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Introduction to differential equation method for Feynman integrals

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Outline

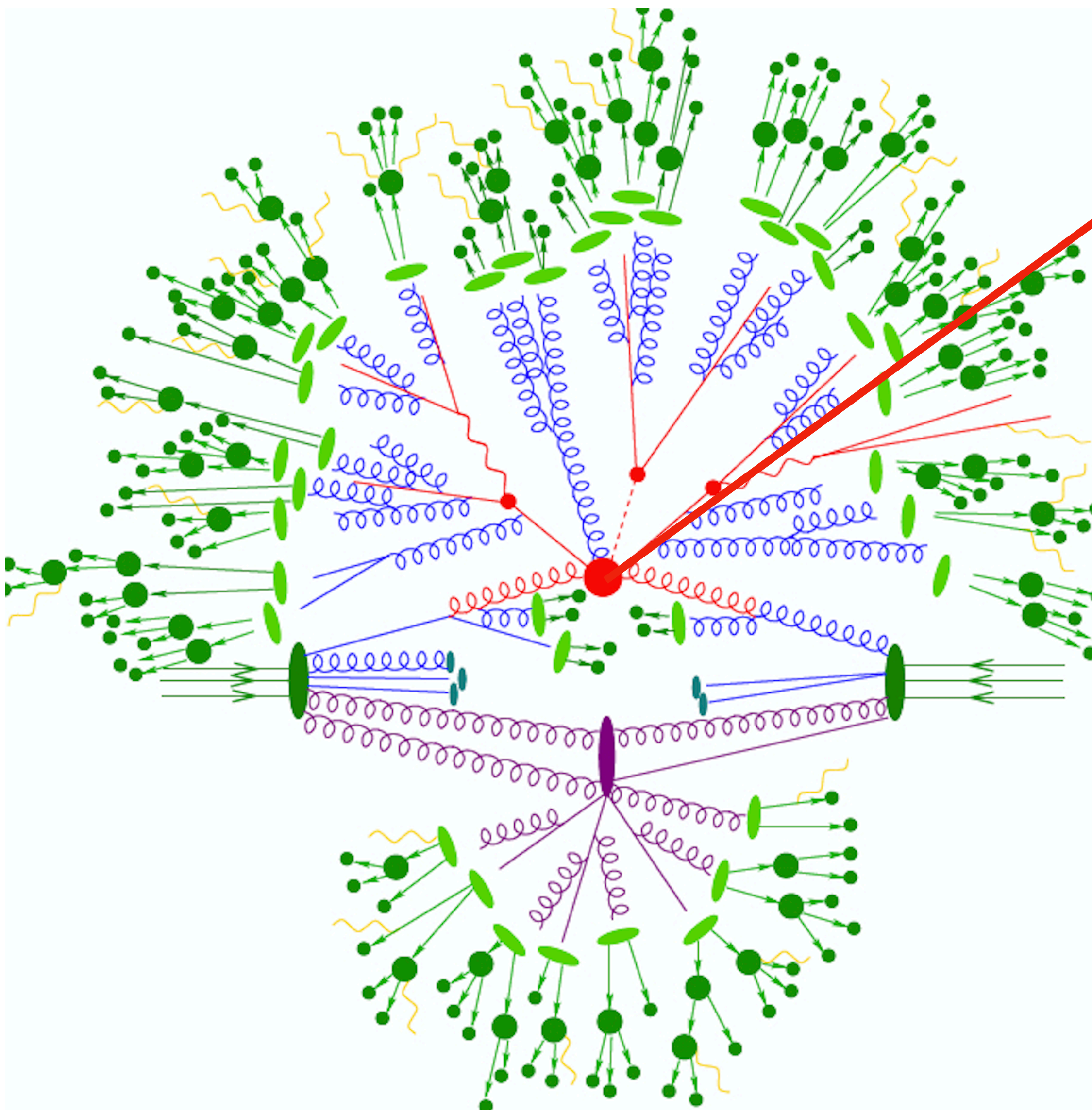
1. Introduction
2. General properties of Feynman integrals
3. Differential equation method
4. Constructing Canonical Feynman Integrals
5. Summary

Introduction

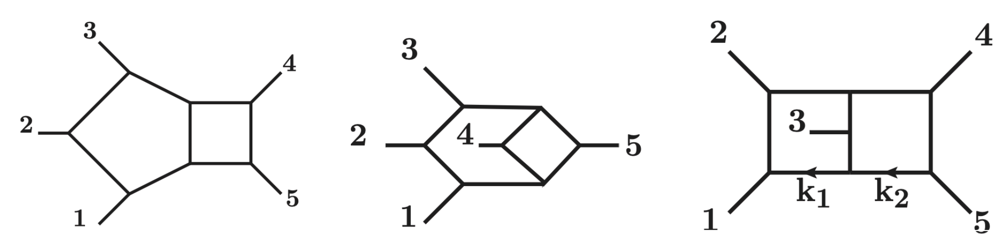
Proton-proton collision at LHC

Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, 1812.11160

Wang, Wang, X.Xu, Xu, Yang, 2010.15649



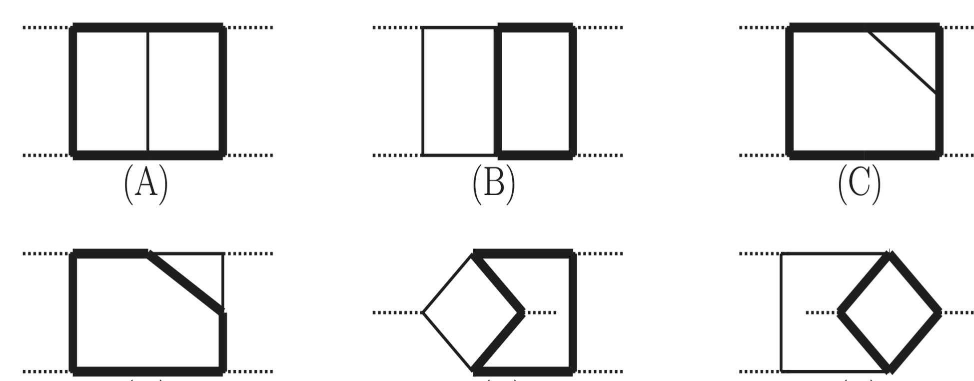
Precise predictions of hard scattering processes require higher order corrections



(a) penta-box

(b) hexa-box

(c) double-pentagon



(A) (B) (C)

(D) (E) (F)

General properties of Feynman integrals

- Feynman integral

General L-loop Feynman integral:

$$F_{a_1,\ldots,a_N} = \int \left[\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}} \right] \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_N^{a_N}}$$

Feynman parameterization representation:

$$F_{a_1,\ldots,a_N} = \frac{(-1)^{N_a}}{\prod_{i=1}^{N} \Gamma(a_i)} \Gamma(N_a - Ld/2)$$

$$\int_0^\infty \prod_{i=1}^{N} dx_i x_i^{a_i - 1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U^{N_a-(L+1)d/2}}{F^{N_a - Ld/2}}$$

Baikov representation:

$$F_{a_1,\ldots,a_N} = \mathcal{N}_\epsilon \int_{\mathcal{C}} \left[\prod_i \left[G_i(\mathbf{z}) \right]^{-\gamma_i - \beta_i \epsilon} \right] \prod_{j=1}^{n} \frac{dz_j}{z_j^{\alpha_j}}$$

"Gram determinant" $G(\mathbf{z}) \equiv \det(q_i \cdot q_j)$

General properties of Feynman integrals

ArXiv:1411.7538

Questions:

1. Can arbitrarily complicated functions appear, e.g., Trigonometric, exponential functions, and etc.
2. Can the arguments of these functions be arbitrarily complicated, e.g., $\log(\log(p^2))$.
3. The definition of Feynman integral involve numbers $e, \gamma_E, \log(\pi)$ why they do not appear in the results for the integral?

Related to "Transcendentality and Periods"

General properties of Feynman integrals

Transcendental and algebraic number:

Definition: a complex number is called an algebraic if it's a root of some polynomials with rational coefficients, otherwise, it's a transcendental number.

Theorem: Let z be a non-zero complex number, either z or e^z is transcendetal.

e , π, π^n, are all transcendental number. $\log(q)$ is transcendental if q is an algebraic number.

General properties of Feynman integrals

Transcendental and algebraic functions:

Definition: A function is algebraic if it is a root of a polynomial with coefficients that are rational functions in the variables, otherwise, it's a transcendental function.

Conjecture: All Multiple polylogarithms (MPLs) are transcendental functions, which is defined as

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

with $G(z) = G(; z) = 1$ and

$$G(\underbrace{0, \ldots, 0}_{n}; z) = \frac{1}{n!} \log^n(z)$$

General properties of Feynman integrals

Period number:

Definition: a complex number is a period if both its real and imaginary parts can be written as integrals of an algebraic function with algebraic coefficients over a domain defined by polynomial inequalities with algebraic coefficients

- every algebraic number is period, $q = \int_0^q dx$
- The logarithm of an algebraic number is period, $\log q = \int_1^q \frac{dx}{x}$
- $\pi = \int_{x^2+y^2\leq 1} dxdy$ is a period.

Theorem: the coefficients of the Laurent expansion of a Feynman integral are periods.

General properties of Feynman integrals

Questions:

1. Can arbitrarily complicated functions appear, e.g., Trigonometric, exponential functions, and etc. ✗

2. Can the arguments of these functions be arbitrarily complicated, e.g., $\log(\log(p^2))$. ✗

3. The definition of Feynman integral involve numbers $e, \gamma_E, \log(\pi)$ why they do not appear in the results for the integral?

Differential equation method

How to calculate Feynman integral?

1. Numerical method:

   sector decomposition, Heinrich 0803.4177

   numerical solving differential equations, Liu, Ma, Wang 1711.09572
   Mandal, Zhao,1812.03060, and etc

2. Analytical method:

   Mellin-Barnes representation, Smirnov, "Evaluating Feynman integrals"

   Differential equation method, Henn,1304.1806

Differential equation method

Henn, 1304.1806

Canonical form of differential equations

$$d\vec{f}_0(\vec{x},\epsilon) = dA_0(\vec{x},\epsilon)\vec{f}_0(\vec{x},\epsilon)$$ differential equations of master integrals

Choose canonical basis $\vec{f}_0(\vec{x},\epsilon) = T(\vec{x},\epsilon)\vec{f}_0(\vec{x},\epsilon)$

$$d\vec{f}(\vec{x},\epsilon) = \epsilon dA(\vec{x})\vec{f}(\vec{x},\epsilon)$$ Canonical form

cast $dA(\vec{x})$ to d-log form

$$d\vec{f}(\vec{x},\epsilon) = \epsilon \sum_k A_k\, d\log\alpha_k(\vec{x})\vec{f}(\vec{x},\epsilon)\,,$$

A_k is constant matrix,

α_k is called a letter.

formal solutions

$$\vec{f}(\vec{x},\epsilon) = \mathcal{P}\exp\left[\epsilon\int\sum_k A_k\, d\log\alpha_k(\vec{x}(t))\right]\vec{f}(\vec{x}_0,\epsilon)$$

The path connecting boundary points to kinematic points is Parameterized by t

Canonical basis are uniform transcendental functions!

Differential equation method

Expand in ϵ

$$\vec{f}(\vec{x},\epsilon) = \sum_{i=0}^{\infty} \vec{f}^{(i)}(\vec{x})\,\epsilon^i\,,$$

the coefficients are determined by

$$\vec{f}^{(0)}(\vec{x}) = \vec{f}^{(0)}(\vec{x}_0)\,,$$

$$\vec{f}^{(i)}(\vec{x}) = \int_0^1 \sum_k A_k\, d\log\big(\alpha_k(\vec{x}(t))\big)\, \vec{f}^{(i-1)}(\vec{x}(t)) + \vec{f}^{(i)}(\vec{x}_0)\,.$$

The solutions are iterated functions

$$F(\vec{x}) = \int_0^1 d\log(\alpha_n(\vec{x}_n(t_n)))\cdots\int_0^{t_3} d\log(\alpha_2(\vec{x}_2(t_2)))\int_0^{t_2} d\log(\alpha_1(\vec{x}_1(t_1)))\,.$$

The functions mapped to “symbols”

$$\mathcal{S}(F(\vec{x})) = \alpha_1(\vec{x})\otimes\alpha_2(\vec{x})\otimes\cdots\otimes\alpha_n(\vec{x})\,.$$

Differential equation method

Properties of symbols

$$\alpha_1(\vec{x}) \otimes \cdots \otimes \big(\alpha_i(\vec{x})\alpha_{i'}(\vec{x})\big) \otimes \cdots \otimes \alpha_n(\vec{x}) = \alpha_1(\vec{x}) \otimes \cdots \otimes \alpha_i(\vec{x}) \otimes \cdots \otimes \alpha_n(\vec{x}) + \alpha_1(\vec{x}) \otimes \cdots \otimes \alpha_{i'}(\vec{x}) \otimes \cdots \otimes \alpha_n(\vec{x})\,,$$

$$\alpha_1(\vec{x}) \otimes \cdots \otimes \big(c\alpha_i(\vec{x})\big) \otimes \cdots \otimes \alpha_n(\vec{x}) = \alpha_1(\vec{x}) \otimes \cdots \otimes \alpha_i(\vec{x}) \otimes \cdots \otimes \alpha_n(\vec{x})\,.$$

We can define "integrable symbol" which is linear combination of symbols

$$\sum C_k \alpha_{k_1}(\vec{x}) \otimes \cdots \otimes \alpha_{k_{i+1}}(\vec{x}) \otimes \cdots \otimes \alpha_{k_n}(\vec{x}).$$

It satisfies condition

$$\sum C_k \alpha_{k_1}(\vec{x}) \otimes \cdots \otimes \alpha_{k_{i-1}}(\vec{x}) \otimes \alpha_{k_{i+2}}(\vec{x}) \otimes \cdots \otimes \alpha_{k_n}(\vec{x}) \times \Big(\frac{\partial \alpha_{k_i}(\vec{x})}{\partial x_m}\frac{\partial \alpha_{k_{i+1}}(\vec{x})}{\partial x_n} - \frac{\partial \alpha_{k_i}(\vec{x})}{\partial x_n}\frac{\partial \alpha_{k_{i+1}}(\vec{x})}{\partial x_m}\Big) = 0\,.$$

Example:

$$\frac{\sqrt{\mu+1}-1}{\sqrt{\mu+1}+1} \otimes \frac{\sqrt{\mu+1}-\sqrt{\mu+\nu+1}}{\sqrt{\mu+1}+\sqrt{\mu+\nu+1}} + \frac{\sqrt{\nu+1}-1}{\sqrt{\nu+1}+1} \otimes \frac{\sqrt{\nu+1}-\sqrt{\mu+\nu+1}}{\sqrt{\nu+1}+\sqrt{\mu+\nu+1}}\,.$$

Differential equation method

Solve integral symbols

$$\frac{\sqrt{\mu+1}-1}{\sqrt{\mu+1}+1}\otimes\frac{\sqrt{\mu+1}-\sqrt{\mu+\nu+1}}{\sqrt{\mu+1}+\sqrt{\mu+\nu+1}}+\frac{\sqrt{\nu+1}-1}{\sqrt{\nu+1}+1}\otimes\frac{\sqrt{\nu+1}-\sqrt{\mu+\nu+1}}{\sqrt{\nu+1}+\sqrt{\mu+\nu+1}}.$$

1. Rationalize the square roots $\quad \mu=\frac{\left(1-x^2\right)\left(1-y^2\right)}{(x-y)^2}, \quad \nu=\frac{4xy}{(x-y)^2}$

$$\frac{y}{x}\otimes\frac{x(-y)+x+y-1}{(x+1)(y+1)}+\frac{(x-1)(y+1)}{(x+1)(y-1)}\otimes(-xy)$$

2. Parameterize a path $\quad x=t(x-y)+y$

$$-\left(\left(t+\frac{y}{x-y}\right)\otimes\left(t-\frac{y-1}{y-x}\right)\right)+\left(t+\frac{y}{x-y}\right)\otimes\left(t-\frac{y+1}{y-x}\right)+$$
$$\left(t-\frac{y-1}{y-x}\right)\otimes\left(t+\frac{y}{x-y}\right)-\left(t-\frac{y+1}{y-x}\right)\otimes\left(t+\frac{y}{x-y}\right)$$

3. Express solutions in MPL

$$G\left(-\frac{y}{x-y},\frac{y-1}{y-x};1\right)-G\left(-\frac{y}{x-y},\frac{y+1}{y-x};1\right)-G\left(\frac{y-1}{y-x},-\frac{y}{x-y};1\right)+G\left(\frac{y+1}{y-x},-\frac{y}{x-y};1\right)$$

Differential equation method

One loop four point integrals:

$$D_1 = k^2 - m_t^2,\ D_2 = (k+p_1)^2 - m_t^2,$$
$$D_3 = (k+p_1+p_2)^2 - m_t^2,\ D_4 = (k+p_3)^2 - m_t^2,$$

with $s = (p_1+p_2)^2\,, t = (p_1-p_3)^2\,, p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0\,.$

The integrals defined as:

$$I_{a_1,a_2,a_3,a_4}(s,t,m_t^2,\epsilon) \equiv \frac{16\pi^2}{i}\left(\frac{m_t^2}{4\pi}\right)^{\epsilon}\Gamma(1+\epsilon)\int\frac{d^dk}{(2\pi)^d}\frac{1}{D_1^{a_1}\,D_2^{a_2}\,D_3^{a_3}\,D_4^{a_4}}$$

The master integrals are

$$\vec{f}_0 = \{I_{0,0,0,1}, I_{0,1,0,1}, I_{1,0,1,0}, I_{1,0,1,1}, I_{0,1,1,1}, I_{1,1,1,1}\}\,.$$

Differential equation method

1. Construct differential operators

$$\frac{\partial}{\partial s} = \left(\frac{1}{2s}p_1^\mu + \frac{2s+t}{2s(s+t)}p_2^\mu - \frac{1}{2(s+t)}p_3^\mu\right)\frac{\partial}{\partial p_2^\mu}\,,$$
$$\frac{\partial}{\partial t} = \left(\frac{1}{2s}p_1^\mu - \frac{1}{2(s+t)}p_2^\mu + \frac{s+2t}{2(s+t)}p_3^\mu\right)\frac{\partial}{\partial p_3^\mu}\,.$$

Act the operators on master integrals, get the differential equations with respect to dimensionless variables

$$\mu \equiv -\frac{4m_t^2}{s}\,, \quad \nu \equiv -\frac{4m_t^2}{t}\,.$$

2. Choose canonical basis

$$\left\{I_{0,0,0,2}, -\frac{4m_t^2\sqrt{\nu+1}I_{0,1,0,2}}{\nu}, -\frac{4m_t^2\sqrt{\mu+1}I_{2,0,1,0}}{\mu}, \frac{m_t^2\epsilon I_{1,0,1,1}}{\mu},\right.$$
$$\left.\frac{m_t^2\epsilon I_{0,1,1,1}}{\nu}, \frac{16m_t^4\epsilon I_{1,1,1,1}\sqrt{\mu+\nu+1}}{\mu\nu}\right\}$$

Differential equation method

3. Cast coefficient matrix into d-log form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -d\log(R_1) & d\log\left(\frac{\nu}{\nu+1}\right) & 0 & 0 & 0 & 0 \\ -d\log(R_2) & 0 & d\log\left(\frac{\mu}{\mu+1}\right) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}d\log(R_2) & 0 & 0 & 0 \\ 0 & \frac{1}{4}d\log(R_1) & 0 & 0 & 0 & 0 \\ 0 & -2d\log(R_3) & -2d\log(R_4) & 8d\log(R_5) & 8d\log(R_5) & d\log\left(\frac{\mu+\nu}{\mu+\nu+1}\right) \end{pmatrix}$$

with

$$R_1 = \frac{\sqrt{\nu+1}-1}{\sqrt{\nu+1}+1}, R_2 = \frac{\sqrt{\mu+1}-1}{\sqrt{\mu+1}+1}, R_3 = \frac{\sqrt{\nu+1}-\sqrt{\mu+\nu+1}}{\sqrt{\nu+1}+\sqrt{\mu+\nu+1}},$$

$$R_4 = \frac{\sqrt{\mu+1}-\sqrt{\mu+\nu+1}}{\sqrt{\mu+1}+\sqrt{\mu+\nu+1}}, R_5 = \frac{\sqrt{\mu+\nu+1}-1}{\sqrt{\mu+\nu+1}+1}.$$

The boundary conditions are $\lim_{\mu,\nu\to\infty} \vec{f}(\mu,\nu,\epsilon) = \{1,0,0,0,0,0\}$.

Differential equation method

3. Solve the differential equations

At $\mathcal{O}(\epsilon^0)$, the solution is $\{1,0,0,0,0,0\}$.

At $\mathcal{O}(\epsilon^1)$, the result is

$$\left\{0, -\log\left(\frac{\sqrt{\nu+1}-1}{\sqrt{\nu+1}+1}\right), -\log\left(\frac{\sqrt{\mu+1}-1}{\sqrt{\mu+1}+1}\right), 0, 0, 0\right\}$$

At $\mathcal{O}(\epsilon^2)$, the solution in symbols

$$\left\{0, \quad \frac{\sqrt{\nu+1}-1}{\sqrt{\nu+1}+1}\otimes(\nu+1)-\frac{\sqrt{\nu+1}-1}{\sqrt{\nu+1}+1}\otimes\nu, \quad \frac{\sqrt{\mu+1}-1}{\sqrt{\mu+1}+1}\otimes(\mu+1)-\frac{\sqrt{\mu+1}-1}{\sqrt{\mu+1}+1}\otimes\mu,\right.$$
$$-\frac{1}{4}\frac{\sqrt{\mu+1}-1}{\sqrt{\mu+1}+1}\otimes\frac{\sqrt{\mu+1}-1}{\sqrt{\mu+1}+1}, \quad -\frac{1}{4}\frac{\sqrt{\nu+1}-1}{\sqrt{\nu+1}+1}\otimes\frac{\sqrt{\nu+1}-1}{\sqrt{\nu+1}+1},$$
$$\left.2\left(\frac{\sqrt{\mu+1}-1}{\sqrt{\mu+1}+1}\otimes\frac{\sqrt{\mu+1}-\sqrt{\mu+\nu+1}}{\sqrt{\mu+1}+\sqrt{\mu+\nu+1}}+\frac{\sqrt{\nu+1}-1}{\sqrt{\nu+1}+1}\otimes\frac{\sqrt{\nu+1}-\sqrt{\mu+\nu+1}}{\sqrt{\nu+1}+\sqrt{\mu+\nu+1}}\right)\right\}$$

Constructing Canonical Feynman Integrals

How to get canonical form of differential equations?

1. Transform differential equations into canonical form

   Argeri, Vita, Mastrolia, and, etc, 1401.2979
   Lee, 1411.0911 Dlapa, Henn, Yan, 2002.02340

   Some packages: Azurite 1612.04252, Fuchsia 1701.04269, epsilon 1701.00725,
   Canonical 1705.06252.

2. Construct canonical basis directly

   Canonical basis are related to d-log form integrand, which is

   $$d\log f_1 \wedge d\log f_2 \wedge \cdots \wedge d\log f_n \,.$$

   Algorithms: Henn, Mistlberger, Smirnov, Wasser, 2002.09492,
   Chen, X. Xu, Yang, 2008.03045

Constructing Canonical Feynman Integrals

- D-log form integrand in Baikov representation

General L-loop Feynman integral:

$$F_{a_1,\ldots,a_N} = \int \left[\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}}\right] \frac{1}{D_1^{a_1}\, D_2^{a_2} \cdots D_N^{a_N}}$$

Baikov representation using loop-by-loop construction

$$F_{a_1,\ldots,a_N} = \mathcal{N}_\epsilon \int_{\mathcal{C}} \left[\prod_i \left[G_i(\mathbf{z})\right]^{-\gamma_i - \beta_i \epsilon}\right] \prod_{j=1}^{n} \frac{dz_j}{z_j^{\alpha_j}}$$

where $\mathbf{z} \equiv \{z_1, \ldots, z_n\}$is subset of propagators and $G(\mathbf{z}) \equiv \det(q_i \cdot q_j)$

Canonical integrals in Baikov representation

$$\int_{\mathcal{C}} \left[\prod_i \left[G_i(\mathbf{z})\right]^{-\beta_i \epsilon}\right] \prod_{j=1}^{n} d\log f_j(\mathbf{z})$$

Constructing Canonical Feynman Integrals

- Hypergeometric function and intersection number

definition of hypergeometric function

$$\int_{\mathcal{C}} u(\mathbf{z})\,\varphi(\mathbf{z})$$

$$u(\mathbf{z}) = \prod_i p_i(\mathbf{z})^{\gamma_i}$$

"multiple-valued fucntion"

$$\varphi(\mathbf{z}) = \frac{q(\mathbf{z})}{\prod_i p_i(\mathbf{z})^{n_i}} dz_1 \wedge \cdots \wedge dz_n$$

"single valued n-form"

$u(\mathbf{z})$ for Feynman integrals

$$u(\mathbf{z}) = \prod_i \left[G_i(\mathbf{z})\right]^{-\gamma_i - \beta_i \epsilon}$$

corresponding n-form

$$\varphi(\mathbf{z}) = \prod_{j=1}^{n} \frac{dz_j}{z_j^{\alpha_j}}$$

Constructing Canonical Feynman Integrals

[Frellesvig, Gasparotto, .etc ‘2019]
[Weinzierl ‘2020]

Consider following integral

$$0 = \int_{\mathcal{C}} d(u\xi) = \int_{\mathcal{C}} u(d\log(u) \wedge + d)\xi \equiv \int_{\mathcal{C}} u\nabla_w \xi$$

where $w = d\log(u)$ and $\nabla_w \equiv d + w\wedge\,.$

Define a twisted cohomology: $H_w^n \equiv \{\langle\varphi| \quad | \quad \langle\varphi| : \varphi \sim \varphi + \nabla_w \xi\}$

Express $\langle\varphi|$ in terms of basis $\{\langle e_i|\}$

$$\langle\varphi| = \sum_i c_i \langle e_i|$$

Introduce dual basis $\{|h_i\rangle\} \in H^n_{-w}$, and the coefficients are determined by

$$c_i = \sum_j \langle\varphi|h_j\rangle(\mathbf{C}^{-1})_{ji}\,, \quad \mathbf{C}_{ij} = \langle e_i|h_j\rangle\,.$$

$\langle\varphi_L|\varphi_R\rangle$ is called an intersection number.

We can project d-log integrand into master integrals using intersection number

Summary

1. Reviewed general properties of Feynman integrals
2. Introduce canonical form of differential equations and canonical basis
3. How to solve the differential equations in terms of "symbols"
4. Advanced topic in constructing canonical basis

Thank you

Back up

Slide from Lorenzo's talk