Regularization and calculation of rapidity divergent Feynman integrals systematically

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Mainly based on M.-x. Luo, T.-Z. Yang, H. X. Zhu and Y. J. Zhu, *Phys. Rev. Lett.* ***124*** *(2020) 092001*

圈积分及相空间积分计算系列讲座

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Standard Feynman integrals

- Standard loop integrals

$$\mathcal{J}(d;\{\nu\},\{p\}) = \left(\prod_{j=1}^{m}\int\frac{d^d l_j}{i\pi^{d/2}}\right)\frac{1}{D_1^{\nu_1}D_2^{\nu_2}\cdots D_f^{\nu_f}}$$

with $D_j = (l_j - p_i)^2$.

- IBP, choose d such that the boundary contribution is zero

$$\prod_{j=1}^{m}\int\frac{d^d l_j}{i\pi^{d/2}}\frac{\partial}{\partial a^\mu}\left(b^\mu\frac{1}{D_1^{\nu_1}D_2^{\nu_2}\cdots D_f^{\nu_f}}\right) = 0$$

- Standard phase-space integrals, reverse unitarity [C. Anastasiou, K, Melnikov, 2002]

$$\delta_+(k^2) = \frac{1}{2i\pi}\left(\frac{1}{k^2 - i0} - \frac{1}{k^2 + i0}\right) = \left(\frac{1}{k^2}\right)_{\text{cut}}$$

Raising or lowering indices of the two propagators simultaneously

Non-standard Feynman integrals

- Thrust T: $\frac{d\Sigma}{dT} = \int \delta\left(T - \max_{\vec{n}} \frac{\sum_i |\vec{n}\cdot\vec{p}_i|}{\sum_i |\vec{p}_i|}\right) d\sigma$, only LO analytic result is available
- Energy-energy correlation $\frac{d\Sigma}{d\cos\chi} = \sum_{i,j}\int \frac{E_i E_j}{Q^2}\delta\left(\cos\chi - \vec{n}_i\cdot\vec{n}_j\right) d\sigma$, NLO analytic result is available in Ref [L.J. Dixon, M.X. Luo, V. Shatabovenko, T.Z. Yang and H.X. Zhu, 2018] using a generalized IBP identities.
- Rapidity divergent integrals

$$dPS_2 = d^dK \exp(-iK_\perp\cdot b_\perp)\delta(\bar{n}\cdot K - (1-z)\bar{n}\cdot P)\, d^dk_1\delta(k_1^2)\delta((K-k_1)^2)$$

$$J_\alpha = \int dPS_2 \frac{1}{\bar{n}\cdot k_1 P\cdot(K-k_1)(P-K)^2}\left(\frac{\nu}{\bar{n}\cdot k_1}\right)^\alpha\left(\frac{\nu}{\bar{n}\cdot(K-k_1)}\right)^\alpha$$

$$J_e = \int dPS_2 \frac{1}{\bar{n}\cdot k_1 P\cdot(K-k_1)(P-K)^2}\exp(-\tau b_0 k_1^0)\exp(-\tau b_0 (K-k_1)^0)$$

$$= \underbrace{\int dPS_2 \frac{1}{\bar{n}\cdot k_1 P\cdot(K-k_1)(P-K)^2}\exp(-\tau b_0 K^0)}_{\text{this talk}}$$

Light-cone coordinate

- Define two light-like vectors

$$n = (1,0,0,-1), \bar{n} = (1,0,0,1),$$

with $n^2 = \bar{n}^2 = 0, n \cdot \bar{n} = 2$.

- Any vector k can be decomposed as

$$k^\mu = \frac{n \cdot k}{n \cdot \bar{n}} \bar{n}^\mu + \frac{\bar{n} \cdot k}{n \cdot \bar{n}} n^\mu + k_\perp^\mu = (\bar{n} \cdot k, n \cdot k, k_\perp)$$
$$k^2 = n \cdot k \bar{n} \cdot k + k_\perp^2 = n \cdot k \bar{n} \cdot k - k_T^2$$

- The measure can be written as

$$\int d^d k = \frac{1}{2} \int dk_\perp^{d-2} d\bar{n} \cdot k dn \cdot k$$

- Convention in this talk

$$P^\mu = \bar{n} \cdot P \frac{1}{2} n^\mu = \frac{1}{2} n^\mu$$

Transverse momentum factorization

- When the transverse momentum is small, Drell-Yan lepton pair production process can be factorized as

$$\frac{d\sigma}{d^2 q_\perp} \sim \sigma_0 H(Q) \int d^2 b_\perp e^{i b_\perp \cdot q_\perp} \mathcal{B} \otimes \bar{\mathcal{B}} \mathcal{S}$$

- $H(Q)$ is the hard function, known to three-loop [Moch, Vermaserren, Vogt(2005); Gehrmann et all(2010)]
- $\mathcal{S}$ is TMD the soft function, known to three-loop[Ye Li, Hua Xing Zhu(2016)]
- $\mathcal{B}$ is the TMD beam function (three-loop in this talk)

Calculation history

- NLO [Collins et all(1985);Ji et all(2005);Becher, Neubert(2011), Aybat et all(2011); Echevarria et all(2012); Chiu et all(2012)]
- NNLO (extract from full theory) [Catani, Grazzini(2012), Catani et all(2013)]
- NNLO (based on operator definition) [Thomas Gehrmann, Thomas Lubbert, Li Lin Yang(2012,2014); Echevarria, Scimemi Vladimirov(2016); Luo, Wang, Xu, Yang, Yang, Zhu(2019); Luo, Yang, Zhu, Zhu(2019);Gutierrez-Reyes, Leal-Gomez, Scimemi,Vladimirov(2019)]

Definition of TMD beam function

- Parton distribution function

$$f_{q/N}(z, b_\perp) = \int \frac{db_-}{2\pi} \, e^{-izb_- \bar{n}\cdot P} \langle N(P)|\bar{\chi}_n(b_-)\frac{\bar{\not{n}}}{2}\chi_n(0)|N(P)\rangle$$

- TMD beam function

$$\mathcal{B}_{q/N}(z, b_\perp) = \int \frac{db_-}{2\pi} \, e^{-izb_- \bar{n}\cdot P}\langle N(P)|\bar{\chi}_n(0, b_-, {\color{red}b_\perp})\frac{\bar{\not{n}}}{2}\chi_n(0)|N(P)\rangle$$

- Outgoing Wilson line

$$\chi_n = W_n^\dagger \xi_n \, , \, W_n^\dagger(x) = \overline{\mathcal{P}} \exp\left(-ig_s \int_{-\infty}^{0} ds\bar{n}\cdot A(x+s\bar{n})\right)$$

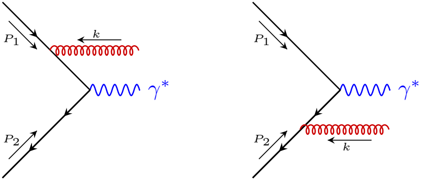
- Gauge invariant

Deriving Feynman rules of the effective vertex

Expanding the Wilson line,

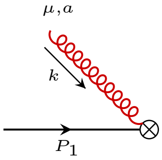
$$
\begin{aligned}
W_n^\dagger(x) &= 1 - ig_s \int_{-\infty}^{0} ds\, \bar{n}\cdot A^a(x+s\bar{n})t^a \; + \mathcal{O}(g_s^2) \\
&= 1 - ig_s \int_{-\infty}^{0} ds \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x+s\bar{n})}\,\bar{n}\cdot\tilde{A}^a(k)\, t^a \; + \mathcal{O}(g_s^2) \\
&= 1 + \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x} \underbrace{\left(g_s \frac{\bar{n}^\mu}{\bar{n}\cdot k + i0}\, t^a \right)}_{\text{Eikonal Feynman rule}} \tilde{A}^a_\mu(k) \; + \mathcal{O}(g_s^2)\,.
\end{aligned}
$$

Or expanding the full theory

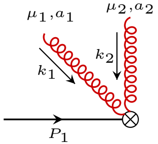


$P_1^\mu = \frac{1}{2}n^\mu, P_2^\mu = \frac{1}{2}\bar{n}^\mu, k \sim P_1, (P_2+k)^2 + i0 = 2P_2\cdot k + i0 = \bar{n}\cdot k + i0$

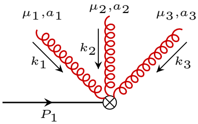
Feynman rules of the effective vertex



$$\rightarrow \; g_s \, \frac{\bar{n}^\mu}{\bar{n}\cdot k + i0} \, t^a$$



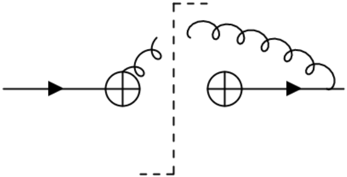
$$\rightarrow \; \frac{g_s^2 \, \bar{n}^{\mu_1} \, \bar{n}^{\mu_2}}{\bar{n}\cdot k_1 + \bar{n}\cdot k_2 + i0} \quad \sum_{P_{\{1,2\}}} \left[\frac{t^{a_1} \, t^{a_2}}{\bar{n}\cdot k_1 + i0} \right]$$



$$\rightarrow \; \frac{g_s^3 \, \bar{n}^{\mu_1} \, \bar{n}^{\mu_2} \, \bar{n}^{\mu_3}}{\bar{n}(\cdot k_1 + k_2 + k_3) + i0} \quad \sum_{P_{\{1,2,3\}}} \left[\frac{t^{a_1} \, t^{a_2} \, t^{a_3}}{\left(\bar{n}\cdot(k_1+k_2) + i0\right)\left(\bar{n}\cdot k_1 + i0\right)} \right]$$

Representative Feynman diagrams

- 1-loop

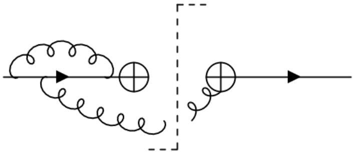


R

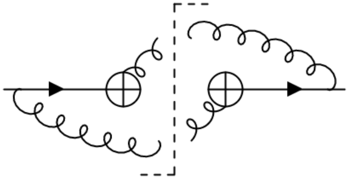
$$\mathcal{B}^{\mathrm{bare}(1)} = R$$

Two-loop

- 2-loop



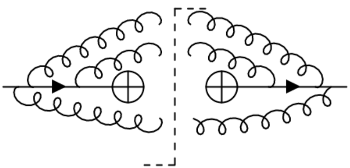
VR



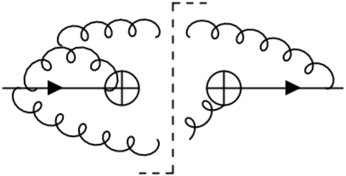
RR

$$\mathcal{B}^{\text{bare}(2)} = \text{RR} + 2\text{Re}(\text{VR})$$

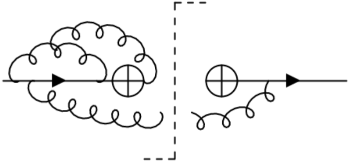
- 3-loop



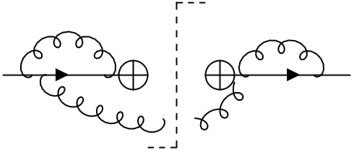
RRR



VRR



VVR



VV*R

$$\mathcal{B}^{\text{bare}(3)} = \text{RRR} + \text{VV*R} + 2\text{Re}(\text{VRR} + \text{VVR})$$

Rapidity divergence

Threshold soft integral : $J_{\text{th}} = \int d^d k \delta_+(k^2) \frac{1}{\bar{n}\cdot k n\cdot k} \delta(\bar{n}\cdot k + n\cdot k - 2E)$

well defined in dimensional regularization

TMD soft integral: $J_{\text{tmd}} = \int d^d k \delta_+(k^2) \frac{1}{\bar{n}\cdot k n\cdot k} \delta(|k_\perp^2| - |\widetilde{k_\perp^2}|)$

$$= \int d^{d-2} k_\perp \frac{1}{|k_\perp^2|} \delta(|k_\perp^2| - |\widetilde{k_\perp^2}|) \int_0^\infty d\bar{n}\cdot k \frac{1}{\bar{n}\cdot k}$$

Even if $\bar{n}\cdot k n\cdot k$ is fixed, still $y = \ln \frac{\bar{n}\cdot k}{n\cdot k} \to \infty$: rapidity divergence

Need rapidity regulator

Rapidity regulator

Several proposed regulator

- analytic regulator [Becher, Neubert(09); Becher, Bell(11)]

$$\int d^dk \to \int d^dk \left(\frac{\nu}{\bar{n}\cdot k}\right)^{\alpha}$$

- Delta regulator [Echevarria, Idilbi and Scimemi(11)]

$$\frac{1}{\bar{n}\cdot k + i\epsilon} \to \frac{1}{\bar{n}\cdot k + \delta}$$

- rapidity regulator [Chiu, Jain, Neill, Rothstein $(11, 12)$]

$$\int d^dk \to \int d^dk \left(\frac{\nu}{|k_z|}\right)^{\eta}$$

- exponential regulator [Y. Li, Neill, H. X. Zhu(16)]

$$\int d^dk \to \int d^dk \exp\left(-b_0 k^0 \tau\right)\ , b_0 = 2\exp\left(\gamma_E\right)$$

From hadronic state to parton asymptotic state

- operator product expansion

$$\mathcal{B}_{q/N}(z, b_\perp) = \sum_i \int_z^1 \frac{d\xi}{\xi} \underbrace{\mathcal{I}_{qi}(\xi, b_\perp)}_{\text{independent of hadronic state}} f_{i/N}(z/\xi) + \mathcal{O}(|b_\perp^2|\Lambda_{\text{QCD}}^2)\,,$$

- Representation in momentum space

$$\mathcal{B}_{qi}^{\text{bare}}(z, b_\perp, \tau) = \int \frac{d^{d-2}\widetilde{K}_\perp}{|\widetilde{K}_\perp^2|^{-\epsilon}} e^{-i\widetilde{K}_\perp \cdot b_\perp} \widetilde{\mathcal{B}}_{qi}$$

$$\widetilde{\mathcal{B}}_{qi} = \lim_{\tau \to 0} \int d^d K \underbrace{\delta(\widetilde{K}_\perp^2 - K_\perp^2)}_{K_\perp^2 \text{ fixed}} \underbrace{\exp\left(-\tau \frac{2P \cdot K}{\bar{n} \cdot P}\right)}_{\text{exponential regulator}} \delta\big(\bar{n} \cdot K - (1 - z)\bar{n} \cdot P\big) \widetilde{\mathcal{B}}_{qi}^{\text{F.D.}}$$

$$\widetilde{\mathcal{B}}_{qi}^{\text{F.D.}} = \int dPS\, \delta^{(d)}(K - \sum_{i=1}^n k_i) \int \prod_j d^d l_j \left|M_{q \leftarrow i}(P, \{l\}, \{k\})\right|^2$$

Calculation method 1

- $\widetilde{\mathcal{B}}_{qi}^{\text{F.D.}}$ is free from rapidity divergence, calculate it first by introducing an extra variable y

$$1 - y = \frac{K^2 \bar{n} \cdot P}{2P \cdot K \bar{n} \cdot K}, \quad 1 - z = \frac{\bar{n} \cdot K}{\bar{n} \cdot P}$$

- Advantage:
  - Without involving non-standard propagator
  - Effectively turn an n-loop problem into an (n-1)-loop problem
  - IBP reduction is direct and fast
- This method was applied in the two-loop calculations[Luo, Wang, Xu, Yang, Yang, Zhu(2019); Luo, Yang, Zhu, Zhu(2019)]
- Shortcoming at the three-loop calculations
  - DE about y, z is complicated
  - The solutions of DE involved multiple roots that are not easy to be rationalized simultaneously

Calculation method 2 : generalized IBP identities

- The first non-standard propagator $\delta(\widetilde{K}_\perp^2 - K_\perp^2)$ has been linearized

$$\delta(\widetilde{K}_\perp^2 - K_\perp^2) = \delta(\widetilde{K}_\perp^2 - K^2 + n\cdot K\bar{n}\cdot K) = \left(\frac{1}{\widetilde{K}_\perp^2 - K^2 + 2P\cdot K(1-z)}\right)_{\text{cut}}$$
$$\delta\big(\bar{n}\cdot K - (1-z)\bar{n}\cdot P\big)$$

- Differentiate with the regulator directly

$$0 = \int d^d q \, \frac{\partial}{\partial q^\mu}\left[e^{-b_0\tau\frac{P\cdot K}{\bar{n}\cdot P}} F(\{\tilde{l}\})\right]$$
$$= \begin{cases} \int d^d q \, e^{-b_0\tau\frac{P\cdot K}{\bar{n}\cdot P}}\left[-b_0\tau\frac{P_\mu}{\bar{n}\cdot P} + \frac{\partial}{\partial q^\mu}\right] F(\{\tilde{l}\}) \, , & q = K \, , \\ \int d^d q \, e^{-b_0\tau\frac{P\cdot K}{\bar{n}\cdot P}} \frac{\partial}{\partial q^\mu} F(\{\tilde{l}\}) & q \neq K \, , \end{cases}$$

- Using the above IBP identities, combine LiteRed and FIRE, we can reduce all integrals as the combination of master integrals

IBP reduction time and the number of master integrals

process	reduction time	number of master integrals
RRR	about one week	~ 500
VRR	about 1 day	~ 500

Differential equations

- z dependence can only come from the following two cut propagators

$$\left(\frac{1}{\widetilde{K}_\perp^2 - K^2 + 2P\cdot K(1-z)}\right)_{\text{cut}}, \left(\frac{1}{\bar{n}\cdot K - \bar{n}\cdot P(1-z)}\right)_{\text{cut}}$$

- τ dependence only comes from the regulator $e^{-\tau\frac{2P\cdot K}{\bar{n}\cdot P}}$
- Differentiate with z and τ to obtain the differential equations

$$\frac{\partial \mathcal{MI}(\nu_1,\nu_2)}{\partial z} = -2P\cdot K\mathcal{MI}(\nu_1+1,\nu_2) + \bar{n}\cdot P\mathcal{MI}(\nu_1,\nu_2+1)\,,$$
$$\frac{\partial \mathcal{MI}(\nu_1,\nu_2)}{\partial \tau} = \frac{-2P\cdot K}{\bar{n}\cdot P}\mathcal{MI}(\nu_1,\nu_2)$$

Expansion of DE in the limit $\tau \to 0$

1. Expanding the master integrals f_i as the following general form(j, n, k are integers)

$$f_i(z,\tau,\epsilon) \overset{\tau\to 0}{=} \sum_j \sum_n \sum_{k=0} g_i^{(j,n,k)}(z,\epsilon)\tau^{j+n\epsilon}\ln^k\tau$$

2. Substituting the above form into the DE about τ, obtaining relations of different expansion coefficients $g_i^{(j,n,k)}$
3. Substituting the form into the DE about z.
4. Constructing DE with respect to z of the independent expansion coefficients $\vec{g}$ (one master integral corresponding to one independent coefficient)

$$\underbrace{\boxed{\frac{d\vec{g}}{dz} = \boldsymbol{A}(z,\epsilon)\vec{g}}}_{\text{single variable differential equations}}$$

Canonical form and appeared letters

- The single variable differential equation can be turned into canonical form [J. M. Henn, 2013; R. N. Lee, 2015; C. Meyer, 2017; O. Gituliar and V. Magerya, 2017]

$$\vec{g} = T\vec{g}', \quad d\vec{g}' = \sum_{\alpha} \epsilon \boldsymbol{A'}_{\alpha} \vec{g}' \, d\ln L_{\alpha}$$

- 5 letters

$$L_{\alpha} : \{z, \quad 1-z, \quad 1+z, \quad \underbrace{\overbrace{2-z, \quad z^2-z+1}^{\text{cancel out after summing up VRR and RRR}}}_{\text{write down the solutions using GPL}}\}$$

- We only need HPL to represent the final results.

Boundary conditions ($z \to 1$)

$z \to 1$ enforces all final state particle ultra-soft,

$$P \sim (1, \lambda^2, \lambda)\, Q\,, \quad K \sim k_i \sim (\lambda^2, \lambda^2, \lambda^2)\, Q\,.$$

For loop momentum,

ultra-soft region: $\quad l \sim (\lambda^2, \lambda^2, \lambda^2)\, Q\,,$

collinear region: $\quad l \sim (1, \lambda^2, \lambda)\, Q\,,$

anti-collinear region: $\quad l \sim (\lambda^2, 1, \lambda)\, Q\,,$

hard region: $\quad l \sim (1, 1, 1)\, Q\,.$

An example

An sample integral from VRR,

$$\mathcal{J} = \int [dPS] d^d l \frac{1}{l^2 (l-P)^2 \bar{n}\cdot(l-k_1) \bar{n}\cdot(P-K+k_1)(l-k_1)^2 (l-P+K-k_1)^2}\,,$$

where

$$\int [dPS] = \int [d^d K] \int [d^d k_1]\,,$$

$$[d^d K] = e^{-2\tau P\cdot K} \delta(\bar{n}\cdot K - (1-z)) \delta(-1+K^2-2(1-z)K\cdot P) \sim \frac{1}{1-z}\,,$$

$$[d^d k_1] = d^d k_1 \delta_+(k_1^2) \delta_+((K-k_1)^2)\,.$$

Scaling in $(1-z)$,

$$P\cdot K \sim \frac{1}{1-z}\,, \quad \bar{n}\cdot K \sim (1-z)\,, K^2 \sim (1-z)^0\,, \quad \delta(\bar{n}\cdot K - (1-z)) \sim \frac{1}{1-z}$$

Anti-collinear region and hard region (scaleless)

Integrals before expansion

$$\mathcal{J} = \int [dPS]\, d^d l \frac{1}{l^2 (l-P)^2 \bar{n}\cdot(l-k_1) \bar{n}\cdot(P-K+k_1)(l-k_1)^2 (l-P+K-k_1)^2}$$

- Anti-collinear region $l \sim (\lambda^2, 1, \lambda) Q$

$$\int [dPS]\, d^d l \frac{1}{l^2 (-2P\cdot l) \bar{n}\cdot l \bar{n}\cdot P (l^2 - 2P\cdot l \bar{n}\cdot k_1)(-2P\cdot l)}$$

- Hard region $l \sim (1,1,1) Q$

$$\int [dPS]\, d^d l \frac{1}{l^2 (l^2 - 2P\cdot l \bar{n}\cdot P) \bar{n}\cdot l \bar{n}\cdot P l^2 (l^2 - 2P\cdot l \bar{n}\cdot P)}$$

Scaleless $\rightarrow 0$

Ultra-soft region

Integrals before expansion

$$\mathcal{J} = \int [dPS]\, d^d l \frac{1}{l^2 (l-P)^2 \bar{n}\cdot(l-k_1)\bar{n}\cdot(P-K+k_1)(l-k_1)^2(l-P+K-k_1)^2}$$

ultra-soft region $l \sim (\lambda^2, \lambda^2, \lambda^2)Q$

$$\mathcal{J}_S = \int [dPS]\, d^d l \frac{1}{l^2 (-2P\cdot l)\bar{n}\cdot(l-k_1)\bar{n}\cdot P(l-k_1)^2\big(-2P\cdot(K-k_1)\big)}$$

$$\sim \int [d^d K] \frac{1}{(P\cdot K)^2 \bar{n}\cdot K} \sim (1-z)^0 \text{[scaling symmetry]}$$

Subtlety: The loop transformation symmetry is not invariant under taking limit

$$l \to l+P: \int [dPS]\, d^d l \frac{1}{2P\cdot l\, l^2\, 2P\cdot(l-k_1)(l+K-k_1)^2} \sim (1-z)^1$$

$$l \to l-P: \int [dPS]\, d^d l \frac{-1}{-P\cdot l(-2P\cdot l)(-2P\cdot(l-k_1))(-2P\cdot(l+K-k_1))} \sim (1-z)^3$$

Collinear region

Integrals before expansion

$$\mathcal{J} = \int [dPS] d^d l \frac{1}{l^2 (l-P)^2 \bar{n}\cdot(l-k_1) \bar{n}\cdot(P-K+k_1)(l-k_1)^2 (l-P+K-k_1)^2}$$

Collinear region $l \sim (1, \lambda^2, \lambda) Q$

$$\mathcal{J}_C = \int [dPS] d^d l \frac{1}{l^2 (l-P)^2 \bar{n}\cdot l \bar{n}\cdot P (l^2 - \bar{n}\cdot l \frac{2P\cdot k_1}{\bar{n}\cdot P}) \left((l-P)^2 + \bar{n}\cdot(l-P) \frac{2P\cdot(K-k_1)}{\bar{n}\cdot P} \right)}$$

$$\sim \int [d^d K] \frac{(2P\cdot K)^{-2-\epsilon}}{(\bar{n}\cdot P)^2} \sim (1-z)^{1+\epsilon}$$

The only external momentum about l is P and $\bar{n}$. After integrating out l, the mass dimension can only be $P\cdot K$ or $P\cdot k_1$

$$\mathcal{J}_C = \frac{e^{2\epsilon\gamma_E}(1-z)^{\epsilon+1}\Gamma(1-\epsilon)^3\Gamma(2\epsilon+2)}{\epsilon^3(\epsilon+1)^2} \left(\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)^2} - \frac{3}{2\Gamma(1-3\epsilon)} \right) + \mathcal{O}(\tau)$$

A one-loop example

$$\widetilde{\mathcal{B}}_{qq}^{(1)} = \int d^d K \, \delta(\widetilde{K}_\perp^2 - K_\perp^2) \exp\left(-\tau \frac{2P\cdot K}{\bar{n}\cdot P}\right) \delta\big(\bar{n}\cdot K - (1-z)\bar{n}\cdot P\big) \delta(K^2) \left|Sp_{qq}^{(R)}\right|^2$$

$$\left|Sp_{qq}^{(R)}\right|^2 = 2\,C_F \frac{\bar{n}\cdot P}{2P\cdot K} \left(\frac{\big(z^2+1\big) - (z-1)^2\epsilon}{1-z} \right)$$

$$\Rightarrow \widetilde{\mathcal{B}}_{qq}^{(1)} = 2\,C_F |\widetilde{K}_\perp^2|^{-1} \left(\frac{\big(z^2+1\big) - (z-1)^2\epsilon}{1-z} \right) \exp\left(-\tau \frac{|\widetilde{K}_\perp^2|}{(1-z)\bar{n}\cdot P} \right)$$

plus-distribution

$$\frac{e^{-\tau/(1-z)}}{1-z} = -(\ln\tau + \gamma_E)\,\delta(1-z) + \frac{1}{(1-z)_+} + \mathcal{O}(\tau)$$

A two-loop example

Consider one of the two-loop processes from RR

$$q(P) \to q^*(P-K) + g(k_1)\ g(K-k_1)\,,$$

After IBP reduction using generalized IBP identities,

$$J_1 = \int [dPS]\,, \quad J_2 = \int [dPS] \frac{1}{P\cdot k_1}\,,$$

$$J_3 = \int [dPS] \frac{1}{(P-K)^2}\,, \quad J_4 = \int [dPS] \frac{1}{\bar{n}\cdot k_1\, P\cdot k_1}\,,$$

$$J_5 = \int [dPS] \frac{1}{\bar{n}\cdot k_1\, P\cdot k_1 (P-K)^2}\,,$$

$$J_6 = \int [dPS] \frac{1}{\bar{n}\cdot k_1\, P\cdot (K-k_1)(P-K)^2}\,,$$

$$J_7 = \int [dPS] \frac{1}{\bar{n}\cdot k_1\, P\cdot k_1\, P\cdot (K-k_1)}\,,$$

$$J_8 = \int [dPS] \frac{1}{\bar{n}\cdot k_1\, P\cdot (K-k_1) K^2}\,,$$

$$
\begin{aligned}
&\widetilde{\mathcal{B}}_{q/q}^{(0,2,\text{non-singlet})}(z,\widetilde{K}_\perp) = C_F^2\Bigg[\frac{2\left(z^2\epsilon - z^2 - 2z\epsilon + \epsilon - 1\right)}{1-z}\left(2zJ_6 + 2J_5 - J_7\right) \\
&\quad + \frac{4(\epsilon-1)\left(2z\epsilon^3 + z\epsilon^2 + 3z\epsilon - z - 2\epsilon^3 + 3\epsilon^2 + \epsilon - 1\right)}{2\epsilon - 1} J_2 \\
&\quad - \frac{8z(\epsilon-1)\left(2z\epsilon^3 - 4z\epsilon^2 - z\epsilon + z - 2\epsilon^3 - 3\epsilon + 1\right)}{\epsilon} J_3\Bigg] + C_F C_A\Bigg[\\
&\quad \frac{\left(z^2\epsilon - z^2 - 2z\epsilon + \epsilon - 1\right)}{1-z}\left(2zJ_8 - 2J_5 + J_7 - 2J_8\right) + \frac{2\epsilon}{(1-z)(2\epsilon-1)}\Big(2z^2\epsilon^3 - z^2\epsilon^2 \\
&\quad - z^2\epsilon - 4z\epsilon^3 + 4z\epsilon^2 + 2z\epsilon + 2\epsilon^3 - 3\epsilon^2 - \epsilon + 3\Big) J_2 \\
&\quad - \frac{4z}{(1-z)\epsilon(2\epsilon-3)}\Big(2z^2\epsilon^5 - 8z^2\epsilon^4 + 12z^2\epsilon^3 - 11z^2\epsilon^2 + 11z^2\epsilon - 6z^2 - 4z\epsilon^5 + 12z\epsilon^4 \\
&\quad - 20z\epsilon^3 + 23z\epsilon^2 - 12z\epsilon + 2\epsilon^5 - 4\epsilon^4 + 8\epsilon^3 - 20\epsilon^2 + 20\epsilon - 6\Big) J_3\Bigg] + \tau\Bigg\{ C_F^2\Bigg[\\
&\quad - \frac{8(1-z)(\epsilon-1)\left(4\epsilon^2 - 3\epsilon + 2\right)}{\epsilon} J_1 + \frac{4(\epsilon-1)\left(2\epsilon^2 - \epsilon + 1\right)}{2\epsilon-1} J_2 - \frac{8(\epsilon-1)\left(2\epsilon^2 - 2\epsilon + 1\right)}{\epsilon} J_3\Bigg] \\
&\quad + C_A C_F\Bigg[\frac{2\epsilon\left(2z\epsilon^2 - 2z\epsilon - 2\epsilon^2 + 2\epsilon - 1\right)}{(1-z)(2\epsilon-1)} J_2 + \frac{8(\epsilon-1)^3}{2\epsilon-3} J_3 + \frac{4}{(1-z)\epsilon^2(2\epsilon-3)}\Big(6z^2\epsilon^5 \\
&\quad - 16z^2\epsilon^4 + 13z^2\epsilon^3 - 8z^2\epsilon^2 + 11z^2\epsilon - 6z^2 - 12z\epsilon^5 + 32z\epsilon^4 - 28z\epsilon^3 + 17z\epsilon^2 - 12z\epsilon \\
&\quad + 6\epsilon^5 - 16\epsilon^4 + 15\epsilon^3 - 11\epsilon^2 + 11\epsilon - 6\Big) J_1\Bigg]\Bigg\}.
\end{aligned}
$$

DE

$$\vec{J} = \{J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8\}^T,$$

$$\frac{\partial \vec{J}}{\partial \tau} = A(\tau, z)\vec{J},$$
$$\frac{\partial \vec{J}}{\partial z} = B(\tau, z)\vec{J},$$

$$A(\tau,z) = \begin{pmatrix} -\frac{\epsilon z - z - \epsilon + \tau + 1}{(1-z)\tau} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2(2\epsilon-1)}{\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{z} & 0 & \frac{1}{z} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2\epsilon}{(1-z)\tau} & 0 & -\frac{2z\epsilon-2\epsilon+\tau}{(1-z)\tau} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{z} & \frac{1}{z} & 0 & 0 & 0 \\ \frac{4(2\epsilon-1)}{z\epsilon} & \frac{2\epsilon}{z\tau} & 0 & 0 & 0 & \frac{1}{z} & 0 & 0 \\ -\frac{8(2\epsilon-1)}{\epsilon} & -\frac{4\epsilon}{\tau} & 0 & -2 & 0 & 0 & 0 & 0 \\ -\frac{4(2\epsilon-1)}{(1-z)\epsilon} & -\frac{2\epsilon}{(1-z)\tau} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{1-z} \end{pmatrix}$$

DE

$$B(\tau, z) =$$

$$\begin{pmatrix}
-\frac{z\epsilon-\epsilon+\tau}{(z-1)^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2(2\epsilon-1)\tau}{(z-1)\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{(2z-1)\tau}{(z-1)z^2} & 0 & \frac{\epsilon z-z-\tau}{z^2} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{2\epsilon}{(z-1)^2} & 0 & -\frac{2\epsilon z+z-2\epsilon+\tau-1}{(z-1)^2} & 0 & 0 & 0 & 0 \\
0 & -\frac{2\epsilon}{(z-1)z} & -\frac{4(2\epsilon-1)}{z-1} & -\frac{(2z-1)\tau}{(z-1)z^2} & \frac{2z\epsilon-\tau}{z^2} & 0 & 0 & 0 \\
-\frac{4(2z-1)(2\epsilon-1)\tau}{(z-1)z^2\epsilon} & -\frac{2\epsilon}{z^2} & \frac{4(2\epsilon-1)}{z-1} & 0 & 0 & -\frac{z+\tau}{z^2} & 0 & 0 \\
\frac{8(2\epsilon-1)\tau}{(z-1)\epsilon} & \frac{4\epsilon}{z-1} & 0 & \frac{2\tau}{z-1} & 0 & 0 & 0 & 0 \\
-\frac{4(2\epsilon-1)\tau}{(z-1)^2\epsilon} & -\frac{2\epsilon}{(z-1)^2} & 0 & 0 & 0 & 0 & 0 & -\frac{z+\tau-1}{(z-1)^2}
\end{pmatrix}$$

Expanding DE in the limit of $\tau \to 0$

$$
\begin{aligned}
J_1 &= c_1(z)\tau^{\epsilon-1} - \frac{c_1(z)\tau^{\epsilon}}{1-z} + \mathcal{O}(\tau)\,,\\
J_2 &= c_2(z) - \frac{2(2\epsilon-1)c_1(z)\tau^{\epsilon}}{\epsilon^2} + \mathcal{O}(\tau)\,,\\
J_3 &= \frac{c_1(z)\tau^{\epsilon}}{z\epsilon} + c_3(z) + \mathcal{O}(\tau)\,,\\
J_4 &= -\frac{4(2\epsilon-1)c_1(z)\tau^{\epsilon}}{(1-z)\epsilon^2} + c_4(z)\tau^{2\epsilon} + \frac{c_2(z)}{1-z} + \mathcal{O}(\tau)\,,\\
J_5 &= c_5(z) + \mathcal{O}(\tau)\,,\\
J_6 &= \frac{2\epsilon c_2(z)\ln(\tau)}{z} + c_6(z) + \mathcal{O}(\tau)\,,\\
J_7 &= c_7(z) - 4\epsilon c_2(z)\ln(\tau) + \mathcal{O}(\tau)\,,\\
J_8 &= c_8(z) - \frac{2\epsilon c_2(z)\ln(\tau)}{1-z} + \mathcal{O}(\tau)\,.
\end{aligned}
$$

Substituting the expansion form into the integrand

$$
\begin{aligned}
&\widetilde{\mathcal{B}}_{q/q}^{(0,2,\text{non-singlet})}(z,\widetilde{K}_\perp) = C_F^2\Bigg[\frac{4(\epsilon-1)}{2\epsilon-1}\left(2z\epsilon^3+z\epsilon^2+3z\epsilon-z-2\epsilon^3+3\epsilon^2+\epsilon-1\right)c_2(z) \\
&+\frac{2\left(-z^2+(1-z)^2\epsilon-1\right)}{1-z}\left(8\epsilon\ln(\tau)c_2(z)+2c_5(z)+2zc_6(z)-c_7(z)\right)-\frac{8z(\epsilon-1)}{\epsilon}\big(2z\epsilon^3 \\
&-4z\epsilon^2-z\epsilon+z-2\epsilon^3-3\epsilon+1\big)c_3(z)\Bigg]+C_AC_F\Bigg[\frac{2\left(-z^2+(1-z)^2\epsilon-1\right)}{1-z}\big(-2c_5(z) \\
&+c_7(z)+2zc_8(z)-2c_8(z)\big)+\frac{2\epsilon}{(1-z)(2\epsilon-1)}\big(2z^2\epsilon^3-z^2\epsilon^2-z^2\epsilon-4z\epsilon^3+4z\epsilon^2+2z\epsilon \\
&+2\epsilon^3-3\epsilon^2-\epsilon+3\big)c_2(z)-\frac{4z}{(1-z)\epsilon(2\epsilon-3)}\big(2z^2\epsilon^5-8z^2\epsilon^4+12z^2\epsilon^3-11z^2\epsilon^2+11z^2\epsilon \\
&-6z^2-4z\epsilon^5+12z\epsilon^4-20z\epsilon^3+23z\epsilon^2-12z\epsilon+2\epsilon^5-4\epsilon^4+8\epsilon^3-20\epsilon^2+20\epsilon-6\big)c_3(z)\Bigg]
\end{aligned}
$$

Free of τ^ϵ

$\ln\tau$ is the genuine rapidity divergence

Constructing single variable DE

Substituting the expansion form into the DE about z

$$\vec{C}(z) = \{c_2(z), c_3(z), c_5(z), c_6(z), c_7(z), c_8(z)\}^T$$

$$\frac{d\vec{C}(z)}{dz} = E(z)\,\vec{C}(z)$$

$$E(z) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon-1}{z} & 0 & 0 & 0 & 0 \\ \frac{2\epsilon}{(1-z)z} & \frac{4(2\epsilon-1)}{1-z} & \frac{2\epsilon}{z} & 0 & 0 & 0 \\ -\frac{2\epsilon}{z^2} & -\frac{4(2\epsilon-1)}{1-z} & 0 & -\frac{1}{z} & 0 & 0 \\ -\frac{4\epsilon}{1-z} & 0 & 0 & 0 & 0 & 0 \\ -\frac{2\epsilon}{(1-z)^2} & 0 & 0 & 0 & 0 & \frac{1}{1-z} \end{pmatrix}$$

Can be easily turned into canonical form

Renormalization

- Zero-bin subtraction and operator product expansion

$$\frac{\mathcal{B}_{ij}^{\text{bare}}}{\mathcal{S}_{0b}} = Z^B \mathcal{B}_{ij} = Z^B \sum_k \mathcal{I}_{ik} \otimes f_{k/j}$$

  - Using exponential regulator, $\mathcal{S}_{0b}$ is just the TMD soft function, known to three loops[Ye Li and Hua Xing Zhu, 2016]
  - Z^B is the multiplicative renormalization factor
  - $\mathcal{I}_{ik}$ is the matching coefficient
- The partonic collinear PDF can be represented through splitting fuctions

$$f_{i/j}(x_B) = \delta_{ij}\delta(1-x_B) - \frac{\alpha_s}{4\pi}\frac{P_{ij}^{S(0)}(x_B)}{\epsilon} + \mathcal{O}(\alpha_s^2)$$

example $$\boxed{\mathcal{B}_{ij}^{\text{bare}(1)} = Z^{B(1)} + \mathcal{S}_{0b}^{(1)} + \mathcal{I}_{ij}^{(1)} - \frac{P_{ij}^{S(0)}}{\epsilon}}$$

Several checks

- All ϵ poles can be absorbed into renormalization factor, repeated the famous 3-loop splitting functions [S. Moch, J.A.M. Vermaseren and A.Vogt, 2004, A. Vogt, S. Moch and J.A.M. Vermaseren, 2004]
- The finite results satisfy the renormalization group equations [J.-Y. Chiu, A. Jain, D. Neill, and I. Z. Rothstein, 2012]

$$\frac{d}{d\ln\mu}\mathcal{I}_{qi} = \left[-\Gamma^{\text{cusp}}(\alpha_s(\mu))L_Q + 2\gamma^B(\alpha_s(\mu))\right]\mathcal{I}_{qi} - 2\sum_j \mathcal{I}_{qj}\otimes P_{ji}(z,\alpha_s(\mu))$$

$$\frac{d}{d\ln\nu}\mathcal{I}_{qi} = -2\left[\int_\mu^{b_0/b_T}\frac{d\bar{\mu}}{\bar{\mu}}\Gamma^{\text{cusp}}(\alpha_s(\bar{\mu})) + \gamma^R(\alpha_s(b_0/b_T))\right]\mathcal{I}_{qi}$$

- Enable us to construct rapidity finite quantities

$$f_{T,ij} = \mathcal{I}_{ij}\sqrt{\mathcal{S}}$$

Several checks

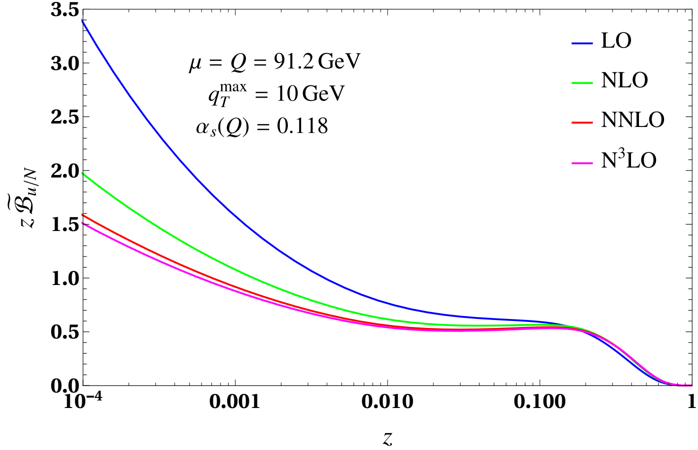
- Correct soft limit $(z \to 1)$ [M. G. Echevarria, I. Scimemi, and A. Vladimirov, 2016; G. Lustermans, W. J. Waalewijn, and L. Zeune, 2016; G. Billis, M. A. Ebert, J. K. L. Michel, and F. J. Tackmann, 2019]

$$I_{qq}^{(3)}(z) = \frac{2\gamma_2^R}{(1-z)_+}$$

- Correct high-energy limit $(z \to 0)$ [S. Marzani, 2016]
- Consistent with another later independent calculation [M. A. Ebert, B. Mistlberger, G. Vita,2020]

Results

$$B_{q/N}(z, q_T^{\max}) = \sum_i \int_0^{q_T^{\max}} dq_T \int_z^1 \frac{d\xi}{\xi} f_{T,qi}(\xi, q_T)\, f_{i/N}(z/\xi)\,,$$



Summary

- Rapidity divergent Feynman integrals and regularization
- Generalized IBP identities
- Expanding DE in the limit of $\tau \to 0$
- Determine the boundary conditions
- A two-loop example
- Renormalization and results

Thanks for your attention!