Humboldt-Universitat zu Berlin

Physics Department - PEP

A Canonical Introduction to Feynman Integral Reduction

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IBP Identities and Applications

- Integration-by-parts (IBP) to reduce the number of Feynman integrals to compute [Chetyrkin, Tkachov, 1981]

- Compute the integrals analytically or numerically with the method of differential equations [Kotikov, 1991; Remiddi, 1997; Henn, 2013]

- Use the method of sector decomposition [Heinrich,2008] (pySecDec [Borowka et al., 2018] and Fiesta4 [Smirnov, 2016]) to compute the Feynman integrals numerically.

Feynman Integrals and Sectors

Feynman integral

$$\int \prod_{j=1}^{L} \frac{d^D l_j}{i\pi^{D/2}} \frac{Poly(l_a \cdot l_b, l_c \cdot k_d)}{\prod_{i=1}^{n} D_i^{\alpha_i}}$$

Sector

$(s_1, \dots, s_n)$, where $s_i = \begin{cases} 1 & n_i > 0 \\ 0 & n_i \leq 0 \end{cases}$

$$S^{(1)} = (s_1^{(1)}, \dots, s_n^{(1)}) \geq S^{(2)} = (s_1^{(2)}, \dots, s_n^{(2)})$$

$$\Leftrightarrow \quad s_i^{(1)} \geq s_i^{(2)}, \forall\, i = 1, \cdots n$$

IBP Identities and Master Integrals

The IBP identities

$$0 = \int \prod_{j=1}^{L} \frac{d^D l_j}{i\pi^{D/2}} \sum_{k=1}^{L} \frac{\partial}{\partial l_k^\mu} \frac{v_k^\mu Poly}{\prod_{i=1}^{n} D_i^{\alpha_i}}$$

$L(E + L)$ IBP equations, since $v_k^\mu \in \mathrm{Span}\left\{l_i^\mu, k_j^\mu\right\}$.

- Express all integrals with the same set of propagators but with different exponents as a linear combination of some basis integrals (**master integrals**)

- Master integrals are finite [Smirnov, 2010]

Laporta Algorithm

- **Consider finitely many IBP identities**
- Boundary conditions to sample the IBP equations

$$r = \sum_{i=1}^{n} \alpha_i \text{ with } \alpha_i > 0\,, \quad s = -\sum_{i=1}^{n} \alpha_i \text{ with } \alpha_i < 0$$

Seed integrals: $r \in [r_{min}, r_{max}],\ \ s \in [s_{min}, s_{max}]$.

- Generate and reduce only a chosen set of integrals to a fixed number of basis integrals
- Public implementations: Air [Lazopoulos, Anastasiou, 2004], FIRE [A. V. Smirnov et al., 2008, 2013, 2014, 2019] Reduze [Studerus, 2010] and Reduze 2 [Studerus, von Manteuffel, 2012] and Kira [Maierhöfer, Usovitsch, Uwer, 2017]

Laporta Ordering

For a given set of Feynman integrals:

1) Count the number of positive power of propagators, if equal, go to 2;
2) Compute abs. sum of powers of propagators, that is $r + s$, if equal, go to 3;
3) Count zero propagators, choose lowest, if equal go to 4;
4) Choose integral with highest power on propagator $D_n, \cdots, D_1$.

Laporta Ordering

Example

a,b,c,d,e represent $D_1, \cdots, D_5$

Rule	$I_{1,1,1,1,1}$	$I_{1,1,2,0,1}$	$I_{1,1,1,-1,1}$	$I_{1,1,1,2,1}$	$I_{2,1,1,1,1}$
1	5	4	4	5	5
2	5	5	5	6	6
3	0	1	0	0	0
4	-	-	-	d ✓	a

Laporta Algorithm

Procedure:

1. Fix boundary r, s and generate finitely many IBP identities;
2. Find the order relation of seed integrals;
3. Gaussian elimination and get the master integrals.

Because of the finiteness of master integrals, if r, s are large enough, the result will be stable.

Reduce Sector-by-Sector

- Choose a specific sector;
- Fix boundary r, s and generate finitely many IBP identities only in this sector;
- Find the order relation of seed integrals in this sector;
- Find the master integrals of this sector.

S-bases

- **A novel approach on IBP reduction**
- **Groebner-type bases**

Define

$$(A_i \cdot F)(a_1, \cdots, a_n) = a_i F(a_1, \cdots, a_n),$$

$$\left(Y_i^{\mp} \cdot F\right)(a_1, \cdots, a_n) = F(a_1, \cdots a_i \mp 1, \cdots, a_n).$$

Ring $A_{1,\cdots n} \equiv k[A_i, Y_i^{+}]$, $f_i \in A_{1,\cdots n}$, f_i correspond to IBP relations

Ideal $I = \{f_i\}$ $\quad (f_i \cdot F)(a_1, \cdots, a_n) = 0$

$$F(a_1, \cdots, a_n) = (Y_1^{a_1-1} \cdots Y_1^{a_n-1} F)(1, \cdots, 1)$$

Reduce $Y_1^{a_1-1} \cdots Y_1^{a_n-1}$, get $Y_1^{a_1-1} \cdots Y_1^{a_n-1} = \sum r_i f_i + \sum c_{i_{1,\cdots} i_n} Y_1^{i_1-1} \cdots Y_1^{i_n-1}$

Apply to F at $a_1 = \cdots = a_n = 1$, $F(a_1, \cdots, a_n) = \sum c_{i_{1,\cdots} i_n} F(i_1, \cdots i_n)$.

Software: LiteRed

- mathematica package
- Algebraic Structure of IBP relations

$$O_{ij} = \frac{\partial}{\partial l_i} k_j$$

$$[O_{ik}, O_{jl}] = \delta_{il} O_{jk} - \delta_{jk} O_{il}$$

- User Friendly
- LiteRed 1.4: arxiv: 1310.1145

Software: LiteRed

```
<<LiteRed`;
```

```
SetDim[d];(*d stands for the dimensionality*)
```

```
Declare[{l1,p1,p2,p4},Vector];
sp[p1,p1]=0;sp[p2,p2]=0;sp[p4,p4]=0;
sp[p1,p2]=1/2 s;sp[p1,p4]=1/2 t;sp[p2,p4]=-1/2 s-1/2 t;
```

Defining the basis and searching for the symmetries & reduction rules

```
(*To get master integrals by IBP relations*)
```

```
NewBasis[onebox,{l1,l1-p1,l1-p1-p2,l1+p4},{l1},Directory->"onebox dir"];
(*Basis definition.*)
GenerateIBP[onebox]
```

```
AnalyzeSectors[onebox]
```

```
FindSymmetries[onebox]
```

```
SolvejSector/@UniqueSectors[onebox]
```

```
DiskSave[onebox];
```

```
IBPReduce[j[onebox,1,1,0,1]]
```

```
MIlist=MIs[onebox](*The master integral*)
```

Software: FIRE

Installation:

- git clone https://bitbucket.org/feynmanIntegrals/fire.git
- cd fire/FIRE6
- ./configure
- Now read the options provided by ./configure and reconfigure with desired options, for example
- ./configure --enable_zlib --enable_snappy --enable_lthreads --enable_tcmalloc --enable_zstd
- make dep
- make

Software: FIRE

- C++ parallel computational software
- mathematica interface
- Laporta Algorithm
- Linux system
- IBP reduction numerically/analytically
- IBP reduction sector-by-sector/entire sectors.
- Work together with LiteRed
- FIRE6: arxiv: 1901.07808

Software: FIRE

```
Get["FIRE6.m"];
Internal = {k1, k2};
External = {p1, p2, p3};
Propagators = {-k1^2, -(k1 + p1 + p2)^2, -k2^2, -(k2 + p1 +
p2)^2, -(k1 + p1)^2, -(k1 - k2)^2, -(k2 - p3)^2, -(k2 + p1)^2,
-(k1 - p3)^2};
Replacements = {p1^2 -> 0, p2^2 -> 0, p3^2 -> 0, p1 p2 -> s/2,
p1 p3 -> t/2, p2 p3 -> -1/2 (s + t)};
PrepareIBP[];
Prepare[];
SaveStart["doublebox"];
Quit[];
```

Optional terms:

- RESTRICTIONS: IBP reduction in a fixed sector
- SYMMETRIES: the symmetry of the diagram

Software: FIRE

- **Reduction in Mathematica**

```
Get["FIRE6.m"];
LoadStart["doublebox", 1];
Burn[]
```

Integral reduction: F[1, {1, 1, 1, 1, 1, 1, 1, -1, -1}]

Find master integrals: MasterIntegrals[]

- **IBP reduction in mathematica is slow.**

Software: FIRE

- ## C++ Reduction

  - Create integrallist file (with .m extension)
  - Create .config file
  - bin/FIRE6 -c examples/doublebox
  - (Optional) #threads number of kernels

```
#variables d, s, t
#start
#folder examples/
#problem 1 doublebox.start
#integrals doublebox.m
#output doublebox.tables
```

Improved Linartas' (PFD) algorithm

- coefficients of IBP reduction: rational functions of Mandelstam variables and Spacetime dimension
- huge size of coefficients is bad for analytical and numerical valuation
- reduce size: partial fraction decomposition (arxiv:2008.13194)

example	input	output	output (indexed)
◇ xbox1m (Laporta)	9.51 MB	2.91 MB (30.7 %)	2.75 MB (29.0 %)
xbox1m (UT)	9.04 MB	1.80 MB (19.9 %)	1.53 MB (16.9 %)
◇ double pentagon (Laporta)	2.42 GB	864 MB (35.7 %)	851 MB (35.2 %)
double pentagon (UT)	712 MB	28.0 MB (3.93 %)	19.8 MB (2.78 %)
◇ dbox elliptic (Laporta)	175 MB	24.1 MB (13.8 %)	23.3 MB (13.3 %)

New Development

- syzygy approach [D. A. Kosower et al.,2011; Y. Zhang et al.,2016, 2017]
- finite-field interpolation [A. von Manteuffel et al.,2015; T. Peraro, 2016, 2019; J. Klappert et al.,2019, 2020]
- module intersection [Y. Zhang et al.,2018, 2020]
- intersection theory [P. Mastrolia et al.,2019, 2020]
- auxilliary mass expansion [Y.-Q. Ma et al., 2018,2019; Z. Li et al., 2020]
- direct solution of IBP recursive relations [D. A. Kosower ,2018]
- …