Syzygy
for Feynman integral reduction

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Recall from the talks in the fall of 2020

Feynman integrals in a family → **IBP reduction** → Master Integrals

2 loop ~100,000 → ~100

Integration-by-parts (IBP)

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{\alpha_1} \dots D_k^{\alpha_k}} = 0$$

Chetyrkin, Tkachov (1981)

IBP relations are usually solved by the Laporta algorithm (Gaussian Elimination)
Frequently, it contains too many integrals which are not targets or masters. Thus the Gaussian Elimination deals with a huge matrix.

IBP reduction is frequently the most time and RAM consuming step for a scattering amplitude computation.

To make integral reduction easier and POSSIBLE

truncated IBP system

Syzygy, module intersection

New type of integral relations

Auxiliary mass flow

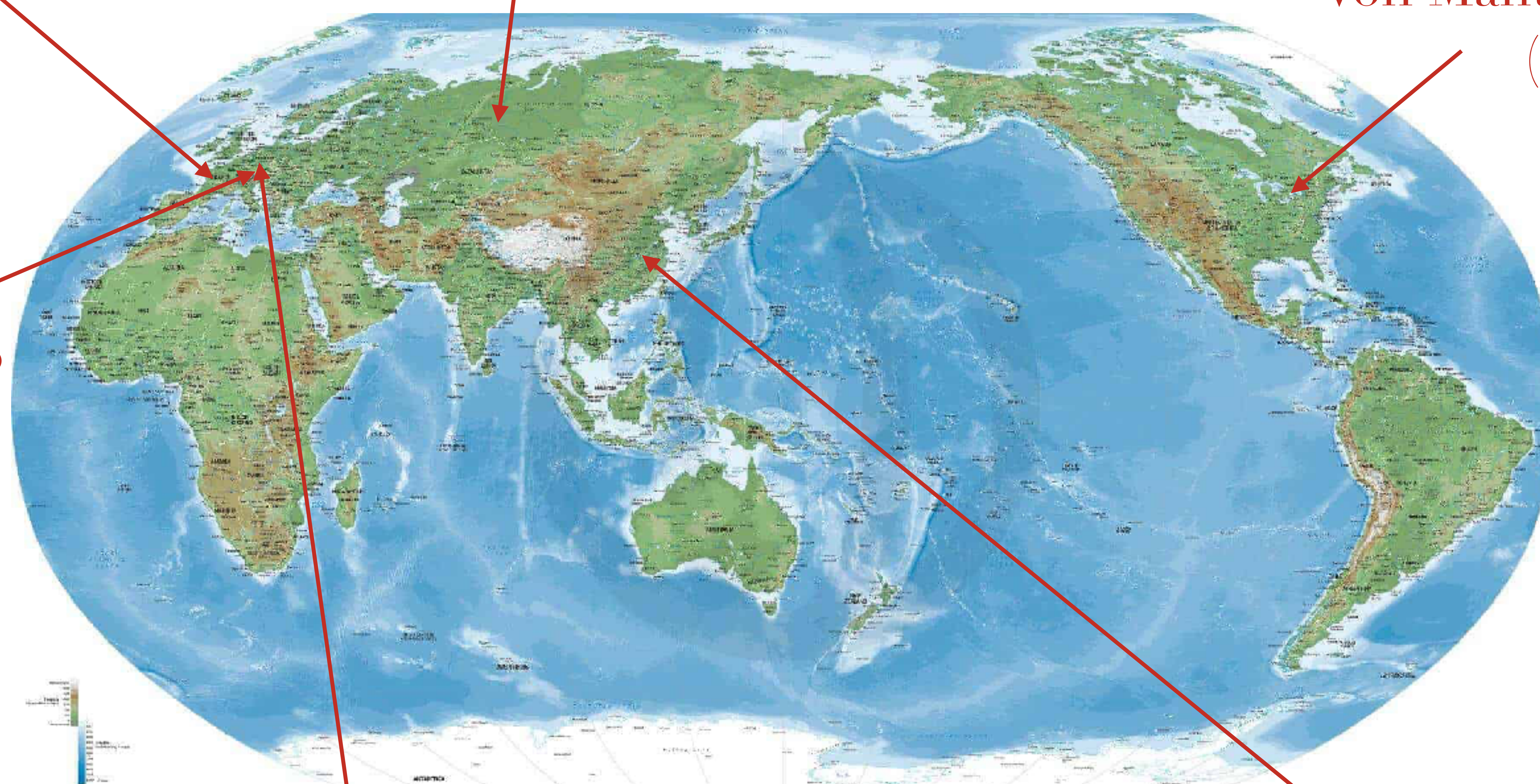
Liu, Ma Phys. Rev. D 99 (2019), no 7 071501
Guan, Liu, Ma, Chin. Phys. C 44(2020) 9, 093106
…

Intersection theory

Mastrolia, Mizera JHEP 02 (2019) 139
Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mizera JHEP 05 (2019) 153
Frellesvig, Gasparotto, Mandal, Mastrolia, Mattiazzi, Mizera PRL. 123.201602
…

Syzygy for IBP reduction

based on works



Outline

1. Syzygy
2. Module intersection
3. Examples

1. Syzygy

“σύζυγος”   Greek word, originally means

a roughly straight-line configuration of three or more celestial bodies

“合冲”

Syzygy for IBP reduction

IBP $$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{\alpha_1} \dots D_k^{\alpha_k}} = 0$$

In traditional Laporta algorithm, the vectors are choose arbitrarily …
By the derivates, there would be a lot of propagators with high power

Suppose that we want to forbid the increase of the propagator index α_i, we can require that,

$$\Big(\sum_{j=1}^{L} v_j^\mu \frac{\partial}{\partial l_j^\mu} D_i\Big) + g_i D_i = 0$$

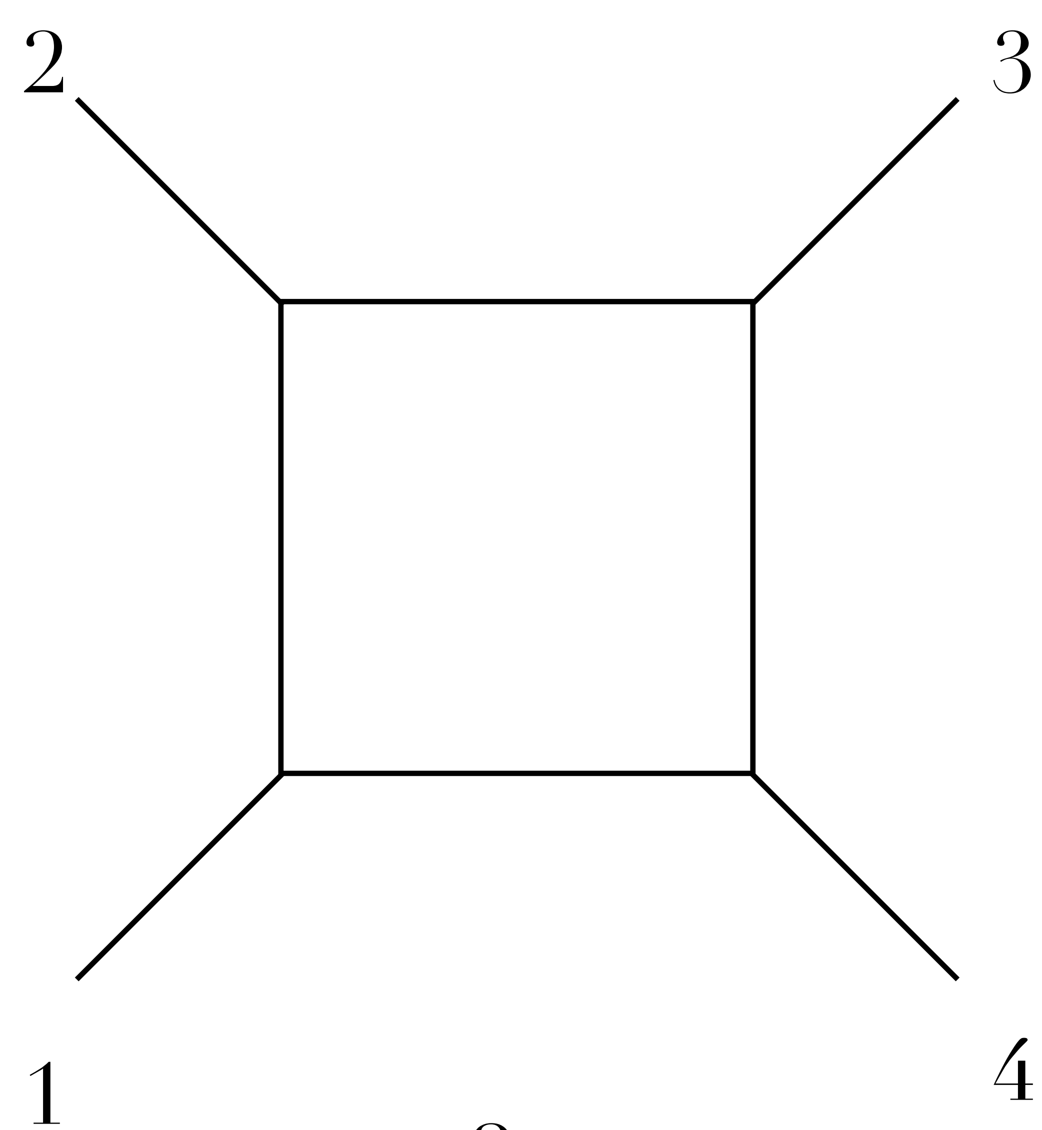
Syzygy equation

where both v_j^μ and g_i contain polynomials in loop momenta.

dramatically reduces the number of IBP relations
speeds the IBP reduction by several order of magnitude

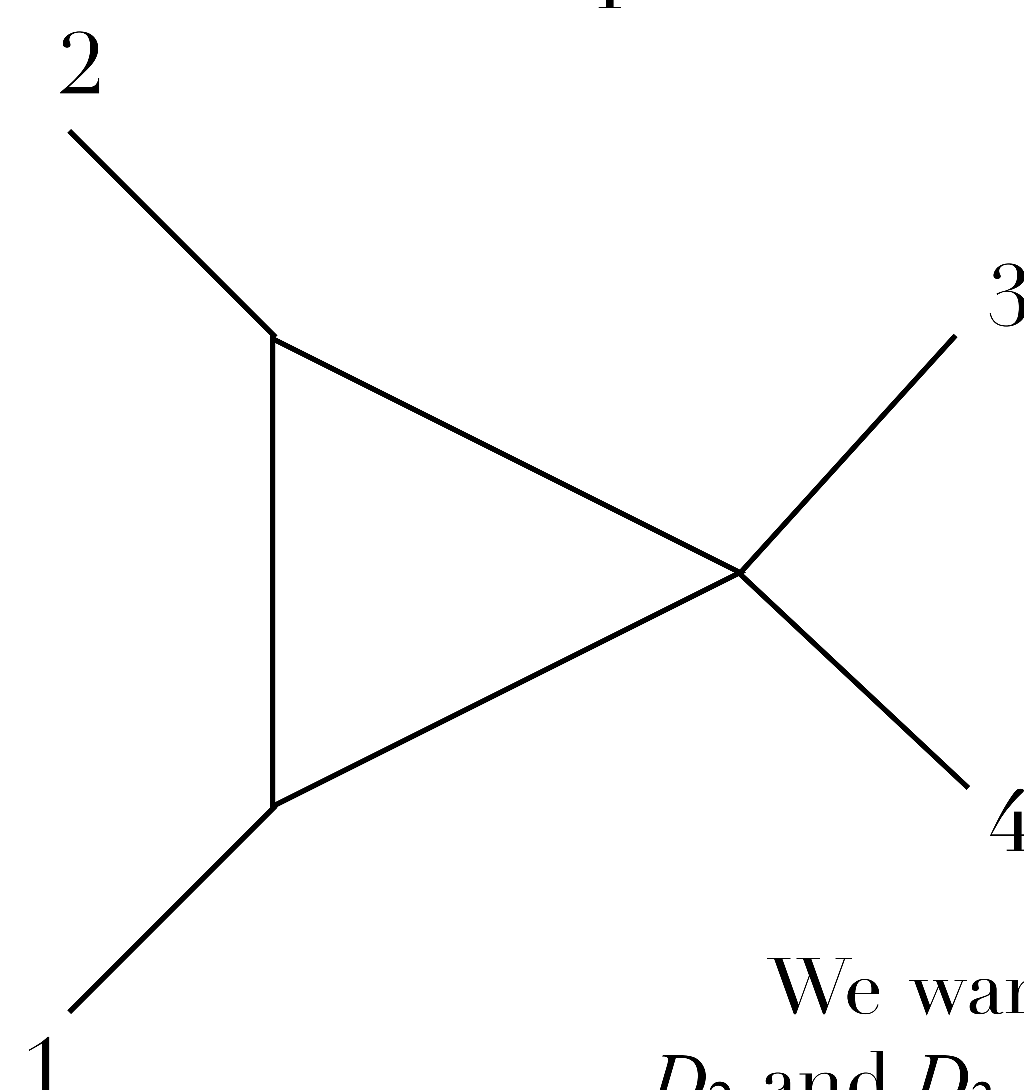
Gluza, Kajda, Kosower, PhysRevD. 83.045012

Syzygy to IBP, a first example



$$D_1 = l^2, \quad D_2 = (l-p_1)^2, \quad D_3 = (l-p_1-p_2)^2, \quad D_4 = (l+p_4)^2$$

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0, \quad p_1 \cdot p_2 = s/2, \quad p_1 \cdot p_4 = t/2$$



Consider a triangular sub-diagram in the sector (1,1,1,0)

We want IBP relations which does not contain double propagator in D_1, D_2 and D_3, via the syzygy method.

Syzygy to IBP, a first example

$$v^\mu = a_1 p_1^\mu + a_2 p_2^\mu + a_3 p_4^\mu + a_4 l^\mu$$

$$v^\mu \frac{\partial D_i}{\partial l^\mu} - b_i D_i = 0, \quad i = 1,2,3$$

$$\boxed{Ay = 0}$$

Syzygy

$$A = \begin{pmatrix} 2l\cdot p_1 & 2l\cdot p_2 & 2l\cdot p_4 & 2l^2 & -l^2 & 0 & 0 \\ 2l\cdot p_1 & 2l\cdot p_2 - s & 2l\cdot p_4 - t & 2l^2 - 2l\cdot p_1 & 0 & 2l\cdot p_1 - l^2 & 0 \\ 2l\cdot p_1 - s & 2l\cdot p_2 - s & 2l\cdot p_4 + s & -2l\cdot p_1 - 2l\cdot p_2 + 2l^2 & 0 & 0 & 2l\cdot p_1 + 2l\cdot p_2 - l^2 - s \end{pmatrix}$$

$$y = (a_1, a_2, a_3, a_4, b_1, b_2, b_3)^{\mathbf{T}}$$

The free variables are $(l\cdot p_1), (l\cdot p_2), (l\cdot p_4), l^2$.

Of course, it looks like a homogenous linear equation,
However, we require the solution has no pole in the free variables

Roughly speaking,
a syzygy computation is to find solutions
of a homogenous linear equation
without poles in free variables

Syzygy to IBP, a first example

suppose that we already have the syzygy solutions …

We get 6 syzygy generators for this case

$$-2(l\cdot p_2)p_1^\mu + (2l\cdot p_1 - 2l^2)p_2^\mu + (4l\cdot p_2 - s)l_1^\mu, \ldots$$

$$0 = \int \frac{d^D l}{i\pi^{D/2}} \frac{\partial}{\partial l^\mu}\left(v_j^\mu \frac{1}{D_1^{m_1} D_2^{m_2} D_3^{m_3} D_4^{m_4}}\right)$$

$$m_1, m_2, m_3 \in \mathbb{Z}_{>0},\ m_4 \in \mathbb{Z}_{\leq 0}$$

$$\begin{aligned}0 = {} & G(m_1,m_2,m_3,m_4)(ds - 2m_2 s - 2m_3 s - m_4 s) + (-2d + 2m_1 + 2m_2 + 2m_3 + 2m_4)G(m_1,m_2,m_3-1,m_4)\\ & + (2d - 2m_1 - 2m_2 - 2m_3 - 2m_4)G(m_1,m_2-1,m_3,m_4) + m_4 st G(m_1,m_2,m_3,m_4+1)\\ & - m_4 s G(m_1,m_2-1,m_3,m_4+1) + m_4(-t)G(m_1-1,m_2,m_3,m_4+1) - m_4 t G(m_1,m_2,m_3-1,m_4+1)\end{aligned}$$

Seeding, $(m_1, m_2, m_3, m_4) = (1,1,1,0)$

$$0 = (-6+2d)G[1,0,1,0] + \underline{(6-2d)G[1,1,0,0]} + (-4s+ds)G[1,1,1,0]$$

zero integral

no double propagator

A very simple IBP which reduces the triangle to bubble immediately

Syzygy, mathematical remarks

Modules

A module M over a ring $R = \mathbb{F}[x_1, \ldots, x_n]$ is an abelian group, such that

- $f(m_1 + m_2) = fm_1 + fm_2$, for $f \in R$ and $m_1, m_2 \in M$,
- $(f_1 + f_2)m = f_1 m + f_2 m$, for $f_1, f_2 \in R$ and $m \in M$,
- $(f_1 f_2)m = f_1(f_2)m$, for $f_1, f_2 \in R$ and $m \in M$,
- $1m = m$, for $1 \in R$, $m \in M$.

Clearly, R^m is a module. Any ideal of R is a module. We mainly consider a sub-module of R^m.

A module is an analogy of linear space, in algebraic geometry. The biggest difference is that for $m \in M$ and $f \in R$, $\frac{1}{f}m$ is not defined.

A basis of a module is a set $\{m_1, \ldots, m_k\}$ in M, such that $m_1, \ldots, m_k$ generate M, and if

$$f_1 m_1 + \ldots + f_k m_k = 0, \quad f_i \in R$$

then $f_1 = \ldots = f_k = 0$.

In most cases, a module does not have a basis. If it has, then such a module is a *free module*. R^m is a free module.

Syzygy, mathematical

Consider $\{m_1, \ldots, m_k\}$ in a module M over R. All tuples $\{f_1, \ldots f_k\}$ such that

$$f_1 m_1 + \ldots + f_k m_k = 0, \quad f_i \in R$$

form the *syzygy* of $\{m_1, \ldots, m_k\}$. The syzygy is a sub-module of R^k.

If M is a sub-module of R^l, then each m_i can be written as a column vector with polynomial components. Define $A = \{m_1, \ldots, m_k\}$ as an $l \times k$ matrix, then the syzygy is,

$$\ker A \cap R^k$$

Usually, the syzygy is not a free module. So the goal would be to compute a generator set instead of the basis.

Syzygy from Schreyer algorithm

The syzygy of elements of a module can computed from Schreyer algorithm

For $G = \{m_1, \ldots, m_s\}$ a Groebner basis with the ordering $\succ$. An S-pair can be reduced on the generators,

$$\frac{m_{ij}}{\mathrm{LT}(m_i)} m_i - \frac{m_{ij}}{\mathrm{LT}(m_j)} m_j = \sum_{k=1}^{s} a_{ij,k} m_k$$

where $m_{ij} = \mathrm{LCM}(\mathrm{LT}(m_i), \mathrm{LT}(m_j))$. So we get a syzygy element,

$$s_{ij} = \frac{m_{ij}}{\mathrm{LT}(m_i)} \mathbf{v_i} - \frac{m_{ij}}{\mathrm{LT}(m_j)} \mathbf{v_j} - \sum_{k=1}^{s} a_{ij,k} \mathbf{v_k}$$

where $\mathbf{v_i}$ is the i-th unit vector in R^s.

Such s_{ij} generate the syzygy of $\{m_1, \ldots, m_s\}$.

Syzygy from Schreyer algorithm

For a set $f_1, \ldots f_k$ which is not a Groebner basis, the Groebner basis can be calculated as well.

$$f_i = A_{ij} m_j, \quad m_i = B_{ij} f_j$$

Then the syzygy of $f_1, \ldots f_k$ is generated by,

$$A s_{ij}, \quad I - AB$$

$k \times s$ matrix

$s \times k$ matrix

s-dimensional vector

Syzygy can be computed in Mathematica with the interface to Singular

Syzygy generators obtained from the Schreyer algorithm,
is again a Groebner basis in a particular ordering

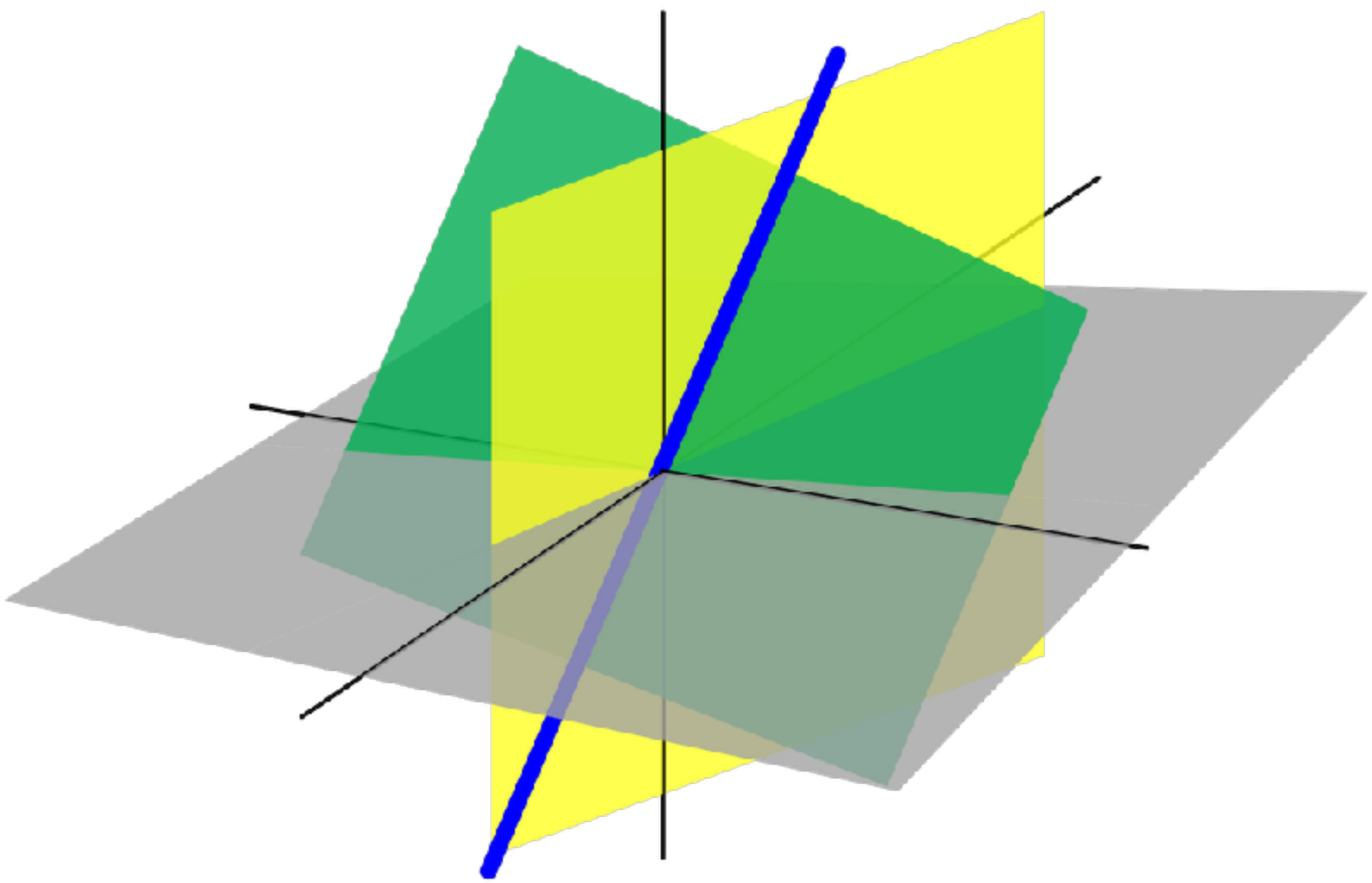
[Cox, Little, O'Shea] Using algebraic geometry 5.3

Syzygy can be computed via Singular program,
or GKK's private package
(Schreyer algorithm fine tuned for multiple parameters.)

However, in practice, it is easier to alternatively use

Module intersection

2. Module intersection



Module intersection
is similar to linear space intersection
but over polynomials only …
an computational algebraic geometry problem

Based on

Larsen, **YZ**
Phys.Rev.D 93 (2016) 4, 041701

Boehm, Schoenemann, Georgoudis
Larsen, **YZ**
JHEP 1809 (2018) 024

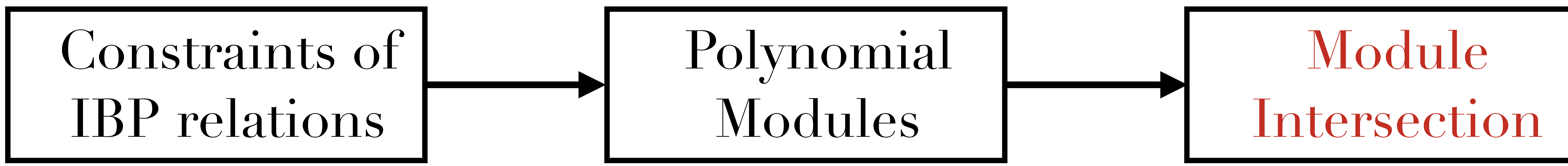
Bendle*, Bendle, Boehm, Decker, Georgoudis,*
Pfreundt, Rahn, Wasser, **YZ**
JHEP 02 (2020) 079

....

Module Intersection

Larsen YZ
Phys.Rev.D 93 (2016) 4, 041701

IBP relations in Baikov representation



- Easily get IBPs without double propagators (or propagator-degree increase)
- Naturally adaptable with unitarity cuts
- Usually much faster than direct syzygy approaches

In parallel with the developments of numeric unitarity
Ita 2015, Abreu, Cordero, Dormans, Ita, Page, Sotnikov
JHEP 1811 (2018) 116, Phys.Rev.Lett. 122 (2019) no.8, 082002, JHEP 1905 (2019) 084

Natural way to construct integrand with IBPs without doubled propagators
very efficient for constructing multi-loop integrand

IBP in Baikov representation

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} \quad \propto \int_\Omega dz_1 \ldots dz_k \frac{F^{\frac{D-L-E-1}{2}}}{z_1^{\alpha_1} \ldots z_k^{\alpha_k}}$$

Baikov

$$0 = \int_\Omega dz_1 \ldots dz_k \sum_{i=1}^{k} \frac{\partial}{\partial z_i} \left(a_i(z) \frac{F^{\frac{D-L-E-1}{2}}}{z_1^{\alpha_1} \ldots z_k^{\alpha_k}} \right)$$

IBP in Baikov representation

No boundary term
easy to set some of z's to zero (unitary cut)

IBP in Baikov representation with constraints

polynomials

Require

1. no shifted exponent: $\sum_{j=1}^{k} a_j(z)\frac{\partial F}{\partial z_j} + \beta(z)F = 0$ These $\big(a_1(z),\ldots a_k(z)\big)$ form a module $M_1 \subset R^k$.

2. no propagator degree increase: $a_i(z) \in \langle z_i \rangle, \quad 1 \leq i \leq m$ These $\big(a_1(z),\ldots a_k(z)\big)$ form a module $M_2 \subset R^k$.

Both M_1 and M_2 are pretty simple ...

Larsen YZ
Phys.Rev.D 93 (2016) 4, 041701

$$M_1 \cap M_2$$

Intersection of two modules

Determine the first module

$$\sum_{j=1}^{k} a_j(z)\frac{\partial F}{\partial z_j} + \beta(z)F = 0$$

More Advanced

- syzygy for the $\{\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_k}, F\}$ Roman Lee's trick
- $\mathrm{Ann}(F^s)$, annihilator of F^s in Weyl algebra. Bitoun, Bogner, Klausen, Panzer Lett.Math.Phys. 109 (2019) no.3, 497-564

If F is a determinant matrix whose elements are free variables, this kind of syzygy module is simple.

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

equivalent to canonical IBP in momentum space

Laplace expansion

$$\sum_j a_{k,j}\frac{\partial(\det A)}{\partial a_{i,j}} - \delta_{k,i}\cdot \det A = 0$$

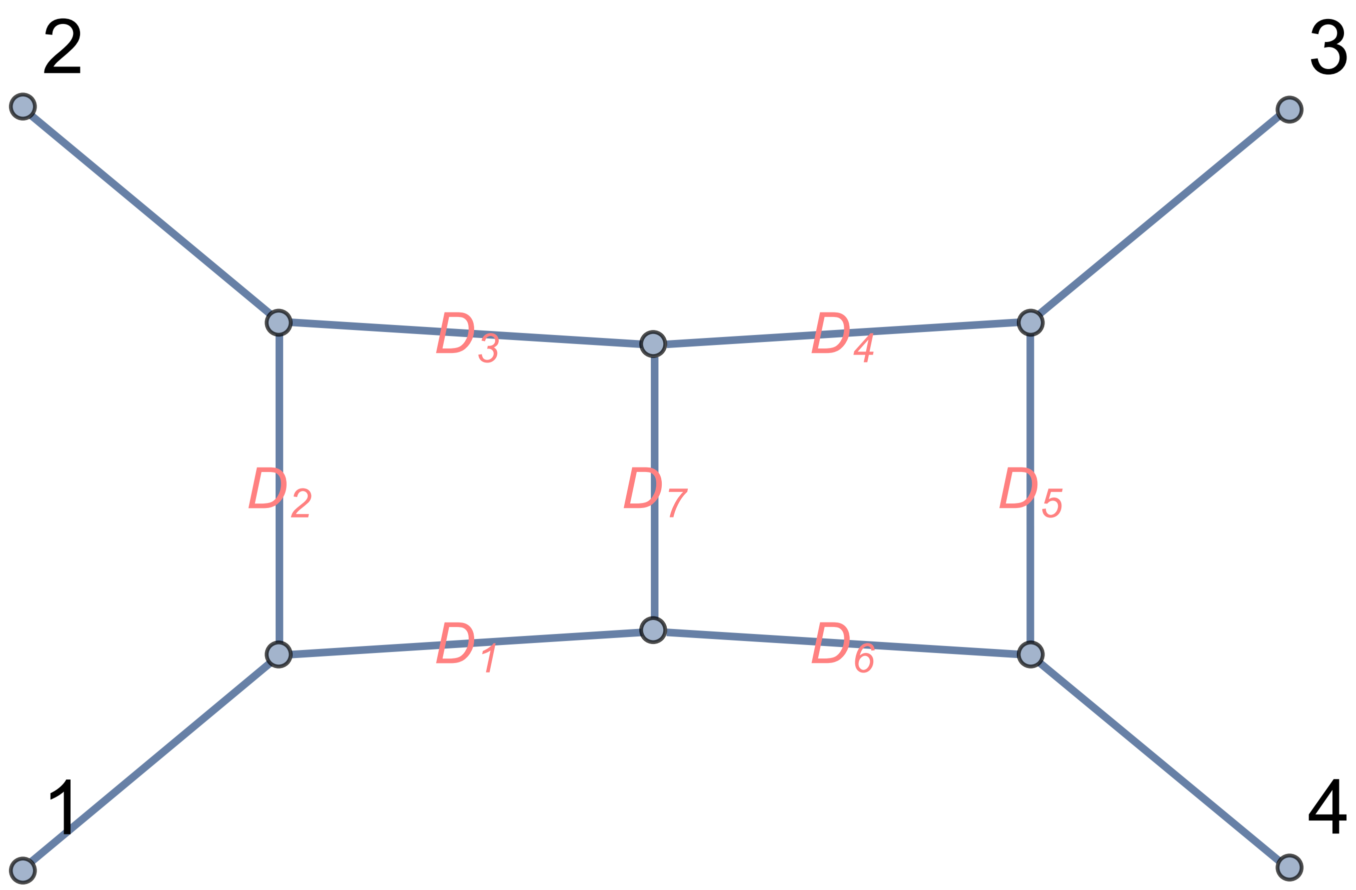
Get all first order annihilator, proved by Gulliksen–Negard and Jozefiak exact sequences

Boehm, Georgoudis, Larsen, Schulze, YZ 2017

Example, massless double box

$\mathbb{Q}(s,t)[z_1,\dots z_9]$: 2 parameters, 9 variables

(Each row is a module generator)



$$M_2 = \begin{pmatrix} z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} z_1-z_2 & z_1-z_2 & -s+z_1-z_2 & 0 & 0 & 0 & z_1-z_2-z_6+z_9 & t+z_1-z_2 & 0 \\ 0 & 0 & 0 & s-z_6+z_9 & -t-z_6+z_9 & -z_6+z_9 & z_1-z_2-z_6+z_9 & 0 & -z_6+z_9 \\ s+z_2-z_3 & z_2-z_3 & z_2-z_3 & 0 & 0 & 0 & z_2-z_3+z_4-z_9 & -t+z_2-z_3 & 0 \\ 0 & 0 & 0 & z_4-z_9 & t+z_4-z_9 & -s+z_4-z_9 & z_2-z_3+z_4-z_9 & 0 & z_4-z_9 \\ -z_1+z_8 & -t-z_1+z_8 & s-z_1+z_8 & 0 & 0 & 0 & -z_1-z_5+z_6+z_8 & -z_1+z_8 & 0 \\ 0 & 0 & 0 & -s-z_5+z_6 & -z_5+z_6 & -z_5+z_6 & -z_1-z_5+z_6+z_8 & 0 & t-z_5+z_6 \\ 2z_1 & z_1+z_2 & -s+z_1+z_3 & 0 & 0 & 0 & z_1-z_6+z_7 & z_1+z_8 & 0 \\ 0 & 0 & 0 & s-z_3-z_6+z_7 & -z_6+z_7-z_8 & -z_1-z_6+z_7 & z_1-z_6+z_7 & 0 & -z_2-z_6+z_7 \\ -z_1-z_6+z_7 & -z_1+z_7-z_9 & s-z_1-z_4+z_7 & 0 & 0 & 0 & -z_1+z_6+z_7 & -z_1-z_5+z_7 & 0 \\ 0 & 0 & 0 & -s+z_4+z_6 & z_5+z_6 & 2z_6 & -z_1+z_6+z_7 & 0 & z_6+z_9 \end{pmatrix}$$

$M_1 \cap M_2$ is computed within seconds, with **Singular** 4.1's **intersect**

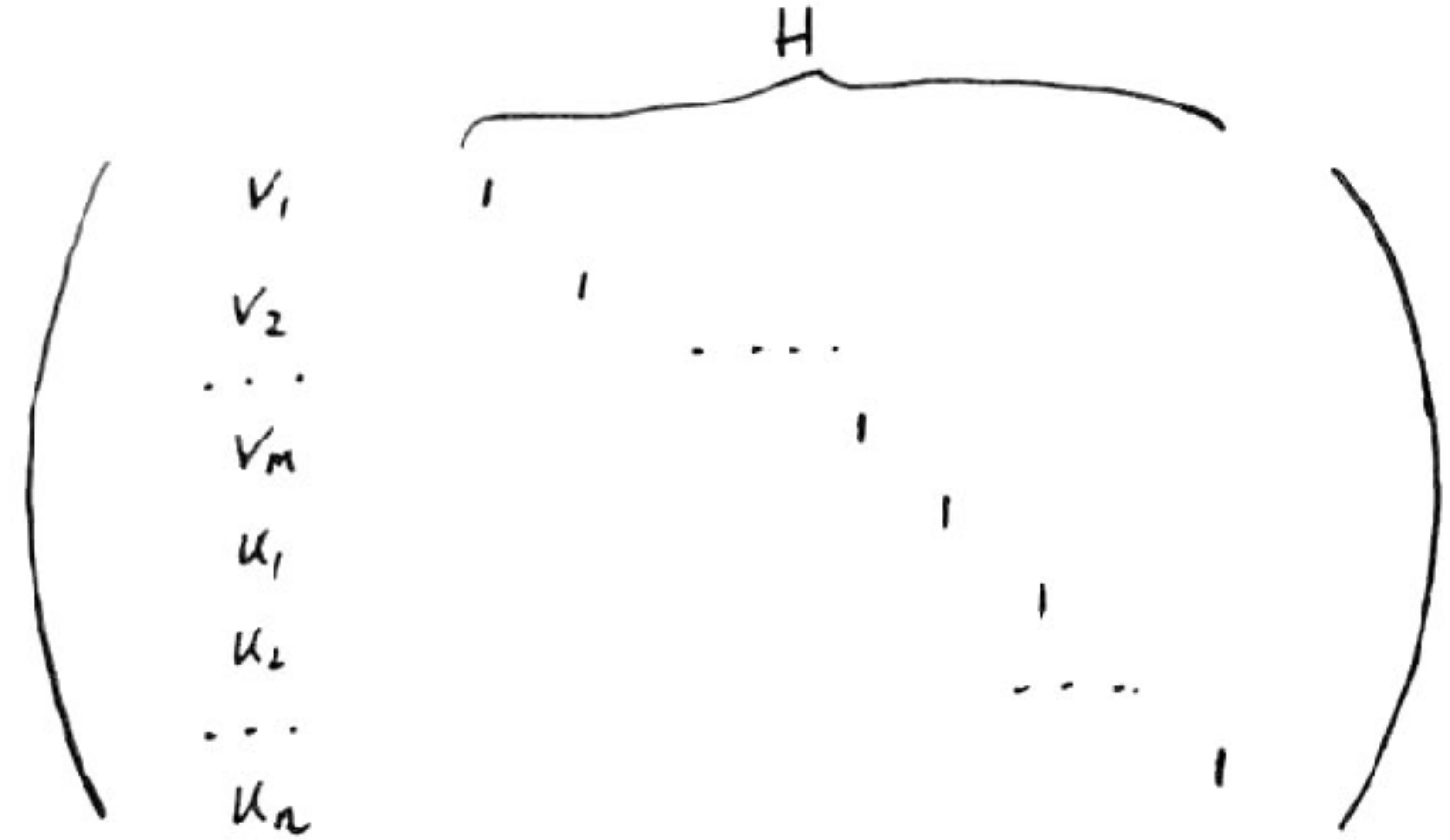
Module Intersection

A better algorithm

Boehm, Schoenemann, Georgoudis
Larsen, **YZ**
JHEP 1809 (2018) 024

$M_1 = \langle v_1, v_2, \ldots v_m \rangle$ each v t-dim row

$M_2 = \langle u_1, u_2, \ldots u_n \rangle$ each u t-dim row



Compute the Gröbner basis w.r.t. rows

Position > Term

Find the Gröbner basis elements in H

Implements

- Use unitarity cuts

- Use degree bound

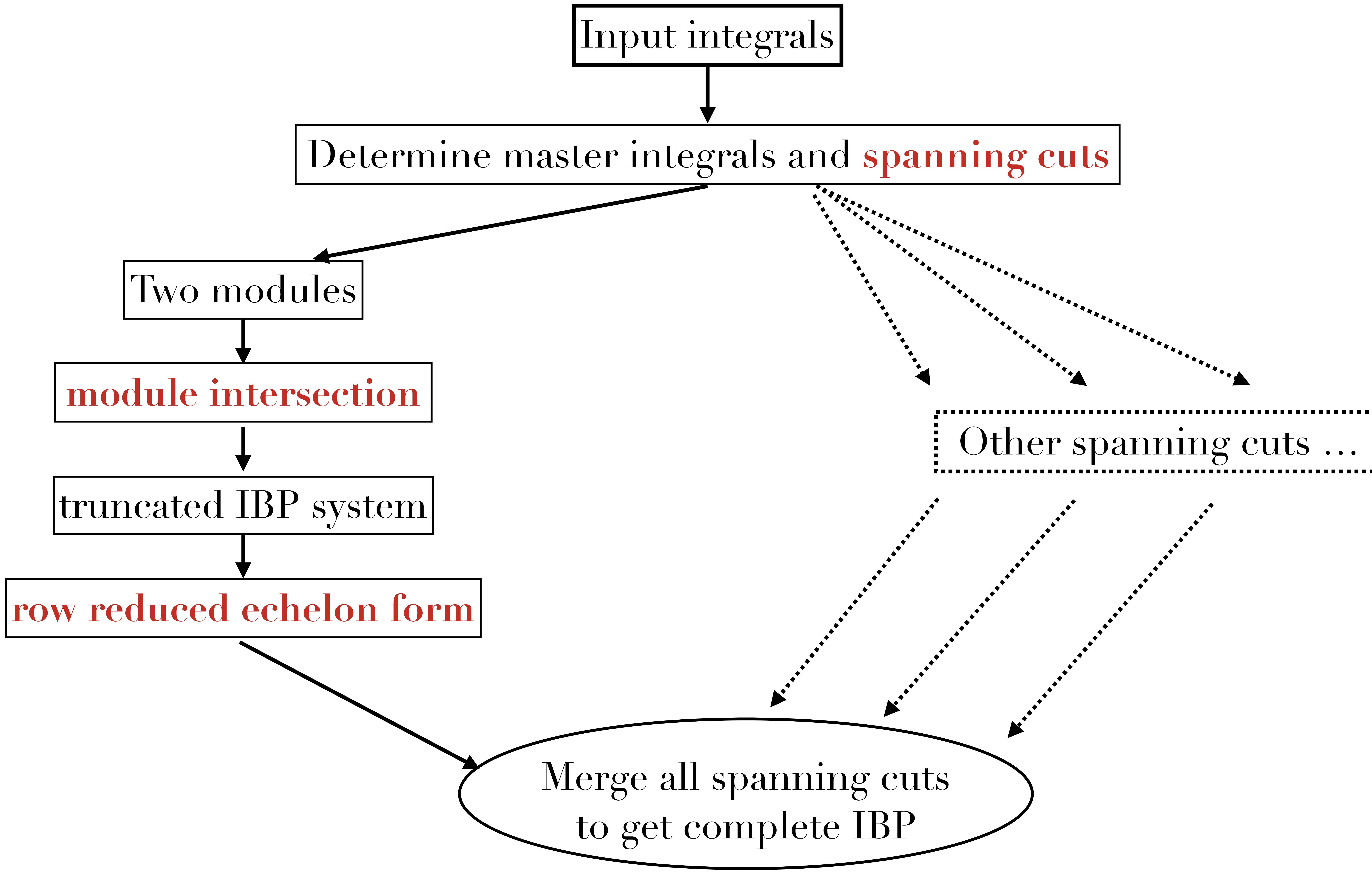
- Localization trick

Treat parameters as variables, and compute in a block ordering

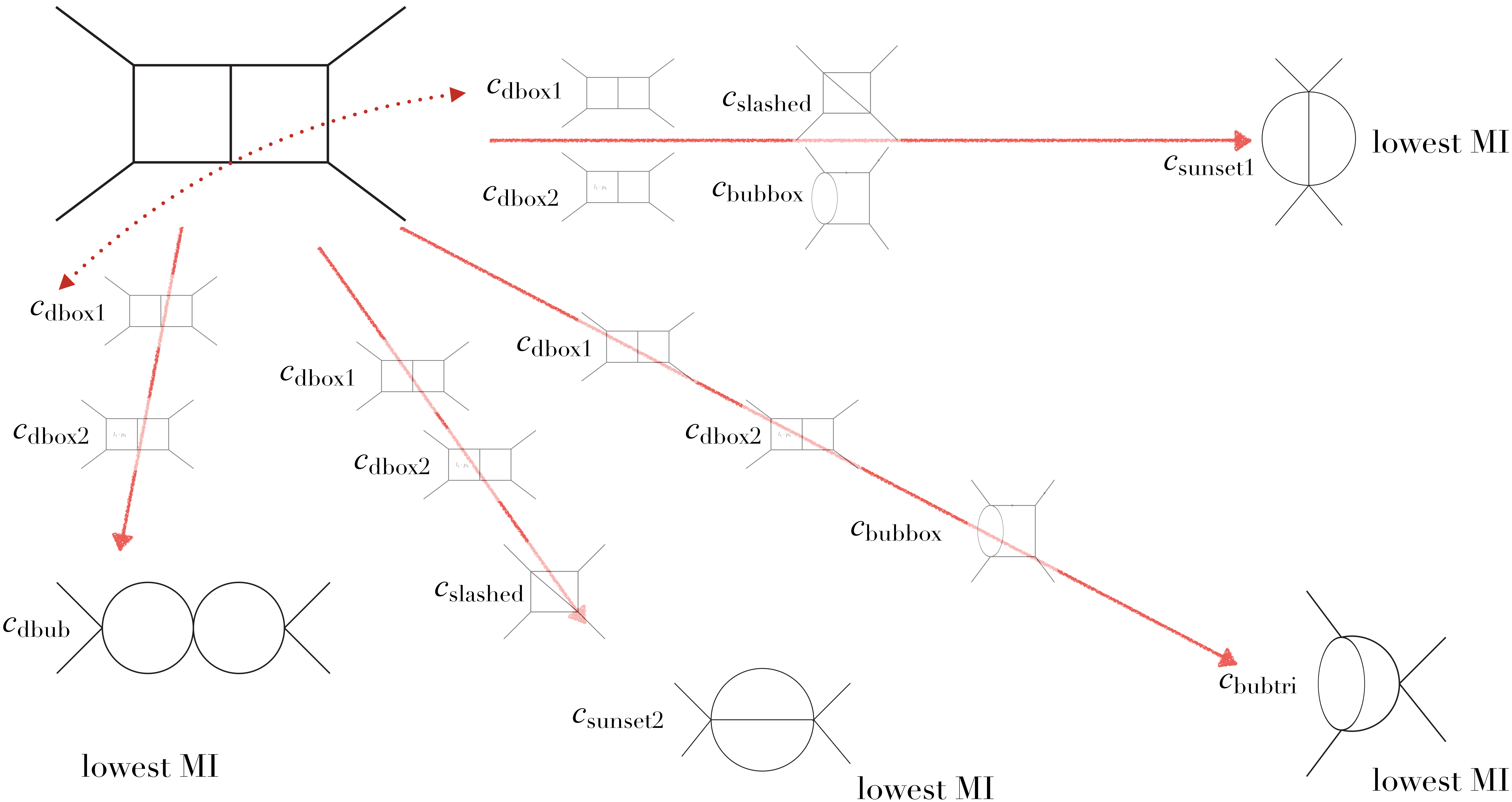
[variables] > [parameters]

A famous trick in computational algebra

Workflow


Input integrals
Determine master integrals and spanning cuts
Two modules
module intersection
truncated IBP system
row reduced echelon form
Other spanning cuts …
Merge all spanning cuts
to get complete IBP

Example, massless double box with spanning cut



Remove cuts overlap

$$\begin{pmatrix} 1 & \boxed{12} & 0 & 0 & \boxed{-124} \\ 0 & 0 & 1 & 0 & 31 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix}$$

Set one non-pivot column (one master integral) to zero before reduction,
does NOT change other non-pivot columns after reduction

Chawdhry, Lim, Mitov Phys. Rev. D 99, 076011 (2019)
also implemented in Kira

If one master integral appears on two cuts, pick up one cut and set this integral to zero.

Can also be used for double propagator integrals

$$0 = \int_\Omega dz_1 \dots dz_4 \sum_{i=1}^{4} \frac{\partial}{\partial z_i} \left(a_i(z) \frac{F^{\frac{D-L-E-1}{2}}}{z_1 z_2 z_3 z_4^2} \right)$$

the fourth index must be less or equal 2

$$(-5+d)G[1,0,1,2] + (5-d)G[1,1,1,1] - tG[1,1,1,2] = 0$$

targeted reduction for integrals with doubled (fourth-)propagator

3. Nontrivial Examples

General applications of syzygy IBPs

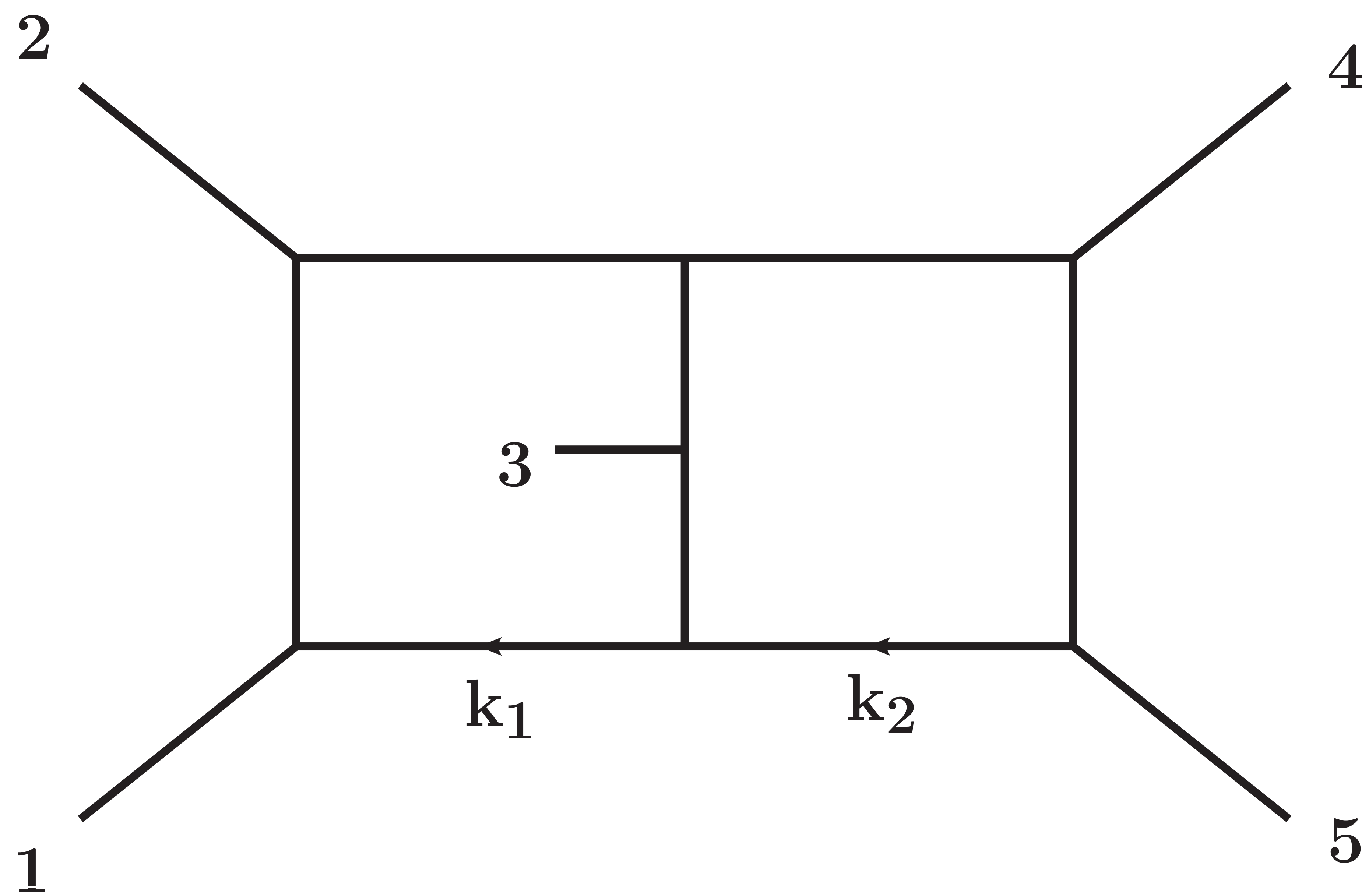
The Five-Loop Four-Point Integrand of N=8 Supergravity as a Generalized Double Copy
Bern, Carrasco, Chen, Johansson, Roiban, Zeng
Phys. Rev. D 96, 126012 (2017)

Ingredients of the IBP methods used therein

- Syzygy
- Unitarity cuts
- Finite fields

Towards an industry-level row-reduction program

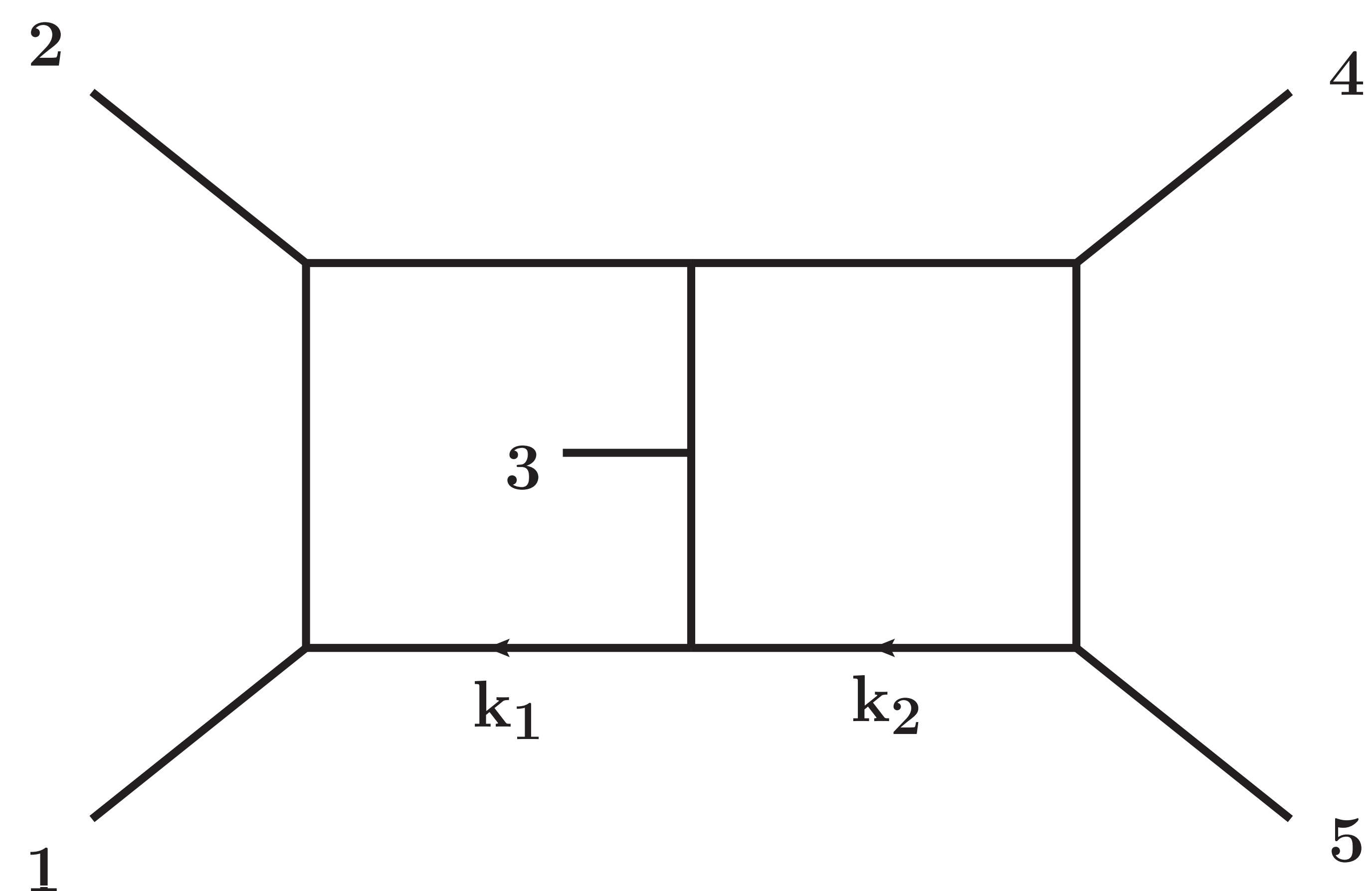
- Row Reduction code written in **Singular**
- With numeric fitting, powered by the large-scale parallelization framework **GPI-space**
- large-scale interpolation
- Bonus: use partial fraction and UT property to simplify the analytic result



Analytic (symbol) Abreu, Dixon, Herrman, Page, Zeng
"*The two-loop five-point amplitude in N=4 sYM theory* ", Phys.Rev.Lett. 122 (2019), no. 12 121603

Analytic (function) Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia
"*All master integrals for three-jet production at NNLO*", Phys.Rev.Lett. 123 (2019), no. 4 041603

Now module intersection is really fast

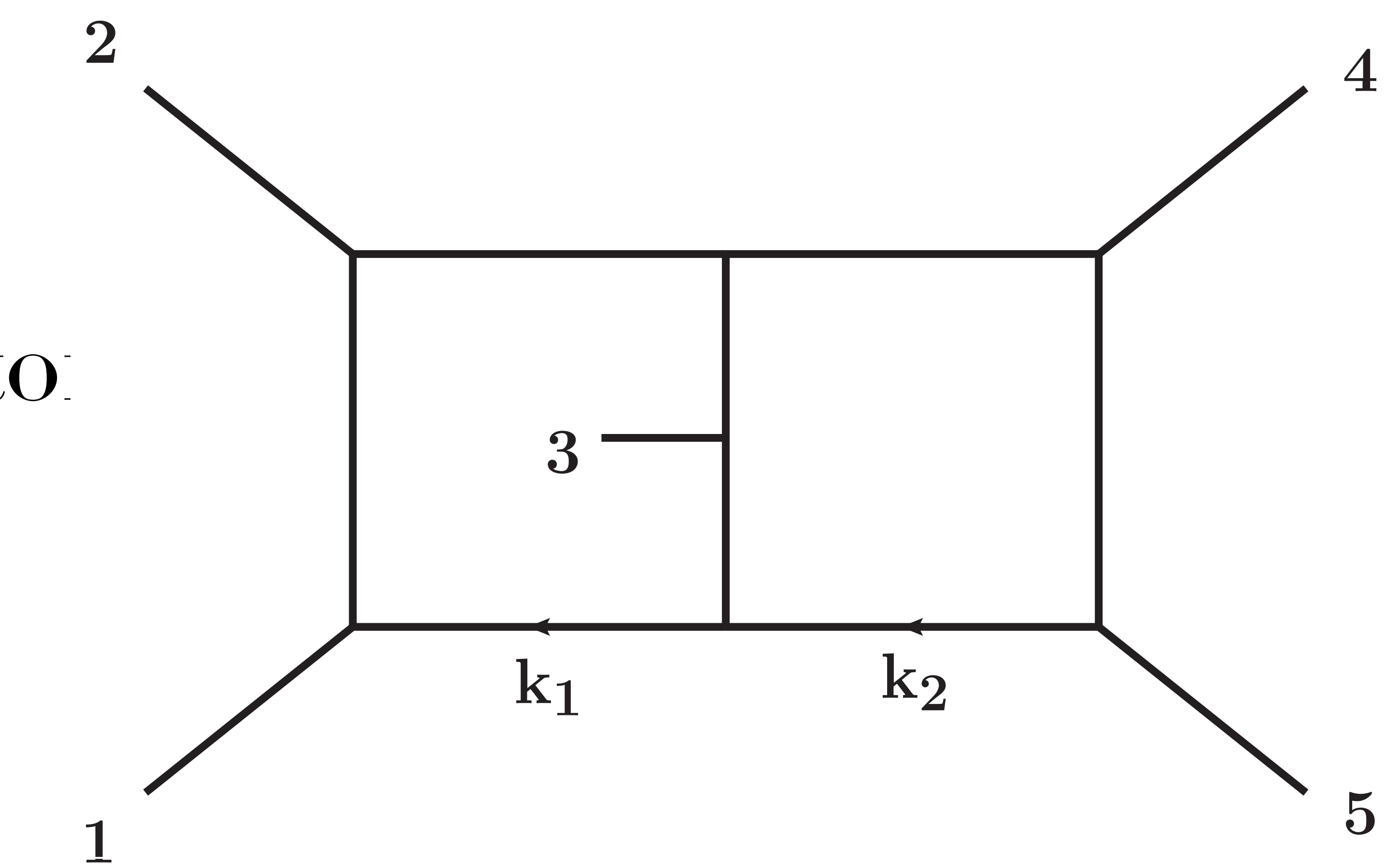


5 Mandelstam variables, with a triple cut
seconds to get the module intersection and truncated IBPs
with our intersection algorithm

Degree-4

Row Reduction

reduce all rank-4 numerato
to master integrals



Cut	# relations	# integrals	size
{1,5,7}	1134	1182	0.77 MB
{1,5,8}	1141	1192	0.85 MB
{1,6,8}	1203	1205	1.1 MB
{2,4,8}	1245	1247	1.1 MB
{2,5,7}	1164	1211	0.84 MB
{2,6,7}	1147	1206	0.62 MB
{2,6,8}	1126	1177	0.83 MB
{3,4,7}	1172	1221	0.78 MB
{3,4,8}	1180	1226	1.0 MB
{3,6,7}	1115	1165	0.82 MB
{1,3,4,5}	721	762	0.43MB

The analytical IBP reduction is actually NOT needed.

Just use these truncated, sparse IBP systems to generate numeric IBP for amplitude, and then interpolate!

Bendle, Boehm, Decker, Georgoudis, Pfreundt, Rahn, Wasser, YZ
JHEP 02 (2020) 079

Row Reduction

reduce all rank-4 numerators analytically
to master integrals

Cut	# relations	# integrals	size
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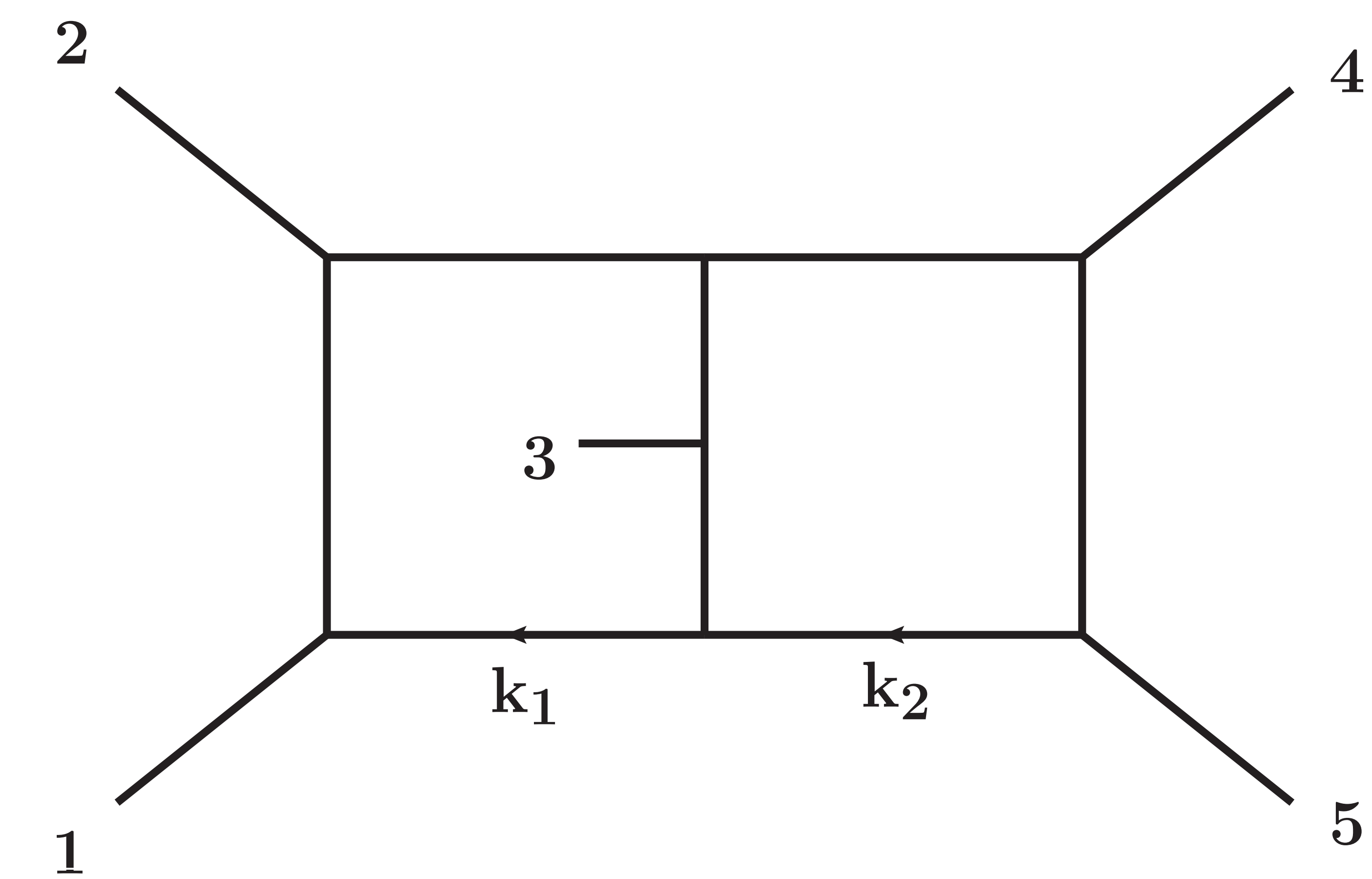


Really want the analytic IBP reduction?

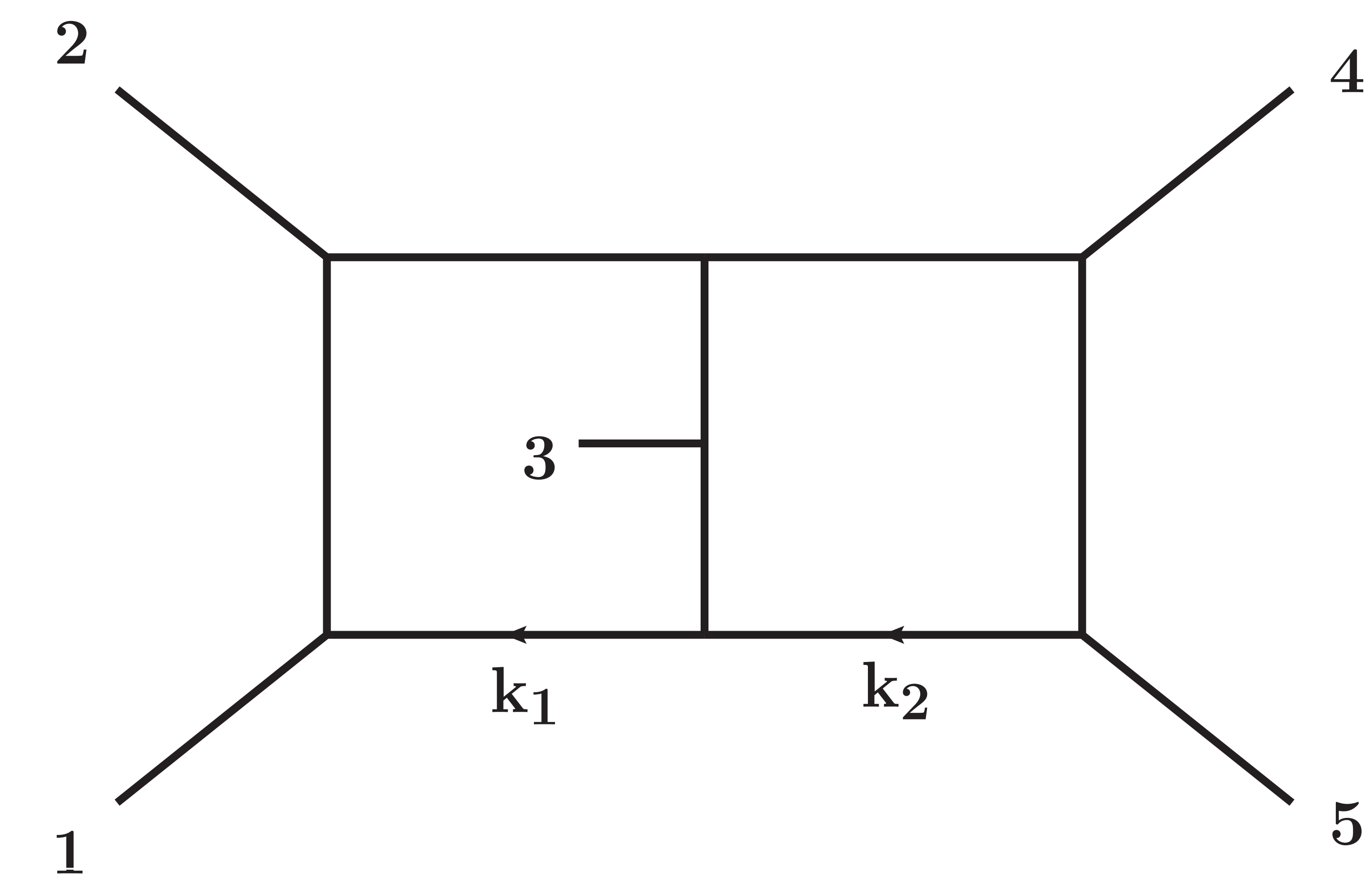
Hardest cut: done within 12 hours, 384 cores

GPI-space

Bendle, Boehm, Decker, Georgoudis, Pfreundt, Rahn, Wasser, YZ
JHEP 02 (2020) 079

Degree-5

deg-5 Row Reduction



The analytic IBP reduction for degree-5
was firstly done by Kira 2.0
result 25GB !!!

Klappert, and Lange and Maierhofer, and Usovitsch
arXiv: 2008.06494

with the block triangular IBPs provides by auxiliary mass flow method

Guan, Liu, Ma, Chin. Phys. C 44(2020) 9, 093106

Here we show how to do this computation with syzygy method (module intersection)
and simplify the result

deg-5 Row Reduction

module intersection

Cut	# relations	# integrals	size
{1,5,7}	2723	2749	1.4 MB
{1,5,8}	2753	2777	1.6 MB
{1,6,8}	2817	2822	2.1 MB
{2,4,8}	2918	2921	2.1 MB
{2,5,7}	2796	2805	1.5 MB
{2,6,7}	2769	2814	1.2 MB
{2,6,8}	2801	2821	1.6 MB
{3,4,7}	2742	2771	1.4 MB
{3,4,8}	2824	2849	1.9 MB
{3,6,7}	2662	2674	1.5 MB
{1,3,4,5}	1600	1650	0.72MB

17.2 MB in total

IBP relations with small size and quite sparse

deg-5 Row Reduction

Row reduce to 108 **UT integrals**

numeric RREF + interpolation with GPI-space

the reduced IBP has the size
$\sim$20 G,

… …

recall Zihao's talk
in Fall 2020

Bonus

Use our partial fraction package "pfd" + UT integrals to simplify the analytic reduced IBP

	reduced IBP size	after pfd	Compression Rate
deg-4	700 MB	20 MB	35
deg-5	20 GB	190 MB	105

Bendle, Boehm, Decker, Georgoudis, Pfreundt, Rahn, Wasser, **YZ**
JHEP 02 (2020) 079

Bendle, Boehm, Heymann Ma, Rahn, Wittmann, Wu, **YZ**
to appear

Summary

- Module intersection + Large-scale parallelzation with GPI-space
- a powerful IBP algorithm
- Since we used Baikov cut form everywhere, some relation to Intersection Theory?
- efficient Lee-Pomeransky IBPs?

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If you have interesting IBP problems,

you may send them to yzhphy@ustc.edu.cn. Thanks!