Analytic calculation
--traditional methods for integrals

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12/17/2020

Catalog

- Introduction to analytic calculation
- Traditional methods for integrals
  - (Feynman) parameterization
  - Mellin-Barnes representation
  - Asymptotic expansions
- More examples
- Summary

Calculation

- Numerical methods:
  - Powerful
  - Time and resource consuming
- Analytic methods:
  - "Time" and resource saving
  - Phenomenological analysis friendly
  - Determining its fundamental parameters
  - Determining limit behavior
  - More difficult

We can do more!
We can do faster!

Firstly, integrals
Then, amplitudes

Analytic calculation

Cross sections

- Total cross sections:

Anti-unitarity relation

$$\hat{\sigma} \sim \left[\prod d^d p_f \delta^+(p_f^2 - m_f^2)\right]\delta^{(d)}\left(\sum p_i - \sum p_f\right)\left[\prod d^d p_l\right]|\mathcal{A}|^2$$

- Differential cross sections:

$$d\hat{\sigma} \sim \left[\prod d^d p_f \delta^+(p_f^2 - m_f^2)\right]\delta^{(d)}\left(\sum p_i - \sum p_f\right)\left[\prod d^d p_l\right]|\mathcal{A}|^2 F(p_i, p_f, \cdots)$$

$\delta -$ functions

$\theta -$ functions

$\cdots$

  - Distributions: $M_{f_1 f_2}$, p_T , $p_{f_i,T}$, y_{f_i} , $\Delta\phi$, τ $\cdots$
  - Polarized distributions.

Squared amplitudes and integrals

Loop- and phase-space integrals

Squared amplitudes

- Squared directly:
  - Low orders and few (vector and/or fermion) legs
- Projection:
  - few (vector and/or fermion) legs

Scalar integrals

IBP

$$g_a^{\mu}(p_1) + g_b^{\nu}(p_2) \to H(p_3) + H(p_4)$$

$$\mathcal{M}_{ab} \sim \epsilon_{\mu}\epsilon_{\nu}\mathcal{M}_{ab}^{\mu\nu} \qquad \mathcal{M}_{ab}^{\mu\nu} \sim A_1^{\mu\nu}F_1 + A_2^{\mu\nu}F_2 \qquad |\mathcal{M}|^2 \sim |F_1|^2 + |F_2|^2$$

$$A_1^{\mu\nu} = g^{\mu\nu} - \frac{2p_1^{\nu}\, p_2^{\mu}}{\hat{s}}$$

- Helicity amplitudes.

$$A_2^{\mu\nu} = g^{\mu\nu} + \frac{2m_H^2\, p_1^{\nu}\, p_2^{\mu} + 2\hat{s}p_3^{\mu}\, p_3^{\nu} + 2\,\hat{u}_1 p_1^{\nu}\, p_3^{\mu} + 2\,\hat{t}_1 p_2^{\mu}\, p_3^{\nu}}{p_T^2\,\hat{s}}$$

Polarized information

Integrals

- Loop integrals:

$$\int \left[\prod_{j=1}^{L} \frac{d^d l_j}{(2\pi)^d} \right] \frac{\mathcal{N}(l_a^2\,,\; l_a \cdot l_b\,,\; l_c \cdot p_d)}{\prod_{i=1}^{n} D_i^{\alpha_i}}$$

UV and IR divergence: dimensional regularization

  - Quadratic: $D_i = (l_j - p_k)^2 - m_i^2 + i\epsilon$
  - Linear: $D_i = \pm l_j \cdot p_k + i\epsilon$
  - Potential: $D_i = q_0 l_0 \pm \boldsymbol{l}^2 \pm y + i\epsilon$

$\mathrm{SCET_I}$

$\mathrm{SCET_{II}}$ Rapidity divergence

- Phase-space integrals:

$$\int \left[\prod_{j=1}^{N} \frac{d^d l_j}{(2\pi)^d} (2\pi)\delta^+(l_j^2 - m_j^2) \right] \delta^{(d)}\left(\sum p_i - \sum l_f\right) \frac{\mathcal{N}(l_a^2\,,\; l_a \cdot l_b\,,\; l_c \cdot p_d)}{\prod_{i=1}^{n} D_i^{\alpha_i}} F(p_i, p_f, \cdots)$$

$$\delta^+(l_j^2 - m_j^2) \to \frac{1}{l_j^2 - m_j^2 \pm i\epsilon}$$

Before integrating loop momenta

Integrals

- Loop integrals: $$\int\left[\prod_{j=1}^{L}\frac{d^d l_j}{(2\pi)^d}\right]\frac{\mathcal{N}(l_a^2\,,\,l_a\cdot l_b\,,\,l_c\cdot p_d)}{\prod_{i=1}^{n}D_i^{\alpha_i}}$$ UV and IR divergence: dimensional regularization
  - Quadratic: $D_i=(l_j-p_k)^2-m_i^2+i\epsilon$
  - Linear: $D_i=\pm\, l_j\cdot p_k+i\epsilon$
  - Potential: $D_i=q_0 l_0\pm \boldsymbol{l}^2\pm y+i\epsilon$

SCET$_{\mathrm{I}}$

SCET$_{\mathrm{II}}$ Rapidity divergence

- Phase-space integrals: $$\int\left[\prod_{j=1}^{N}\frac{d^d l_j}{(2\pi)^d}(2\pi)\delta^+(l_j^2-m_j^2)\right]\delta^{(d)}\left(\sum p_i-\sum l_f\right)\frac{\mathcal{N}}{\prod_{i=1}^{n}D_i^{\alpha_i}}F$$

$$\frac{8\,\epsilon\,\Gamma(1-4\epsilon)\,\omega^{1+4\epsilon}}{\pi^{2-2\epsilon}\,\Gamma^2(-\epsilon)}\int[dk_1]\,[dk_2]\,\frac{\delta\big(\omega-v_0\cdot(k_1+k_2)\big)}{(k_1+k_2)^2\;v_1\cdot k_2\;v_2\cdot k_1}$$

$$=-\frac{4\,\Gamma(-2\epsilon)\,\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)\,\Gamma(-3\epsilon)}\,{}_3F_2(-\epsilon,-\epsilon,-\epsilon;1-\epsilon,-3\epsilon;1)\,.$$

Regularization

- UV and IR divergence:
  - Dimensional regularization

CDR
HV
FDH
DRED

C. Gnendiger, *et al.* 1705.01827

Regularization

- UV and IR divergence:
  - Dimensional regularization

CDR
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DRED

C. Gnendiger, *et al.* 1705.01827

✓the integral of a linear combination of integrands equals the same linear combination of the corresponding integrals

✓one may cancel the same factors in the numerator and denominator of integrands.

✓<u>a derivative of an integral with respect to a mass or momentum equals the corresponding integral of the derivative.</u>

✓<u>any diagram with a detachable massless subgraph is zero.</u>

IBP
DEs

EFTs

Regularization

- UV and IR divergence:
  - Dimensional regularization
  - Pauli-Villars regularization
  - Mass regularization

CDR
HV
FDH
DRED

C. Gnendiger, *et al.* 1705.01827

- Rapidity divergence:
  - Analytic regularization $D_i^{\alpha_i} \to (\nu^2)^{\alpha} D_i^{\alpha_i - \alpha}$ T. Becher, *et al.* 1112.3907
  - Eta-regularization add $\omega^2 \nu^{\eta} |n \cdot l|^{-\eta}$ Jui-yu Chiu, *et al.* 1202.0814
  - Exponential regularization add $\exp(-b_0 \tau l^0)$ Y. Li, *et al.* 1604.00392

Scheme independent!

Methods for integrals

- Parameterizations:
  - Alpha Parameterization $$\frac{i}{A}=\int_0^\infty ds\,e^{isA}\,,\quad \mathrm{Im}(A)>0$$
  - Feynman Parameterization $$\frac{1}{AB}=\int_0^1 dxdy\,\delta(1-x-y)\frac{1}{(xA+yB)^2}$$
- Mellin-Barnes representation: $$\frac{1}{(X+Y)^\lambda}=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}dz\,\frac{\Gamma(\lambda+z)\Gamma(-z)}{\Gamma(\lambda)}\frac{Y^z}{X^{\lambda+z}}$$
- Asymptotic expansions: $$\frac{1}{(q-l)^2}=\frac{1}{q^2}+\frac{2q\cdot l-l^2}{(q^2)^2}+\cdots$$
- Residue theorem.

For some boundaries of differential equations

⋮

V. A. Smirnov, *Analytic Tools for Feynman Integrals*
J. Blümlein, *et al.* 1809.02889

Parameterization

- Alpha Parameterization: $\frac{i}{A} = \int_0^\infty ds\, e^{isA}\,, \quad \mathrm{Im}(A) > 0$

$$\frac{1}{AB} = -\int_0^\infty ds \int_0^\infty dt\; e^{isA + itB}$$

$$\frac{(m-1)!}{A_1^m A_2 \cdots A_n} = (-i)^{n+m-1} \int_0^\infty \prod_{i=1}^n ds_i\; s_1^{m-1} \exp\left(i \sum_{i=1}^n s_i A_i\right)$$

Parameterization

- Alpha Parameterization: $\frac{i}{A}=\int_0^{\infty} ds\, e^{isA}\,,\quad \mathrm{Im}(A)>0$

$$\frac{1}{AB}=-\int_0^{\infty} ds \int_0^{\infty} dt\, e^{isA+itB}$$

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- Feynman Parameterization: $\frac{1}{AB}=\int_0^1 dxdy\,\delta(1-x-y)\frac{1}{(xA+yB)^2}$

$$\frac{1}{A_1A_2\cdots A_n}=\int_0^1\prod_{i=1}^{n} dx_i\,\delta\left(1-\sum_{i=1}^{n} x_i\right)\frac{(n-1)!}{(x_1A_1+x_2A_2+\cdots+x_nA_n)^n}$$

$$\frac{(m-1)!}{A_1^m A_2\cdots A_n}=\int_0^1\prod_{i=1}^{n} dx_i\,\delta\left(1-\sum_{i=1}^{n} x_i\right)\frac{(n+m-2)!\,x_1^{m-1}}{(x_1A_1+x_2A_2+\cdots+x_nA_n)^{n+m}}$$

Parameterization

$$\frac{1}{AB} = -\int_0^\infty ds \int_0^\infty dt\; e^{isA+itB}$$

$$\frac{1}{AB} = \int_0^1 dx dy\, \delta(1-x-y)\, \frac{1}{(xA+yB)^2}$$

$$\tau = s+t\,,\ x = \frac{t}{s+t}$$

$$\frac{1}{AB} = -\int_0^1 dx \int_0^\infty d\tau\, \tau\, e^{i\tau[A+(B-A)x]}$$

$$= \int_0^1 dx\, \frac{1}{[A+(B-A)x]^2}$$

$$= \int_0^1 dx\, \frac{1}{[A(1-x)+Bx]^2}$$

$$\tau = \frac{x}{1-x}$$

$$\frac{1}{AB} = \int_0^\infty d\tau\, \frac{1}{(A+B\tau)^2}$$

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^\infty \prod_{i=2}^{n} d\tau_i\, \frac{(n-1)!}{(A_1+\tau_2 A_2+\cdots+\tau_n A_n)^n}$$

Parameterization

$$\frac{1}{AB} = -\int_0^\infty ds \int_0^\infty dt\, e^{isA+itB}$$

$$\frac{1}{AB} = \int_0^1 dx dy\, \delta(1-x-y) \frac{1}{(xA+yB)^2}$$

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$$= \int_0^1 dx \frac{1}{[A+(B-A)x]^2}$$

$$= \int_0^1 dx \frac{1}{[A(1-x)+Bx]^2}$$

$$\int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h \exp\left[\mathrm{i}\left(\sum_{i,j} A_{ij} k_i \cdot k_j + 2\sum_i q_i \cdot k_i\right)\right]$$

$$= \mathrm{e}^{\mathrm{i}\pi h(1-d/2)/2} \pi^{hd/2} (\det A)^{-d/2} \exp\left[-\mathrm{i}\sum_{i,j} A^{-1}_{ij} q_i \cdot q_j\right]$$

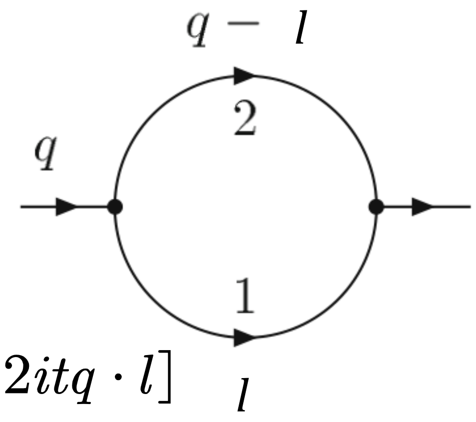
$$\tau = \frac{x}{1-x}$$

$$\frac{1}{AB} = \int_0^\infty d\tau \frac{1}{(A+B\tau)^2}$$

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^\infty \prod_{i=2}^n d\tau_i \frac{(n-1)!}{(A_1+\tau_2 A_2+\cdots+\tau_n A_n)^n}$$

Example 1

$$I(m_1^2, m_2^2, q^2; \lambda_1, \lambda_2; d) = \mathcal{C}\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - m_1^2 + i\varepsilon)^{\lambda_1}[(l+q)^2 - m_2^2 + i\varepsilon]^{\lambda_2}}$$



$$\mathcal{C} = -i(4\pi)^{2-\epsilon}, \; d = 4 - 2\epsilon$$

$$I = \frac{\mathcal{C}(-i)^{\lambda_1+\lambda_2}}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^\infty ds dt\, s^{\lambda_1} t^{\lambda_2} \exp[i(tq^2 - tm_2^2 - sm_1^2)] \int \frac{d^d l}{(2\pi)^d} \exp[i(s+t)l^2 + 2itq \cdot l]$$

Gauss integral

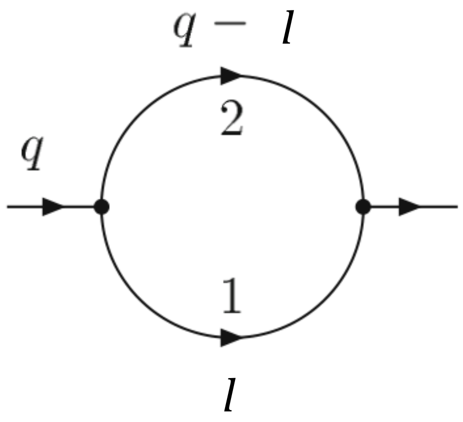
$$I = \frac{\mathcal{C}(-i)^{\lambda_1+\lambda_2+1}}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \frac{e^{i\pi\epsilon/2}}{(4\pi)^{2-\epsilon}} \int_0^\infty ds dt\, s^{\lambda_1} t^{\lambda_2} (s+t)^{(-2+\epsilon)} \exp\left[i\left(\frac{st}{s+t} q^2 - tm_2^2 - sm_1^2\right)\right]$$

$$s = \eta\xi, \; t = \eta(1-\xi)$$

$$I = \frac{\Gamma(\lambda_1 + \lambda_2 + \epsilon - 2)}{(-1)^{\lambda_1+\lambda_2}\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^1 d\xi \frac{\xi^{\lambda_1-1}(1-\xi)^{\lambda_2-1}}{[-q^2\xi(1-\xi) + \xi m_1^2 + (1-\xi)m_2^2 - i\varepsilon]^{\lambda_1+\lambda_2+\epsilon-2}}$$

Example 1

$$I=\frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)}{(-1)^{\lambda_1+\lambda_2}\Gamma(\lambda_1)\Gamma(\lambda_2)}\int_0^1 d\xi\,\frac{\xi^{\lambda_1-1}(1-\xi)^{\lambda_2-1}}{[-q^2\xi(1-\xi)+\xi m_1^2+(1-\xi)m_2^2-i\varepsilon]^{\lambda_1+\lambda_2+\epsilon-2}}$$



$$\Big\downarrow\; m_1=0\,,\; m_2=0$$

$$I=\frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)}{(-1)^{\lambda_1+\lambda_2}\Gamma(\lambda_1)\Gamma(\lambda_2)}\frac{\Gamma(2-\lambda_1-\epsilon)\Gamma(2-\lambda_2-\epsilon)}{\Gamma(4-\lambda_1-\lambda_2-2\epsilon)}(-q^2-i\varepsilon)^{2-\lambda_1-\lambda_2-\epsilon}$$

$$I(m_1^2,m_2^2,q^2;\lambda_1,\lambda_2;d)$$

$$I(m^2,m^2,q^2;1,1;d)$$

$$\Big\downarrow\; q^2=4m^2$$

$$I(m^2,m^2,4m^2;1,1;d)=\Gamma(\epsilon)\frac{m^{-2\epsilon}}{1-2\epsilon}$$

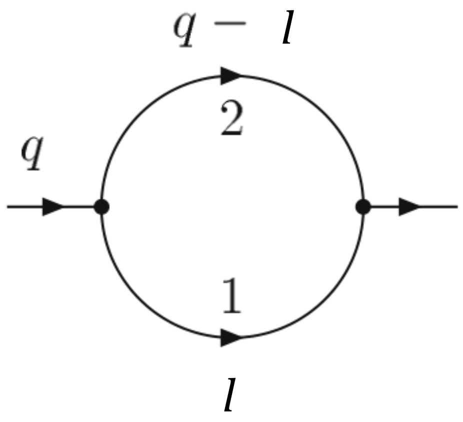
$$I(m^2,m^2,q^2;2,1;d)$$

$$\cancel{\Big\downarrow}\; q^2=4m^2$$

$$I(m^2,m^2,4m^2;2,1;d)=-\Gamma(1+\epsilon)\frac{-m^{-2-2\epsilon}}{2+4\epsilon}$$

Example 1

$$I=\frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)}{(-1)^{\lambda_1+\lambda_2}\Gamma(\lambda_1)\Gamma(\lambda_2)}\int_0^1 d\xi\,\frac{\xi^{\lambda_1-1}(1-\xi)^{\lambda_2-1}}{[-q^2\xi(1-\xi)+\xi m_1^2+(1-\xi)m_2^2-i\varepsilon]^{\lambda_1+\lambda_2+\epsilon-2}}$$



$m_1=0\ ,\ m_2=0$

$$I=\frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)}{(-1)^{\lambda_1+\lambda_2}\Gamma(\lambda_1)\Gamma(\lambda_2)}\frac{\Gamma(2-\lambda_1-\epsilon)\Gamma(2-\lambda_2-\epsilon)}{\Gamma(4-\lambda_1-\lambda_2-2\epsilon)}(-q^2-i\varepsilon)^{2-\lambda_1-\lambda_2-\epsilon}$$

- Linear propagators

$$\frac{i}{A}=\int_0^\infty ds\,e^{isA}\,,\quad \mathrm{Im}(A)>0$$

$$\frac{1}{AB}=\int_0^\infty d\tau\,\frac{1}{(A+B\tau)^2}$$

$$I_1=\mathcal{C}\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2)^{\lambda_1}[-2n\cdot l]^{\lambda_2}[(q+r_\perp)^2]^{\lambda_3}(2\bar{n}\cdot l)^{\lambda_4}}$$

Analytic regulator

Eta-regulator

$$I_1=\mathcal{C}\int\frac{d^dl}{(2\pi)^d}\frac{|2l_z|^{-\eta}}{(l^2)\,[-2n\cdot l]\,(q+r_\perp)^2(2\bar{n}\cdot l)}$$

Some formulas

$$\frac{1}{A^\alpha B^\beta}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dx\frac{x^{\alpha-1}(1-x)^{\beta-1}}{\{xA+(1-x)B\}^{\alpha+\beta}}\ ,$$

$$\frac{1}{A^\alpha B^\beta C^\gamma}=\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_0^1 xdx\int_0^1 dy\frac{u_1^{\alpha-1}u_2^{\beta-1}u_3^{\gamma-1}}{\{u_1A+u_2B+u_3C\}^{\alpha+\beta+\gamma}}$$

$(u_1=xy\ ,u_2=x(1-y)\ ,u_3=1-x)$,

$$\frac{1}{A^\alpha B^\beta C^\gamma D^\delta}=\frac{\Gamma(\alpha+\beta+\gamma+\delta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}$$

$$\times\int_0^1 x^2dx\int_0^1 ydy\int_0^1 dz\frac{u_1^{\alpha-1}u_2^{\beta-1}u_3^{\gamma-1}u_4^{\delta-1}}{\{u_1A+u_2B+u_3C+u_4D\}^{\alpha+\beta+\gamma+\delta}}\ ,$$

$(u_1=1-x\ ,u_2=xyz\ ,u_3=x(1-y)\ ,u_4=xy(1-z))$ 。

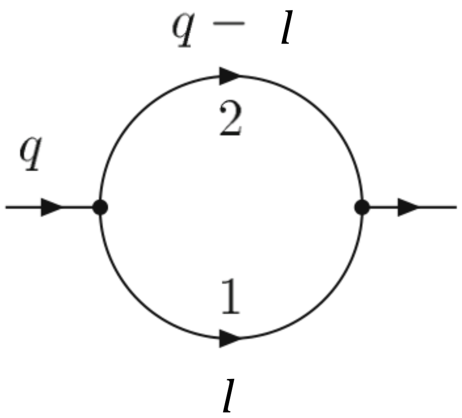
Mellin-Barnes representation

https://mbtools.hepforge.org/

$$\boxed{\frac{1}{(X+Y)^{\lambda}} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \frac{\Gamma(\lambda+z)\Gamma(-z)}{\Gamma(\lambda)}\frac{Y^{z}}{X^{\lambda+z}}}$$

AMBRE.m

J. Gluza, *et. al.* : 0704.2423



$$I(m_1^2, m_2^2, q^2; \lambda_1, \lambda_2; d) = \mathcal{C}\int \frac{d^d l}{(2\pi)^d}\frac{1}{(l^2 - m_1^2 + i\varepsilon)^{\lambda_1}[(l+q)^2 - m_2^2 + i\varepsilon]^{\lambda_2}}$$

- Applied to massive propagators: $X = (l+q)^2,\ Y = m_2^2$

Mellin-Barnes representation

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$$\boxed{\frac{1}{(X+Y)^{\lambda}} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \frac{\Gamma(\lambda+z)\Gamma(-z)}{\Gamma(\lambda)} \frac{Y^{z}}{X^{\lambda+z}}}$$

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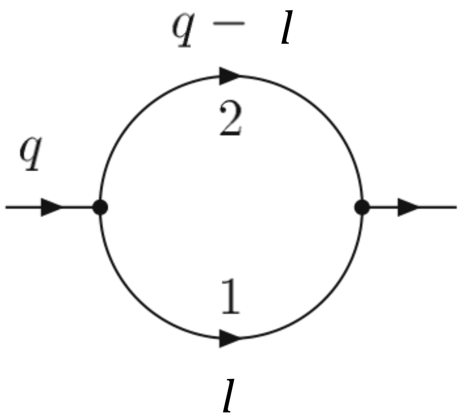


$$I(m_1^2, m_2^2, q^2; \lambda_1, \lambda_2; d) = \mathcal{C}\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - m_1^2 + i\varepsilon)^{\lambda_1}[(l+q)^2 - m_2^2 + i\varepsilon]^{\lambda_2}}$$

- Applied to massive propagators: $X = (l+q)^2,\ Y = m_2^2$

$$I = \frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)}{(-1)^{\lambda_1+\lambda_2}\Gamma(\lambda_1)\Gamma(\lambda_2)}\int_0^1 d\xi \frac{\xi^{\lambda_1-1}(1-\xi)^{\lambda_2-1}}{[-q^2\xi(1-\xi)+\xi m_1^2+(1-\xi)m_2^2 - i\varepsilon]^{\lambda_1+\lambda_2+\epsilon-2}}$$

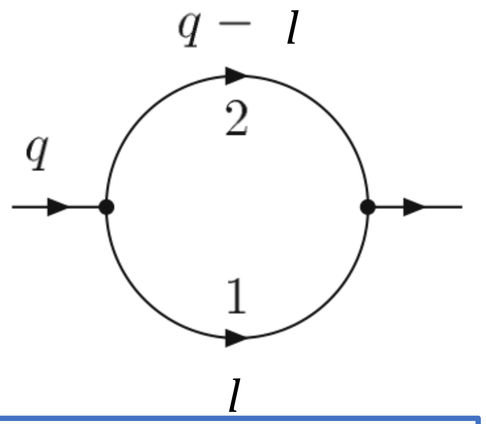
- Applied to integrals of Feynman parameters:

$$X = \xi m_1^2 + (1-\xi)m_2^2,\ Y = -q^2\xi(1-\xi)$$

- The poles from $\Gamma(\cdots+z)$ and $\Gamma(\cdots-z)$ at different sides of contour.

Mellin-Barnes representation

$$I(m^2,m^2,q^2;1,1;d)=\Gamma(\epsilon)\int_0^1 d\xi\,\frac{1}{[-q^2\xi(1-\xi)+m^2-i\varepsilon]^{\epsilon}}$$



$$I=\frac{\Gamma(\epsilon)}{2\pi i}\int_{-i\infty}^{i\infty}dz\,\frac{\Gamma(\epsilon+z)\Gamma(-z)}{\Gamma(\epsilon)\,(-q^2-i\varepsilon)^{\epsilon}}\left[\frac{m^2}{-q^2-i\varepsilon}\right]^z\int_0^1 d\xi\,\frac{1}{[\xi(1-\xi)]^{\epsilon+z}}$$

$$\boxed{\frac{1}{(X+Y)^{\lambda}}=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}dz\,\frac{\Gamma(\lambda+z)\Gamma(-z)}{\Gamma(\lambda)}\frac{Y^z}{X^{\lambda+z}}}$$

$$I=\frac{(-q^2-i\varepsilon)^{-\epsilon}}{2\pi i}\int_{-i\infty}^{i\infty}dz\,\frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma^2(1-\epsilon-z)}{\Gamma(2-2\epsilon-2z)}\left[\frac{m^2}{-q^2-i\varepsilon}\right]^z$$

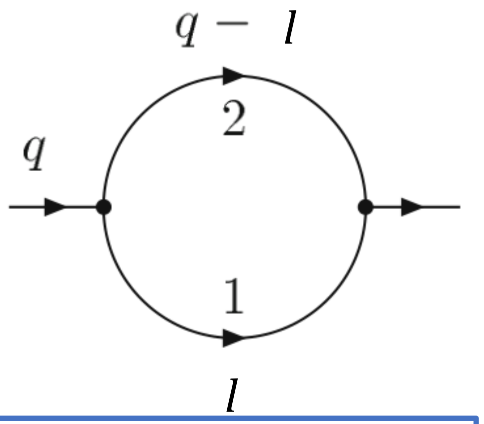
$$\Gamma(2+x)=\frac{2^{1+x}}{\sqrt{\pi}}\Gamma\left(1+\frac{x}{2}\right)\Gamma\left(\frac{3}{2}+\frac{x}{2}\right)$$

$$I=\frac{\Gamma(\epsilon)}{(m^2-i\varepsilon)^{\epsilon}}\,{}_2F_1\left(1,\epsilon,\frac{3}{2};\frac{q^2}{4m^2}\right)$$

$$\boxed{\frac{\left(4\,\mathrm{m}^2-\mathrm{q}^2\right)^{\frac{1}{2}-\mathrm{eps}}\left(-\mathrm{Beta}\left[\frac{1}{2}\left(1-\sqrt{-\frac{\mathrm{q}^2}{4\,\mathrm{m}^2-\mathrm{q}^2}}\right),1-\mathrm{eps},1-\mathrm{eps}\right]+\mathrm{Beta}\left[\frac{1}{2}\left(1+\sqrt{-\frac{\mathrm{q}^2}{4\,\mathrm{m}^2-\mathrm{q}^2}}\right),1-\mathrm{eps},1-\mathrm{eps}\right]\right)\mathrm{Gamma}[\mathrm{eps}]}{\sqrt{-\mathrm{q}^2}}}$$

Mellin-Barnes representation

$$I(m^2,m^2,q^2;1,1;d)=\Gamma(\epsilon)\int_0^1 d\xi \frac{1}{[-q^2\xi(1-\xi)+m^2-i\varepsilon]^{\epsilon}}$$



$$I=\frac{\Gamma(\epsilon)}{2\pi i}\int_{-i\infty}^{i\infty}dz\frac{\Gamma(\epsilon+z)\Gamma(-z)}{\Gamma(\epsilon)\,(-q^2-i\varepsilon)^{\epsilon}}\left[\frac{m^2}{-q^2-i\varepsilon}\right]^z\int_0^1 d\xi\frac{1}{[\xi(1-\xi)]^{\epsilon+z}}$$

$$\boxed{\frac{1}{(X+Y)^{\lambda}}=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}dz\frac{\Gamma(\lambda+z)\Gamma(-z)}{\Gamma(\lambda)}\frac{Y^z}{X^{\lambda+z}}}$$

$$I=\frac{(-q^2-i\varepsilon)^{-\epsilon}}{2\pi i}\int_{-i\infty}^{i\infty}dz\frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma^2(1-\epsilon-z)}{\Gamma(2-2\epsilon-2z)}\left[\frac{m^2}{-q^2-i\varepsilon}\right]^z$$

$$\Gamma(2+x)=\frac{2^{1+x}}{\sqrt{\pi}}\Gamma\left(1+\frac{x}{2}\right)\Gamma\left(\frac{3}{2}+\frac{x}{2}\right)$$

$$I=\frac{\Gamma(\epsilon)}{(m^2-i\varepsilon)^{\epsilon}}\,{}_2F_1\left(1,\epsilon,\frac{3}{2};\frac{q^2}{4m^2}\right)$$

$$I(m^2,m^2,q^2;\lambda,1;d)=\frac{(-1)^{1+\lambda}\Gamma(\epsilon+\lambda-1)}{(m^2-i\varepsilon)^{\epsilon+\lambda-1}\Gamma(1+\lambda)}\,{}_3F_2\left(1,\lambda,\epsilon+\lambda-1;\frac{1}{2}+\frac{\lambda}{2},1+\frac{\lambda}{2};\frac{q^2}{4m^2}\right)$$

$\lambda=2$, $q^2\to 4m^2$ singularity

Mellin-Barnes representation

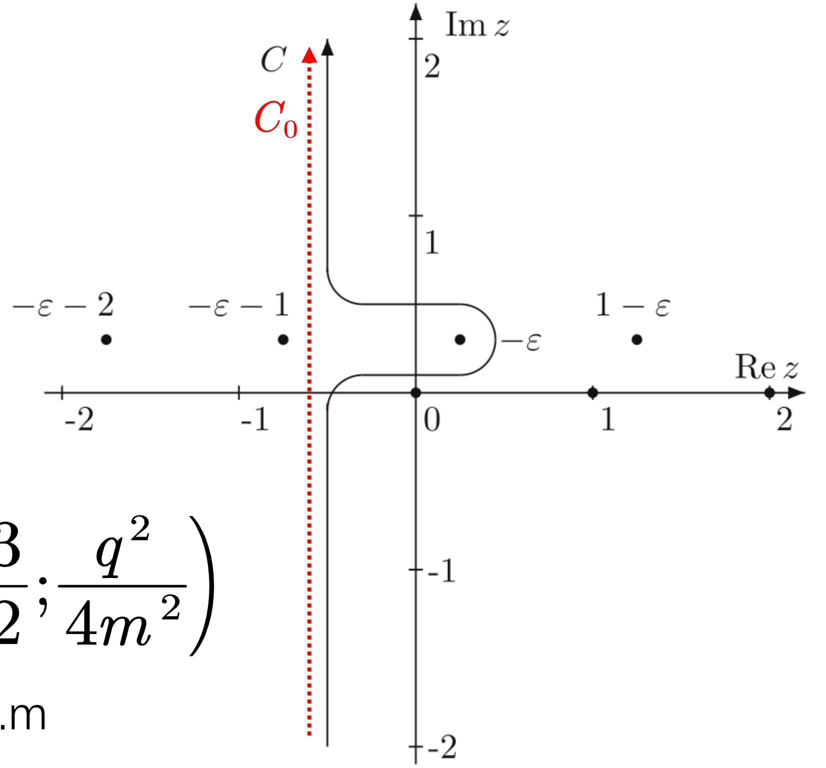
- Resolving singularities

$$I = \frac{(-q^2 - i\varepsilon)^{-\epsilon}}{2\pi i}\int_{-i\infty}^{i\infty} dz \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma^2(1-\epsilon-z)}{\Gamma(2-2\epsilon-2z)}\left[\frac{m^2}{-q^2-i\varepsilon}\right]^z$$

  - Strategy A: shift the contour

$$I = \Gamma(\epsilon)\int_0^1 d\xi \frac{1}{[-q^2\xi(1-\xi)+m^2-i\varepsilon]^\epsilon}$$

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}}\int_C f(z,\varepsilon)\mathrm{d}z &= \frac{1}{2\pi \mathrm{i}}\int_{C_0} f(z,\varepsilon)\mathrm{d}z \\ &+ \left(\frac{1}{2\pi \mathrm{i}}\int_C f(z,\varepsilon)\mathrm{d}z - \frac{1}{2\pi \mathrm{i}}\int_{C_0} f(z,\varepsilon)\mathrm{d}z\right) \\ &= \frac{1}{2\pi \mathrm{i}}\int_{C_0} f(z,\varepsilon)\mathrm{d}z + \mathrm{res}_{z=-\varepsilon} f(z,\varepsilon)\end{aligned}$$



$$I_\epsilon = \frac{\Gamma(\epsilon)}{(m^2)^\epsilon} = \frac{1}{\epsilon} - \gamma_E - \ln(m^2) + \mathcal{O}(\epsilon)$$

$$I = \frac{\Gamma(\epsilon)}{(m^2-i\varepsilon)^\epsilon}\,{}_2F_1\left(1,\epsilon,\frac{3}{2};\frac{q^2}{4m^2}\right)$$

$$I_0 = 2 - \beta\ln\left(\frac{\beta+1}{\beta-1}\right) + \mathcal{O}(\epsilon)$$

$$\beta = \sqrt{1-\frac{4m^2}{q^2}}$$

MBresolve.m

A. V. Smirnov, *et,al* : 0901.0386

Mellin-Barnes representation

- Resolving singularities

$$I = \frac{(-q^2 - i\varepsilon)^{-\epsilon}}{2\pi i} \int_{-i\infty}^{i\infty} dz \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma^2(1-\epsilon-z)}{\Gamma(2-2\epsilon-2z)} \left[\frac{m^2}{-q^2-i\varepsilon}\right]^z$$

  - Strategy B: shift the regulator ϵ

$$I = \Gamma(\epsilon) \int_0^1 d\xi \frac{1}{[-q^2\xi(1-\xi) + m^2 - i\varepsilon]^\epsilon}$$

    a) choose a straight contour and fixed ϵ

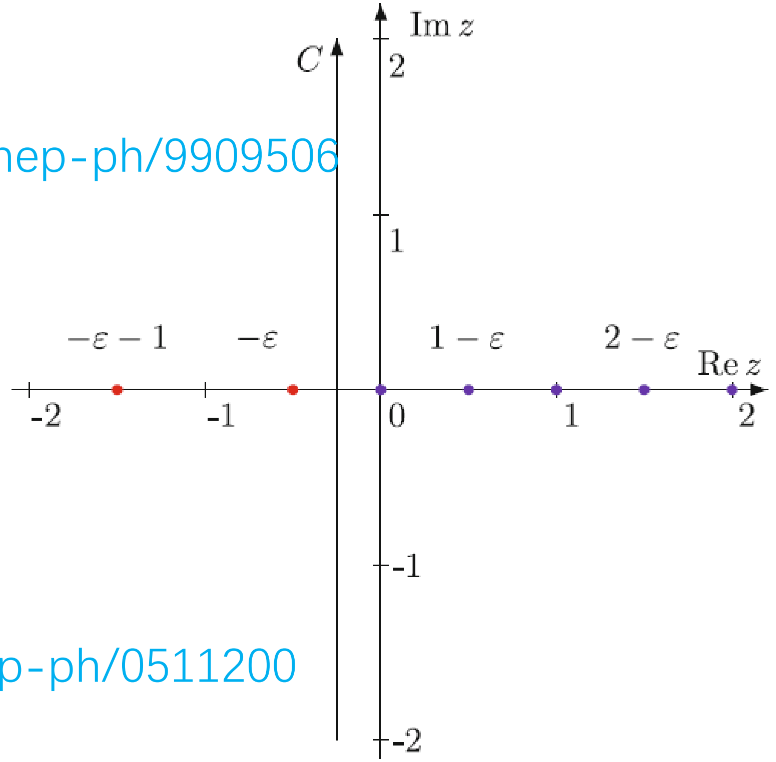
$$\Re(z) = -\frac{1}{4}, \quad \epsilon = \frac{1}{2}$$

    b) keep contour fixed and tend ϵ to zero:

J.B. Tausk: hep-ph/9909506

the first pole of $\Gamma(\epsilon + z)$ cross the contour,

add the residue $I_\epsilon = \dfrac{\Gamma(\epsilon)}{(m^2)^\epsilon}$



MB.m

M. Czakon: hep-ph/0511200

Mellin-Barnes representation

- the first Barnes lemma:

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z) = \frac{\Gamma(\lambda_1+\lambda_3)\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_3)\Gamma(\lambda_2+\lambda_4)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}.$$

- the second Barnes lemma:

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma(\lambda_4-z)\Gamma(\lambda_5-z)}{\Gamma(\lambda_6+z)} = \frac{\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_4)\Gamma(\lambda_3+\lambda_4)\Gamma(\lambda_1+\lambda_5)}{\Gamma(\lambda_1+\lambda_2+\lambda_4+\lambda_5)\Gamma(\lambda_1+\lambda_3+\lambda_4+\lambda_5)} \times \frac{\Gamma(\lambda_2+\lambda_5)\Gamma(\lambda_3+\lambda_5)}{\Gamma(\lambda_2+\lambda_3+\lambda_4+\lambda_5)},$$

$$\lambda_6 = \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5.$$

$$K = \frac{1}{(2\pi i)^4}\int_{-i\infty}^{i\infty} \mathrm{d}\xi_1 \int_{-i\infty}^{i\infty} \mathrm{d}\xi_2 \int_{-i\infty}^{i\infty} \mathrm{d}\alpha \int_{-i\infty}^{i\infty} \mathrm{d}\beta \times (-s)^{-3-2\epsilon-\xi_1-\xi_2}(-t)^{\xi_1}(-u)^{\xi_2}\frac{\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7\Gamma_8\Gamma_9\Gamma_{10}\Gamma_{11}\Gamma_{12}\Gamma_{13}}{\Gamma_{14}^2}$$

J.B. Tausk: hep-ph/9909506

Mellin-Barnes representation

- Special summation:

$$
\begin{aligned}
&{}_2F_1(a, b; c; x)\\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(-z)}{\Gamma(c+z)} (-x)^z dz\\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}\\
&\quad \times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \Gamma(a+z)\Gamma(b+z)\,\Gamma(c-a-b-z)\Gamma(-z)(1-x)^z dz.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \Gamma(a+z)\Gamma(b+z)\,\Gamma(c-z)\Gamma(-z)\,x^z dz\\
&= \Gamma(a+c)\Gamma(b+c) \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(-z)}{\Gamma(a+b+c+z)} (x-1)^z dz.
\end{aligned}
$$

XSummer S. Moch, *et.al.*: math-ph/0508008

SUMMER J.A.M. Vermaseren: hep-ph/9806280

Mellin-Barnes representation

- Special summation:

$$
\begin{aligned}
{}_2F_1(a,b;c;x) \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma(a+z)\Gamma(b+z)\Gamma(-z)}{\Gamma(c+z)}(-x)^z dz \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \\
&\quad\times \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\Gamma(a+z)\Gamma(b+z)\,\Gamma(c-a-b-z)\Gamma(-z)(1-x)^z dz.
\end{aligned}
$$

$$
\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1)\frac{z^n}{n} = -\frac{1}{2}\mathrm{Li}_2(z)^2 + \ln(1-z)(S_{1,2}(z)-\mathrm{Li}_3(z)) + \frac{1}{2}\ln^2(1-z)\mathrm{Li}_2(z),
$$

$$
S_i(n) = \sum_{j=1}^{n}\frac{1}{j^i},
$$

$$
\psi(n) = S_1(n-1) - \gamma_E,
$$

$$
\psi^{(k)}(n) = (-1)^k k!\,(S_{k+1}(n-1) - \zeta(k+1)), \quad k = 1,2,\ldots,
$$

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\Gamma(a+z)\Gamma(b+z)\,\Gamma(c-z)\Gamma(-z)\,x^z dz \\
&= \Gamma(a+c)\Gamma(b+c)\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma(a+z)\Gamma(b+z)\Gamma(-z)}{\Gamma(a+b+c+z)}(x-1)^z dz.
\end{aligned}
$$

XSummer S. Moch, *et.al.*: math-ph/0508008

SUMMER J.A.M. Vermaseren: hep-ph/9806280

Asymptotic expansions

- Expansion by regions:
  - Divide the integration domain into various regions, and expand the integrand into a Taylor.
  - Integrate every expanded integrand over the whole integration domain and sum them all.

$$\begin{aligned} \text{hard:} \quad & q^\mu \sim \hat{s}\,, \\ \text{soft:} \quad & q^\mu \sim \hat{s}\lambda\,, \\ \text{collinear:} \quad & q^\mu \sim \hat{s}(1\,,\lambda^2,\lambda)\,, \\ \text{Glauber:} \quad & q^\mu \sim \hat{s}(\lambda^2,\lambda^2,\lambda)\,, \end{aligned}$$

- Expansion by Subgraphs:
  - Confirm the large and small variables and expand.
  - Integrate every expanded integrand and sum them all.

Overlapping and divergences!

Asymptotic expansions

- Expansion by regions:
  - Divide the integration domain into various regions, and expand the integrand into a Taylor.
  - Integrate every expanded integrand over the whole integration domain and sum them all.

$$\begin{aligned}
&\text{hard:} && q^\mu \sim \hat{s}\,, \\
&\text{soft:} && q^\mu \sim \hat{s}\lambda\,, \\
&\text{collinear:} && q^\mu \sim \hat{s}(1,\lambda^2,\lambda)\,, \\
&\text{Glauber:} && q^\mu \sim \hat{s}(\lambda^2,\lambda^2,\lambda)\,,
\end{aligned}$$

- Expansion b... f... ... ...
  - Confirm $I = \frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)}{(-1)^{\lambda_1+\lambda_2}\Gamma(\lambda_1)\Gamma(\lambda_2)}\int_0^1 d\xi \frac{\xi^{\lambda_1-1}(1-\xi)^{\lambda_2-1}}{[-q^2\xi(1-\xi)+\xi m_1^2+(1-\xi)m_2^2-i\varepsilon]^{\lambda_1+\lambda_2+\epsilon-2}}$
  - Integrate

$$I = \frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)}{(-1)^{\lambda_1+\lambda_2}\Gamma(\lambda_1)\Gamma(\lambda_2)}\frac{\Gamma(2-\lambda_1-\epsilon)\Gamma(2-\lambda_2-\epsilon)}{\Gamma(4-\lambda_1-\lambda_2-2\epsilon)}(-q^2-i\varepsilon)^{2-\lambda_1-\lambda_2-\epsilon}$$

If $\lambda_1=1\,,\ \lambda_2=2$

Overlapping and divergences!

V. A. Smirnov, *Applied asymptotic expansions in momenta and masses*

Asymptotic expansions

$$F(q,m,\varepsilon)=\int_0^\infty \frac{k^{-\varepsilon}\,\mathrm{d}k}{(k+m)(k+q)} \quad \text{with } \ 0<m\ll q$$

$$F(q,m,\varepsilon)=-\frac{\pi}{\sin\pi\varepsilon}\,\frac{q^{-\varepsilon}-m^{-\varepsilon}}{q-m}$$

$$F(q,m,0)=\frac{\ln(q/m)}{q-m}$$

- Naively expand with m: LP term is divergent!

$$F_{\text{large}}\sim\int_0^\infty \mathrm{d}k\,\frac{k^{-\varepsilon-1}}{k+q}-m\int_0^\infty \mathrm{d}k\,\frac{k^{-\varepsilon-2}}{k+q}+\ldots \quad \boxed{k>m}$$

$$=\sum_{n=0}^\infty(-1)^n m^n\int_0^\infty\frac{k^{-n-\varepsilon-1}\,\mathrm{d}k}{k+q}=\frac{-\pi}{q^{1+\varepsilon}\sin\pi\varepsilon}\sum_{n=0}^\infty\left(\frac{m}{q}\right)^n$$

$$F_{\text{small}}\sim\frac{1}{q}\int_0^\infty \mathrm{d}k\,\frac{k^{-\varepsilon}}{k+m}-\frac{1}{q^2}\int_0^\infty \mathrm{d}k\,\frac{k^{-\varepsilon+1}}{k+m}+\ldots \quad \boxed{k<q}$$

$$=\sum_{n=0}^\infty\frac{(-1)^n}{q^{n+1}}\int_0^\infty\frac{k^{n-\varepsilon}\,\mathrm{d}k}{k+m}=\frac{\pi}{qm^\varepsilon\sin\pi\varepsilon}\sum_{n=0}^\infty\left(\frac{m}{q}\right)^n$$

Asymptotic expansions

$$F(q,m,\varepsilon)=\int_0^\infty \frac{k^{-\varepsilon}\,\mathrm{d}k}{(k+m)(k+q)} \quad \text{with } \ 0<m\ll q$$

- Naively expand with m: LP term is divergent!

$$F_{\text{large}}\sim\int_0^\infty \mathrm{d}k\,\frac{k^{-\varepsilon-1}}{k+q}-m\int_0^\infty \mathrm{d}k\,\frac{k^{-\varepsilon-2}}{k+q}+\ldots \quad \boxed{k>m}$$

$$=\sum_{n=0}^{\infty}(-1)^n m^n\int_0^\infty \frac{k^{-n-\varepsilon-1}\,\mathrm{d}k}{k+q}=\frac{-\pi}{q^{1+\varepsilon}\sin\pi\varepsilon}\sum_{n=0}^{\infty}\left(\frac{m}{q}\right)^n$$

$$F_{\text{small}}\sim\frac{1}{q}\int_0^\infty \mathrm{d}k\,\frac{k^{-\varepsilon}}{k+m}-\frac{1}{q^2}\int_0^\infty \mathrm{d}k\,\frac{k^{-\varepsilon+1}}{k+m}+\ldots \quad \boxed{k<q}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{q^{n+1}}\int_0^\infty \frac{k^{n-\varepsilon}\,\mathrm{d}k}{k+m}=\frac{\pi}{qm^{\varepsilon}\sin\pi\varepsilon}\sum_{n=0}^{\infty}\left(\frac{m}{q}\right)^n$$

Sum to

$$F(q,m,\varepsilon)=-\frac{\pi}{\sin\pi\varepsilon}\,\frac{q^{-\varepsilon}-m^{-\varepsilon}}{q-m}$$

$$F(q,m,0)=\frac{\ln(q/m)}{q-m}$$

$$F\sim F_{\text{small}}+F_{\text{large}}$$

$$\boxed{m<\Lambda<q}$$

$$F=f_{\text{small}}+f_{\text{large}}$$

$$f_{\text{small}}(q,m,\Lambda)=\int_0^\Lambda \frac{k^{-\varepsilon}\,\mathrm{d}k}{(k+m)(k+q)}$$

$$f_{\text{large}}(q,m,\Lambda)=\int_\Lambda^\infty \frac{k^{-\varepsilon}\,\mathrm{d}k}{(k+m)(k+q)}$$

Asymptotic expansions

$$f_{\text{small}} \sim F_{\text{small}} - \left(\frac{1}{q} \int_{\Lambda}^{\infty} \mathrm{d}k \, \frac{k^{-\varepsilon}}{k+m} - \frac{1}{q^2} \int_{\Lambda}^{\infty} \mathrm{d}k \, \frac{k^{-\varepsilon+1}}{k+m} + \ldots \right) ,$$

$$f_{\text{large}} \sim F_{\text{large}} - \left(\int_{0}^{\Lambda} \mathrm{d}k \, \frac{k^{-\varepsilon-1}}{k+q} - m \int_{0}^{\Lambda} \mathrm{d}k \, \frac{k^{-\varepsilon-2}}{k+q} + \ldots \right) ,$$

Expand with m and q, again

$$\frac{1}{q} \int_{\Lambda}^{\infty} \mathrm{d}k \, k^{-\varepsilon-1} = \frac{1}{\varepsilon q \Lambda^{\varepsilon}} ,$$

$$\frac{1}{q} \int_{0}^{\Lambda} \mathrm{d}k \, k^{-\varepsilon-1} = -\frac{1}{\varepsilon q \Lambda^{\varepsilon}} ,$$

Asymptotic expansions

$$f_{\text{small}} \sim F_{\text{small}} - \left(\frac{1}{q}\int_\Lambda^\infty \mathrm{d}k\,\frac{k^{-\varepsilon}}{k+m} - \frac{1}{q^2}\int_\Lambda^\infty \mathrm{d}k\,\frac{k^{-\varepsilon+1}}{k+m} + \ldots\right),$$

$$f_{\text{large}} \sim F_{\text{large}} - \left(\int_0^\Lambda \mathrm{d}k\,\frac{k^{-\varepsilon-1}}{k+q} - m\int_0^\Lambda \mathrm{d}k\,\frac{k^{-\varepsilon-2}}{k+q} + \ldots\right),$$

Expand with m and q, again

$$\frac{1}{q}\int_\Lambda^\infty \mathrm{d}k\,k^{-\varepsilon-1} = \frac{1}{\varepsilon q \Lambda^\varepsilon},$$

$$\frac{1}{q}\int_0^\Lambda \mathrm{d}k\,k^{-\varepsilon-1} = -\frac{1}{\varepsilon q \Lambda^\varepsilon},$$

- Expansion by Subgraphs:

$$\mathcal{R}^n F(q,m;\varepsilon) = \int_0^\infty \mathrm{d}k\,k^{-\varepsilon}\underbrace{\boxed{\left(\frac{1}{k+m} - \frac{1}{k} + \frac{m}{k^2} - \ldots - (-1)^n\frac{m^n}{k^{n+1}}\right)}}_{\propto \frac{1}{k^{n+2}}}$$

$$\times \underbrace{\boxed{\left(\frac{1}{k+q} - \frac{1}{q} + \frac{k}{q^2} - \ldots - (-1)^n\frac{k^n}{q^{n+1}}\right)}}_{\propto k^{n+1}}.$$

$$\mathcal{T}_x^n f(x) = \sum_{j=0}^n \frac{1}{j!} f^{(j)}(0) x^j$$

$$\mathcal{R}^n F(q,m;\varepsilon) = \int_0^\infty \mathrm{d}k\,k^{-\varepsilon}\left((1-\mathcal{T}_m^n)\frac{1}{k+m}\right)\left((1-\mathcal{T}_k^n)\frac{1}{k+q}\right) \Rightarrow \mathcal{R}^n F(q,m;\varepsilon) = \left(\frac{m}{q}\right)^{n+1} F(q,m;\varepsilon)$$

$$\mathcal{R}^n = (1-\mathcal{M}_{\text{large}}^n)(1-\mathcal{M}_{\text{small}}^n)$$

$$\boxed{\begin{aligned} F &= (1-\mathcal{R}^n)F + \mathcal{R}^n F \\ &\equiv (\mathcal{M}_{\text{large}}^n + \mathcal{M}_{\text{small}}^n - \mathcal{M}_{\text{large}}^n\mathcal{M}_{\text{small}}^n)F + \mathcal{R}^n F \end{aligned}}$$

Example 3

- Expansion by Subgraphs: $I(m^2,0,q^2;2,1;d)=\mathcal{C}\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-m^2+i\varepsilon)^2[(l+q)^2+i\varepsilon]}$

$$\left\{\frac{1}{(l+q)^2}\right\}=\frac{1}{q^2}+\frac{-l^2-2q\cdot l}{(q^2)^2}+\frac{(-l^2-2q\cdot l)^2}{(q^2)^3}+\cdots$$

$$\left\{\frac{1}{(l^2-m^2)^2}\right\}=\frac{1}{(l^2)^2}+\frac{2m^2}{(l^2)^3}+\frac{3(m^2)^2}{(l^2)^4}+\cdots$$

$$I=\mathcal{C}\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-m^2+i\varepsilon)^2}\left\{\frac{1}{(l+q)^2+i\varepsilon}\right\}+\left\{\frac{1}{(l^2-m^2+i\varepsilon)^2}\right\}\frac{1}{(l+q)^2+i\varepsilon}$$

$$\Rightarrow$$

$$I=-\frac{1}{q^2}\frac{\Gamma^2(1-\epsilon)\Gamma(\epsilon)}{(q^2)^\epsilon\Gamma(1-2\epsilon)}\left(1+2\epsilon\frac{m^2}{q^2}+\cdots\right)+\frac{1}{q^2}\frac{\Gamma(\epsilon)}{(m^2)^\epsilon}\left(1+\frac{\epsilon}{1+\epsilon}\frac{m^2}{q^2}+\cdots\right)$$

$$\nearrow$$

$$I=-\frac{1}{q^2}\ln\left(\frac{m^2}{m^2-q^2}\right)$$

$$I=-\Gamma(1+\epsilon)\int_0^1 d\xi\,\frac{\xi^{-\epsilon}}{[-q^2(1-\xi)+m^2-i\varepsilon]^{1+\epsilon}}$$

More examples

$$I_1 = \int_0^1 db\, (b^2)^{1-\epsilon} (1-b^2)^{\frac{\phi}{2}-2\epsilon} {}_2F_1\left(\frac{1}{2}, 1, \frac{3}{2}-\epsilon, b^2\right) {}_2F_1\left(\frac{1}{2}-\epsilon, 1-\epsilon, \frac{3}{2}+\psi-\epsilon, b^2\right)$$

⊛ $\phi = \psi = 0$, just expand the integrand

$$I_1(\phi=0\,,\psi=0) = \frac{\pi^2}{12} + \left(-\frac{\ln(2)\pi^2}{6} + \frac{7\zeta(3)}{2}\right)\epsilon + \left(\frac{\pi^3}{3} + \frac{\pi^4}{9} + \frac{\ln^2(2)\pi^2}{6} - 7\ln(2)\zeta(3)\right)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$$\begin{aligned} H(^n0;x) &= \frac{1}{n!}\log^n x \\ H(a, a_{1,\ldots,k};x) &= \int_0^x f_a(t) H(a_{1,\ldots,k};t)\mathrm{d}t\,, \end{aligned}$$

$$f_1(x) = \frac{1}{1-x}, \quad f_0(x) = \frac{1}{x}, \quad f_{-1}(x) = \frac{1}{1+x}$$

$$H(p_1,\ldots,p_{w_1};x)H(q_1,\ldots,q_{w_2};x) = H(\mathbf{p};x)H(\mathbf{q};x) = \sum_{r\in\mathbf{p}\uplus\mathbf{q}} H(\mathbf{r};x)$$

$$\begin{aligned} H(a,b;x)H(y,z;x) = \; & H(a,b,y,z;x) + H(a,y,b,z;x) \\ & + H(a,y,z,b;x) + H(y,a,b,z;x) \\ & + H(y,a,z,b;x) + H(y,z,a,b;x) \end{aligned}$$

HypExp T. Huber, *et.al., :hep-ph/0507094*

More examples

$$I_1 = \int_0^1 db\,(b^2)^{1-\epsilon}(1-b^2)^{\frac{\phi}{2}-2\epsilon}{}_2F_1\left(\frac{1}{2},1,\frac{3}{2}-\epsilon,b^2\right){}_2F_1\left(\frac{1}{2}-\epsilon,1-\epsilon,\frac{3}{2}+\psi-\epsilon,b^2\right)$$

⊛ $\phi=\psi=0$, just expand the integrand

$$I_1(\phi=0\,,\psi=0)=\frac{\pi^2}{12}+\left(-\frac{\ln(2)\pi^2}{6}+\frac{7\zeta(3)}{2}\right)\epsilon+\left(\frac{\pi^3}{3}+\frac{\pi^4}{9}+\frac{\ln^2(2)\pi^2}{6}-7\ln(2)\zeta(3)\right)\epsilon^2+\mathcal{O}(\epsilon^3)$$

⊛ $\phi=-1\,,\ \psi=0$, expand with plus $-$ distribution

$$\frac{1}{(1-b)^{-1+\phi\epsilon}}=-\frac{1}{\phi\epsilon}\delta(1-b)+\left(\frac{1}{1-b}\right)_+-\phi\epsilon\left(\frac{\ln(1-b)}{1-b}\right)_++\mathcal{O}(\epsilon^2)$$

⊛ $\phi=0\,,\ \psi=-1$, handle Hypergeometric function

e.g. ${}_2F_1(a,b,c,z)=(1-z)^{c-a-b}{}_2F_1(c-a,c-b,c,z)$

HypExp T. Huber, *et.al., :hep-ph/0507094*

More examples

$$I=\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+i\epsilon}\frac{1}{(q-p_1)^2+i\epsilon}\frac{1}{(q+r_\perp)^2+2q\cdot r_+ +i\epsilon}\frac{1}{\bar{n}\cdot q+i\epsilon}\frac{\nu^\eta}{|\bar{n}\cdot q|^\eta}$$

$$\boxed{\begin{aligned}&d^dq=\tfrac{1}{2}dq^+dq^-d^{d-2}q_\perp\\ &q^+=n\cdot q\quad q^-=\bar{n}\cdot q\end{aligned}}$$

$$\sim\int_{-\infty}^{+\infty}dq^+dq^-d^{d-2}q_\perp\frac{1}{q^+q^-+q_\perp^2+i\epsilon}\frac{1}{q_+q_-+q_\perp^2-q_+p_1^-+i\epsilon}\frac{1}{q_+q_-+q_\perp^2+2q_\perp\cdot r_\perp+r_\perp^2+q_-\cdot r_++i\epsilon}\frac{1}{q^-+i\epsilon}\frac{\nu^\eta}{|q^-|^\eta}$$

$$\boxed{\text{Residue}\begin{cases}q^+=*+\frac{i\epsilon}{-q^-}\\ q^+=*+\frac{i\epsilon}{p_1^--q^-}\\ q^+=*+\frac{i\epsilon}{-q^-}\end{cases}\Rightarrow\begin{cases}-q^->0\\ p_1^--q^-<0\end{cases}\ \because p_1^->0\quad\text{or}\begin{cases}-q^-<0\\ p_1^--q^->0\end{cases}\ \therefore 0<q^-<p_1^-}$$

$$\sim\int_0^{p_1^-}dq^-d^{d-2}q_\perp\frac{1}{q_\perp^2+i\epsilon}\frac{p_1^--q^-}{q_\perp^2+\frac{2}{p_1^-}(p_1^--q^-)q_\perp\cdot r_\perp+\frac{1}{p_1^-}(p_1^--q^-)(r_\perp^2+q_-\cdot r_+)+i\epsilon}\frac{1}{q_-+i\epsilon}\frac{\nu^\eta}{(q_-)^\eta}\qquad q^-=yp_1^-$$

$$\sim\int_0^1dy\int_0^1dx\int d^{d-2}q_\perp\frac{1-y}{\{q_\perp^2-[r_\perp^2x^2(1-y)^2-x(1-y)(r_\perp^2+yp_1^-r_+)]+i\epsilon\}^2}\frac{1}{y+i\epsilon}\frac{\nu^\eta}{(y)^\eta}$$

$$\sim\left(\frac{p_1^-}{\nu}\right)^\eta\Gamma(1+\epsilon)\int_0^1dy\,dx\,x^{-1-\epsilon}(1-y)^{-\epsilon}y^{-1-\eta}\left[-r_\perp^2(1-x)(1-y)-ym^2-i\epsilon\right]^{-1-\epsilon}$$

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More examples

If $m \equiv 0$

$$I \sim \Gamma(1+\epsilon) \int_0^1 dy \int_0^1 dx\; x^{-1-\epsilon} (1-x)^{-1-\epsilon} (1-y)^{-1-2\epsilon} y^{-1-\eta} \left(-r_\perp^2 - i\epsilon\right)^{-1-\epsilon}$$

$$\sim \frac{\Gamma(1+\epsilon)}{\Gamma(-2\epsilon-\eta)} \Gamma^2(-\epsilon)\Gamma(-\eta)$$

First expand with η, then ϵ

$$\sim \frac{1}{\epsilon^2} + \frac{2}{\epsilon\eta} - \frac{3}{4}\pi^2 + O(\epsilon, \eta)$$

$$\sim \left(\frac{p_1^-}{\nu}\right)^\eta \Gamma(1+\epsilon) \int_0^1 dy\, dx\; x^{-1-\epsilon} (1-y)^{-\epsilon} y^{-1-\eta} \left[-r_\perp^2 (1-x)(1-y) - y m^2 - i\epsilon\right]^{-1-\epsilon}$$

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More examples

If $m \neq 0$

$$\int dx dy\, x^{-1-\epsilon}(1-y)^{-\epsilon} y^{-1-\eta}\left[(1-x)(1-y)+yR-i\epsilon\right]^{-1-\epsilon}$$

$$=\int dz\, \Gamma(1+\epsilon+z)\,\Gamma(-z)\,\Gamma(-\epsilon-z)\,\Gamma(-\eta+z)\,R^{z}\,\frac{\Gamma(-\epsilon)}{\Gamma(-2\epsilon-\eta)}$$

$$\sim \frac{\Gamma(-\epsilon)}{\Gamma(-2\epsilon-\eta)}\left\{-R^{-\epsilon}\Gamma(\epsilon)\Gamma(-\epsilon-\eta)\,{}_2F_1[1,-\epsilon-\eta,1-\epsilon,R]-(1-R)^{\eta}\Gamma(-\epsilon)\Gamma(1+\epsilon)\Gamma(-\eta)\right\}$$

$$\Rightarrow I \sim \frac{\Gamma(-\epsilon)}{\Gamma(2\epsilon-\eta)}\left\{-R^{-\epsilon}\Gamma(\epsilon)\Gamma(-\epsilon-\eta)\,{}_2F_1[1,-\epsilon-\eta,1-\epsilon,R]-(1-R)^{\eta}\Gamma(-\epsilon)\Gamma(1+\epsilon)\Gamma(-\eta)\right\}$$

$$\sim \frac{1}{\epsilon^2}-\frac{2}{\eta\epsilon}-\frac{2}{\epsilon}\ln R+\frac{7}{12}\pi^2-2\ln(1-R)\ln R+\ln^2 R-2\,\mathrm{Li}_2 R$$

$$\sim\left(\frac{p_1^-}{\nu}\right)^{\eta}\Gamma(1+\epsilon)\int_0^1 dy dx\, x^{-1-\epsilon}(1-y)^{-\epsilon}y^{-1-\eta}\left[-p_\perp^2(1-x)(1-y)-ym^2-i\epsilon\right]^{-1-\epsilon}$$

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Summary

- Introduction to analytic calculation
- Traditional methods for integrals
  - (Feynman) parameterization
  - Mellin-Barnes representation
  - Asymptotic expansions
- Some simple examples