{1}^-\mathbf{3}^+ - \alpha_4\mathbf{1}^-\mathbf{4}^+ + \alpha_3 s\mathbf{3}^+]\frac{1}{D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}D_4^{\alpha_4}}$$

❖ Example: One loop massless box

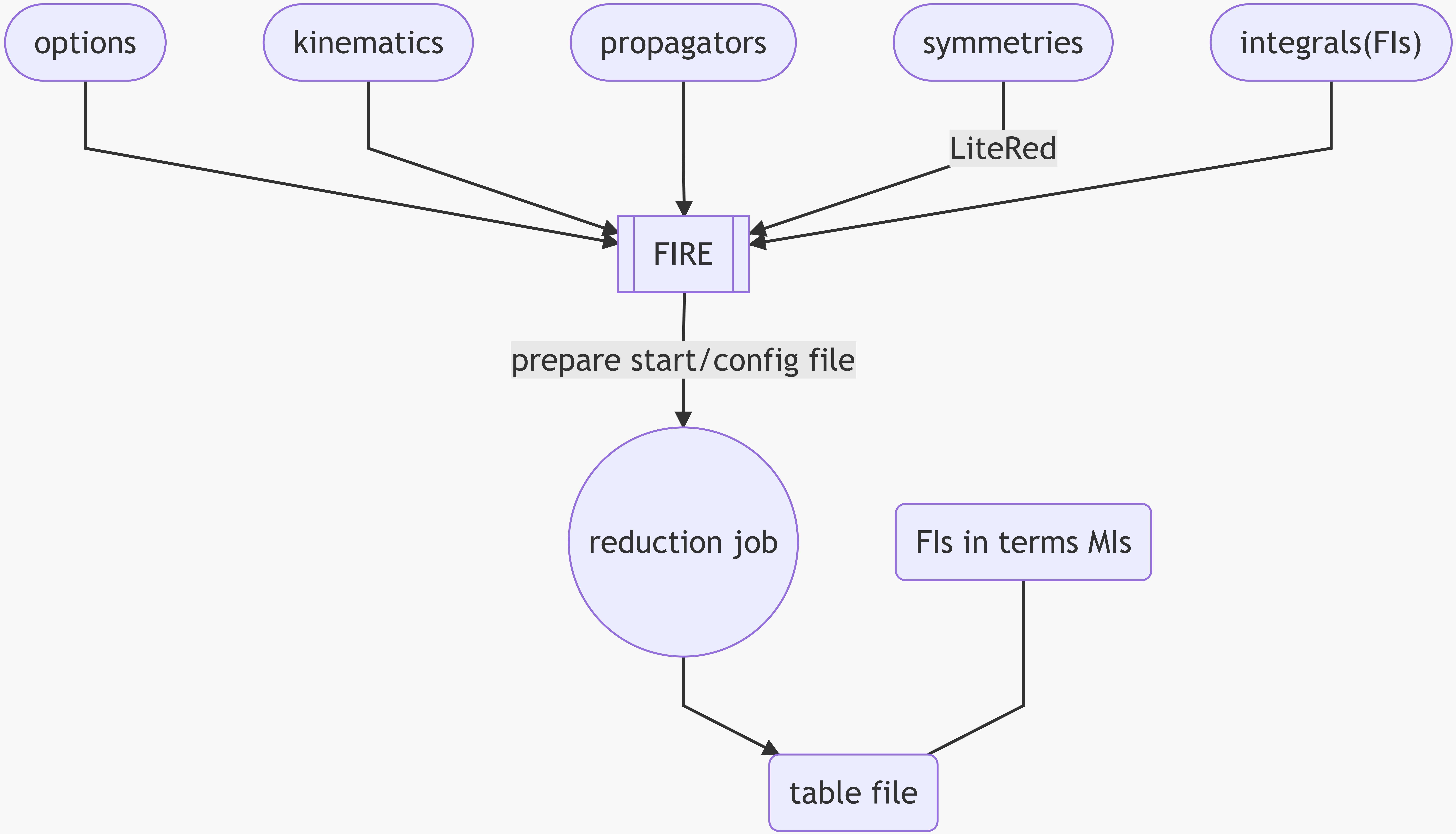
4 primary IBPs:

$$\begin{cases} 0 = [(d - 2\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) - \alpha_2 \mathbf{1}^-\mathbf{2}^+ - \alpha_3 \mathbf{1}^-\mathbf{3}^+ - \alpha_4 \mathbf{1}^-\mathbf{4}^+ + \alpha_3 s \mathbf{3}^+] \circ F[\alpha_1, \alpha_2, \alpha_3, \alpha_4] \\ 0 = [(\alpha_1 - \alpha_2) - \alpha_1 \mathbf{1}^+\mathbf{2}^- + \alpha_2 \mathbf{1}^-\mathbf{2}^+ + \alpha_3 \mathbf{1}^-\mathbf{3}^+ - \alpha_3 \mathbf{2}^-\mathbf{3}^+ + \alpha_4 \mathbf{1}^-\mathbf{4}^+ - \alpha_4 \mathbf{2}^-\mathbf{4}^+ - \alpha_3 s \mathbf{3}^+ + \alpha_4 t \mathbf{4}^+] \circ F[\alpha_1, \alpha_2, \alpha_3, \alpha_4] \\ 0 = [(\alpha_2 - \alpha_3) + \alpha_1 \mathbf{2}^-\mathbf{1}^+ - \alpha_1 \mathbf{3}^-\mathbf{1}^+ - \alpha_2 \mathbf{3}^-\mathbf{2}^+ + \alpha_3 \mathbf{2}^-\mathbf{3}^+ + \alpha_4 \mathbf{2}^-\mathbf{4}^+ - \alpha_4 \mathbf{3}^-\mathbf{4}^+ + \alpha_1 s \mathbf{1}^+ - \alpha_4 t \mathbf{4}^+] \circ F[\alpha_1, \alpha_2, \alpha_3, \alpha_4] \\ 0 = [(\alpha_3 - \alpha_4) + \alpha_1 \mathbf{3}^-\mathbf{1}^+ - \alpha_1 \mathbf{4}^-\mathbf{1}^+ + \alpha_2 \mathbf{3}^-\mathbf{2}^+ - \alpha_2 \mathbf{4}^-\mathbf{2}^+ - \alpha_3 \mathbf{4}^-\mathbf{3}^+ + \alpha_4 \mathbf{3}^-\mathbf{4}^+ - \alpha_1 s \mathbf{1}^+ + \alpha_2 t \mathbf{2}^+] \circ F[\alpha_1, \alpha_2, \alpha_3, \alpha_4] \end{cases}$$

- Stored in ‘start’ file of FIRE

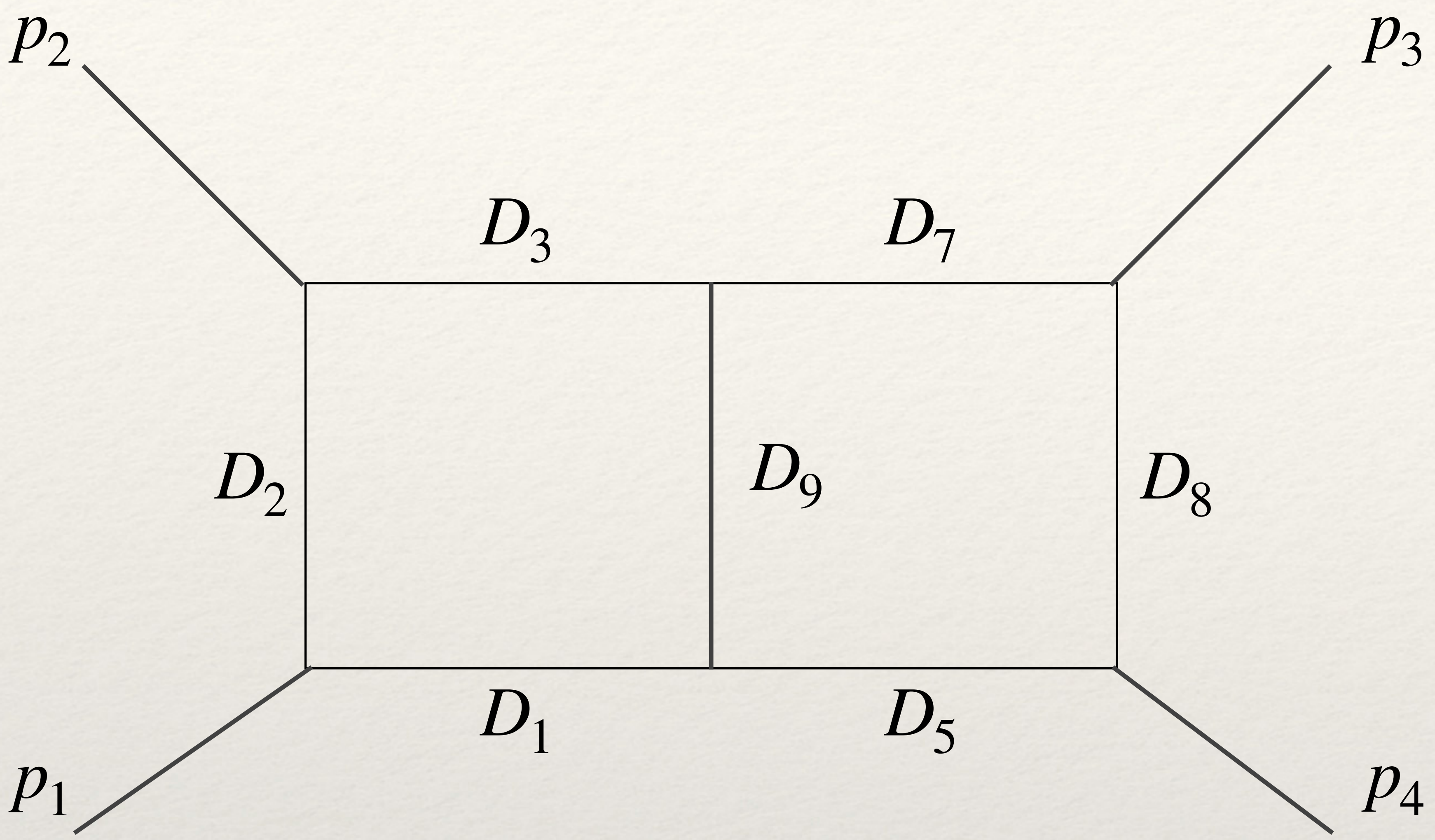
FIRE6::usage

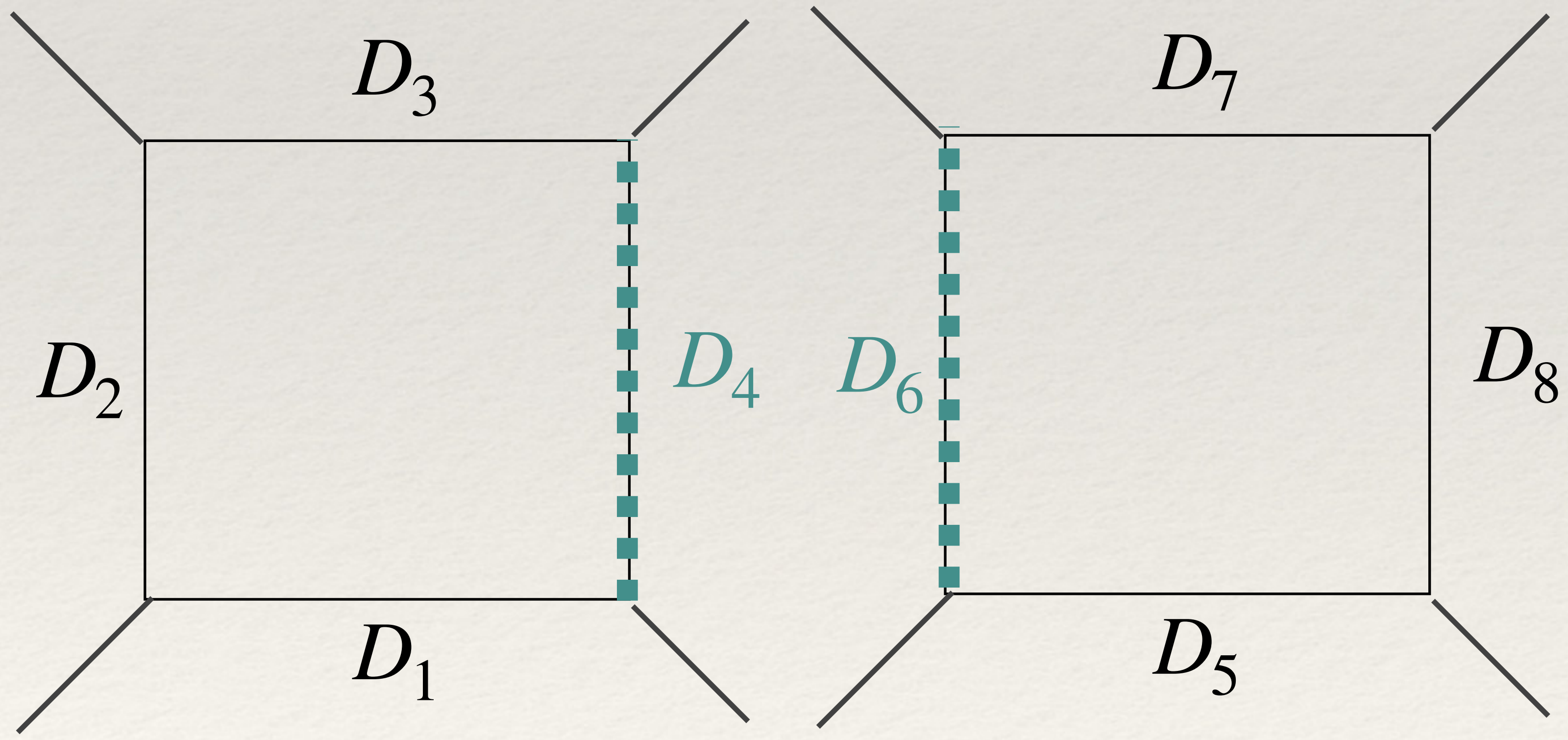
Input & Output



IBP in Action #1

- Massless double box-Convention





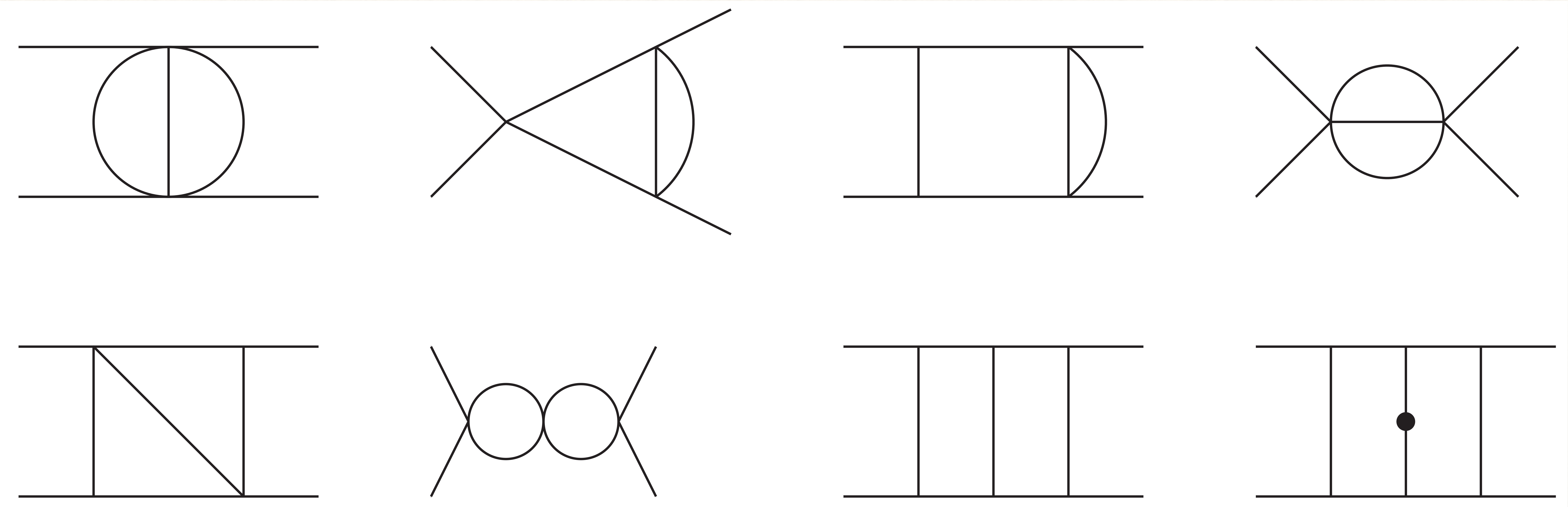
$$\sum_i^4 p_i = 0,\ s = (p_1 + p_2)^2,\ t = (p_2 + p_3)^2$$

$$D_1 = l_1^2,$$
$$D_2 = (l_1 + p_1)^2,$$
$$D_3 = (l_1 + p_{12})^2,$$
$$D_4 = (l_1 + p_{123})^2,$$
$$D_5 = l_2^2,$$
$$D_6 = (l_2 + p_1)^2,$$
$$D_7 = (l_2 + p_{12})^2,$$
$$D_8 = (l_2 + p_{123})^2,$$
$$D_9 = (l_1 - l_2)^2$$

Switch to shell

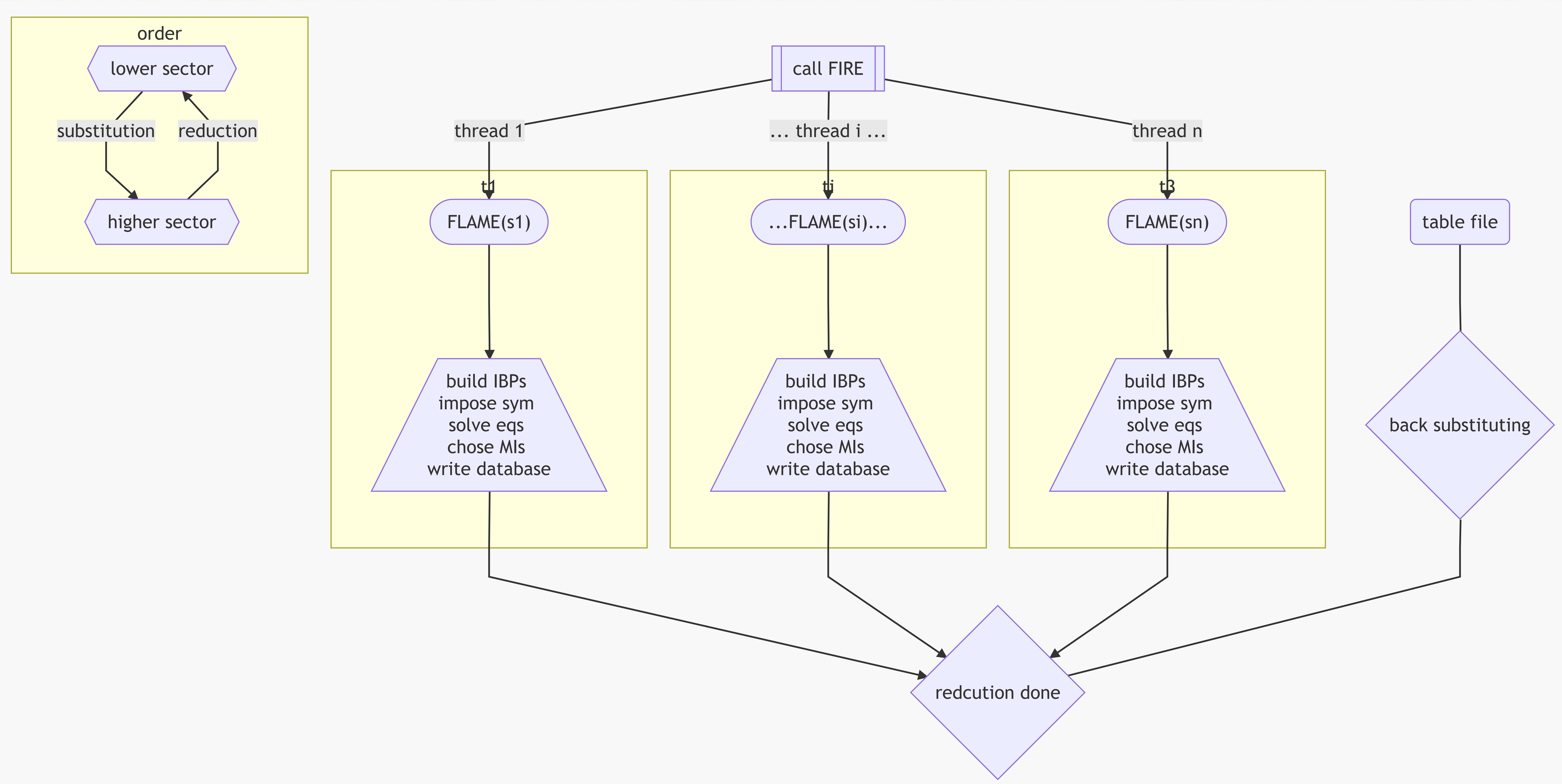
IBP in Action #1

- Massless double box-MIs



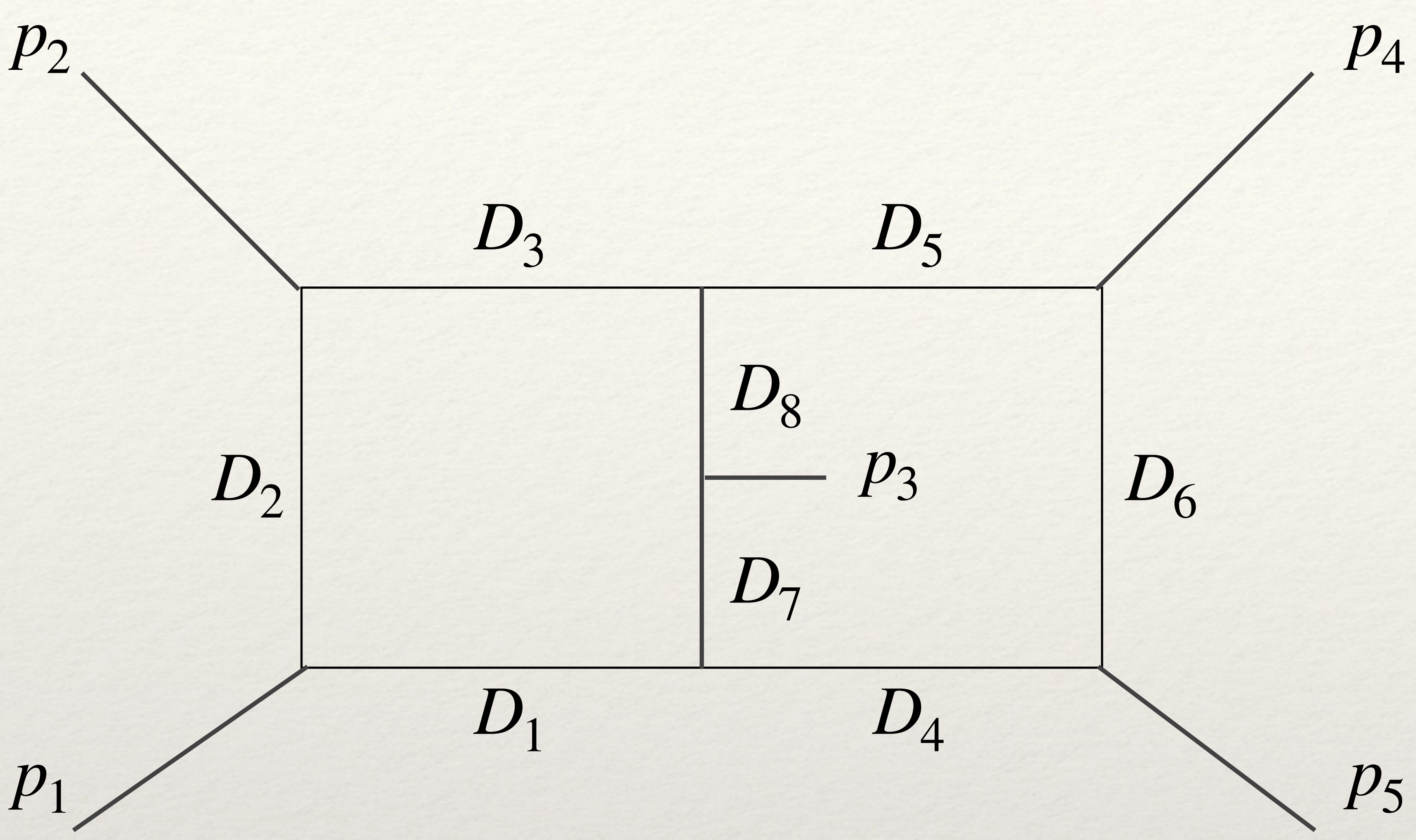
$$\left\{ \begin{array}{l} F[0,0,0,1,0,1,0,0,1], F[0,0,0,1,1,0,1,0,1], F[0,0,0,1,1,1,1,0,1], F[0,0,1,0,1,0,0,0,1] \\ F[0,0,1,1,1,1,0,0,1], F[1,0,1,0,1,0,1,0,0], F[1,0,1,1,1,1,1,0,1], F[1,0,1,1,1,1,1,0,2] \end{array} \right.$$

Sketch of frame



IBP in Action #2

- Massless double pentagon-Convention



$$\sum_i^5 p_i = 0, \quad p_1p_2 = s_{12}/2,$$

$$p_2p_3 = s_{23}/2, \; p_1p_5 = s_{15}/2,$$

$$p_3p_5 = (s_{12} - s_{34} - s_{45})/2,$$

$$p_1p_3 = (s_{45} - s_{12} - s_{23})/2,$$

$$p_2p_5 = (s_{34} - s_{12} - s_{15})/2.$$

$$D_1 = l_1^2,$$

$$D_2 = (l_1 - p_1)^2,$$

$$D_3 = (l_1 - p_{12})^2,$$

$$D_4 = l_2^2,$$

$$D_5 = (l_2 - p_{123})^2,$$

$$D_6 = (l_2 + p_5)^2,$$

$$D_7 = (l_1 - l_2)^2,$$

$$D_8 = (l_1 - l_2 + p_3)^2,$$

$$D_9 = (l_1 + p_5)^2,$$

$$D_{10} = (l_2 - p_1)^2,$$

$$D_{11} = (l_2 - p_{12})^2$$

Switch to shell

IBP in Action #2

- Massless double pentagon-MIs of top sector

$$\{1,1,1,1,1,1,1,1,0,0,0\} \quad \{1,1,1,1,1,1,1,1,0,0,-1\} \quad \{1,1,1,1,1,1,1,1,0,-1,0\}$$
$$\{1,1,1,1,1,1,1,1,-1,0,0\} \quad \{1,1,1,1,1,1,1,2,0,0,0\} \quad \{1,1,1,1,1,1,2,1,0,0,0\}$$
$$\{1,1,1,1,1,2,1,1,0,0,0\} \quad \{1,1,1,1,2,1,1,1,0,0,0\} \quad \{1,1,1,2,1,1,1,1,0,0,0\}$$

- Trick: Impose restrictions by hand to focus on one(or several) sector(s), i.e. to partition reduction jobs.
- Helpful especially while resources are limited.
- Sometimes might make infeasible tasks possible.
- **Q:** Do we lose any information about this sector?

Start file

- Box

$$[(d-2\alpha_1-\alpha_2-\alpha_3-\alpha_4)-\alpha_2\mathbf{1}^-\mathbf{2}^+-\alpha_3\mathbf{1}^-\mathbf{3}^+-\alpha_4\mathbf{1}^-\mathbf{4}^++\alpha_3 s\mathbf{3}^+]$$

$$-\alpha_4\mathbf{1}^-\mathbf{4}^+$$

$$-\alpha_3\mathbf{1}^-\mathbf{3}^+$$

$$-\alpha_2\mathbf{1}^-\mathbf{2}^+$$

$$\{-2\alpha_1,\ -\alpha_2,\ -\alpha_3,\ -\alpha_4,\ d\}$$

$$+\alpha_3 s\mathbf{3}^+$$

$$\text{SBasis0D}[0,1]=\begin{pmatrix}-1&0&0&1\\-1&0&1&0\\-1&1&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}$$

$$\text{SBasis0C}[0,1\{-1,0,0,1\}]=\begin{pmatrix}-1&4\end{pmatrix}$$

$$\text{SBasis0C}[0,1\{-1,0,1,0\}]=\begin{pmatrix}-1&3\end{pmatrix}$$

$$\text{SBasis0C}[0,1\{-1,1,0,0\}]=\begin{pmatrix}-1&2\end{pmatrix}$$

$$\text{SBasis0C}[0,1\{0,0,0,0\}]=\begin{pmatrix}-2&1\\-1&2\\-1&3\\-1&4\\d&0\end{pmatrix}$$

$$\text{SBasis0C}[0,1\{0,0,0,0\}]=\begin{pmatrix}s&3\end{pmatrix}$$

One more thing

- FIRE6 supports finite field method

FIRE6: Feynman Integral REduction with Modular Arithmetic

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Failed when trying to run the examples…

Thank you!