Applications of conformal symmetry in QCD

Yao Ji

University of Siegen

Zoominar, Loop and phase space integrals

Conformal group

- **classically in $d = 4$, many theories enjoy Poincaré + Conformal symmetry**

	#	finite action	generator of inf. action
translation	4	$x_\mu \mapsto x_\mu + a_\mu$	$\mathbf{P}_\mu = -i\partial_\mu$
rotation	6	$x_\mu \mapsto \omega_{\mu\nu} x^\nu$	$\mathbf{M}_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu - \Sigma_{\mu\nu})$
dilatation	1	$x_\mu \mapsto \lambda x_\mu$	$\mathbf{D} = -i(x \cdot \partial + \Delta_\varphi)$
SCT	4	$x_\mu \mapsto \frac{x_\mu - a_\mu x^2}{1 - 2a\cdot x + a^2 x^2}$	$\mathbf{K}_\mu = -i(2x_\mu x \cdot \partial - x^2 \partial_\mu + 2\Delta_\varphi x_\mu - 2ix^\nu \Sigma_{\mu\nu})$

Poincaré group: *protected from quantum corrections*

$$i[\mathbf{P}_\mu, \mathbf{P}_\nu] = 0\,, \qquad i[\mathbf{M}_{\alpha\beta}, \mathbf{P}_\mu] = g_{\alpha\mu}\mathbf{P}_\beta - g_{\beta\mu}\mathbf{P}_\alpha\,,$$
$$i[\mathbf{M}_{\alpha\beta}, \mathbf{M}_{\mu\nu}] = g_{\alpha\mu}\mathbf{M}_{\beta\nu} - g_{\alpha\nu}\mathbf{M}_{\beta\mu} - g_{\beta\mu}\mathbf{M}_{\alpha\nu} + g_{\beta\nu}\mathbf{M}_{\alpha\mu}\,,$$

Poincaré + conformal group: *broken by quantum corrections* $\beta \neq 0$

$$i[\mathbf{D}, \mathbf{P}_\mu] = \mathbf{P}_\mu\,, \quad i[\mathbf{D}, \mathbf{K}_\mu] = -\mathbf{K}_\mu\,, \quad i[\mathbf{D}, \mathbf{M}_{\alpha\beta}] = 0\,, \quad i[\mathbf{K}_\mu, \mathbf{K}_\nu] = 0\,,$$
$$i[\mathbf{M}_{\alpha\beta}, \mathbf{K}_\mu] = g_{\alpha\mu}\mathbf{K}_\beta - g_{\beta\mu}\mathbf{K}_\alpha\,, \quad i[\mathbf{P}_\mu, \mathbf{K}_\nu] = -2g_{\mu\nu}\mathbf{D} + 2\mathbf{M}_{\mu\nu}\,.$$

UNIVERSITÄT SIEGEN

Conformal group on the light-cone

- **light-cone coordinate**

$$x^\mu = x_+ \bar{n}^\mu + x_- n^\mu + x_\perp^\mu \,, \qquad n^2 = \bar{n}^2 = 0$$

- **ultra-relativistic particles** [V. Braun, G. Korchemsky, D. Müller, (2003)]

$$\varphi(x) \mapsto \boxed{\varphi(zn) \equiv \varphi(z)} \quad \Longrightarrow \quad \text{full conformal group} \mapsto \boxed{SL(2,\mathbb{R}) \times \mathbf{E} \times \mathbf{H}}$$

◇ **collinear subgroup** $SL(2,\mathbb{R})$

- action

$$z \mapsto \tilde{z} = \frac{az+b}{cz+d} \,, \qquad a,b,c,d \in \mathbb{R} \,, \qquad ad - bc = 1 \,,$$

$$\varphi(z) \mapsto T^j \tilde{\varphi}(z) = \frac{1}{(cz+d)^{2j}} \varphi\left(\frac{az+b}{cz+d}\right) \,, \qquad \boxed{j = \frac{1}{2}(l+s) \,, \quad \text{conformal spin}}$$

- generators

$$S_-^i \equiv -\partial_{z_i} \,, \qquad S_+^i \equiv z_i^2 \partial_{z_i} + 2j z_i \,, \qquad S_0^i = z_i \partial_{z_i} + j \,,$$

$$[S_0^i, S_\pm^i] = \pm S_\pm^i \,, \qquad [S_+^i, S_-^i] = 2S_0^i \,, \qquad SL(2,\mathbb{R}) \text{ algebra}$$

$$n-\text{particle:} \quad S_{\pm,0} = \sum_i S_{\pm,0}^i \,, \quad \widehat{\mathbb{C}} = S_0(S_0 - 1) + S_+ S_- \Leftarrow \text{Casimir operator}$$

UNIVERSITÄT SIEGEN

Scale invariance in QCD

- **massless (bare) QCD in $d = 4 - 2\epsilon$**

$$S_{\text{bare}} = \int d^d x \left\{ \bar{q} i \not{D} q - \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} - \bar{c}^a \partial_\mu (D^\mu c)^a + \frac{1}{2\xi} (\partial_\mu A^{a,\mu})^2 \right\}_b$$

- **renormalized QCD**

$$\text{fundamental fields: } \varphi_b = \sqrt{Z_\varphi}\,\varphi\,, \qquad \text{vertices: } Z_1\, g\bar{q}\not{A} t^a q\,, \ldots$$

$$\text{charges: } a_b = \mu^{2\epsilon} a \frac{Z_1^2}{Z_q^2 Z_A} \equiv \mu^{2\epsilon} a\, \widetilde{Z}\,, \qquad a \equiv g^2/(4\pi)^2\,, \qquad \xi_b = Z_A \xi$$

in $\overline{\text{MS}}$-scheme

$$\ln Z = \sum_{j=1}^{\infty} \frac{z_j(a)}{\epsilon^j}\,, \qquad z_j(a) \text{ are } \epsilon - \textit{independent!}$$

β-function: $$\beta_a \equiv \mu \frac{da}{d\mu} = -2a\left(\epsilon + \bar{\beta}(a)\right) = -2a\left(\epsilon - \frac{\partial \tilde{z}_1(a)}{\partial \ln a}\right)\,, \qquad \beta_\xi = -2\xi\gamma_A$$

breaking of scale invariance (dilatation current)

$$D_\mu = \Theta_{\mu\nu} x^\nu\,, \qquad \partial \cdot D = \beta(a)\partial\mathcal{L}/\partial a\,, \qquad \Longrightarrow \text{ no scale inv. for } d = 4\,, \beta \neq 0$$

Wilson-Fisher point

- **stay in non-integer dimensions $d = 4 - 2\epsilon$** [V. Braun, A. Manashov, (2013)]

$$\exists\epsilon \text{ for } \forall a_*, \quad \text{s.t.} \quad \boxed{\beta(a_*) = 0} \qquad \text{Wilson-Fisher fix point}$$

scale invariance is restored! technically, $\bar{\beta}(a_*) \mapsto -\epsilon$

- **consider leading-twist (renormalized) light-ray operator**

$$\mathcal{O}_{\not{n}}(x; z_1, z_2) \equiv \bar{q}(x + z_1 n)\not{n} q(x + z_2 n), \quad \mathcal{O}_{\not{n}}(z_1, z_2) \equiv \mathcal{O}_{\not{n}}(0; z_1, z_2) \quad \textit{main subject}$$

satisfying RGE

$$(\mu\partial_\mu + \beta(a)\partial_a + \mathbb{H}(a))\, \mathcal{O}_{\not{n}}(z_1, z_2) = 0$$
$$\left(\mu\partial_\mu + \widetilde{\mathbb{H}}(a_*)\right) \mathcal{O}_{\not{n}}(z_1, z_2) = 0$$

$\widetilde{\mathbb{H}} \sim$ simple pole in Laurent ϵ expansion

- **Z is ϵ-independent in $\overline{\text{MS}}$-like schemes by construction**

$$\mathbb{H}(a) = a\mathbb{H}^{(1)} + a^2\mathbb{H}^{(2)} + \cdots, \qquad \widetilde{\mathbb{H}}(a_*) = a_*\mathbb{H}^{(1)} + a_*^2\mathbb{H}^{(2)} + \cdots$$

QCD at critical point

- at the critical point with coupling a_*

$$[S_{\pm,0}(a_*), \mathbb{H}(a_*)] = 0$$

with $S_{\pm,0}(a_*)$ obeying $SL(2,R)$ algebra

$$[S_+(a_*), S_-(a_*)] = 2S_0(a_*)$$
$$[S_0(a_*), S_\pm(a_*)] = \pm S_\pm(a_*)$$

- $S_-(a_*) \leftrightarrow \mathbf{P}_+ \implies \boxed{S_-(a_*) = S_-}$, i.e., no quantum corrections!

- however, $S_+(a_*) \sim \mathbf{K}_-$ and $S_0(a_*) \sim \mathbf{D}$ receive quantum corrections!

$$S_0(a_*) = S_0 - \epsilon + \frac{1}{2}\mathbb{H}(a_*)$$
$$S_+(a_*) = S_+^{(0)} + (z_1 + z_2)\Big(-\epsilon + \frac{1}{2}\mathbb{H}(a_*)\Big) + z_{12}\Delta(a_*)$$

$$z_{12} \equiv z_1 - z_2$$

- $\Delta(a_*)$ conformal anomaly: nontrivial information require diagrammatic calculations

What's the advantage?

an example

- $\mathcal{O}_{\not{n}}(z_1, z_2)$ is $SL(2)$ invariant classically (oblivious to n_f: BZ fix point)

[Bukhvostov, Frolov, Kuraev, Lipatov (1985)]

$SL(2)$ invariance $\Longrightarrow [S_{0,\pm}, \mathbb{H}^{(1)}] = 0 \Longrightarrow \mathbb{H}^{(1)}_{\bar{q}q} = h(\widehat{\mathbb{C}})$

$$\widehat{\mathbb{C}} = S_+S_- + S_0(S_0 - 1)$$

quadratic Casimir operator

- z_{12}^N is the eigenfunction of $\mathbb{H}(a)$ to all orders (translational inv. + local OPE)

$$\mathbb{H}^{(1)} z_{12}^N = \gamma_N^{(1)} z_{12}^N \,,$$

forward lim. $\mapsto$ splitting function

$$\Downarrow$$

$$\mathbb{H}^{(1)} = 2C_F\left[\psi(\widehat{J}+1) + \psi(\widehat{J}-1) - 2\gamma_E - \frac{3}{2}\right] \qquad \widehat{\mathbb{C}} = \widehat{J}(\widehat{J}-1) \quad \sim C_N^{3/2}$$

DGLAP+ERBL+GPD evolution

- diagrammatic calculation only needed for local operators. easy!

What's the advantage?

- **recipe for obtaining $\mathbb{H}^{(\ell)}$**
  - (1) find $\Delta^{(\ell-1)}$, (for $\mathbb{H}^{(1)}$, $\Delta^{(0)} = 0$, trivial)
  - (2) find anomalous dimension $\gamma_N^{(\ell)}$ from forward kinematics (available upto high orders, e.g, splitting functions)
  - (3) from the nontrivial identities

$$[S_+(a_*), \mathbb{H}(a_*)] = 0$$

obtain a series of commutation relations to the desired order in a, e.g.,

$$\begin{aligned} [S_+^{(0)}, \mathbb{H}^{(1)}] &= 0\,, \\ [S_+^{(0)}, \mathbb{H}^{(2)}] &= [\mathbb{H}^{(1)}, \Delta S_+^{(1)}]\,, \\ &\vdots \end{aligned}$$

  - (4) write $\mathbb{H}^{(\ell)} = \mathbb{H}^{(\ell)}_{\text{inv.}} + \mathbb{H}^{(\ell)}_{\text{ninv}}$ with $[S_+^{(0)}, \mathbb{H}^{(\ell)}_{\text{inv}}] = 0$, solve $\mathbb{H}^{(\ell)}_{\text{ninv}}$ from comm. relations, finally obtain $\mathbb{H}_{\text{inv}}$ by matching (e.g., moments of splitting function)

$$\mathbb{H}^{(\ell)} z_{12}^N = \gamma_N^{(\ell)} z_{12}^N$$

Conformal anomaly and how to find it

- **nonfoward information of $\mathbb{H}$ contained in conformal anomaly Δ**
- **at WF critical point, two additional symmetries $\Longrightarrow$ two (exact) Ward identities**

consider

$$\mathcal{G}(x; z, w) \equiv \langle \mathcal{O}_{\not n}(0; z_1, z_2) \mathcal{O}_{\bar{\not n}}(x; w_1, w_2) \rangle$$

  - **scale Ward identity (SWI) ($\mathbb{R}^d$)** [A. Belitsky, D. Müller, (1998)]

$$0 = \delta_{\mathbf{D}} \mathcal{G}(x; z, w) = \langle [\delta_{\mathbf{D}} \mathcal{O}_{\not n}(0; z)] \mathcal{O}_{\bar{\not n}}(x; w) \rangle + \langle \mathcal{O}_{\not n}(0; z) [\delta_{\mathbf{D}} \mathcal{O}_{\bar{\not n}}(x; w)] \rangle - \langle [\delta_{\mathbf{D}} S_R] \mathcal{O}_{\not n}(0; z) \mathcal{O}_{\bar{\not n}}(x; w) \rangle$$

$$\Downarrow$$

$$0 = \left(\mu \partial_\mu + \beta(a) \partial_a + \mathbb{H}^{(z)}(a) + \mathbb{H}^{(w)}(a) \right) \mathcal{G}(x; z, w)$$

RGE; β restored

  - **conformal Ward identity (CWI)** [A. Belitsky, D. Müller, (1998)]

$$0 = \delta_{\mathbf{K}} \mathcal{G}(x; z, w) = \langle [\delta_{\mathbf{K}} \mathcal{O}_{\not n}(0; z)] \mathcal{O}_{\bar{\not n}}(x; w) \rangle + \langle \mathcal{O}_{\not n}(0; z) [\delta_{\mathbf{K}} \mathcal{O}_{\bar{\not n}}(x; w)] \rangle - \langle [\delta_{\mathbf{K}} S_R] \mathcal{O}_{\not n}(0; z) \mathcal{O}_{\bar{\not n}}(x; w) \rangle$$

Conformal anomaly and how to find it

- **implication of (exact) CWI** [V. Braun, A. Manashov, (2013), (2014)]

$$S_+\mathcal{G}(x;z,w) = ZS_+^{(\epsilon)}Z^{-1}\mathcal{G}(x;z,w) + 2\int d^dy\,(\bar{n}\cdot y)\,\langle \mathcal{N}(y)\mathcal{O}_{\not{n}}(0;z)\mathcal{O}_{\not{\bar{n}}}(x;w)\rangle$$

- $ZS_+^{(\epsilon)}Z^{-1}$ generates terms independent of Δ in S_+
- second term generates conformal anomaly, Δ [V. Braun, A. Manashov, S. Moch, M. Strohmaier, (2018)]

$$\mathcal{N} = -\frac{\beta(a)}{a}\mathcal{L}^{\mathrm{YM+gf}} + \mathrm{EOM} + \mathrm{BRST}$$

EOM contribution $\propto \gamma_\varphi \mathcal{G}(x;z,w)$ (functional IBP)

BRST operators have no contribution

$\mathcal{L}^{\mathrm{YM+gf}}$ insertion is difficult, utilize renormalizability in $d=4$ to simplify $\langle\cdots\rangle$

$$\Delta S_+(a)\mathcal{G}(x;z,w) = \cdots - \frac{2\beta(a)}{a}\int d^dy\,(\bar{n}\cdot y)\langle[\mathcal{L}^{\mathrm{YM+gf}}(y)]_r[\mathcal{O}_{\not{n}}(0,z)]_r[\mathcal{O}_{\not{\bar{n}}}(x;w)]_r\rangle$$

$[\cdots]_r[\cdots]_r$ are separately renormalized

Conformal anomaly and how to find it

- **explicit expression for ΔS_+ at ℓ-order**

$$\Delta S_+^{(\ell)} = \left(\beta^{(\ell-1)} + \frac{1}{2}\mathbb{H}^{(\ell)}\right)(z_1 + z_2) - \frac{1}{2\ell}\left[\mathbb{H}^{(\ell)}, z_1 + z_2\right] + z_{12}\widetilde{\Delta}^{(\ell)}$$

- **$\widetilde{\Delta}(a)$ from modified Feynman rules** [V. Braun, A. Manashov, S. Moch, M. Strohmaier, (2016)]

calculating $\widetilde{\Delta}^{(\ell)}$ from Green's function with $\mathcal{L}^{\mathrm{YM+gf}}$ (effective vertex) insertion

$$\widetilde{\Delta}(a) = 2\mathrm{KR}\left\{\left\langle q(x_1)\bar{q}(x_2)\int d^d y\,(\bar{n}\cdot y)\mathcal{L}^{\mathrm{YM+gf}}(y)\mathcal{O}_{\not{n}}(0,z)\right\rangle\right\}$$

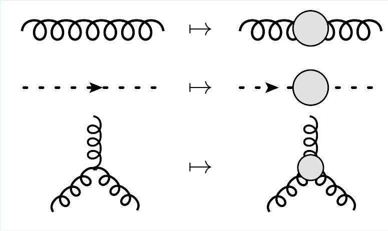
$\mathrm{KR} \implies$ extracting coeff. of simple pole

modified effective Feynman rules

gluon propagator

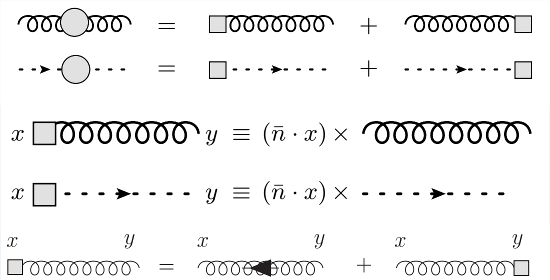
ghost propagator

three-gluon vertex



Conformal anomaly and how to find it

modified effective Feynman rules (continued)



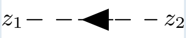
shift operation

$$\bar{n}\cdot x = \bar{n}\cdot(x-y) + \bar{n}\cdot y$$ **translational invariant**

$$= 2ig_{\mu\nu}\int\frac{d^dk}{(2\pi)^d}\mathrm{e}^{-ik\cdot(x-y)}\frac{\bar{n}\cdot k}{k^4} = -g_{\mu\nu}D_g(x-y), \quad (\mathbb{R}^d, \xi=1)$$

sometime convenient to shift $\bar{n}\cdot x$ along quark and/or Wilson line

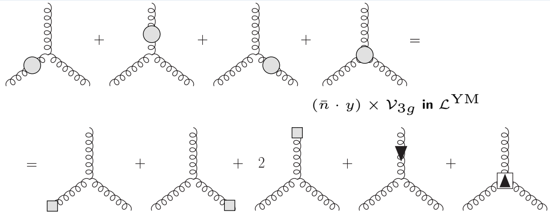
$$= -\int\frac{d^dk}{(2\pi)^d}\mathrm{e}^{-ik\cdot(x-y)}\frac{\not{k}\not{\bar{n}}\not{k}}{k^4}, \quad (\mathbb{R}^d)$$

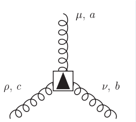


$$= (n\cdot\bar{n})z_{12}[z_1 n, z_2 n]$$ **Gauge link**

Conformal anomaly and how to find it

modified effective Feynman rules (continued)





$= g f^{abc} \left(g^{\mu\rho} \bar{n}^{\nu} - g^{\mu\nu} \bar{n}^{\rho} \right)$ **modified three-gluon vertex**

One-loop conformal anomaly

- **One-loop anomaly diagrams (+conjugated)** [V. Braun, A. Manashov, S. Moch, M. Strohmaier, (2016)]

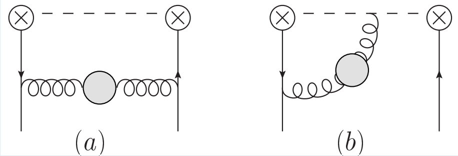
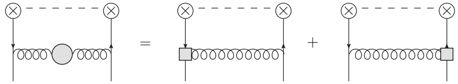
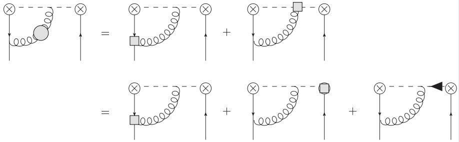


diagram (a):



external (constant) positions (trivial) $\propto \mathbb{H}^{(1)}_{\mathcal{D}_1}$

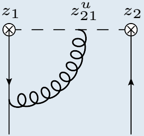
diagram (b):



(two) external $\propto \mathbb{H}^{(1)}_{\mathcal{D}_2}$ **+ modified gauge-link**

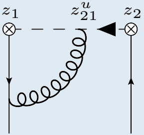
One-loop conformal anomaly

diagram (b) (continued)



$z_{21}^u \equiv \bar{u} z_2 + u z_1$

$$= \int_0^1 d\alpha\, h^{(b)}(\alpha) \int_0^1 du\, \frac{d}{du} \mathcal{G}(z_{12}^{\alpha\bar{u}}, z_2)\,, \quad h^{(b)}(\alpha) = 4C_F \frac{\bar{\alpha}}{\alpha}$$



$z_{21}^u \equiv \bar{u} z_2 + u z_1$

$$= \frac{1}{2} z_{12}(n\bar{n}) \int_0^1 d\alpha\, h^{(b)}(\alpha) \int_0^1 du\, u \frac{d}{du} \mathcal{G}(z_{12}^{\alpha\bar{u}}, z_2)$$ ($\frac{1}{2}$ due to different norm.)

- S_+ **at one-loop order: summing up $(a) + (b) +$ (conjugated):**

$$S_+^{(1)}(a) = S_+^{(0)} + a\left[(z_1 + z_2)\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) + z_{12}\widetilde{\Delta}^{(1)}\right]$$

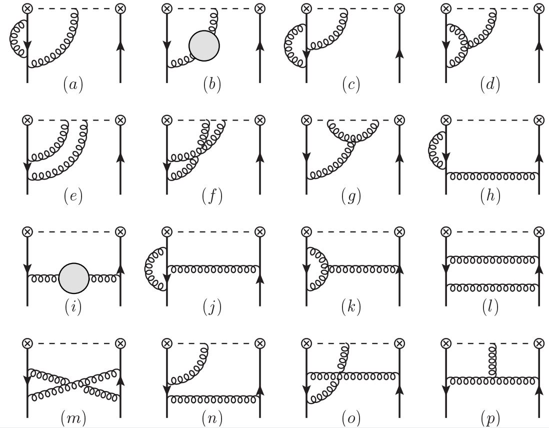
$$\widetilde{\Delta}^{(1)}\mathcal{O}(z_1, z_2) = -2C_F \int_0^1 d\alpha \left(\frac{\bar{\alpha}}{\alpha} + \ln\alpha\right)[\mathcal{O}(z_{12}^{\alpha}, z_2) - \mathcal{O}(z_1, z_{21}^{\alpha})]$$

at the critical point $a \mapsto a_*$, $\beta \mapsto -\epsilon$

Two-loop anomaly

- **diagrams for two-loop conformal anomaly are generated from + (conjugated)**

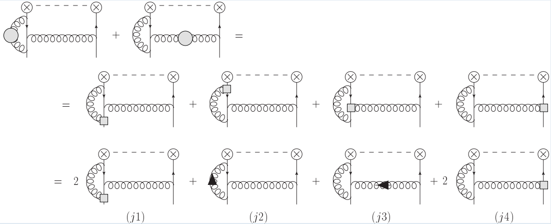
[V. Braun, A. Manashov, S. Moch, M. Strohmaier, (2016)]



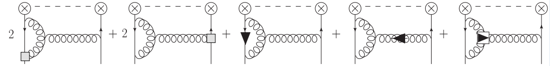
by inserting a "bubble" on one gluon/ghost line/three-gluon vertex each time

Two-loop anomaly

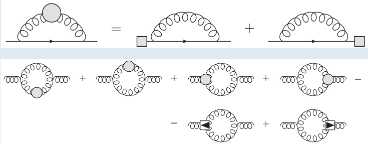
- **example: diagram (j)**



- **example: diagram (k) with three-gluon vertex**



- **simplifications for quark and gluon self-energy**



◇ **key message: most diagrams are proportional to $\mathcal{H}_{\mathcal{D}}^{(2)}$**

Two-loop anomaly

- **conformal anomaly to $\mathcal{O}(a^2)$**

$$S_+(a) = S_+^{(0)} + \sum_{\ell=1}^{\infty} a^\ell \Delta S_+^{(\ell)}\,, \qquad \mathbb{H}(a) = \sum_{\ell=1}^{\infty} a^\ell \mathbb{H}^{(\ell)}$$

$$\Delta S_+^{(\ell)} = \left(\beta^{(\ell-1)} + \frac{1}{2}\mathbb{H}^{(\ell)}\right)(z_1+z_2) - \frac{1}{2\ell}\left[\mathbb{H}^{(\ell)}, z_1+z_2\right] + z_{12}\widetilde{\Delta}^{(\ell)}$$

$$\widetilde{\Delta}^{(1)} f(z_1,z_2) = -2C_F \int_0^1 d\alpha \left(\frac{\bar\alpha}{\alpha} + \ln\alpha\right)\left[f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha)\right]$$

$$\widetilde{\Delta}^{(2)} f(z_1,z_2) = \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \left[\omega(\alpha,\beta)\left(f(z_{12}^\alpha, z_{21}^\beta) - f(z_{12}^\beta, z_{21}^\alpha)\right) + (z_1 \leftrightarrow z_2)\right]$$
$$+ \int_0^1 du \int_0^1 dt\, \varkappa(t)\left[f(z_{12}^{ut}, z_2) - f(z_1, z_{21}^{ut})\right]$$

explicit expressions for $\omega(\alpha,\beta)$ and $\varkappa(t)$ in [V. Braun, A. Manashov, S. Moch, M. Strohmaier, (2016)]

From conformal anomaly to evolution kernel

- **commutation relation for three-loop evolution kernel $\epsilon(a_*) \mapsto -\bar{\beta}(a)$ in 4d**

$$S_+(a) = S_+^{(0)} + \sum_{\ell=1}^{\infty} a^\ell \Delta S_+^{(\ell)}, \qquad \mathbb{H}(a) = \sum_{\ell=1}^{\infty} a^\ell \mathbb{H}^{(\ell)}$$

$$[S_+^{(0)}, \mathbb{H}^{(3)}] = [\mathbb{H}^{(2)}, \Delta S_+^{(1)}] + [\mathbb{H}^{(1)}, \Delta S_+^{(2)}]$$

- **generically to all orders**

$$[\mathbb{H}f](z_1,z_2) = H_{\text{const}}\, f(z_1,z_2) + \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, h(\alpha,\beta) f(z_{12}^{\alpha}, z_{21}^{\beta}) + \int_0^1 d\alpha\, \frac{\bar{\alpha}}{\alpha} g(\alpha)\left(2f(z_1,z_2) - f(z_{12}^{\alpha},z_2) - f(z_1,z_{21}^{\alpha})\right)$$

therefore $[S_+^{(0)}, \mathbb{H}^{(\ell)}] \mapsto$ diff. eqns. on $h(\alpha,\beta)$ and $g(\alpha)$

- $\mathbb{H} = \mathbb{H}_{\text{inv}} + \mathbb{H}_{\text{ninv}}$ **HDEs. on $h_{\text{inv}}(\alpha,\beta), g_{\text{inv}}(\alpha)$** $\Longrightarrow h_{\text{inv}}(\alpha,\beta) = T\left(\frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}\right), g_{\text{inv}} = \Gamma_{\text{cusp}}$

[B. Basso and G. Korchemsky, (2007)]

- $\mathbb{H}_{\text{ninv}}$ **from commutation relations**
- **finally, fix $h_{\text{inv}}(\alpha,\beta)$ by matching forward A.D. $\gamma(N) = \gamma_{\text{inv}}(N) + \gamma_{\text{ninv}}(N)$ with testing function $f(z_1,z_2) = z_{12}^N$, Mellin transform**

From conformal anomaly to evolution kernel

- **Similarity transform** [V. Braun, A. Manashov, S. Moch, M. Strohmaier, (2017)]

$$\mathbb{U} \ : \ [\mathcal{O}(z_1, z_2)] \mapsto [\mathcal{O}(z_1, z_2)]^{\mathbb{U}} \equiv \mathbb{U}\mathcal{O}(z_1, z_2) \equiv \mathrm{e}^{\mathbb{X}(a)}\mathcal{O}(z_1, z_2)$$

such that

$$\begin{aligned}
\mathbf{S}_-(a) &\equiv \mathbb{U}S_-(a)\mathbb{U}^{-1} = S_-(a)\,,\\
\mathbf{S}_0(a) &\equiv \mathbb{U}S_0(a)\mathbb{U}^{-1} = S_0^{(0)} + \beta(a) + \frac{1}{2}\mathbb{U}\mathbb{H}(a)\mathbb{H}^{-1} = S_0^{(0)} + \beta(a) + \frac{1}{2}\mathbf{H}(a)\,,\\
\mathbf{S}_+(a) &\equiv \mathbb{U}S_+(a)\mathbb{U}^{-1} = S_+^{(0)} + (z_1 + z_2)\left(\beta(a) + \frac{1}{2}\mathbf{H}(a)\right)
\end{aligned}$$

therefore

$$\begin{aligned}
&[S_+^{(0)}, \mathbb{X}^{(1)}] = z_{12}\Delta^{(1)}\,,\\
&[S_+^{(0)}, \mathbb{X}^{(2)}] = z_{12}\Delta^{(2)} + \left[\mathbb{X}^{(1)}, z_1 + z_2\right]\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) + \frac{1}{2}\left[\mathbb{X}^{(1)}, z_{12}\Delta^{(1)}\right]
\end{aligned}$$

no conformal anomaly for bold-font operators, simpler commutation relations

From conformal anomaly to evolution kernel

- further define operators $\mathbb{T}_j^{(i)}$ with $[S_\pm^{(0)}, \mathbf{H}_{\text{inv}}^{(i)}] = 0$

$$[S_{0,-}^{(0)}, \mathbb{T}^{(i)}] = 0\,, \qquad [S_+^{(0)}, \mathbb{T}^{(1)}] = [\mathbf{H}_{\text{inv}}^{(1)}, z_1 + z_2]\,,$$
$$[S_+^{(0)}, \mathbb{T}^{(2)}] = [\mathbf{H}_{\text{inv}}^{(2)}, z_1 + z_2]\,, \qquad [S_+^{(0)}, \mathbb{T}_1^{(1)}] = [\mathbb{T}^{(1)}, z_1 + z_2]$$

choosing $\mathbb{H}_{\text{inv}}^{(i)} = \mathbf{H}_{\text{inv}}^{(i)}$

$$\mathbb{H}_{\text{ninv}}^{(1)} = 0\,,$$
$$\mathbb{H}_{\text{ninv}}^{(2)} = \mathbb{T}^{(1)}\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) + [\mathbb{H}^{(1)}, \mathbb{X}^{(1)}]\,,$$
$$\mathbb{H}_{\text{ninv}}^{(3)} = F\left(\mathbb{H}^{(1)}, \mathbb{H}^{(2)}, \mathbb{H}_{\text{inv}}^{(2)}, \mathbb{T}^{(1)}, \mathbb{T}^{(2)}, \mathbb{T}_1^{(1)}, \mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \beta_0, \beta_1\right)$$

- complete analytical expression for $\mathbb{H}^{(2)}$ (checked)
- semi-analytical expression for $\mathbb{H}_{\text{inv}}^{(3)} \Leftrightarrow$ diagonal in ADM. in Gegenbauer basis from splitting functions [S. Moch, J. Vermaseren, A. Vogt, (2004)]
- analytic off-diagonal ADM γ_{MN} obtained with $1 \le M, N \le 7$, sufficient for pheno. studies [V. Braun, A. Manashov, S. Moch, M. Strohmaier, (2017)]

One-loop kernel beyond leading-twist

- one-loop kernel upto twist-4 obtained from conformal symmetry (without diagram calc!)

  [V. Braun, A. Manashov, J. Rohrwild, (2009)]

  complication: operator mixing of equal twist

  - $2 \to 2$ operator mixing: $\mathbb{H}_{\rm ht}$ in 2×2 matrix form, commutes with $SO(4,2)$ (full) conformal Casimir $\mathbb{C}^2_{\rm ht} \implies \mathbb{H}_{\rm ht} = f(\mathbb{C}^2_{\rm ht})$. spectrum of $\mathbb{H}_{\rm ht}$ uniquely determined by leading-twist kernel, fixing $\mathbb{H}_{\rm ht}$ completely in terms of $SL(2,R)$ invariants and intertwining operators [N. Beisert, G. Ferretti, R. Heise and K. Zarembo, (2005)]
  - $2 \to 3$ evolution kernel: remarkably completely fixed by corresponding $2 \to 2$ kernels, EOM and Poincarè symmetry [V. Braun, A. Manashov, J. Rohrwild, (2009)]

- confirmed by explicit diagrammatic calculations, results available in mom. space

  [YJ, A. Belitsky, (2014)]

Coefficient functions from conformal symmetry

Conformal symmetry also applicable beyond evolution kernels

- leading-twist Wilson coefficient functions for DVCS analytically at two-loop
  - exploiting conformal OPE of two EM currents $T\{j_\mu(x_1)j_\nu(x_2)\}$ at critical point and matching to DIS coefficient functions (C_2, C_L)
    ℓ-loop DVCS CFs require ℓ-loop DIS CFs and $(\ell-1)$-loop conformal infos. from $\mathbb{H}$ calc.
  - checked explicitly for one-loop DVCS CFs
  - directly applicable to EIC; numerically $\sim 10\%$ two-loop corrections

[V. Braun, A. Manashov, S. Moch, J. Schoenleber, (2020)]

- all twist kinematic contribution to DVCS from conformal OPE [V. Braun, YJ, A. Manashov, (2020)]
  - nontrivial for nonfoward kinematics, completely fixed through DIS CFs
  - important for evaluating target-mass corrections and checking factorization

[V. Braun, YJ, A. Manashov, (to appear)]

Leading-twist distribution amplitude in HQET

Definition

[A. Grozin, M. Neubert (1997)]

$$\langle 0|\left[\bar{q}(zn)\, \not{n}[zn,0]\gamma_5 h_v(0)\right]_R|\bar{B}(v)\rangle = iF_B(\mu)\ \Phi_+(z,\mu)$$

- v_μ is the heavy quark velocity
- n_μ is the light-like vector, $n^2=0$, such that $n\cdot v=1$
- The twist-2 LCDA $\Phi_+(z-i0,\mu)$ is an analytic function of z in the lower half-plane

Fourier transform

$$\phi_+(\omega,\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, e^{i\omega z}\Phi_+(z-i0,\mu)\,,$$

$$\Phi_+(z,\mu) = \int_0^{\infty} d\omega\, e^{-i\omega z}\phi_+(\omega,\mu)\,.$$

- $\omega>0$ is the $(2\times)$ light quark energy in the b–quark rest frame

One-loop evolution of leading twist DA

- **RGE**

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(a)\frac{\partial}{\partial a}+\mathcal{H}(a)\right)\Phi_+(z,\mu)=0\,,$$

[B. Lange, M. Neubert (2003)]

with $\mathcal{H}$ being the evolution kernel, usually presented as an integral operator.

- **One-loop evolution kernel**

$$\mathcal{H}^{(1)}\Phi_+(z,\mu)=4C_F\left\{[\ln(i\widetilde{\mu}z)+1/2]\Phi_+(z,\mu)+\int_0^1 du\frac{\bar{u}}{u}\left[\Phi_+(z,\mu)-\Phi_+(\bar{u}z,\mu)\right]\right\}$$

where $\widetilde{\mu}=\mathrm{e}^{\gamma_E}\mu_{\overline{\mathrm{MS}}}$ and $\bar{u}=1-u$. [A. Grozin, M. Neubert, (1997); V. Braun, D. Ivanov, G. Korchemsky, (2004)]

- **Solution to one-loop RGE** [G. Bell, T. Feldmann, Y.-M. Wang, M. W. Y. Yip, (2013); V. Braun, A. Manashov (2014)]

$$\Phi_+(z,\mu)=-\frac{1}{z^2}\int_0^\infty ds\, s\, e^{is/z}\,\eta_+(s,\mu)\,,$$

$$\eta_+(s,\mu)=R(s,\mu,\mu_0)\eta_+(s,\mu_0)\,,\qquad R(s,\mu,\mu_0)\propto s^{\frac{2C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}}$$

Come back later!

Solution to one-loop evolution of higher-twist DAs (1)

- **One-loop evolution kernels of three-particle DAs are pairwise.**
  Example

$$-2iF(\mu)\Phi_3(z_1,z_2,\mu)=\langle 0|\,\bar q(z_1)gG_{\mu\nu}(z_2)n^\nu\sigma^{\mu\rho}n_\rho\gamma_5 h_v(0)\,|\bar B(v)\rangle$$

where the one-loop kernel takes the form $\mathcal{H}^{(1)}_{\Phi_3}=\mathcal{H}^{(1)}_{qg}+\mathcal{H}^{(1)}_{gh}+\mathcal{H}^{(1)}_{qh}$ with

$$[\mathcal{H}^{(1)}_{qh}f](z_1)=\frac{-1}{N_c}\left\{\int_0^1\frac{d\alpha}{\alpha}\left[f(z_1)-\bar\alpha f(\bar\alpha z_1)\right]+\left[\ln(i\mu z_1)-\frac{5}{4}\right]f(z_1)\right\},$$

$$[\mathcal{H}^{(1)}_{gh}f](z_2)=N_c\left\{\int_0^1\frac{d\alpha}{\alpha}\left[f(z_2)-\bar\alpha^2 f(\bar\alpha z_2)\right]+\left[\ln(i\mu z_2)-\frac{1}{2}\right]f(z_2)\right\},$$

$$[\mathcal{H}^{(1)}_{qg}\varphi](z_1,z_2)=N_c\left\{\int_0^1\frac{d\alpha}{\alpha}\left[2\varphi(z_1,z_2)-\bar\alpha\varphi(z^\alpha_{12},z_2)-\bar\alpha^2\varphi(z_1,z^\alpha_{21})\right]\right.$$
$$\left.-\frac{3}{4}\varphi(z_1,z_2)\right\}-\frac{2}{N_c}\int_0^1 d\alpha\int_{\bar\alpha}^1 d\beta\,\bar\beta\,\varphi(z^\alpha_{12},z^\beta_{21}),$$

where $z^\alpha_{12}=\bar\alpha z_1+\alpha z_2$. [M. Knödlseder, N. Offen (2011); V. Braun, A. Manashov, J Rohrwild, (2009); YJ, A. Belitsky (2014)]

- **Solution?**

UNIVERSITÄT SIEGEN

One-loop evolution of higher-twist DAs (2)

- **RGE for $\Phi_3(z_1, z_2, \mu)$ is integrable at large N_c limit.**

  Two conserved charges (hidden symmetries)

$$\left[\mathbb{Q}_1, \mathbb{Q}_2\right] = \left[\mathbb{Q}_1, \mathcal{H}^{(1)}_{\Phi_3}\right] = \left[\mathbb{Q}_2, \mathcal{H}^{(1)}_{\Phi_3}\right] = 0$$

  explicitly [V. Braun, A. Manashov, N. Offen (2015)]

$$\begin{aligned} \mathbb{Q}_1 &= i(S_q^+ + S_g^+)\,, \\ \mathbb{Q}_2 &= \frac{9}{4} i S_g^+ - i S_g^+ \left(S_g^+ S_q^- + S_g^0 S_q^0\right) - i S_g^0 \left(S_q^0 S_g^+ - S_g^0 S_q^+\right) \end{aligned}$$

  from **Quantum Inverse Scattering Method (QISM)** [E. Sklyanin (1992)]
  $S^+ = z^2\partial_z + 2jz\,, S^0 = z\partial_z + j\,, S^- = -\partial_z$, j conformal spin.

- **Two DOF in $\Phi^{(1)}_{\Phi_3}$ $\Longrightarrow$ $\mathcal{H}^{(1)}_{\Phi_3}$ and $\{\mathbb{Q}_1, \mathbb{Q}_2\}$ share the same eigenfunction.**

- **Integrability of RGE $\Leftrightarrow$ Integrable spin chains** [V. Braun, YJ, A. Manashov (2018)]

One-loop evolution of higher-twist DAs (3)

- **Solving for eigenfunction of $\{\mathbb{Q}_1, \mathbb{Q}_2\}$ leads to (complete orthonormal basis)**

$$\phi_-(\omega,\mu) = \int_\omega^\infty \frac{d\omega'}{\omega'}\phi_+(\omega',\mu) + \int_0^\infty ds\, J_0(2\sqrt{\omega s})\, \eta_3^{(0)}(s,\mu)$$

$$\phi_3(\underline{\omega},\mu) = \int_0^\infty ds\Big[\eta_3^{(0)}(s,\mu)\, Y_3^{(0)}(s\,|\,\underline{\omega}) + \frac{1}{2}\int_{-\infty}^{\infty} dx\, \eta_3(s,x,\mu)\, Y_3(s,x\,|\,\underline{\omega})\Big],$$

where $Y_3^{(0)}(s\,|\,\underline{\omega}) = Y_3(s, x = i/2\,|\,\underline{\omega})$ and

$$Y_3(s,x\,|\,\underline{\omega}) = -\int_0^1 du\, \sqrt{us\omega_1}\, J_1(2\sqrt{us\omega_1})\, \omega_2\, J_2(2\sqrt{\bar{u}s\omega_2})\, {}_2F_1\left(\begin{matrix}-\frac{1}{2}-ix, -\frac{1}{2}+ix \\ 2\end{matrix}\,\Big|\, -\frac{u}{\bar{u}}\right)$$

Solving RGE for $\phi_3(\underline{\omega},\mu)$ up to $1/N_c^2$ gives

$$\eta_3(s,x,\mu) = L^{\gamma_3(x)/\beta_0} R(s;\mu,\mu_0)\, \eta_3(s,x,\mu_0)$$

$$\eta_3^{(0)}(s,\mu) = L^{N_c/\beta_0} R(s;\mu,\mu_0)\eta_3^{(0)}(s,\mu_0)$$

where $L = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$ and $\gamma_3(x) = N_c[\psi(3/2+ix) + \psi(3/2-ix) + 2\gamma_E]$.

- $1/N_c^2 \sim \mathcal{O}(10^{-1})$ **taken perturbatively**

One-loop evolution of higher-twist DAs (4)

- **RGEs for twist-4 DAs are also integrable** [V. Braun, YJ, A. Manashov (2017)]

  Light fields mixing: kernels of 2×2 matrices. [V. Braun, A. Manashov, J. Rohrwild (2009); YJ, A. Belitsky (2014)]

  Three conserved charges $\{\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3\}$

$$\Phi_4(\underline{\omega}) = \frac{1}{2}\int_0^\infty ds \int_{-\infty}^\infty dx\, \eta_4^{(+)}(s,x,\mu)\, Y_{4;1}^{(+)}(s,x\,|\underline{\omega}),$$

$$(\Psi_4 + \widetilde{\Psi}_4)(\underline{\omega}) = -\int_0^\infty ds \int_{-\infty}^\infty dx\, \eta_4^{(+)}(s,x,\mu)\, Y_{4;2}^{(+)}(s,x\,|\underline{\omega}),$$

$$(\Psi_4 - \widetilde{\Psi}_4)(\underline{\omega}) = 2\int_0^\infty \frac{ds}{s}\left(-\frac{\partial}{\partial\omega_2}\right)\left\{\eta_3^{(0)}(s,\mu)\, Y_3^{(0)}(s\,|\underline{\omega}) + \frac{1}{2}\int_{-\infty}^{\infty} dx\, \eta_3(s,x,\mu)\, Y_3(s,x\,|\underline{\omega})\right\}$$
$$-\int_0^\infty ds \int_{-\infty}^\infty dx\, \varkappa_4^{(-)}(s,x,\mu)\, Z_{4;2}^{(-)}(s,x\,|\underline{\omega}),$$

$$\eta_4^{(+)}(s,x,\mu) \overset{\mathcal{O}(1/N_c^2)}{=} L^{\gamma_4(x)/\beta_0} R(s;\mu,\mu_0)\, \eta_4^{(+)}(s,x,\mu_0)$$
$$\varkappa_4^{(-)}(s,x,\mu) \overset{\mathcal{O}(1/N_c^2)}{=} L^{\gamma_4(x)/\beta_0} R(s;\mu,\mu_0)\, \varkappa_4^{(-)}(s,x,\mu_0)$$

- **Redundant operators are traded for others using EOMs and Lorentz symmetry.**

Conformal symmetry of heavy kernels

what about heavy kernels??

- $\mathcal{H}_h^{(1)}$ commute with special conformal generator of light field $S_+ \sim v^\mu \mathbf{K}_\mu$ but not S_0

$$[S_+, \mathcal{H}_h^{(1)}] = 0\,, \quad [S_0, \mathcal{H}_h^{(1)}] = 4C_F = \Gamma_{\text{cusp}}^{(1)}$$

[M. Knödlseder, N. Offen (2011)]

solution:

$$\mathcal{H}_h^{(1)} = \Gamma_{\text{cusp}}^{(1)} \ln(i\mu S_+) + \text{const}$$

(for $\mathcal{O}_{\bar{q}h} = \bar{q}(nz)\gamma^+ h(0)$, $j = 1$)

- light $\mapsto$ heavy reduction

[V. Braun, YJ, A. Manashov, (2018)]

$$S_+^{(h)} \mapsto \lambda^{-1} S_+^{(h)}\,, \qquad S_-^{(h)} \mapsto \lambda S_-^{(h)} \mapsto \mu\,, \qquad S_0^{(h)} \mapsto S_0^{(h)}\,, \qquad \lambda \sim m_b \to \infty$$

$$\Longrightarrow \quad \mathcal{H}_{\bar{q}q}^{(1)} \mapsto \mathcal{H}_{\bar{q}h} = \Gamma_{\text{cusp}}^{(1)} \ln(i\mu S_+) + \text{const}$$

- eigenfunction of $\mathcal{H}_{\bar{q}h}^{(1)}$ coincides with that of S_+:

$$Q_s(z) = -\frac{e^{is/z}}{z^2}$$

[V. Braun, A. Manashov (2014)]

Conformal symmetry of heavy kernels

- **Proposition:** $\boxed{\mathcal{H}_h^{t=2} = \Gamma_{\text{cusp}}(a)\ln(i\bar{\mu}S_+) + \Gamma_+(a)}$ **to all orders** **why?**

Evolution kernels in the MS-like schemes are ϵ-independent

Exact conformal symmetry in $d = 4 - 2\epsilon$ at the critical point $\beta(a_*) = 0$

$$(1) \quad \left[S_+^{\text{full}}, \mathcal{H}_h^{t=2}(a_*)\right] = 0$$

Conformal generators receive quantum corrections:

$$S_+^{(0)} = z^2\partial_z + 2z \mapsto S_+^{\text{full}}(a_*) = S_+^{(0)} + z\left[-\epsilon + \Delta(a_*)\right],$$
$$S_0^{(0)} = z\partial_z + 1 \mapsto S_0^{\text{full}}(a_*) = S_0^{(0)} - \epsilon + \mathcal{H}_h^{t=2}(a_*)$$

$\Delta(a_*) = a_*\Delta^{(1)} + a_*^2\Delta^{(2)} + \ldots$ is called conformal anomaly satisfying

from (1) and $SL(2)$ algebra $\Longrightarrow$ $\boxed{(2) \quad [z\partial_z, S_+^{\text{full}}(a_*)] = S_+^{\text{full}}(a_*)}$

$\ln\mu z$ enters $\mathcal{H}_h^{t=2}$ only linearly with coefficient Γ_{cusp} [G. Korchemsky, A. Radyushkin (1992)]

$$(3) \quad \left[z\partial_z, \mathcal{H}_h^{t=2}(a_*)\right] = \Gamma_{\text{cusp}}(a_*)$$

$$(1) \Longrightarrow \mathcal{H}_h^{t=2}(a_*) = f(S_+^{\text{full}}(a_*)) \overset{(2),(3)}{\Longrightarrow} zf'(z) = \Gamma_{\text{cusp}}(a_*) \Longrightarrow \textbf{Proposition}$$

Conformal generators at one-loop

- **Two-loop evolution of twist-2 DA** [V. Braun, YJ, A. Manashov (2019)]

$$\mathcal{H}_h^{(2)}(a_*) = \Gamma_{\text{cusp}}^{(2)}(a_*)\ln(i\bar{\mu}S_+^{(1)}(a_*)) + \Gamma_+^{(2)}(a_*)\,,$$
$$S_+^{(1)}(a_*) = S_+^{(0)} + z\big(-\epsilon(a_*) + a_*\Delta^{(1)}\big)$$

$$\bar{\mu} = \widetilde{\mu}e^{\gamma_E} = \mu_{\overline{\text{MS}}}\,e^{2\gamma_E}$$

$$\epsilon(a_*) = -\beta_0 a_* + O(a_*^2)$$

One-loop conformal anomaly *four one-loop diagrams*

$$\Delta^{(1)}\mathcal{O}(z) = C_F\left\{3\mathcal{O}(z) + 2\int_0^1 d\alpha\left(\frac{2\bar{\alpha}}{\alpha} + \ln\alpha\right)[\mathcal{O}(z) - \mathcal{O}(\bar{\alpha}z)]\right\}$$

- **The scheme-dependent constant $\Gamma_+^{(2)}(a)$ is found from Feynman diagrams**

Two-loop kernel in integral representation

- **Integral representation for $\mathcal{H}_h^{t=2}$ is usually preferred**

Ansatz

$$\mathcal{H}(a)\mathcal{O}(z) = \Gamma_{\text{cusp}}(a)\left[\ln(i\widetilde{\mu}z)\mathcal{O}(z) + \int_0^1 d\alpha\,\frac{\bar{\alpha}}{\alpha}\big(1+h(a,\alpha)\big)\big(\mathcal{O}(z)-\mathcal{O}(\bar{\alpha}z)\big)\right] + \gamma_+(a)\mathcal{O}(z)$$

- $\Delta^{(1)}$ **and** $\epsilon(a_*) = -\beta_0 a_* + O(a_*^2)$ **dictate** $h(a,\alpha)$ *going to Mellin space*

$$h(a,\alpha) = a\ln\bar{\alpha}\left\{\beta_0 - 2C_F\left(\frac{3}{2} + \ln\frac{\alpha}{\bar{\alpha}} + \frac{\ln\alpha}{\bar{\alpha}}\right)\right\} + O(a^2)$$

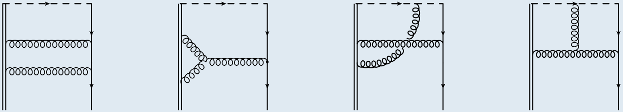
- γ_+ **requires additional calculation** *scheme-dependent*, $\gamma_{\phi_+} = \gamma_+ - \gamma_F$

$$\gamma_+^{\overline{\text{MS}}}(a) = -aC_F + a^2C_F\left\{4C_F\left[\frac{21}{8}+\frac{\pi^2}{3}-6\zeta_3\right] + C_A\left[\frac{83}{9}-\frac{2\pi^2}{3}-6\zeta_3\right] + \beta_0\left[\frac{35}{18}-\frac{\pi^2}{6}\right]\right\} + \dots$$

Two-loop kernel from Feynman diagrams

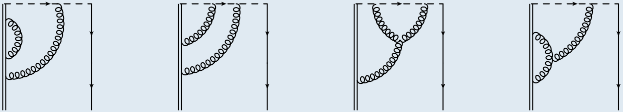
There are ~ 30 diagrams in three categories:

- **Exchange diagrams**



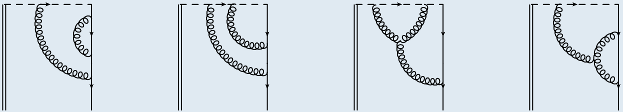
...

- **Cusp diagrams**



...

- **Light vertices**



...

Two-loop kernels from Feynman diagrams

- **Exchange diagrams contribute to both $h(a, \alpha)$ and γ_+** *(many are UV-finite)*
- **Cusp diagrams generate $\sim \ln z$ and contribute to γ_+**
- **Light vertices contribute to $h(a, \alpha)$ only, known**

[V. Braun, A. Manashov, S. Moch, and M. Strohmaier (2016)]

- $h(a, \alpha)$ **confirmed by explicit Feynman diagram calculation!**

Light-heavy reduction

- **Evolution kernel of $\mathcal{O} = \bar{q}(nz_1)\gamma^+ q(nz_2)$ in integral form**

$$[\mathcal{H}_l\varphi](z_1,z_2) \propto \int_0^1 du\, h(u)\Big[2\varphi(z_1,z_2) - \varphi(z_{12}^u, z_2) - \boxed{\varphi(z_1, z_{21}^u)}\Big] + \boxed{\int_0^1 du \int_0^{\bar{u}} dv\, \chi(u,v)[\varphi(z_{12}^u, z_{21}^v) + \varphi(z_{12}^v, z_{21}^u)]} + c\varphi(z_1,z_2)$$

- **drop terms in boxes and $z_2 \to 0$ to obtain $\mathcal{H}_h^{\text{ex}} + \mathcal{H}_h^{\text{lv}}$.**
  ↰ **Location of the heavy quark is fixed!**

  Explicit expressions for $\mathbb{H}_l^{(2)}$ available [V. Braun, A. Manashov, S. Moch, M. Strohmaier (2016)]
- **Adding contribution of cusp diagrams again gives us $\mathcal{H}_h^{(2)}$.**

Evolution of the coefficient function at two-loop

Reminder ($Q_s(z)$ form a complete orthonormal basis)

$$\Phi_+(z,\mu) = \int_0^\infty ds\, s\, Q_s(z)\,\eta_+(s,\mu) = -\frac{1}{z^2}\int_0^\infty ds\, s\, \mathrm{e}^{is/z}\,\eta_+(s,\mu)$$

- **RGE of $\phi_+(z,\mu) \mapsto$ integro-differential eq. over $\eta_+(s,\mu)$** [V. Braun, YJ, A. Manashov (2019)]

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} + \Gamma_{\text{cusp}}(a)\ln(\widetilde{\mu}\mathrm{e}^{\gamma_E} s) + \gamma_\eta(a)\right)\eta_+(s,\mu) = 4C_F a\int_0^1 du\frac{\bar{u}}{u}h(a,u)\eta_+(\bar{u}s,\mu)$$

$$\gamma_\eta = \gamma_+^{\overline{\text{MS}}} - \gamma_F - \Gamma_{\text{cusp}}^{(2)}\left[1 - a\left(C_F\left(\frac{\pi^2}{6} - 3\right) + \beta_0\left(1 - \frac{\pi^2}{6}\right)\right)\right]$$

- **NNLL resummation requires Γ_{cusp} to $O(a^3)$ since numerically $\ln(s) \sim 1/a$**

Analytic solution of the two-loop RGE

- **Operator $\mathcal{O}(z)$ in Mellin space**

$$\mathcal{O}(z) = \int_{-i\infty}^{+i\infty} dj \, (i\mu_{\overline{\text{MS}}} \, e^{\gamma_E} \, z)^j \, \mathcal{O}(j)$$

gives rise to the Mellin-space RGE:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a} - \Gamma_{\text{cusp}}(a) \frac{\partial}{\partial j} + V(j, a)\right) \mathcal{O}(j, a, \mu) = 0$$

explicit expression for $V(j, a)$ at two-loop available in [V. Braun, YJ, A. Manashov, 1912.03210]

- **Mellin moment j as the second coupling, with Γ_{cusp} as the β-function**

Conclusion and Outlook

Conclusion

- conformal symmetry is extremely powerful for loop/OPE calculations in nonforward kinematics (DVCS, meson production/decays): one-loop less
- applicable to calculating kernel and solving RGE in HQET

Outlook for future work

- two-loop coefficient functions for axial-vector currents **to appear soon**
- coefficient functions for heavy meson decays from conformal symmetry?
- one-loop correction to kinematic higher-power/twist