PEKING UNIVERSITY

Calculation of phase-space integration of sufficient inclusive processes

Peng Zhang
Peking University

January 7, 2021

Liu, Ma, Tao, Zhang, *Chin. Phys. C* **45**, (2021) 013115
Zhang, Wang, Liu, et al. *JHEP* **04** (2019) 116

Outline

- Examples: $\gamma^* \to t\bar{t} + X$ at NNLO
  - Reverse unitarity relation
  - Differential equations
  - Auxiliary mass flow
- Examples: $g \to Q\bar{Q}({}^1S_0^{[1,8]}) + X$ at NLO
- Summary

General integral form

General phase-space and loop integral

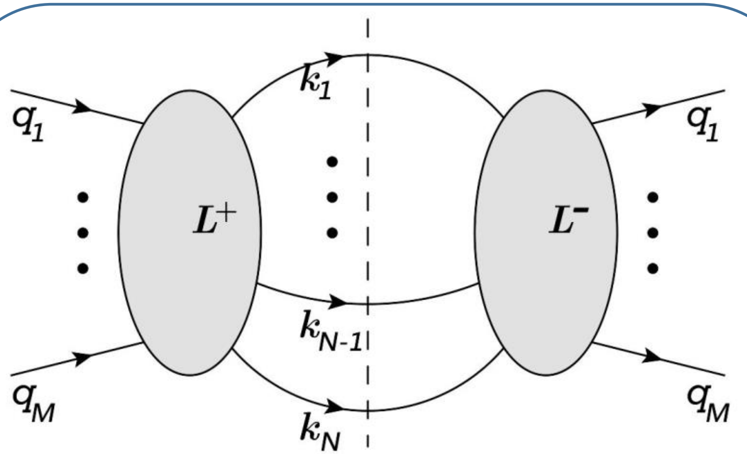
- form

$$F(\vec{\nu};\vec{s}) \equiv \int \mathrm{dPS}_N \prod_\alpha \frac{1}{(\mathcal{D}^{\mathrm{t}}_\alpha)^{\nu^{\mathrm{t}}_\alpha}}$$

$$\int \prod_{i=1}^{L^+} \frac{\mathrm{d}^D l_i^+}{(2\pi)^D} \prod_\beta \frac{1}{(\mathcal{D}^+_\beta + \mathrm{i}0^+)^{\nu^+_\beta}}$$

$$\int \prod_{j=1}^{L^-} \frac{\mathrm{d}^D l_j^-}{(2\pi)^D} \prod_\gamma \frac{1}{(\mathcal{D}^-_\gamma - \mathrm{i}0^+)^{\nu^-_\gamma}}$$

$$\prod_{i,j} (l_i^+ \cdot l_j^-)^{-\nu^\pm_{ij}},$$



$$Q = \sum_{i=1}^M q_i = \sum_{i=1}^N k_i$$

- $\vec{\nu} \equiv (\nu^{\mathrm{t}}_1, \nu^{\mathrm{t}}_2, \cdots, \nu^+_1, \nu^+_2, \cdots, \nu^-_1, \nu^-_2, \cdots, \nu^\pm_{11}, \nu^\pm_{12}, \nu^\pm_{21}, \cdots)$
- $\vec{s}$: kinematical invariants (including Q^2)
- phase-space

$$\mathrm{dPS}_N \equiv (2\pi)^D \delta^D(Q - \sum_{i=1}^N k_i) \prod_{i=1}^N \frac{\mathrm{d}^D k_i}{(2\pi)^D} (2\pi)\delta(\mathcal{D}^{\mathrm{c}}_i)\Theta(k_i^0 - m_i)$$

PEKING UNIVERSITY

Examples: VRR of $\gamma^* \to t\bar{t} + X$

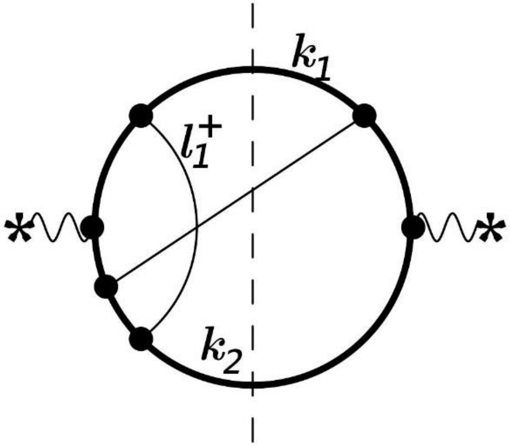
Integral form

- notation: $\mathrm{V}^{L^+}\mathrm{R}^{N-1}\mathrm{V}^{L^-}$ to distinguish sub-processes
- the phase-space integrals

$$\hat{F}(\vec{\nu};\vec{s}) = \int \mathrm{dPS}_3 \prod_{\alpha=1}^{2} \frac{1}{(\mathcal{D}_\alpha^{\mathrm{t}})^{\nu_\alpha^{\mathrm{t}}}} \int \frac{\mathrm{d}^D l_1^+}{(2\pi)^D} \prod_{\beta=1}^{6} \frac{1}{(\mathcal{D}_\beta^+ + \mathrm{i}0^+)^{\nu_\beta^+}}$$

- inverse propagators

$$\mathcal{D}_1^{\mathrm{c}} = k_1^2 - m_t^2, \mathcal{D}_2^{\mathrm{c}} = k_2^2 - m_t^2, \mathcal{D}_3^{\mathrm{c}} = (Q - k_1 - k_2)^2;$$
$$\mathcal{D}_1^{\mathrm{t}} = (Q - k_2)^2 - m_t^2, \mathcal{D}_2^{\mathrm{t}} = (Q - k_1)^2 - m_t^2;$$
$$\mathcal{D}_1^{+} = (k_1 + l_1^+)^2 - m_t^2, \mathcal{D}_2^{+} = (k_2 - l_1^+)^2 - m_t^2,$$
$$\mathcal{D}_3^{+} = l_1^{+2}, \mathcal{D}_4^{+} = (Q - k_1 - k_2 + l_1^+)^2,$$
$$\mathcal{D}_5^{+} = (Q - k_2 + l_1^+)^2 - m_t^2, \mathcal{D}_6^{+} = (Q - k_1 - l_1^+)^2 - m_t^2,$$



- phase-space

$$\mathrm{dPS}_N \equiv (2\pi)^D \delta^D(Q - \sum_{i=1}^{N} k_i) \prod_{i=1}^{N} \frac{\mathrm{d}^D k_i}{(2\pi)^D} (2\pi) \boxed{\delta(\mathcal{D}_i^{\mathrm{c}})} \boxed{\Theta(k_i^0 - m_i)}$$

a typical Feynman diagrams

Reverse unitarity relation

Dirac delta function

Anastasiou, Melnikov, *Nucl. Phys. B* **646** (2002), 220-256

- transformation

$$\mathcal{D}_i^c = k_i^2 - m_i^2$$

$$(2\pi)\delta(\mathcal{D}_i^{\mathrm{c}}) = \frac{\mathrm{i}}{\mathcal{D}_i^{\mathrm{c}} + \mathrm{i}0^+} + \frac{-\mathrm{i}}{\mathcal{D}_i^{\mathrm{c}} - \mathrm{i}0^+}$$

- mass shell condition ⟶ propagator
- map phase-space integrals onto pure loop integrals
- use the techniques for loop integration
    - integration-by-parts (IBP) reduction
    - dimensional recurrence
    - differential equations (DEs) w.r.t. kinematical invariants
        - auxiliary mass flow (AMF) : DEs w.r.t. auxiliary mass

Reverse unitarity relation

For 1 delta function case

- with these high loop techniques, relations are the same whether the imaginary part is $\mathcal{D}_1^c + i0^+$ or $\mathcal{D}_1^c - i0^+$
- two parts reduce to the similar loop master integrals (MIs) (except the signature of imaginary part)
- linear function of MIs
- coefficients of MIs of two parts are the same

Reverse unitarity relation

For 1 delta function case

- choose MIs in which power of $\mathcal{D}_1^c \pm i0^+$ is no more than 1
- plus reduction results of two parts again
- for MIs in which power of $\mathcal{D}_1^c \pm i0^+$ is smaller than 1, set them to 0
- inverse propagator ⟶ mass shell condition
- loop integrals reduction ⟶ phase-space integrals reduction

Reverse unitarity relation

For 2 delta function case

$$(2\pi)^2\delta(\mathcal{D}_1^c)\delta(\mathcal{D}_2^c) = \frac{\mathrm{i}}{\mathcal{D}_1^c + \mathrm{i}0^+}\frac{\mathrm{i}}{\mathcal{D}_2^c + \mathrm{i}0^+} + \frac{\mathrm{i}}{\mathcal{D}_1^c + \mathrm{i}0^+}\frac{-\mathrm{i}}{\mathcal{D}_2^c - \mathrm{i}0^+} + \frac{-\mathrm{i}}{\mathcal{D}_1^c - \mathrm{i}0^+}\frac{\mathrm{i}}{\mathcal{D}_2^c + \mathrm{i}0^+} + \frac{-\mathrm{i}}{\mathcal{D}_1^c - \mathrm{i}0^+}\frac{-\mathrm{i}}{\mathcal{D}_2^c - \mathrm{i}0^+}$$

- similar with 1 delta function
- four parts have the same reduction relations
- linear function of similar MIs with same coefficients
- inverse propagator ⟶ mass shell condition
- loop integrals reduction ⟶ phase-space integrals reduction

Reverse unitarity relation

Heaviside Function

- $\Theta(k_i^0 - m_i)$ are equivalent to $\Theta(k_i^0)$ here
- its derivative is $\delta(k_i^0 - m_i)$
- all space components of k_i to be at the origin
- well regularized by dimensional regularization
- set to 0

Differential equations

Reduction and DEs

- IBP reduction

Chetyrkin, Tkachov, *Nucl. Phys. B* **192**, (1981) 159-204

- set up DEs w.r.t $\vec{s}$ among MIs ($\vec{I}(\vec{s})$)

$$\frac{\partial}{\partial s_i}\vec{I}(\vec{s}) = M_i(\vec{s})\vec{I}(\vec{s})$$

Kotikov, *Phys. Lett. B* **254**, (1991) 158-164
Gehrmann, Remiddi, *Nucl. Phys. B* **580**, (2000) 485-518

- for $\gamma^* \to t\bar{t} + X$, only two kinematical: Q^2 and m_t^2
- take

$$s = Q^2\,, \quad x = \frac{4m_t^2}{s}\,, \quad \nu = \sum \vec{\nu}$$

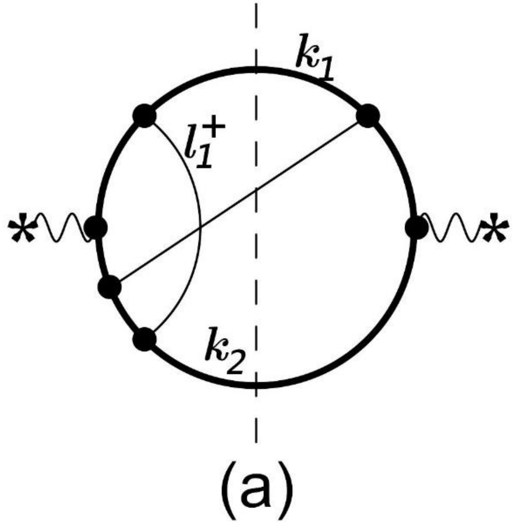
- dimensionless integrals

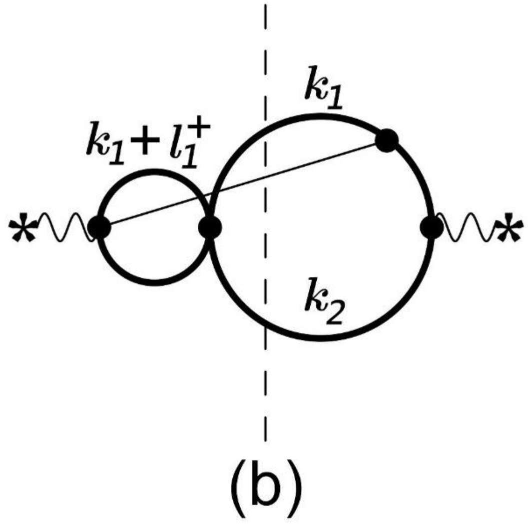
$$\hat{F}(\vec{\nu};x) \equiv s^{N-\frac{N-1+L}{2}D+\nu}F(\vec{\nu};\vec{s})$$

- set up DEs w.r.t. x

VRR for example

some typical Feynman diagrams





- (a) is the most complicated diagram
- (b) is (a)'s sub-diagram, take it as an example

Differential equations

VRR: sub-diagram (b)

- define MIs as

$$\hat{F}(\{\nu_1^{\mathrm{t}}, \nu_1^{+}, \nu_2^{+}\}; x) = s^{3-\frac{3}{2}D+\nu_1^{\mathrm{t}}+\nu_1^{+}+\nu_2^{+}} \int \mathrm{dPS}_3 \frac{1}{\mathcal{D}_1^{\mathrm{t}\,\nu_1^{\mathrm{t}}}} \int \frac{\mathrm{d}^D l_1^{+}}{(2\pi)^D} \frac{1}{(\mathcal{D}_1^{+} + \mathrm{i}0^{+})^{\nu_1^{+}} (\mathcal{D}_2^{+} + \mathrm{i}0^{+})^{\nu_2^{+}}}$$

$$\mathcal{D}_1^{\mathrm{t}} = (Q - k_2)^2 - m_t^2,$$
$$\mathcal{D}_1^{+} = (k_1 + l_1^{+})^2 - m_t^2,$$
$$\mathcal{D}_2^{+} = (k_2 - l_1^{+})^2 - m_t^2,$$

- 6 MIs

$$\Big\{\hat{F}(\{0,0,1\};x), \hat{F}(\{-1,0,1\};x), \hat{F}(\{0,1,1\};x),$$
$$\hat{F}(\{0,1,2\};x), \hat{F}(\{1,1,1\};x), \hat{F}(\{1,1,2\};x)\Big\}.$$

Differential equations

VRR: sub-diagram (b)

- Set up DEs w.r.t. x

$$\frac{\partial}{\partial x}\hat{F}(\{\nu_1^{\mathrm{t}},\nu_1^{+},\nu_2^{+}\};x) =$$

$$\begin{bmatrix}
\frac{2+3x-2\epsilon-4x\epsilon}{2(-1+x)x} & \frac{6(-1+\epsilon)}{(-1+x)x} & 0 & 0 & 0 & 0 \\
\frac{-1+\epsilon}{x} & \frac{-4(-1+\epsilon)}{x} & 0 & 0 & 0 & 0 \\
\frac{-2(-1+\epsilon)(-4-x+4\epsilon+2x\epsilon)}{(-1+x)x^2(-1+2\epsilon)} & \frac{24(-1+\epsilon)^2}{(-1+x)x^2(-1+2\epsilon)} & 0 & 1 & 0 & 0 \\
\frac{4(2-5\epsilon+3\epsilon^2)}{(-1+x)x^2} & 0 & \frac{-2(1-2\epsilon)^2}{(-1+x)x} & \frac{2(x+\epsilon-3x\epsilon)}{(-1+x)x} & 0 & 0 \\
\frac{6(-1+\epsilon)(-4-x+4\epsilon+2x\epsilon)}{(-1+x)^2x^2(-1+2\epsilon)} & \frac{-72(-1+\epsilon)^2}{(-1+x)^2x^2(-1+2\epsilon)} & \frac{-2(-1+2\epsilon)}{(-1+x)x} & \frac{-2}{-1+x} & \frac{-\epsilon}{x} & \frac{1}{2} \\
\frac{-4(4+3x-8\epsilon-8x\epsilon+4\epsilon^2+5x\epsilon^2)}{(-1+x)^2x^3} & \frac{48(-1+\epsilon)^2}{(-1+x)^2x^3} & \frac{2(1-2\epsilon)^2}{(-1+x)^2x} & \frac{2(-1+2\epsilon)}{(-1+x)^2} & \frac{(1-2\epsilon)(3\epsilon-1)}{(-1+x)x} & \frac{x+4\epsilon-10x\epsilon}{2(-1+x)x}
\end{bmatrix}$$

$$\hat{F}(\{\nu_1^{\mathrm{t}},\nu_1^{+},\nu_2^{+}\};x)$$

- pole: 0, 1

Differential equations

Boundary conditions

- At ordinary point
  - sector decomposition
  - auxiliary mass flow
- At singularity
  - analyze regions

Binoth, Heinrich, *Nucl. Phys. B* **585** (2000) 741–759
Heinrich, *Int. J. Mod. Phys. A* **23** (2008) 1457–1486

Auxiliary mass flow

Add auxiliary mass to inverse propagators

- general phase-space and loop integral with auxiliary masses

$$F(\vec{\nu};\vec{s},\vec{\eta}) \equiv \int \mathrm{dPS}_N \prod_\alpha \frac{1}{(\mathcal{D}^{\mathrm{t}}_\alpha + \eta^{\mathrm{t}}_\alpha)^{\nu^{\mathrm{t}}_\alpha}}$$

$$\int \prod_{i=1}^{L^+} \frac{\mathrm{d}^D l_i^+}{(2\pi)^D} \prod_\beta \frac{1}{(\mathcal{D}^+_\beta + \mathrm{i}\eta^+_\beta)^{\nu^+_\beta}}$$

$$\int \prod_{j=1}^{L^-} \frac{\mathrm{d}^D l_j^-}{(2\pi)^D} \prod_\gamma \frac{1}{(\mathcal{D}^-_\gamma - \mathrm{i}\eta^-_\gamma)^{\nu^-_\gamma}}$$

$$\prod_{i,j} (l_i^+ \cdot l_j^-)^{-\nu^\pm_{ij}}$$

$$\vec{\eta} \equiv (\eta^{\mathrm{t}}_1, \eta^{\mathrm{t}}_2, \cdots, \eta^+_1, \eta^+_2, \cdots, \eta^-_1, \eta^-_2, \cdots)$$

$$\mathrm{dPS}_N \equiv (2\pi)^D \delta^D(Q - \sum_{i=1}^N k_i) \prod_{i=1}^N \frac{\mathrm{d}^D k_i}{(2\pi)^D} (2\pi)\delta(\mathcal{D}^{\mathrm{c}}_i)\Theta(k_i^0 - m_i)$$

Auxiliary mass flow

Direction choice of $\vec{\eta} \to 0$

- Rule of Feynman prescription for Feynman propagators

  take $\eta_\beta^+ \to 0^+$ and $\eta_\gamma^- \to 0^+$

- $\eta_\alpha^t \to 0^+$ or $\eta_\alpha^t \to 0^-$ are both fine

  take $\eta_\alpha^t \to 0^+$ for convenient

$$F(\vec{\nu}; \vec{s}, \vec{\eta}) \equiv \int \mathrm{dPS}_N \prod_\alpha \frac{1}{(\mathcal{D}_\alpha^\mathrm{t} + \eta_\alpha^\mathrm{t})^{\nu_\alpha^\mathrm{t}}}$$
$$\int \prod_{i=1}^{L^+} \frac{\mathrm{d}^D l_i^+}{(2\pi)^D} \prod_\beta \frac{1}{(\mathcal{D}_\beta^+ + \mathrm{i}\eta_\beta^+)^{\nu_\beta^+}}$$
$$\int \prod_{j=1}^{L^-} \frac{\mathrm{d}^D l_j^-}{(2\pi)^D} \prod_\gamma \frac{1}{(\mathcal{D}_\gamma^- - \mathrm{i}\eta_\gamma^-)^{\nu_\gamma^-}}$$
$$\prod_{i,j} (l_i^+ \cdot l_j^-)^{-\nu_{ij}^\pm}$$

Auxiliary mass flow

Choice of finite $\vec{\eta}$

- either related to each others or completely independent
- all choices are workable
  - all same
  - a strong ordering

$$F(\vec{\nu};\vec{s},\vec{\eta}) \equiv \int \mathrm{dPS}_N \prod_{\alpha} \frac{1}{(\mathcal{D}_\alpha^{\mathrm{t}} + \eta_\alpha^{\mathrm{t}})^{\nu_\alpha^{\mathrm{t}}}} \int \prod_{i=1}^{L^+} \frac{\mathrm{d}^D l_i^+}{(2\pi)^D} \prod_{\beta} \frac{1}{(\mathcal{D}_\beta^+ + \mathrm{i}\eta_\beta^+)^{\nu_\beta^+}} \int \prod_{j=1}^{L^-} \frac{\mathrm{d}^D l_j^-}{(2\pi)^D} \prod_{\gamma} \frac{1}{(\mathcal{D}_\gamma^- - \mathrm{i}\eta_\gamma^-)^{\nu_\gamma^-}} \prod_{i,j} (l_i^+ \cdot l_j^-)^{-\nu_{ij}^\pm}$$

- our choice: either 0^+ or η
  - if $\{\mathcal{D}_\beta^+\}$ or $\{\mathcal{D}_\gamma^-\}$ depend on $\vec{s}$, choose $\eta_\alpha^t \to 0^+$ and $\eta_\beta^+ = \eta_\gamma^- = \eta$
  - else, choose $\eta_\alpha^t = \eta$
  - introduce one auxiliary mass η

PEKING UNIVERSITY

Auxiliary mass flow

VRR: sub-diagram (b)

$$\mathcal{D}_1^{\mathrm{t}} = (Q-k_2)^2 - m_t^2,$$
$$\mathcal{D}_1^{+} = (k_1+l_1^+)^2 - m_t^2,$$
$$\mathcal{D}_2^{+} = (k_2-l_1^+)^2 - m_t^2,$$

- take $y = \eta/s$
- define dimensional integrals

$$\hat{F}(\{\nu_1^{\mathrm{t}},\nu_1^+,\nu_2^+\};x,y) = s^{3-\frac{3}{2}D+\nu_1^{\mathrm{t}}+\nu_1^++\nu_2^+}\int \mathrm{dPS}_3 \frac{1}{\mathcal{D}_1^{\mathrm{t}\,\nu_1^{\mathrm{t}}}}\int \frac{\mathrm{d}^D l_1^+}{(2\pi)^D}\frac{1}{(\mathcal{D}_1^+ + \mathrm{i}\eta)^{\nu_1^+}(\mathcal{D}_2^+ + \mathrm{i}\eta)^{\nu_2^+}}$$

- after reduction, 7 MIs for finite η (or y)

$$\Big\{\hat{F}(\{0,0,1\};x,y), \hat{F}(\{-1,0,1\};x,y), \hat{F}(\{0,1,1\};x,y), \hat{F}(\{-1,1,1\};x,y),$$
$$\hat{F}(\{0,1,2\};x,y), \hat{F}(\{1,1,1\};x,y), \hat{F}(\{1,1,2\};x,y)\Big\},$$

- 6 MIs for $\eta \to 0^+$(as show in page 12)

$$\Big\{\hat{F}(\{0,0,1\};x,0^+), \hat{F}(\{-1,0,1\};x,0^+), \hat{F}(\{0,1,1\};x,0^+),$$
$$\hat{F}(\{0,1,2\};x,0^+), \hat{F}(\{1,1,1\};x,0^+), \hat{F}(\{1,1,2\};x,0^+)\Big\}.$$

Auxiliary mass flow

VRR: sub-diagram (b)

- choose $x = 1/2$ (ordinary point) and set up DEs

$$
\frac{\partial}{\partial y}\hat{F}(\{\nu_1^{\mathrm{t}},\nu_1^{+},\nu_2^{+}\};\frac{1}{2},y) =
$$

$$
\begin{bmatrix}
\frac{-8(-1+\epsilon)}{\mathrm{i}+8y} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-8(-1+\epsilon)}{\mathrm{i}+8y} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2\mathrm{i} & 0 & 0 \\
\frac{8(-1+\epsilon)}{\mathrm{i}+8y} & 0 & \mathrm{i}(-1+2\epsilon) & 0 & \frac{1}{2(-\mathrm{i}+8y)} & 0 & 0 \\
\frac{-16\binom{5\mathrm{i}+20y-11\mathrm{i}\epsilon-}{52y\epsilon+6\mathrm{i}\epsilon^2+32y\epsilon^2}}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)^2} & \frac{192\mathrm{i}(-1+\epsilon)^2}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)^2} & \frac{2(1-2\epsilon)\binom{5-20\mathrm{i}y-}{6\epsilon+40\mathrm{i}y\epsilon}}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{8(1-2\epsilon)(-3+4\epsilon)}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{\binom{1+16\mathrm{i}y+128y^2-}{2\epsilon-16\mathrm{i}y\epsilon-384y^2\epsilon}}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2\mathrm{i} \\
\frac{-64(5-11\epsilon+6\epsilon^2)}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{768(-1+\epsilon)^2}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{8(1-2\epsilon)\binom{5-4\mathrm{i}y-}{6\epsilon+8\mathrm{i}y\epsilon}}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{32(1-2\epsilon)(-3+4\epsilon)}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{-4(-\mathrm{i}+4y)(-1+2\epsilon)}{y(-\mathrm{i}+8y)} & \frac{16\mathrm{i}(1-2\epsilon)(3\epsilon-1)}{(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{4(\mathrm{i}+8y-4\mathrm{i}\epsilon-64y\epsilon)}{(-\mathrm{i}+8y)(\mathrm{i}+8y)}
\end{bmatrix}
$$

$$
\hat{F}(\{\nu_1^{\mathrm{t}},\nu_1^{+},\nu_2^{+}\};\frac{1}{2},y)
$$

- take boundaries at $\eta \to \infty$

Auxiliary mass flow

Expansion for tree propagators at $\eta \to \infty$

- scalar products among external momenta and cut momenta are finite

$$\frac{1}{\mathcal{D}_\alpha^{\mathrm{t}} + \eta} \xlongequal{\eta \to \infty} \frac{1}{\eta} \sum_{j=0}^{+\infty} \left(\frac{-\mathcal{D}_\alpha^{\mathrm{t}}}{\eta} \right)^j$$

$$\frac{1}{\mathcal{D}_\alpha^{\mathrm{t}}} \xlongequal{\eta \to \infty} \frac{1}{\mathcal{D}_\alpha^{\mathrm{t}}},$$

- If η is introduced, tree propagators are removed
- else, tree propagators remain

Auxiliary mass flow

Expansion for loop propagators at $\eta \to \infty$

- loop momenta can be any large value
- at $\eta \to \infty$, linear combinations of loop momenta can be either at the order of $|\eta|^{1/2}$ or much smaller than it
- decompose $\mathcal{D}_\alpha^+$ into two parts

$$\mathcal{D}_\alpha^+ = \widetilde{\mathcal{D}}_\alpha^+ + K_\alpha$$

  - $\widetilde{\mathcal{D}}_\alpha^+$: only including the part at order $|\eta|$
  - K_α: other parts

Auxiliary mass flow

Expansion for loop propagators at $\eta \to \infty$

$$\frac{1}{\mathcal{D}_\alpha^+ + \mathrm{i}\eta} \overset{\eta\to\infty}{=\!=\!=} \frac{1}{\widetilde{\mathcal{D}}_\alpha^+ + \mathrm{i}\eta} \sum_{j=0}^{+\infty} \left(\frac{-K_\alpha}{\widetilde{\mathcal{D}}_\alpha^+ + \mathrm{i}\eta} \right)^j,$$

$$\frac{1}{\mathcal{D}_\alpha^+ + \mathrm{i}0^+} \overset{\eta\to\infty}{=\!=\!=} \begin{cases} \dfrac{1}{\widetilde{\mathcal{D}}_\alpha^+ + \mathrm{i}0^+} \displaystyle\sum_{j=0}^{+\infty} \left(\frac{-K_\alpha}{\widetilde{\mathcal{D}}_\alpha^+ + \mathrm{i}0^+} \right)^j & \text{if } \widetilde{\mathcal{D}}_\alpha^+ \neq 0, \\ \dfrac{1}{\mathcal{D}_\alpha^+ + \mathrm{i}0^+} & \text{if } \widetilde{\mathcal{D}}_\alpha^+ = 0, \end{cases}$$

- expansion for $\mathcal{D}_\alpha^-$ are similar
- propagators (η exits and $\widetilde{\mathcal{D}}_\alpha^+ = 0$) removed
- decouple some loop momenta at order $|\eta|^{1/2}$

$\longrightarrow$ single-scale vacuum integrals factored

eg: $l_\alpha^+ \cdot k_i$, $l_\alpha^+ \cdot q_i$

PEKING UNIVERSITY

Auxiliary mass flow

Expansion at $\eta \to \infty$

- $F(\vec{v};\vec{s},\eta)$ is simplified to a linear combination of integrals with fewer inverse propagators

$$F(\vec{v};\vec{s},\eta) \longrightarrow \sum c \times F^{\text{cut}} \times F^{\text{bub}}$$

- c are rational functions of $\vec{s}$ and η
- F^{bub}: single-scale vacuum bubble integrals
  - studied up to five-loop order

Luthe, Maier, Marquard et al., *JHEP* **03**, (2017) 020

- F^{cut} : basal phase-space integrations with the integrands being polynomials of scalar products between cut momenta.
  - also studied for $m_i = 0$ or m (no more than 2)

Bernreuther, Bogner, Dekkers, *JHEP* **06**, (2011) 032
Liu, Ma, Tao, Zhang, *Chin. Phys. C* **45**, (2021) 013115

Auxiliary mass flow

VRR: sub-diagram (b)

$$F^{\rm bub}_{1,1}(D) \equiv \int \frac{{\rm d}^D l_1^+}{(2\pi)^D} \frac{1}{{l_1^+}^2 + {\rm i}}$$

$$\hat{F}(\{0,0,1\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{4-\frac{3}{2}D}\eta^{\frac{D}{2}-1} F^{\rm bub}_{1,1}(D)\left(\int {\rm dPS}_3\right)_{x=1/2}$$

$$\hat{F}(\{-1,0,1\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{3-\frac{3}{2}D}\eta^{\frac{D}{2}-1} F^{\rm bub}_{1,1}(D)\left(\int {\rm dPS}_3 \mathcal{D}_1^{\rm t}\right)_{x=1/2}$$

$$\hat{F}(\{0,1,1\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{5-\frac{3}{2}D}\eta^{\frac{D}{2}-2} \frac{{\rm i}(D-2)}{2} F^{\rm bub}_{1,1}(D)\left(\int {\rm dPS}_3\right)_{x=1/2}$$

$$\hat{F}(\{-1,1,1\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{4-\frac{3}{2}D}\eta^{\frac{D}{2}-2} \frac{{\rm i}(D-2)}{2} F^{\rm bub}_{1,1}(D)\left(\int {\rm dPS}_3 \mathcal{D}_1^{\rm t}\right)_{x=1/2}$$

$$\hat{F}(\{0,1,2\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{6-\frac{3}{2}D}\eta^{\frac{D}{2}-3} \frac{(4-D)(D-2)}{8} F^{\rm bub}_{1,1}(D)\left(\int {\rm dPS}_3\right)_{x=1/2}$$

$$\hat{F}(\{1,1,1\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{6-\frac{3}{2}D}\eta^{\frac{D}{2}-2} \frac{{\rm i}(D-2)}{2} F^{\rm bub}_{1,1}(D)\left(\int {\rm dPS}_3 \frac{1}{\mathcal{D}_1^{\rm t}}\right)_{x=1/2}$$

$$\hat{F}(\{1,1,2\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{7-\frac{3}{2}D}\eta^{\frac{D}{2}-3} \frac{(4-D)(D-2)}{8} F^{\rm bub}_{1,1}(D)\left(\int {\rm dPS}_3 \frac{1}{\mathcal{D}_1^{\rm t}}\right)_{x=1/2}$$

Auxiliary mass flow

basal phase-space integrations

- $\int \mathrm{dPS}_3(\mathcal{D}_1^t)^i$ can be reduced to two MIs of RR process
- $F_{r,N,n}^{\mathrm{cut}}$ denote the *n*-th MI for *N*-particle-cut integrals with $m_1 = \cdots = m_r = m$ and $m_{r+1} = \cdots = m_N = 0$
- for $N = 3$, two MIs: $F_{2,3,1}^{\mathrm{cut}}$ and $F_{2,3,2}^{\mathrm{cut}}$
- definition

$$F_{2,N,n}^{\mathrm{cut}} \equiv \int \mathrm{dPS}_N \left((k_1 + k_2)^2\right)^{n-1}$$

Auxiliary mass flow

basal phase-space integrations

- MI result of RR

$$F^{\text{cut}}_{2,N,n} \equiv \int \mathrm{dPS}_N \left((k_1+k_2)^2\right)^{n-1} = \frac{2^{5+2N(\epsilon-2)-2\epsilon}\pi^{3+N(\epsilon-2)-\epsilon}\Gamma(1+n-2\epsilon)\Gamma(1-\epsilon)^{N-1}\Gamma(n-\epsilon)}{\Gamma(2-2\epsilon)\Gamma(n-1+N-N\epsilon)\Gamma(n-2+N+\epsilon-N\epsilon)} s^{n-3+N+\epsilon-N\epsilon}$$

$$\times {}_3F_2\left(\epsilon-\frac{1}{2}, 2-n-N+N\epsilon, 3-n-N-\epsilon+N\epsilon; 1-n+\epsilon, 2\epsilon-n; \frac{4m^2}{s}\right)$$

$$+\frac{2^{4+2N(\epsilon-2)}\pi^{\frac{7}{2}+N(\epsilon-2)-\epsilon}\Gamma(1-\epsilon)^{N-1}\Gamma(\epsilon-n)}{\Gamma\left(\frac{3}{2}-n\right)\Gamma((N-1)(1-\epsilon))\Gamma((N-2)(1-\epsilon))} s^{n-3+N+\epsilon-N\epsilon}\left(\frac{4m^2}{s}\right)^{n-\epsilon}$$

$$\times {}_3F_2\left(n-\frac{1}{2}, 3-N-2\epsilon+N\epsilon, 2-N-\epsilon+N\epsilon; 1+n-\epsilon, \epsilon; \frac{4m^2}{s}\right)$$

$$+\frac{2^{4+2N(\epsilon-2)}\pi^{\frac{7}{2}+N(\epsilon-2)-\epsilon}\Gamma(1-\epsilon)^{N-2}\Gamma(\epsilon-1)\Gamma(2\epsilon-1-n)}{\Gamma\left(\frac{1}{2}-n+\epsilon\right)\Gamma((N-2)(1-\epsilon))\Gamma((N-3)(1-\epsilon))} s^{n-3+N+\epsilon-N\epsilon}\left(\frac{4m^2}{s}\right)^{1+n-2\epsilon}$$

$$\times {}_3F_2\left(\frac{1}{2}+n-\epsilon, 4-N-3\epsilon+N\epsilon, 3-N-2\epsilon+N\epsilon; 2+n-2\epsilon, 2-\epsilon; \frac{4m^2}{s}\right).$$

Comment

basal phase-space integrations

- without η, MIs of RRR are not the basal phase-space integrations
- add η and make expansion at $\eta \to \infty$ (see Page 17)
- all $\mathcal{D}_\alpha^t$ come to the numerators
- then MIs are all basal phase-space integrations

Auxiliary mass flow

Flow of η (y)

- Set up DEs w.r.t. y (as shown in page 19)
- boundary condition: fixed x= $1/2$ and $y \to \infty$
- solve DEs with the flow of y from ∞ to 0^+
- eg: $\hat{F}(\{1,1,2\}; 1/2,0)$

$$
\begin{aligned}
\hat{F}\left(\{1,1,2\};\frac{1}{2},0\right) =& (7.7879044672106926250285009377 4\times 10^{-6} + 2.9131946977223739413535630834 8\times 10^{-6}\mathrm{i}) \\
&+ (0.00013043037301578765560448819886 1 + 0.000068404169458201184291920092123\mathrm{i})\epsilon \\
&+ (0.0010774348138281919097871863624 32 + 0.00075092687625074547221027758943 0\mathrm{i})\epsilon^2 \\
&+ (0.0058427815092083906261561250813 6 + 0.0052757010138215866158903106169 1\mathrm{i})\epsilon^3 \\
&+ (0.023346128001244437233449421912 3 + 0.027085973695161752456396628286 8\mathrm{i})\epsilon^4 \\
&+ (0.073091853943714866707610465480 0 + 0.10951652497432042525899338696 72\mathrm{i})\epsilon^5 \\
&+ (0.18597537388312598648861388152 0 + 0.36639377004270844333156480150 9\mathrm{i})\epsilon^6 \\
&+ (0.39309398618851907651242469456 4 + 1.0521721707656381162578254106 32\mathrm{i})\epsilon^7 \\
&+ (0.69775277299606861048706250047 + 2.6728254612200838302261510428 9\mathrm{i})\epsilon^8 + \cdots .
\end{aligned}
$$

Auxiliary mass flow

Flow of x

- $\widehat{F}(\{1,1,2\}; 1/2,0)$ at fixed x$=$ $1/2$ is the boundary condition of DEs w.r.t. x (as shown in Page 13)
- solve DEs w.r.t. x to obtain MIs at different values of x

PEKING UNIVERSITY

Section Summary

- Reverse unitarity relation transform the delta function to inverse propagators on cut
- with IBP reduction, complex integrals can be reduced to linear combination of MIs
- set up DEs w.r.t. kinematical invariants
- use AMF to calculate the boundary conditions
  - add auxiliary mass on inverse propagators
  - set up DEs w.r.t. η
  - at $\eta \to \infty$, integrals are reduced to a linear combination of basal phase-space integrals multiplied by single-scale vacuum bubble integrals
  - flow $\eta \to \infty$ to $\eta \to 0$ with DEs

Comment

Rapidity

- introduce $\delta(y - k_i \cdot p_1 / k_i \cdot p_2)$ in dPS_N
  - Rapidity distribution of i-th particle

Anastasiou, Dixon, Melnikov, *Nucl. Phys. Proc. Suppl.* **116**, (2003) 193-197
Anastasiou, Dixon, Melnikov et al., *Phys. Rev. Lett.* **91**, (2003) 182002

- add auxiliary mass in the delta functions
  - solve rapidity divergence
  - discussed in next section

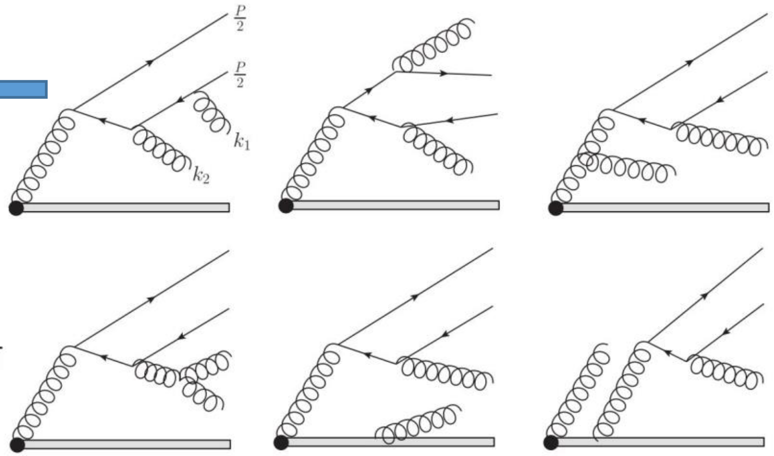
Outline

- Examples: $\gamma^* \to t\bar{t} + X$ at NNLO
  - Reverse unitarity relation
  - Differential equations
  - Auxiliary mass flow
- Examples: $g \to Q\bar{Q}({}^1S_0^{[1,8]}) + X$ at NLO
- Summary

Real Calculation

- Feynman Diagram
- Form of SDCs

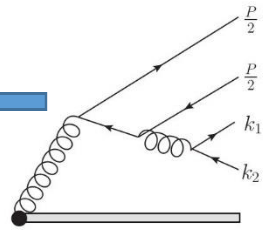
$S = 2$



$$\int \mathrm{d}\Phi_{\text{real}} \prod_i \frac{1}{E_i^{a_i}}$$
$$= \frac{P\cdot n}{2z^2} \int \frac{\mathrm{d}^D k_1}{(2\pi)^{D-1}} \frac{\mathrm{d}^D k_2}{(2\pi)^{D-1}} \delta_+(k_1^2)\delta_+(k_2^2)\delta\left(k_1\cdot n + k_2\cdot n - \frac{1-z}{z} P\cdot n\right) \prod_i \frac{1}{E_i^{a_i}}$$

in which

$S = 1$



$$E_1 = k_1\cdot k_2\,, \qquad E_2 = k_1\cdot P\,, \qquad E_3 = k_2\cdot P\,,$$
$$E_4 = 2\,k_1\cdot P + 1\,, \qquad E_5 = 2\,k_2\cdot P + 1\,,$$
$$E_6 = 2\,k_1\cdot k_2 + k_1\cdot P + k_2\cdot P\,,$$
$$E_7 = 2\,k_1\cdot k_2 + 2\,k_1\cdot P + 2\,k_2\cdot P + 1\,,$$
$$E_8 = k_1\cdot n\,, \qquad E_9 = k_1\cdot n + P\cdot n\,,$$
$$E_{10} = k_2\cdot n\,, \qquad E_{11} = k_2\cdot n + P\cdot n\,.$$

$$E_{12} = k_1^2\,,$$
$$E_{13} = k_2^2\,,$$
$$E_{14} = k_1\cdot n + k_2\cdot n - \frac{1-z}{z} P\cdot n$$

Real Calculation

Problem in real IBP reduction

- Unregularized rapidity divergence
  - For MI $\int \mathrm{d}\Phi_{\text{real}} \frac{1}{E_1 E_4}$ → $E_1 = k_1 \cdot k_2$, $E_4 = 2k_1 \cdot P + 1$
- integrated out k_1^-, k_2^-, k_2^+, we get

$$\frac{1}{(4\pi)^2 z^2}\int_0^1 \frac{\mathrm{d}z_1}{z_1}\int \frac{\mathrm{d}^{D-2}k_{1\perp}}{(2\pi)^{D-2}}\frac{\mathrm{d}^{D-2}k_{2\perp}}{(2\pi)^{D-2}}\frac{1}{(k_{2\perp}-k_{1\perp})^2\left(k_{1\perp}^2+\left(\frac{1-z}{z}\right)^2 z_1(1-z_1)+\frac{1-z}{z}(1-z_1)\right)}$$

- rapidity divergence
- unregularized in dimensional regularization
- Problems
  - IBP relation?
  - Value of MI?

Real Calculation

Problem in real IBP reduction

- Gluon mass regularization
  - Transform the phase space integral

$$\mathrm{d}\Phi' = \frac{P\cdot n}{z^2 2!}\frac{\mathrm{d}^D k_1}{(2\pi)^{D-1}}\frac{\mathrm{d}^D k_2}{(2\pi)^{D-1}}\delta_+(k_1^2-m_g^2)\delta_+(k_2^2-m_g^2)\delta\left(k_1\cdot n + k_2\cdot n - \frac{1-z}{z}P\cdot n\right)$$

  - Take the limit of $m_g \to 0$
- Calculation of the MI
  - integrated out $k_{1\perp}, k_{2\perp}$, we get

$$t = \frac{1-z}{z}$$

$$(4\pi)^{-4+2\epsilon} m_g^{-2\epsilon}\Gamma(\epsilon)^2 z^{-2}\int_0^1 \mathrm{d}z_1\, z_1^{-1+\epsilon}(1-2z_1+2z_1^2)^{-\epsilon}(t^2 z_1 + t + m_g^2/z_1)^{-\epsilon}$$

  - Only $z_1 \sim m_g^2$ values in the limit of $m_g \to 0$
  - The final result is

$$(4\pi)^{-4+2\epsilon}\Gamma(\epsilon)^2 z^{-2}\int_0^\infty \mathrm{d}y\, y^{-1+\epsilon}(t+1/y)^{-\epsilon}$$
$$=(4\pi)^{-4+2\epsilon} z^{-2+2\epsilon}(1-z)^{-2\epsilon}\Gamma(2\epsilon)\Gamma(\epsilon)\Gamma(-\epsilon)\,.$$

Real Calculation

Problem in real IBP reduction

- Divide the origin express two parts
  - Integrals that can be regularized:
    naive IBP reduction (ignore $i\eta$ directly)
  - Integrals that can not be regularized:
    gluon mass regularization method
- The unregularized integrals cancelled finally
- Test the IBP reduction of $\int \mathrm{d}\Phi \frac{1}{E_1 E_4 E_7}$ with naive IBP method
  - One of MIs is $\int \mathrm{d}\Phi_{\text{real}} \frac{1}{E_1\, E_4}$, but IBP relation values once we take the gluon mass regularization method in the calculation of this MI
  - gluon mass regulator can indeed give correct result
  - naïve IBP reduction values once the initial integrals are regularized
- Finally obtain 95 MIs

Real Calculation

Calculation of MIs

Henn, *J. Phys.* **A48** (2015) 153001

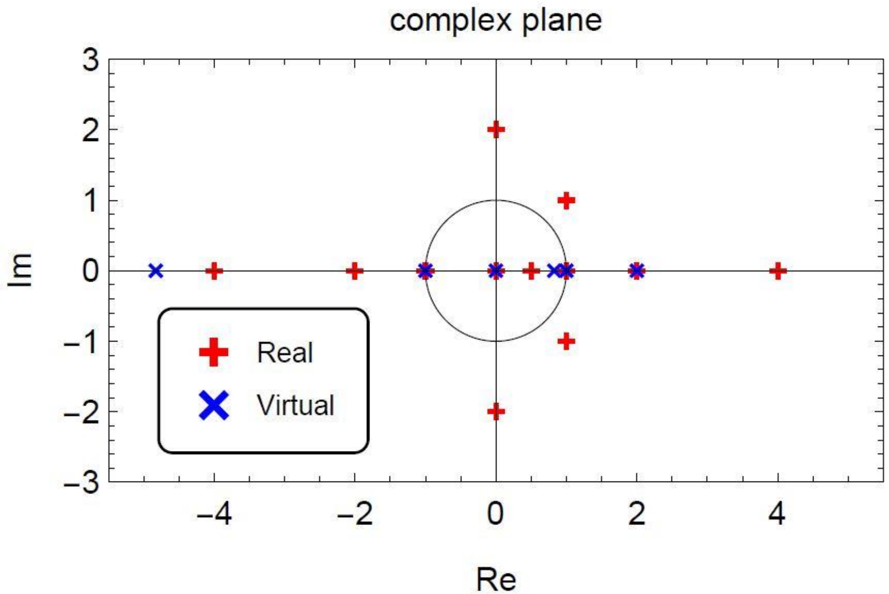
- set up differential equations (DEs)

$$\frac{\mathrm{d}\boldsymbol{I}(\epsilon,z)}{\mathrm{d}z} = A(\epsilon,z)\boldsymbol{I}(\epsilon,z)$$

- asymptotic expansions

$$I_k(z,\epsilon)|_{z_0} = \sum_s \sum_{i=0}^{n_s} (z-z_0)^s \ln^i(z-z_0) \sum_{j=0}^{\infty} I_k^{s\,i\,j}(\epsilon)(z-z_0)^j$$

- singularities in DEs: $0, 1/2, 1$



Real Calculation

Calculation of MIs

Henn, *J. Phys.* **A48** (2015) 153001

- set up differential equations (DEs)

$$\frac{\mathrm{d}\boldsymbol{I}(\epsilon, z)}{\mathrm{d}z} = A(\epsilon, z)\boldsymbol{I}(\epsilon, z)$$

- asymptotic expansions

$$I_k(z,\epsilon)|_{z_0} = \sum_{s}\sum_{i=0}^{n_s}(z-z_0)^s \ln^i(z-z_0)\sum_{j=0}^{\infty} I_k^{s\,i\,j}(\epsilon)(z-z_0)^j$$

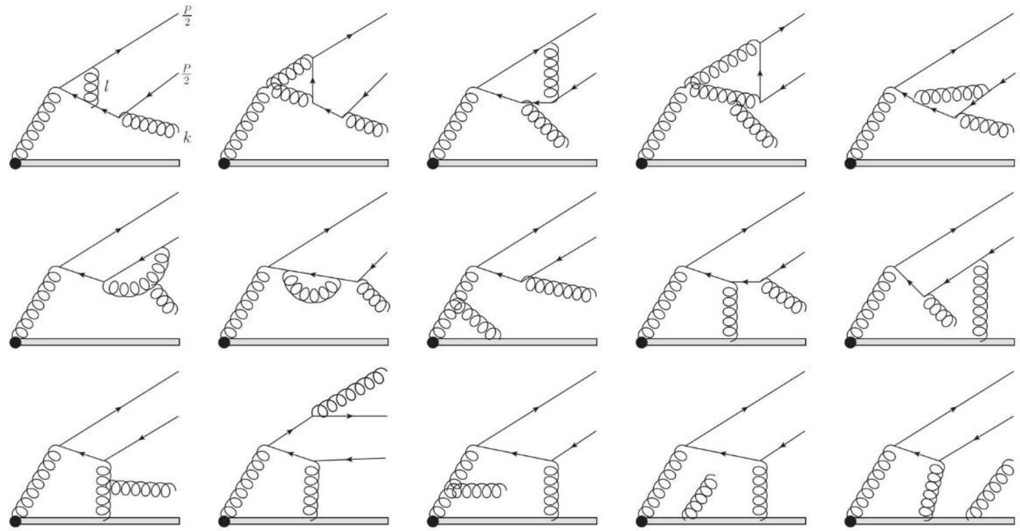
- singularities in DEs: $0, 1/2, 1$
- estimate values of MIs in regions $0\sim\frac{1}{4}, \frac{1}{4}\sim\frac{3}{4}, \frac{3}{4}\sim 1$ respectively by the asymptotic expansions of MIs at $z = 0,1/2,1$
- coefficients at high order are related with those at lower order
- calculate the boundary at $z \to 1$
  - Sector analyzation
  - Sector decomposition

Virtual Calculation

- Feynman Diagram
- Form of SDCs

$S = 1$

$$\int \mathrm{d}\Phi_{\text{loop}} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \prod_i \frac{1}{F_i^{a_i}}$$
$$= \frac{P\cdot n}{z^2} \int \frac{\mathrm{d}^D k}{(2\pi)^{D-1}} \frac{\mathrm{d}^D l}{(2\pi)^D} \delta_+(k^2)\delta\left(k\cdot n - \frac{1-z}{z} P\cdot n\right) \prod_i \frac{1}{F_i^{a_i}}$$



In which

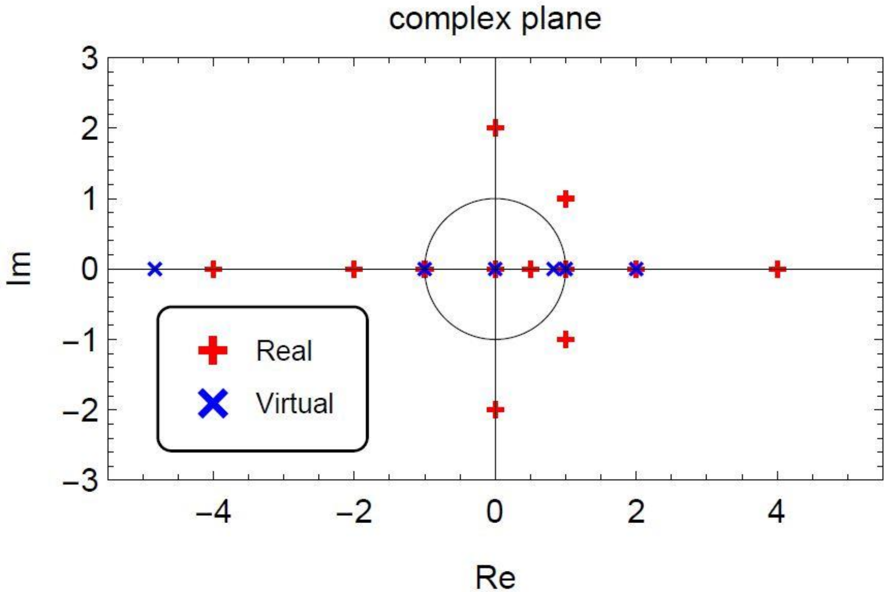
$$F_1 = k\cdot P, \qquad F_2 = 2\,k\cdot P + 1,$$
$$F_3 = l^2, \qquad F_4 = (l+k)^2, \qquad F_5 = (l+P)^2,$$
$$F_6 = (l + \frac{P}{2})^2 - \frac{1}{4}, \qquad F_7 = (l - \frac{P}{2})^2 - \frac{1}{4}, \qquad F_8 = (l + k + \frac{P}{2})^2 - \frac{1}{4},$$
$$F_9 = (l+k+P)^2, \qquad F_{10} = l\cdot n.$$

- Use naïve IBP reduction
- Obtain 66 MIs

Virtual Calculation

Calculations of MIs

- Calculate the asymptotic expansions at singularities
- Singularities in DEs: $0, 2(\sqrt{2}-1), 1$



- $z = 2(\sqrt{2}-1)$ does not affect the radius of convergence
- Boundaries at $z \to 1$ are difficult to calculate

Virtual Calculation

Calculations of MIs

- Use AMF method to calculate boundaries at $z = z_0$
- estimate values of MIs in regions $0 \sim \frac{1}{4}, \frac{1}{4} \sim \frac{3}{4}, \frac{3}{4} \sim 1$ respectively by the asymptotic expansions of MIs at $z = 0,1/2,1$

Final results

After renormalization

$$b_0 = \frac{11N_c - 2n_f}{6}$$

$$d^{(0)}_{\rm LO}(z) = \lim_{\epsilon\to 0} d_{\rm LO}(z) = (3-2z)z + 2(1-z)\ln(1-z)$$

$$d^{[1]}_{\rm NLO}(z) = \frac{\alpha_s^3}{2\pi N_c m_Q^3} \times \left(d^{[1]}(z) + \ln\left(\frac{\mu_r^2}{4m_Q^2}\right) b_0\, d^{(0)}_{\rm LO}(z) + \ln\left(\frac{\mu_f^2}{4m_Q^2}\right) f(z) \right),$$

$$d^{[8]}_{\rm NLO}(z) = \frac{\alpha_s^3 (N_c^2-4)}{4\pi N_c (N_c^2-1) m_Q^3} \times \left(d^{[8]}(z) + \ln\left(\frac{\mu_r^2}{4m_Q^2}\right) b_0\, d^{(0)}_{\rm LO}(z) + \ln\left(\frac{\mu_f^2}{4m_Q^2}\right) f(z) \right),$$

$$f(z) = -\frac{n_f}{6} d^{(0)}_{\rm LO}(z) + N_c \Bigg(-2(z+2)\mathrm{Li}_2(z) - 2(z-1)\ln^2(1-z) + 2(z-1)\ln(z)\ln(1-z) + (z-4)z\ln(z) - \frac{(2z+1)\left(9z^2-5z-6\right)\ln(1-z)}{6z} + \frac{46z^3 + \left(8\pi^2-3\right)z^2 + 4\left(\pi^2-9\right)z + 4}{12z} \Bigg),$$

$$d^{[1/8]}(z) = \begin{cases} -\dfrac{N_c}{2z} + \displaystyle\sum_{i=0}^{2}\sum_{j=0}^{\infty} \ln^i z\,(2z)^j \left(A^f_{ij}\, n_f + A^{[1/8]}_{ij}\, N_c + \frac{A^N_{ij}}{N_c} \right), & \text{for } 0 < z < \frac{1}{4} \\ \displaystyle\sum_{j=0}^{\infty} (2z-1)^j \left(B^f_j\, n_f + B^{[1/8]}_j\, N_c + \frac{B^N_j}{N_c} \right), & \text{for } \frac{1}{4} \le z \le \frac{3}{4} \\ \displaystyle\sum_{i=0}^{3}\sum_{j=0}^{\infty} \ln^i(1-z)\,(2-2z)^j \left(C^f_{ij}\, n_f + C^{[1/8]}_{ij}\, N_c + \frac{C^N_{ij}}{N_c} \right), & \text{for } \frac{3}{4} < z < 1 \end{cases}$$

- 530 orders --> 160-digit precision

Outline

- Examples: $\gamma^* \to t\bar{t} + X$ at NNLO
  - Reverse unitarity relation
  - Differential equations
  - Auxiliary mass flow
- Examples: $g \to Q\bar{Q}({}^1S_0^{[1,8]}) + X$ at NLO
- Summary

Summary

- Reverse unitarity relation transform phase-space integrals to pure loop integrals
- With IBP reduction, complex integrals can be reduced to linear combination of MIs
- Set up DEs w.r.t. kinematical invariants
- Use AMF to calculate the boundary conditions
- Final results can be expressed by a piecewise function of the asymptotic expansions at singularities, which gives a high precision
- The method is systematic and efficient
- Its high-precision nature makes it possible to obtain analytical results with a proper ansatz

Thank you!