# Feynman integrals calculation by difference equations

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# Outline

**I. Overview**

**II. The method and examples**

**III. Summary**

# Procedure of high order Calculation

- Generate scattering amplitudes

Relatively easier

- Express the amplitudes in terms of master integrals (MIs) by integration-by-part(IBP) identities

Much harder

$$\mathcal{M} \xrightarrow{\text{IBP}} \sum_i c_i \times I_i$$

- Calculate MIs

Hard

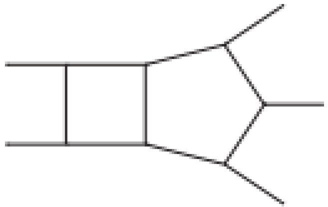
  - One-loop order: solved problem ['t Hooft, Veltman 1979] [Oldenborgh, Vermaseren 1990]
  - Higher-loop order?

# Differential equation

## ➢ Differential equation method

[A.V.Kotikov 1991]
[E.Remiddi 1997]

- Powerful for multi-scale integrals

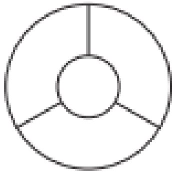


Multi-scale

$$\frac{\partial}{\partial x}\vec{I}(x;\epsilon) = A(x;\epsilon)\vec{I}(x;\epsilon)$$

[D.Chicherin, et. al. 2019]

- Can't handle single scale integrals directly
- Handle single scale by auxiliary mass flow



Single scale

$$\frac{\partial}{\partial \eta}\vec{I}(\eta) = A(\eta)\vec{I}(\eta)$$

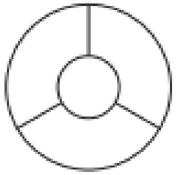
[X. Liu, Y.Q. Ma, C.Y. Wang 2017]

# Difference equation

## ➢ Difference equation method

[O.V.Tarasov 1996]
[S.Laporta 2000]

- Powerful for single scale integrals
- Good supplement to differential equation method
- Good for numerical calculating



Single scale

$$\vec{J}(v-1) = M(v)\vec{J}(v)$$

# Overview

- Dimension recurrence relation(DRR) was first researched by O.V.Tarasov in1996 with 1-2 loop calculations.
- DRR was developed by R.N.Lee in 2009 to calculate 3 loop integrals.
- S.Laporta propose another difference equations in 2000, with many 3 loop calculating.
- Widely used for calculating 4-loop g-2 in QED, 4-5 loop QCD Beta function.

[S.Laporta 2017]

[P.A.Baikov 2016]

# Difference equation method

- **Difference variable: space-time dimension**

[O.V.Tarasov 1996]
[R.N.Lee 2009]

$$I(D)=\frac{1}{\pi^{D/2}}\int\frac{d^Dk}{k^2+m^2},I(D-2)=m^{-2}(1-D/2)I(D)$$

IBP reduction

- **Difference variable: the power of one massive propagator**

[S.Laporta 2000]

$$J(x)=\int\frac{d^Dk}{(k^2+m^2)^x},m^2(x-1)J(x)-(x-1-D/2)J(x-1)=0$$

IBP identities

# DRR method

- **Dimension(d) difference equations**

$$J(d-2) = C(d)J(d) + R(d)$$

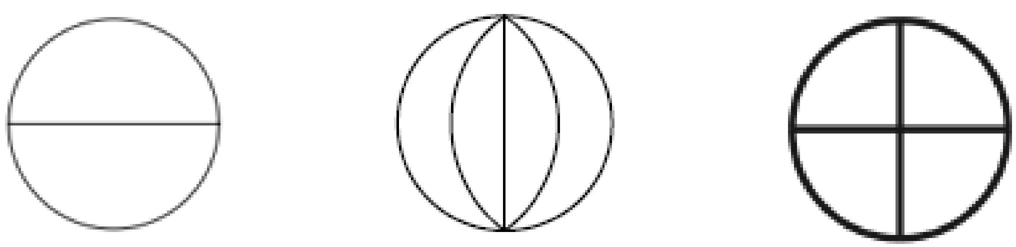
$$J(d) = w(d)J_{sh}(d) + J_{sih}(d)$$

$$w(d-2) = w(d)$$

$$J_{sh}(d-2) = C(d)J_{sh}(d)$$

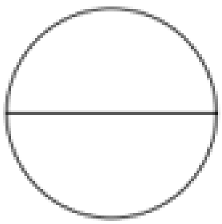
$$J_{sih}(d-2) = C(d)J_{sih}(d) + R(d)$$

- **Tadpoles as examples**



1.single scale
2.w(d) is easy

# An example

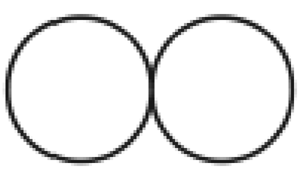


$$I_{111}(d) = \frac{1}{\pi^d}\int \frac{d^d k_1 d^d k_2}{(k_1^2+1)(k_2^2+1)\left((k_1+k_2)^2+1\right)}$$

$k_1$, $k_2$ is Euclidean

- **Step1: Set up difference equation of $I_{111}(d)$**

$$I_{111}(d-2) = C_1(d)I_{111}(d) + C_2(d)I_{110}(d)$$



$$I_{110}(d) = \Gamma(1-d/2)^2$$

# Parametric representation

$$\mathcal{I}(\nu_1,\ldots,\nu_N) = \left(\prod_{j=1}^{L}\int \frac{\mathrm{d}^d \ell_j}{\mathrm{i}\pi^{d/2}}\right)\prod_{a=1}^{N} \mathsf{D}_a^{-\nu_a}.$$

$$\frac{1}{\mathsf{D}_e} = \frac{1}{-k_e^2 + m_e^2 - \mathrm{i}\epsilon} \quad (1 \le e \le N),$$

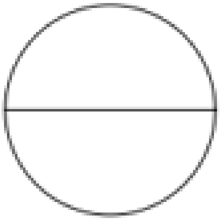
$$\mathcal{I}(\nu_1,\ldots,\nu_N) = \left(\prod_{i=1}^{N}\int_0^\infty \frac{x_i^{\nu_i-1}\mathrm{d}x_i}{\Gamma(\nu_i)}\right)\frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2}},$$

$$\mathcal{I}(\nu_1,\ldots,\nu_N) = \Gamma(\omega)\left(\prod_{i=1}^{N}\int_0^\infty \frac{x_i^{\nu_i-1}\mathrm{d}x_i}{\Gamma(\nu_i)}\right)\frac{\delta\left(1-\sum_{j=1}^{N} x_j\right)}{\mathcal{U}^{d/2-\omega}\mathcal{F}^{\omega}}$$

$$\mathcal{I}(\nu_1,\ldots,\nu_N) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}-\omega\right)}\left(\prod_{i=1}^{N}\int_0^\infty \frac{x_i^{\nu_i-1}\mathrm{d}x_i}{\Gamma(\nu_i)}\right)\mathcal{G}^{-d/2}.$$

[T.Bitoun, et. al. 2018]

# Construct difference equation



$$I_{111}(d-2) = \prod_{i=1}^{3} \int_0^\infty dx_i \, U \frac{e^{-F/U}}{U^{d/2}}, U = x_1 x_2 + x_2 x_3 + x_3 x_1$$

$$I_{111}(d-2) = I_{221}(d) + I_{122}(d) + I_{212}(d) = 3 I_{221}(d)$$

$\downarrow$ **IBP reduction**

$$I_{111}(d-2) = \frac{4}{3}(3/2 - d/2)(1 - d/2) I_{111}(d) + (1 - d/2)^2 I_{110}(d)$$

$$\begin{pmatrix} I_{110}(d-2) \\ I_{111}(d-2) \end{pmatrix} = \begin{pmatrix} (1-d/2)^2 & 0 \\ (1-d/2)^2 & \frac{4}{3}(3/2 - d/2)(1-d/2) \end{pmatrix} \begin{pmatrix} I_{110}(d) \\ I_{111}(d) \end{pmatrix}$$

# Inhomogeneous solution

- **Step2: Construct inhomogeneous solution**

$$J(d-2) = C(d)J(d) + R(d)$$

**If** $$r = \left| \lim_{d \to -\infty} \frac{R(d-2)}{C(d)R(d)} \right| < 1$$

**Then** $$J(d) = C^{-1}(d)\big(J(d-2) - R(d)\big)$$

$$J(d-2) = C^{-1}(d-2)\big(J(d-4) - R(d-2)\big)$$

$$J_{sih}(d) = -C^{-1}(d)R(d) - C^{-1}(d)C^{-1}(d-2)R(d-2) - \cdots$$

**Series behavior** $$\sum_{k=1}^{\infty} r^k$$ **Convergent**

# Inhomogeneous solution

$$I_{111}(d-2)=\frac{4}{3}(3/2-d/2)(1-d/2)I_{111}(d)+(1-d/2)^2 I_{110}(d)$$

$$r=\frac{3}{4},I_{111}^{(sih)}\sim\sum_{k=1}^{\infty}(3/4)^k$$

## Numerically calculate it

**Set** $$I_{111}^{(sih)}(d-2000)=0,d=4-2\varepsilon$$

$$I_{111}^{(sih)}(4-2\varepsilon)=\frac{-1.5}{\varepsilon^2}-\frac{8.20975}{\varepsilon}-9.32062-48.9345\varepsilon+\cdots$$

**Error=$(3/4)^{1000}$ ~$10^{-125}$**

**Time=O(n)**

# Homogeneous solution

- **Step3: Construct homogeneous solution**

$$I_{111}(d-2)=\frac{4}{3}(3/2-d/2)(1-d/2)I_{111}(d)+(1-d/2)^2 I_{110}(d)$$

$$I_{111}^{(sh)}(d-2)=\frac{4}{3}(3/2-d/2)(1-d/2)I_{111}^{(sh)}(d)$$

**By the property of Gamma function, we get**

$$I_{111}^{(sh)}(d)=(4/3)^{-d/2}\Gamma(3/2-d/2)\Gamma(1-d/2)$$

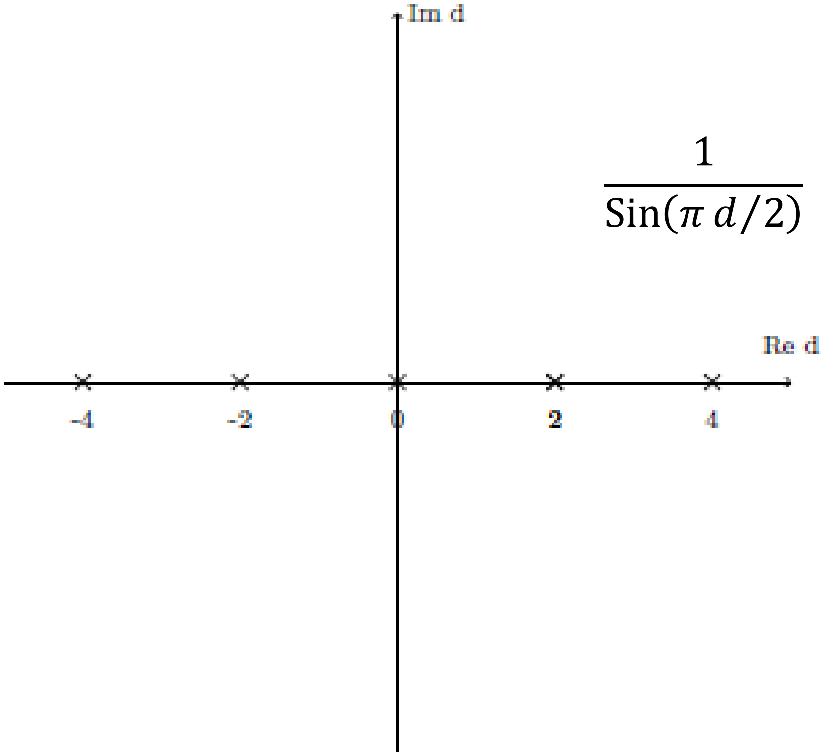
**1/(summing factor)**

**[R.N.Lee 2009]**

# Periodic function

- **Step4: Determine periodic function w(d)**

**A series of poles crossing the whole d complex plane would give rise to periodic function in Feynman integral.**




$$\frac{1}{\mathrm{Sin}(\pi\, d/2)}$$

# Periodic function

1.For tadpoles, they have only poles on the right half d complex plane.

2.We only introduce UV poles when construct inhomogeneous and homogeneous solution, so w(d) is a constant.

$$I_{111}(d) = w(d) I_{111}^{(sh)}(d) + I_{111}^{(sih)}(d), w(d) = const$$

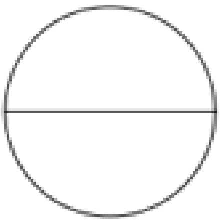
3.With a trivial boundary condition, w(d) is obtained.

$$I_{111}(d = 0) = 1$$

$$w = 2.72887$$

# An example

- **Combine the above**



$$I_{111}(d) = \frac{1}{\pi^d} \int \frac{d^d k_1 d^d k_2}{(k_1^2+1)(k_2^2+1)\left((k_1+k_2)^2+1\right)}$$

$$I_{111}(4-2\varepsilon) = -\frac{1.5}{\varepsilon^2} - \frac{2.76835}{\varepsilon} - 5.25613 - 15.9117\varepsilon + \cdots$$

# The method

1.Set up difference equations by IBP reduction.

2.Construct inhomogeneous solution.

3.Construct homogenous solution.

4.Determine periodic function w(d).

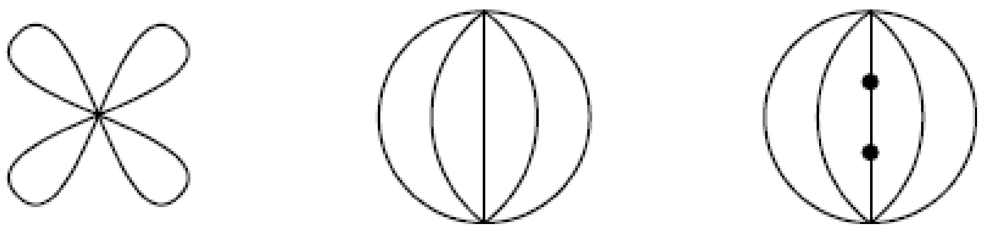
$$J(d-2) = C(d)J(d) + R(d)$$

$$J(d) = w(d)J_{sh}(d) + J_{sih}(d)$$

$$w(d) = const, J(0) = 1$$ For tadpoles

# Multi-component MIs

- Topology more than one master integral
- Construct  homogeneous is hard
- Other method is needed



$$\begin{pmatrix} (v-1)^4 & 0 & 0 \\ (v-1)^4 & \frac{2}{75}(v-1)(2v-3)(4v-5)(18v-31) & -\frac{14}{5}(v-1)(2v-3) \\ -\frac{1}{225}(v-1)^4\left(78v^2-22v-281\right) & \frac{2(v-1)(2v-3)(4v-5)\left(2646v^3-19341v^2+44676v-33091\right)}{16875} & -\frac{2(v-1)(2v-3)\left(1629v^2-7566v+8827\right)}{1125} \end{pmatrix}$$

$$v = d/2$$

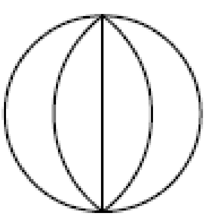
# Inhomogeneous solution

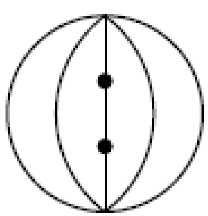
$$\begin{pmatrix} (v-1)^4 & 0 & 0 \\ (v-1)^4 & \frac{2}{75}(v-1)(2v-3)(4v-5)(18v-31) & -\frac{14}{5}(v-1)(2v-3) \\ -\frac{1}{225}(v-1)^4\left(78v^2-22v-281\right) & \frac{2(v-1)(2v-3)(4v-5)\left(2646v^3-19341v^2+44676v-33091\right)}{16875} & -\frac{2(v-1)(2v-3)\left(1629v^2-7566v+8827\right)}{1125} \end{pmatrix}$$

$$(v-1)^4 \to 1$$

$$\begin{pmatrix} \frac{2}{75}(v-1)(2v-3)(4v-5)(18v-31) & -\frac{14}{5}(v-1)(2v-3) \\ \frac{2(v-1)(2v-3)(4v-5)\left(2646v^3-19341v^2+44676v-33091\right)}{16875} & -\frac{2(v-1)(2v-3)\left(1629v^2-7566v+8827\right)}{1125} \end{pmatrix} \to \begin{pmatrix} \frac{96}{25} & -\frac{28}{5} \\ \frac{1568}{625} & -\frac{724}{125} \end{pmatrix} \to \begin{pmatrix} -4 & 0 \\ 0 & \frac{256}{125} \end{pmatrix}$$

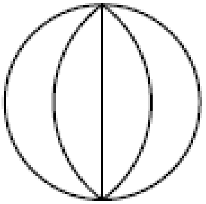
$$r = \max\{1/4, 125/256\} = 125/256 < 1$$

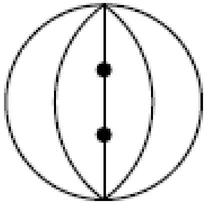




$$J_{sih}(d) \sim \sum_k (125/256)^k$$

# Homogeneous solution





$$\vec{J}(d-2) = C(d)\vec{J}(d) + \vec{R}(d)$$

$$\vec{J}_{sh}(d-2) = C(d)\vec{J}_{sh}(d)$$

$$C(d) = \begin{pmatrix} \frac{2}{75}(v-1)(2v-3)(4v-5)(18v-31) & -\frac{14}{5}(v-1)(2v-3) \\ \frac{2(v-1)(2v-3)(4v-5)\left(2646v^3-19341v^2+44676v-33091\right)}{16875} & -\frac{2(v-1)(2v-3)\left(1629v^2-7566v+8827\right)}{1125} \end{pmatrix}$$

- **Can’t get homogeneous function by Gamma functions**
- **Factorial series method**

# Factorial series method

$$p_0(x)U^{HO}(x) + p_1(x)U^{HO}(x+1) + \ldots + p_R(x)U^{HO}(x+R) = 0$$

$$\vec{J}_{sh}(d) \to U^{HO}(x), x = -d/2$$

$$\rho^m U(x) = \frac{\Gamma(x+1)}{\Gamma(x-m+1)} U(x-m) \; .$$

$$\rho^m 1 = \rho^m = \frac{\Gamma(x+1)}{\Gamma(x-m+1)} \; .$$

$$\pi U(x) = x(U(x) - U(x-1)) \; .$$

$$[\pi, \rho]U(x) = \rho U(x)$$

[L.M.Milne-Thomson 1951]
[S.Laporta 2000]

# Factorial series method

$$p_0(x)U^{HO}(x) + p_1(x)U^{HO}(x+1) + \ldots + p_R(x)U^{HO}(x+R) = 0$$

$$\downarrow x \to x - R$$

$$q_0(x)U^{HO}(x) + q_1(x)U^{HO}(x-1) + \ldots + q_R(x)U^{HO}(x-R) = 0$$

$$\downarrow \; U^{HO}(x) = \mu^x V^{HO}(x)$$

$$\mu^R q_0(x)V^{HO}(x) + \mu^{R-1} q_1(x)V^{HO}(x-1) + \ldots + q_R(x)V^{HO}(x-R) = 0$$

$$\downarrow \; xV(x-1) = \rho V(x)$$

$$\left[\phi_0(x,\mu) + \phi_1(x,\mu)\rho + \ldots + \phi_R(x,\mu)\rho^R\right] V^{HO}(x) = 0$$

$$\downarrow \; x = \pi + \rho$$

$$\left[f_0(\pi,\mu) + f_1(\pi,\mu)\rho + f_2(\pi,\mu)\rho^2 + \ldots + f_{m+1}(\pi,\mu)\rho^{m+1}\right] V^{HO}(x) = 0$$

**First canonical form**

$$f_{m+1}(\mu) = 0$$

**Characteristic equation**

# Factorial series method

$$\left[f_0(\pi,\mu) + f_1(\pi,\mu)\rho + f_2(\pi,\mu)\rho^2 + \ldots + f_{m+1}(\pi,\mu)\rho^{m+1}\right] V^{HO}(x) = 0$$

$$\downarrow\ \mu = \mu_i$$

$$\left[f_0(\pi) + f_1(\pi)\rho + f_2(\pi)\rho^2 + \ldots + f_m(\pi)\rho^m\right] V^{HO}(x) = 0$$

$$V^{HO}(x) = \sum_{s=0}^{\infty} \frac{a_s \Gamma(x+1)}{\Gamma(x-K+s+1)} = \sum_{s=0}^{\infty} a_s \rho^{K-s} = a_0\rho^K + a_1\rho^{K-1} + \ldots$$

$$\downarrow$$

$$a_0 f_m(K+m) = 0\ ,$$

$$a_1 f_m(K+m-1) + a_0 f_{m-1}(K+m-1) = 0\ ,$$

$$\ldots$$

$$a_s f_m(K+m-s) + a_{s-1} f_{m-1}(K+m-s) + \ldots + a_{s-m} f_0(K+m-s) = 0 \quad (s \geq m)\ .$$

$$\downarrow\ a_0 \neq 0$$

$$f_m(K+m) = 0$$

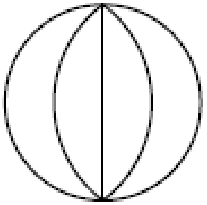
**indicial equation**

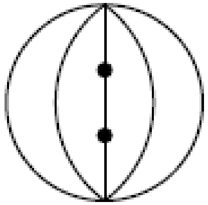
# Factorial series method

$$p_0(x)U^{HO}(x) + p_1(x)U^{HO}(x+1) + \ldots + p_R(x)U^{HO}(x+R) = 0$$

$$U^{HO}(x) = \sum_{i=1}^{\lambda}\sum_{j=1}^{\nu_i} \tilde{\omega}_{ij}(x)\mu_i^x V_{ij}^{HO}(x)$$

$$\sum_{i=1}^{\lambda} \nu_i = R$$



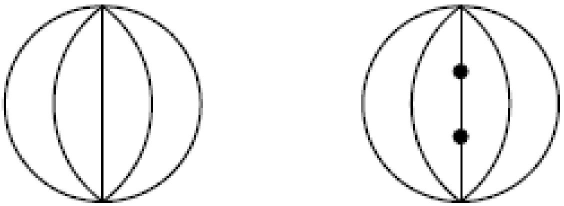


$$\begin{pmatrix} \frac{2}{75}(v-1)(2v-3)(4v-5)(18v-31) & -\frac{14}{5}(v-1)(2v-3) \\ \frac{2(v-1)(2v-3)(4v-5)(2646v^3-19341v^2+44676v-33091)}{16875} & -\frac{2(v-1)(2v-3)(1629v^2-7566v+8827)}{1125} \end{pmatrix}$$

$$\downarrow$$

$$\frac{125}{2}(-2+3v)(-1+3v)T(v) + 2(16+549v^2)T(v+1) - 288(1+4v)(3+4v)T(v+2) = 0$$

# Factorial series method



$$\frac{125}{2}(-2+3v)(-1+3v)T(v)+2(16+549v^2)T(v+1)-288(1+4v)(3+4v)T(v+2)=0$$

$$\mu_1=-4, \mu_2=256/125$$

$$K_1=-1, K_2=-1$$

$$-\frac{(n-1)\left(2853\,n^2-10\,260\,n+9380\right)a(n-2)}{1701\,n}+\frac{\left(7072\,n^2-13\,120\,n+5929\right)a(n-1)}{3024\,n}+\frac{64\,(n-2)\,(n-1)\,(3\,n-8)\,(3\,n-7)\,a(n-3)}{1701\,n}=0$$

$$a_0=1, a_1=-17/432$$

**Error=1/N$^K$**

**Time=O(n)~O(n$^2$)**

# Factorial series method

- **Can also be used to construct inhomogeneous solution**
- **Convergence condition**

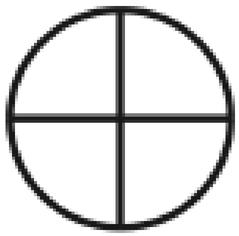
$$U^{(\alpha)}(x) = \left(\mu^{(\alpha)}\right)^x \sum_{s=0}^{\infty} a_s^{(\alpha)} \frac{\Gamma(x+1)}{\Gamma(x+1-K^{(\alpha)}+s)}$$

$$0 < |\mu_j/\mu^{(\alpha)} - 1| < 1 \ , \qquad j = 1, \ldots, R$$

**Divergence condition**

$$|-4/(256/125) - 1| = 189/64 > 1$$

# Example



$$C(v) = \begin{pmatrix} \frac{2(v-2)^2(v-1)(2v-3)(6v-13)}{3(2v-5)} & \frac{2(v-2)(v-1)(2v-3)(4v-9)}{3(2v-5)} \\ -\frac{4(v-2)^2(v-1)(2v-3)(4v^2-22v+29)}{9(2v-5)} & -\frac{2(v-2)(v-1)(2v-3)(6v^2-33v+44)}{9(2v-5)} \end{pmatrix}$$

$$C(v) \to \begin{pmatrix} 4 & \frac{8}{3} \\ -\frac{16}{9} & -\frac{4}{3} \end{pmatrix} \to \begin{pmatrix} \frac{4}{9}\left(3+2\sqrt{3}\right) & 0 \\ 0 & \frac{4}{9}\left(3-2\sqrt{3}\right) \end{pmatrix} = \begin{pmatrix} 2.87293 & 0. \\ 0. & -0.206267 \end{pmatrix}$$

$$\mu = 27/16$$

$$(27/16)/\left\{\frac{4}{9}(3+2\sqrt{3}), \frac{4}{9}(3-2\sqrt{3})\right\} = \{0.587, -8.18\}$$

$$\left\{\frac{4}{9}(3+2\sqrt{3}), \frac{4}{9}(3-2\sqrt{3})\right\}/(27/16) - 1 = \{0.702, -1.12\}$$

**Divergent, can't get convergent inhomogeneous solution by above methods**

# Laplace's transformation method

$$p_0(x)U(x) + p_1(x)U(x+1) + \ldots + p_N(x)U(x+N) = 0$$

[S.Laporta 2000]

$$U(x) = \int_l dt\, t^{x-1} v(t)$$

$$p_k(x) = A_{k0} + \sum_{i=1}^{P} A_{ki} \prod_{j=0}^{i-1} (x+k+j)$$

$\downarrow$ Integrating by parts

$$\sum_{k=0}^{N} p_k(x) U(x+k) = \int_l dt\, t^{x-1} \sum_{i=0}^{P} \Phi_i(t)(-t)^i v^{(i)}(t) + [I(x,t)]_l$$

$$\Phi_i(t) = \sum_{k=0}^{N} A_{ki} t^k \;,$$

$$I(x,t) = \sum_{i=0}^{P-1} (-1)^i v^{(i)}(t) \sum_{m=0}^{P-1-i} \left(\frac{d}{dt}\right)^m \left(\Phi_{m+i+1}(t) t^{x+m+i}\right) \;.$$

# Laplace's transformation method

$\downarrow$ **Choose integration l so that** $[I(x,t)]_l = 0$

$$\sum_{i=0}^{P} \Phi_i(t)(-t)^i v^{(i)}(t) = 0$$

$$\Phi_P(t) = 0$$ **Characteristic equation**

$$t_i = \mu_i$$

$$U_{ij}(x) = \int_{l_i} dt\ t^{x-1} v_{ij}(t) \qquad j = 1, \ldots, m_i$$

$$\sum_i m_i = N$$

# Laplace's transformation method

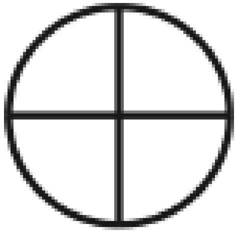
## For nonhomogeneous equation

$$\sum_{k=0}^{N} p_k(x)U(x+k) = \sum_{k=0}^{N'} q_k(x)T(x+k)$$

$$T(x) = \int_{l_T} dt\, t^{x-1} w(t) \;, \qquad U^{NH}(x) = \int_{l_T} dt\, t^{x-1} v_{NH}(t)$$

↓ $l_T$ is known

$$\sum_{i=0}^{P} \Phi_i(t)(-t)^i v_{NH}^{(i)}(t) = \sum_{i=0}^{P'} \Psi_i(t)(-t)^i w^{(i)}(t)$$



$$U^{NH}(x) = \int_0^{27/16} t^{x-1} v_{NH}(t)\, dt, x = -d/2$$

# Summary

$$J(d-2) = C(d)J(d) + R(d)$$

$$J(d) = w(d)J_{sh}(d) + J_{sih}(d)$$

- $J_{sih}(d)$ is obtained by recurrence relations or factorial series, Laplace's transformation method.
- $J_{sh}(d)$ is obtained by factorial series or Laplace's transformation method.
- w(d) is fixed by analysis of analytic properties of Feynman integral.
- Take care of convergence, Laplace's transformation method holds more.

# Summary

1. Difference equation method is suitable for single Scale integrals, which can't be calculated directly by Differential equation method.

2. Difference equation method is good for numerical Calculations with high precision and efficiency.

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# Thank you!