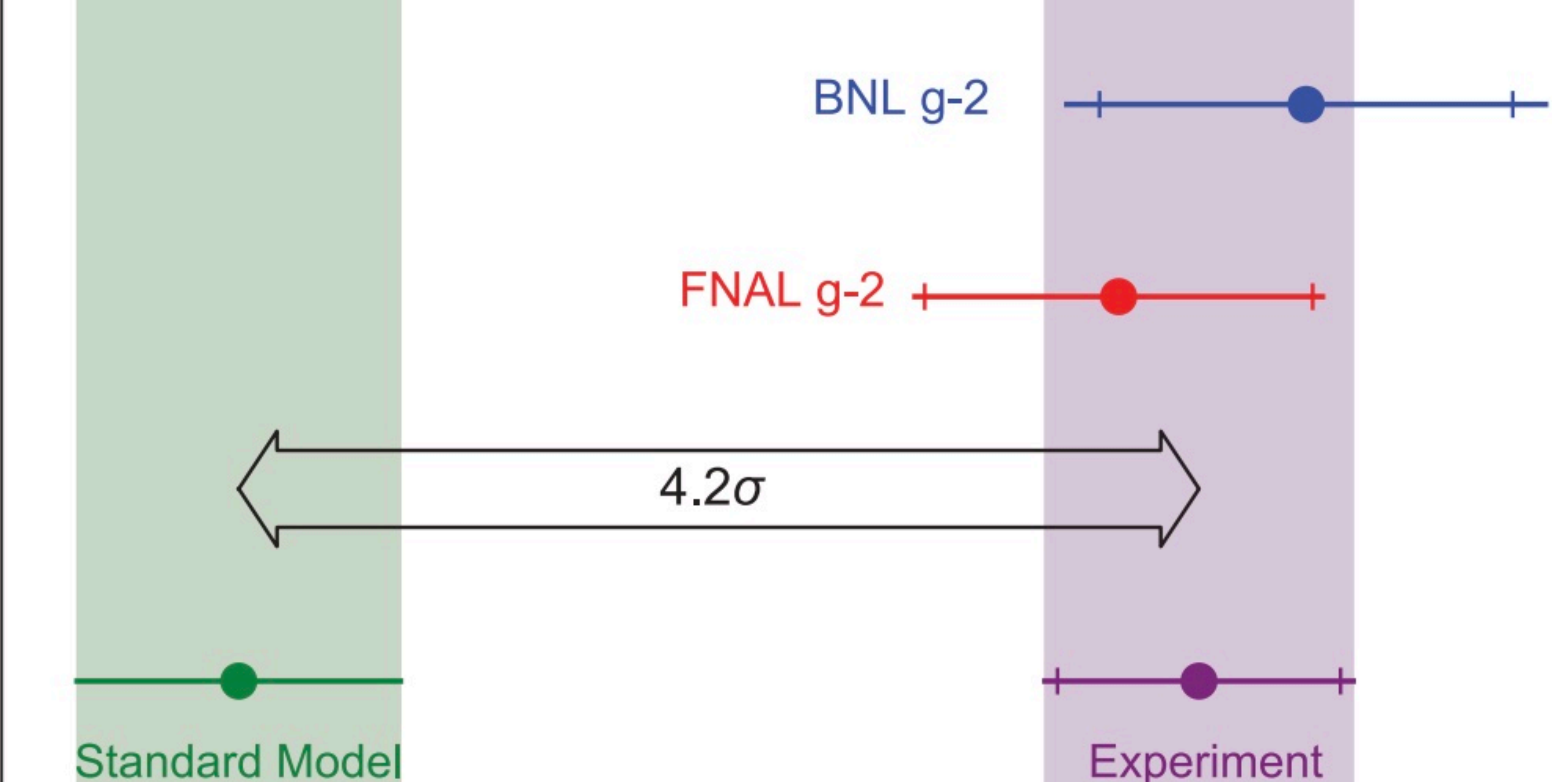

HIGH PRECISION PHENOMENOLOGY AND SUBTRACTION METHODS

BEIJING NORMAL UNIVERSITY SEMINARS

Karlsruhe Institute of Technology

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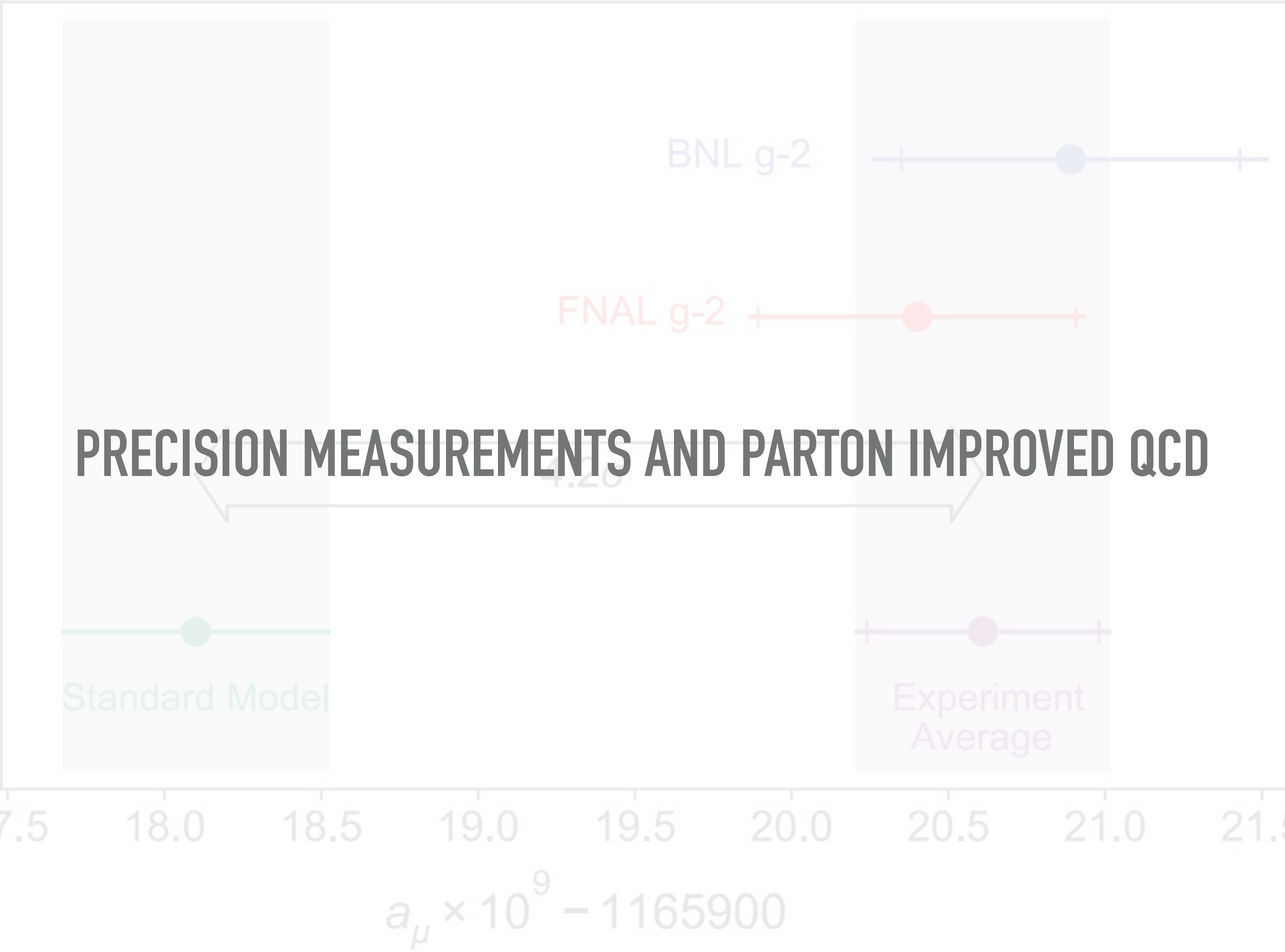
Beijing, China, April 08, 2021

OUTLINE

- **Precision measurements and Parton Improved QCD**
  - Real scattering events at LHC
  - Theoretical tools to describe/approximate particle scattering
  - Roadmap of NNLO
  - Roadmap of N3LO
- **NLO and NNLO subtraction methods**
  - Colour ordered amplitudes
  - Jet algorithm
  - Dynamic IR divergence
  - Explicit IR divergence (pole structure)
  - Antenna subtraction method at NLO
    - X30 antenna functions
    - Momentum mapping
    - Integrated antenna functions
  - Antenna subtraction method at NNLO
    - X40 and X31 antenna functions
    - 4 to 2 momentum mapping
    - NNLO antenna tool box
  - **Application to precision phenomenology at LHC**
    - H+J, photon+J, V+J
- **N3LO subtraction methods and application**
  - qT subtraction and projection to born subtraction
- **Outlook & Summary**

PRECISION MEASUREMENTS AND PARTON IMPROVED QCD

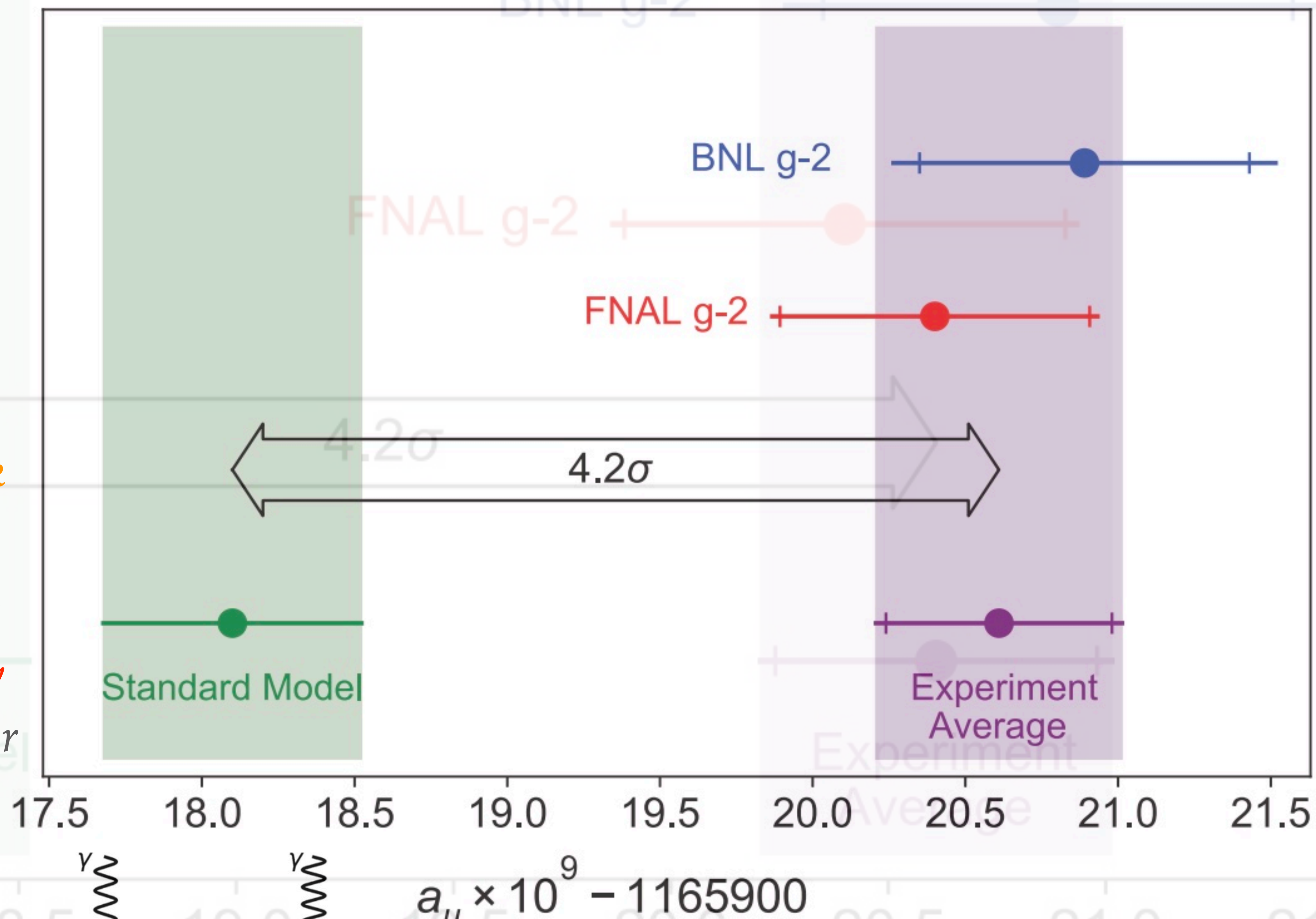


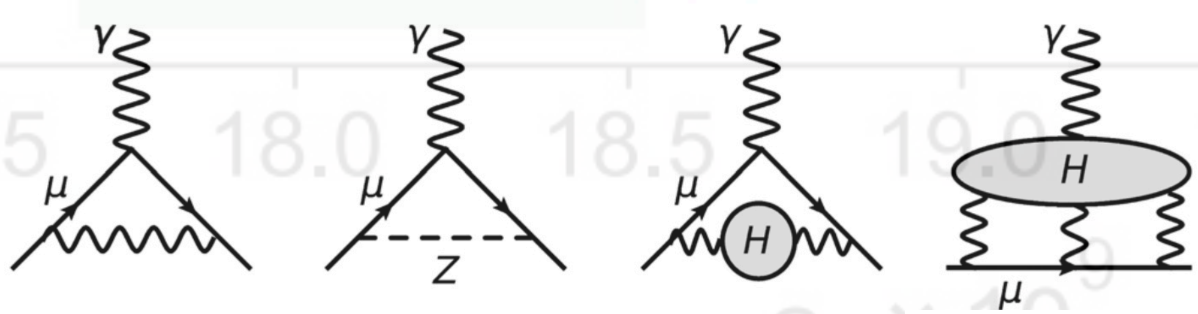
WHY DO WE CARE?

PRECISION MEASUREMENTS AND THEORY

SM prediction is based on evaluations of the contributions from QED to tenth order, hadronic vacuum polarisation, hadronic light-by-light, and electroweak processes

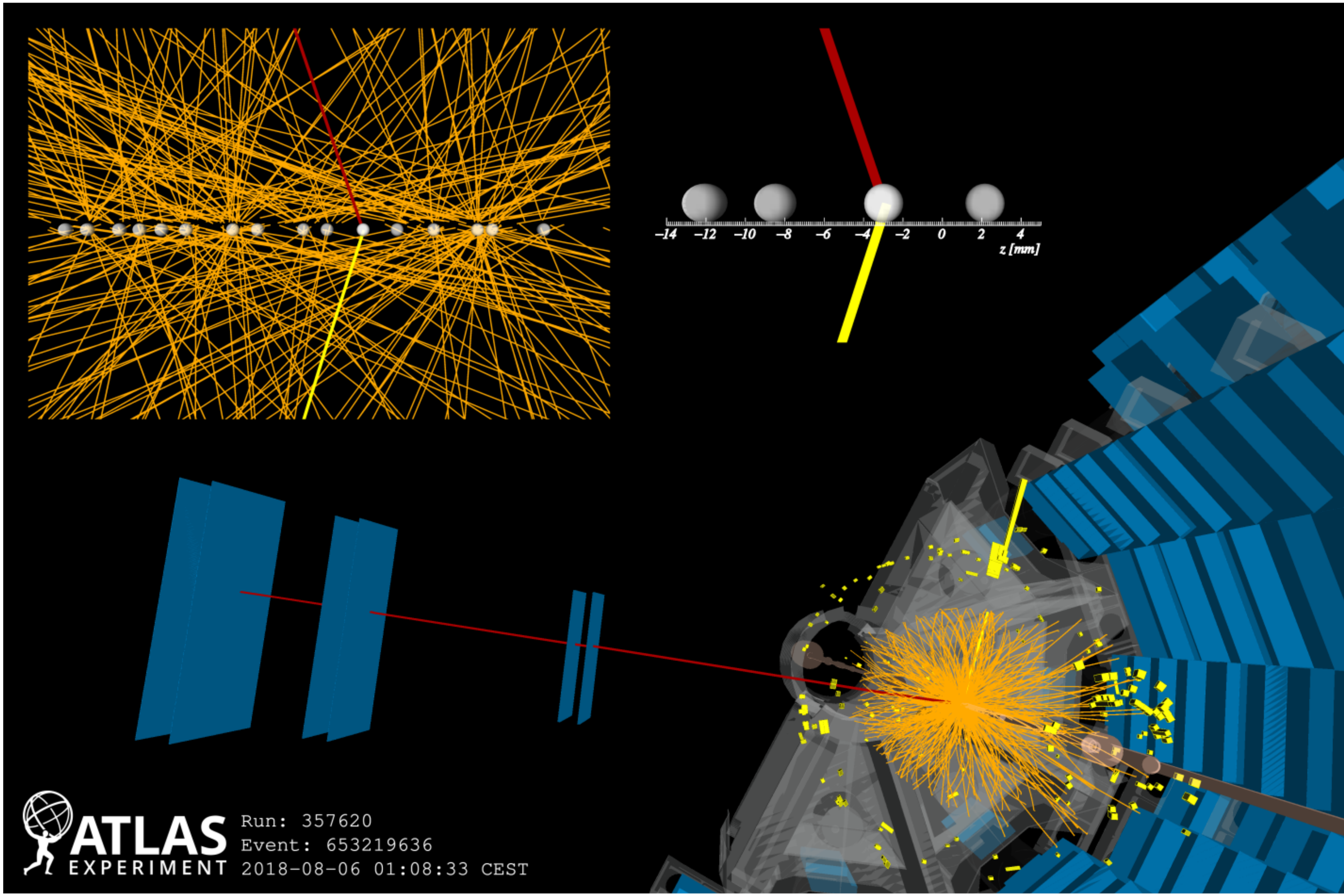
29 of 45 new preprint on arXiv hep-ph today is about new theory for g-2 anomaly



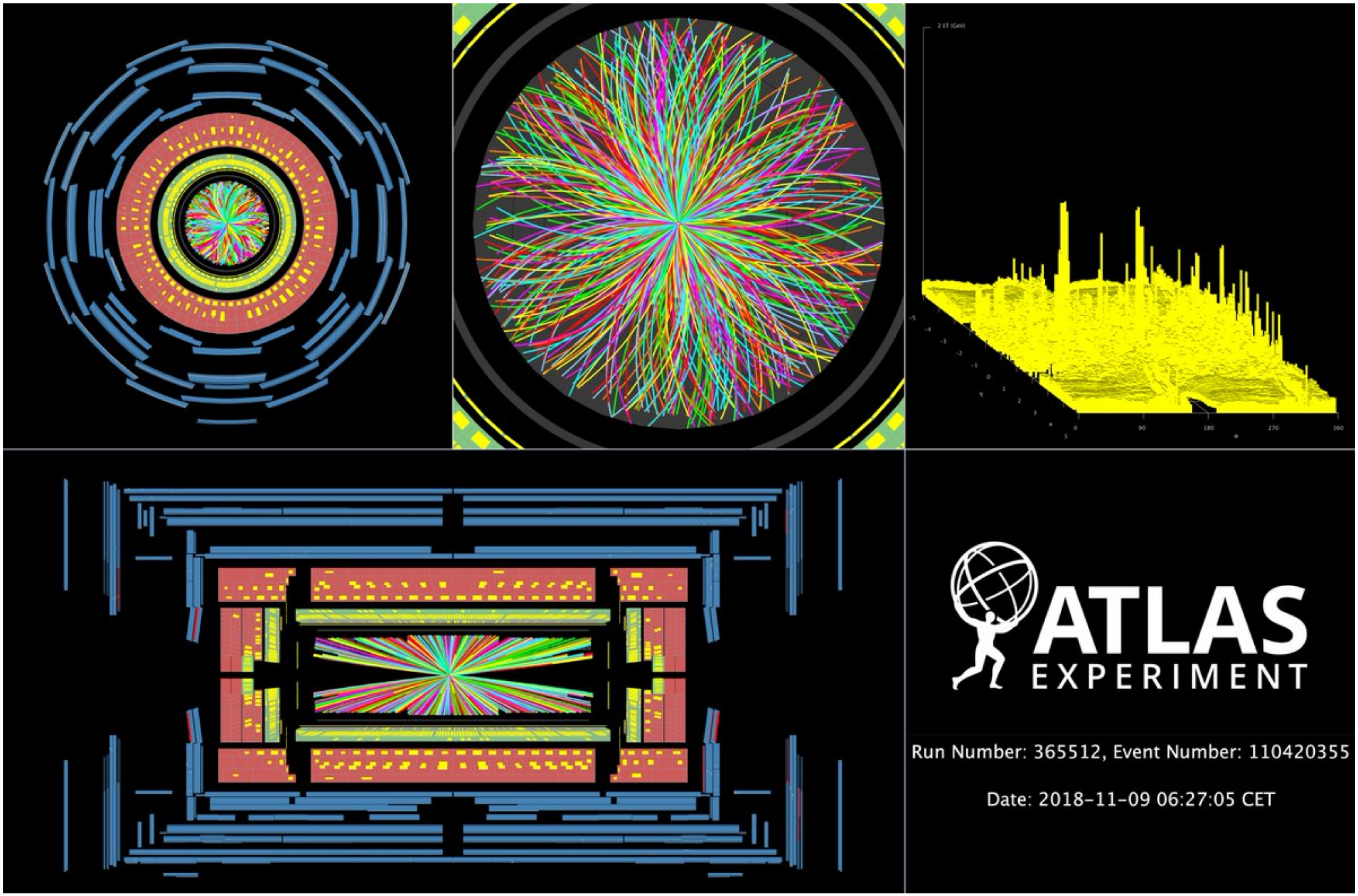


Phys. Rev. Lett. **126**, 141801

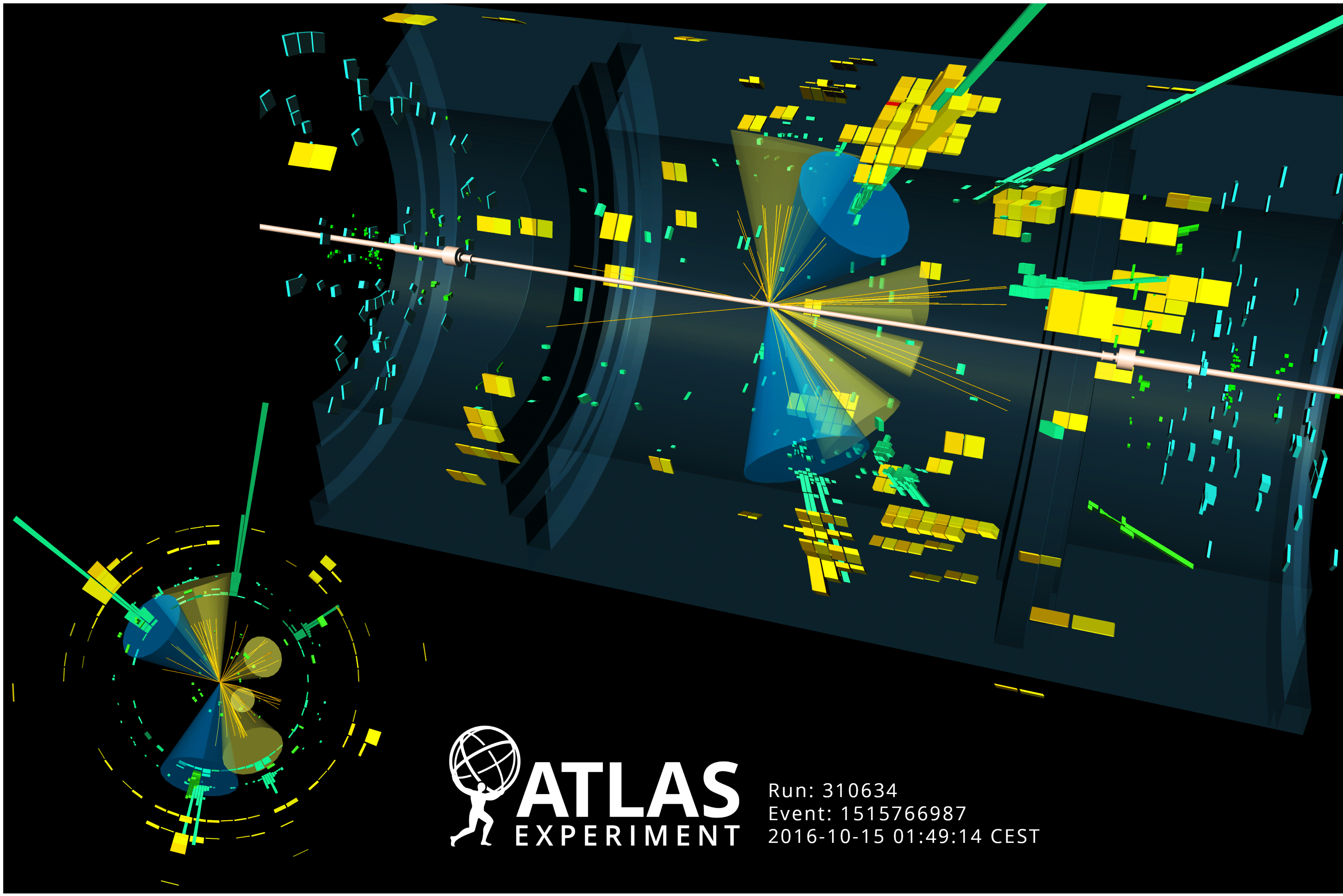
REAL SCATTERING EVENTS AT LHC



REAL SCATTERING EVENTS AT LHC



REAL SCATTERING EVENTS AT LAC



THEORETICAL TOOLS TO DESCRIBE/APPROXIMATE PARTICLE SCATTERING

QCD improved parton model

$$\hat{\sigma}_{ab} = \hat{\sigma}^{(0)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)\hat{\sigma}^{(1)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)^2\hat{\sigma}^{(2)}_{ab} + \dots$$

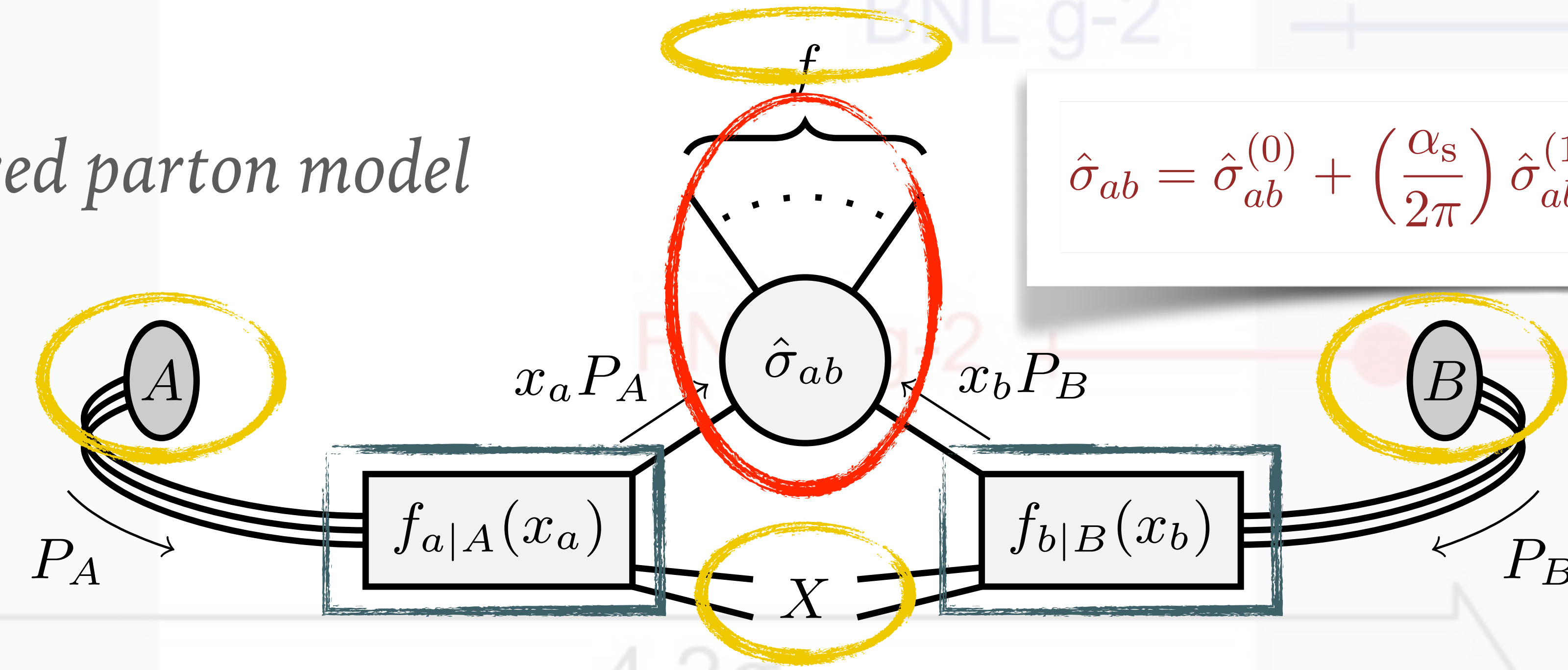


Figure by A. Huss

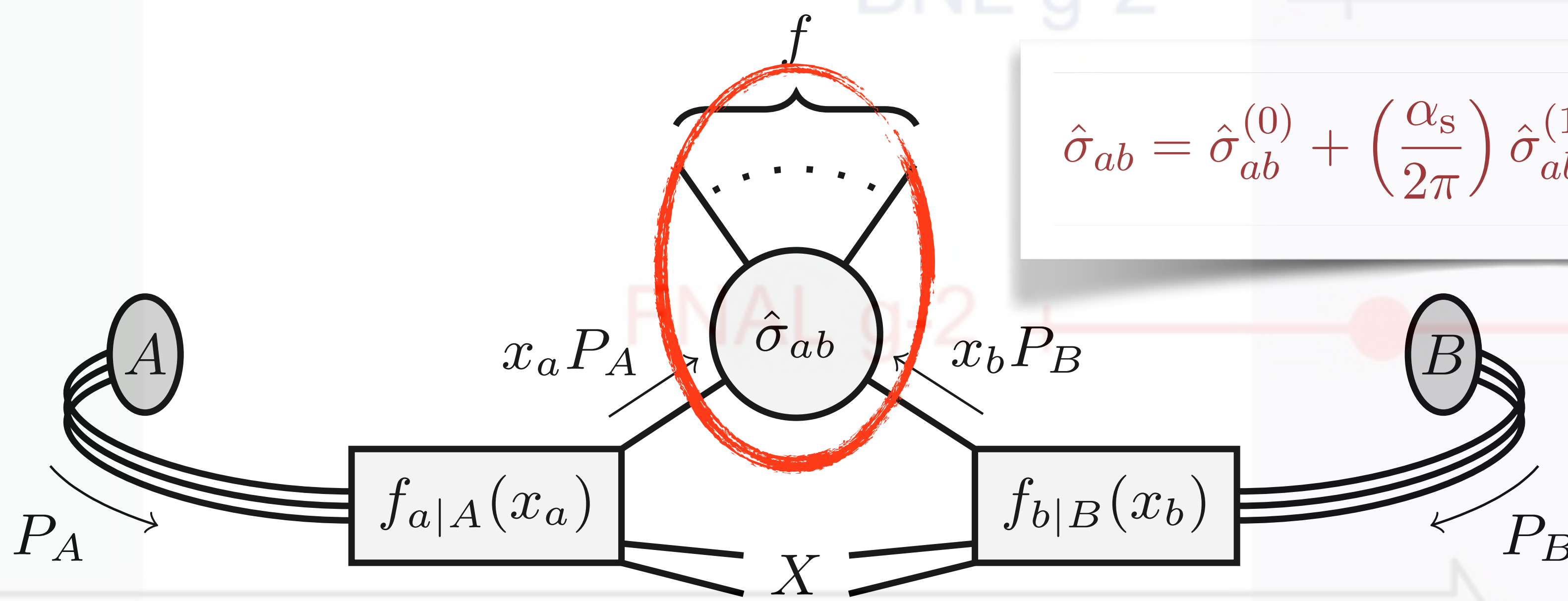
$$\sigma_{AB} = \sum_{ab} \int_0^1 \mathrm{d}x_a \int_0^1 \mathrm{d}x_b \; f_{a|A}(x_a)\; f_{b|B}(x_b)\; \hat{\sigma}_{ab}(x_a, x_b)\; (1+\mathcal{O}(\Lambda_{\mathrm{QCD}}/Q))$$

parton distribution functions
(systematically, improvable)
~5 % at the LHC

hard scattering
(systematically improvable)
~10 % level!

non-perturbative effects
(no good understanding)
~ few %?

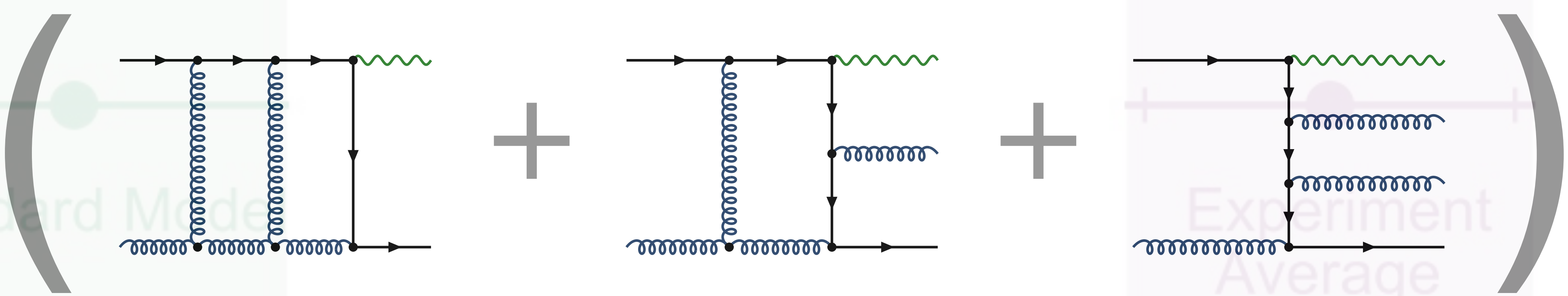
QCD improved Parton model



$$\hat{\sigma}_{ab} = \hat{\sigma}^{(0)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)\hat{\sigma}^{(1)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}^{(2)}_{ab} + \ldots$$

next-to-next-to-leading order (NNLO)

2 to 2 scattering



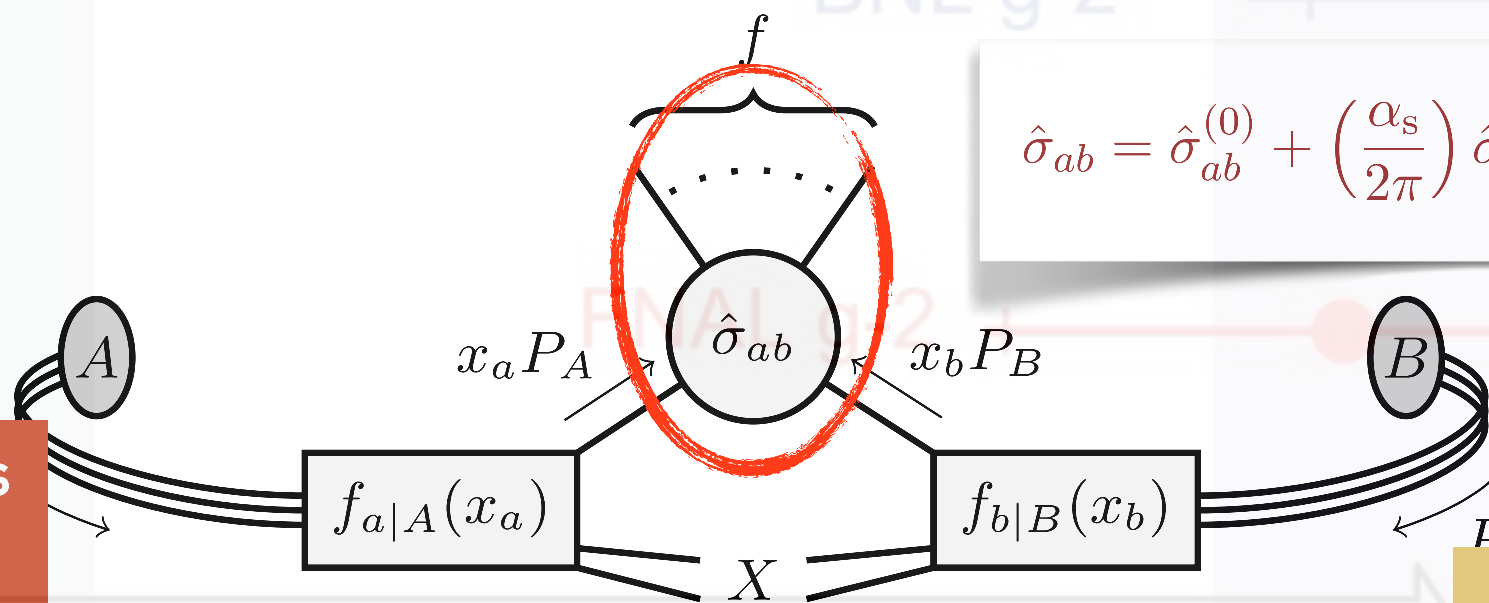
"double virtual" — *"real-virtual"* — *"double real"*

"double virtual"	"real-virtual"	"double real"
• $1/\varepsilon^4, 1/\varepsilon^3, 1/\varepsilon^2, 1/\varepsilon$	• $1/\varepsilon^2, 1/\varepsilon$	• double unresolved
	• single unresolved	• single unresolved

Slide from A. Huss in LH19

QCD improved Parton model

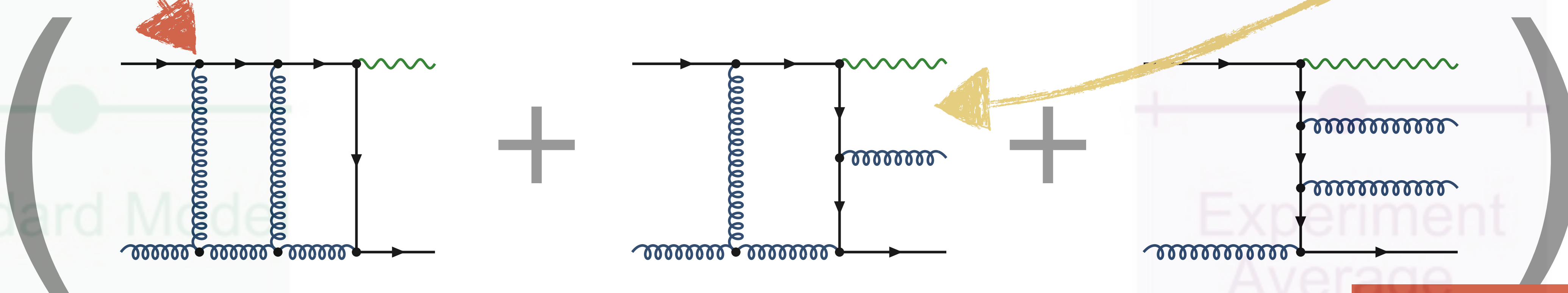
$$\hat{\sigma}_{ab} = \hat{\sigma}^{(0)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)\hat{\sigma}^{(1)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)^2\hat{\sigma}^{(2)}_{ab} + \ldots$$



two-loop amplitudes
(new class of functions, combinatoric & algebraic complexity)

one-loop amplitudes
(evaluation in singular & unstable regions)

next-to-next-to-leading order (NNLO)
2 to 2 scattering



"double virtual" *"real-virtual"* *"double re*

IR subtraction
(involved IR structure, numerical stability, construction)

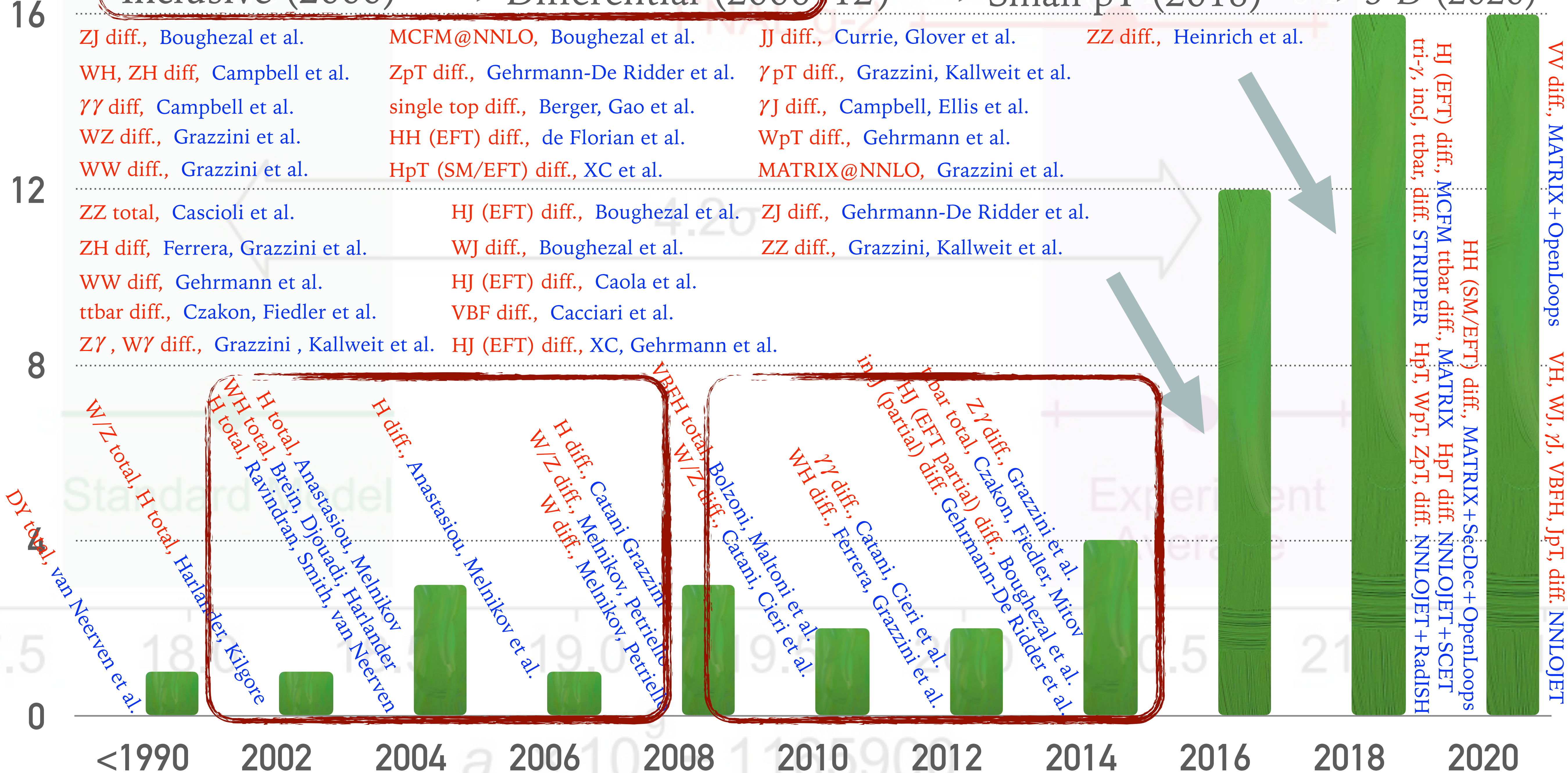
infrared singularities

Slide from A. Huss in LH19

6~8 years for a "small" step

THE STANDARD NNLO @ LHC

- LHC processes at NNLO QCD accuracy (include secondary confirmations)
  - One colourless (2000) ⟶ Two colourless (2004) ⟶ One or two Colourful (2014)
    Three colourless (2020)
  - Inclusive (2000) ⟶ Differential (2006-12) ⟶ Small pT (2018) ⟶ 3-D (2020)

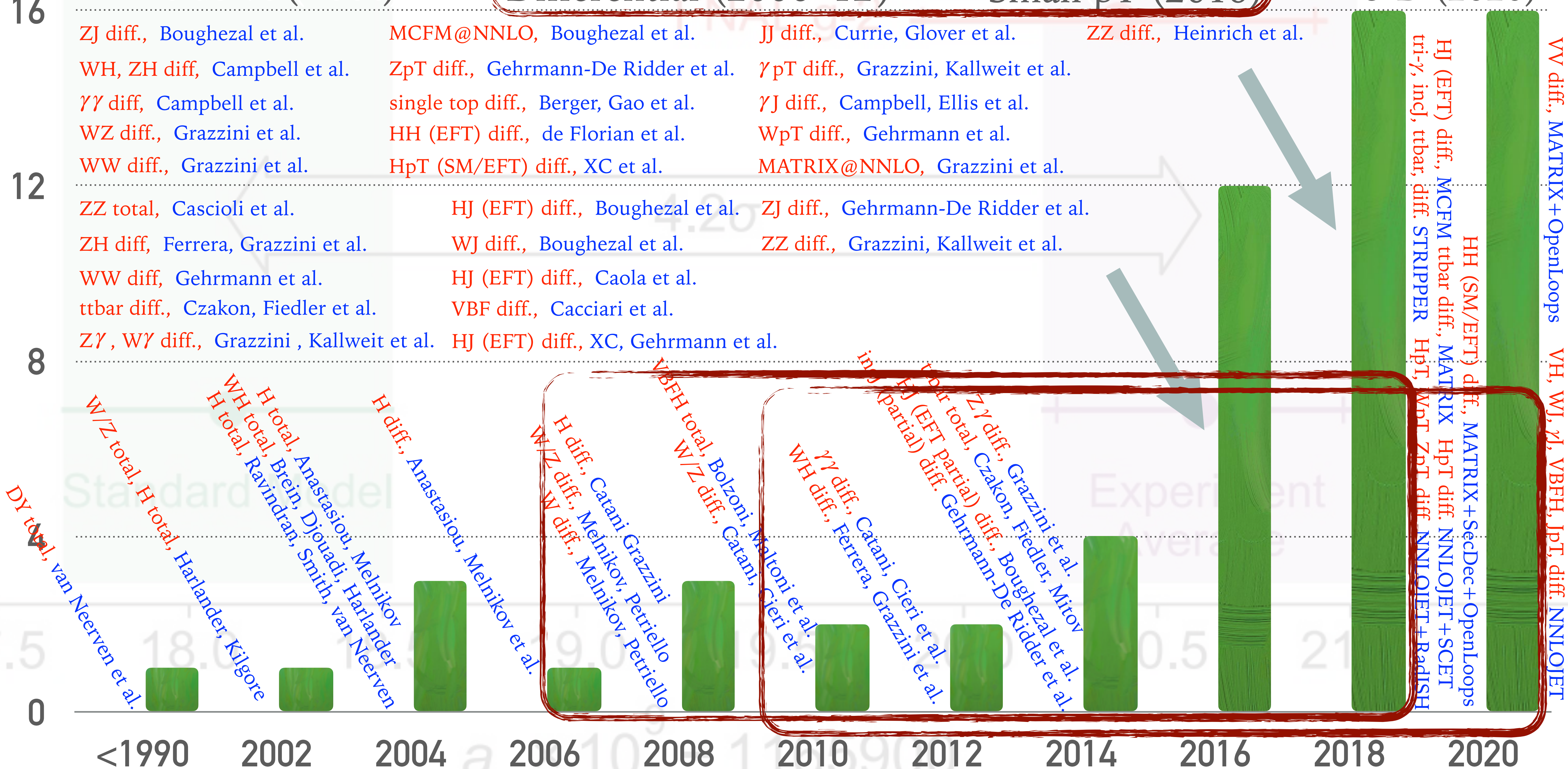


6~8 years for a “small” step

10~12 years for a “big” step

THE STANDARD NNLO @ LHC

- LHC processes at NNLO QCD accuracy (include secondary confirmations)
  - One colourless (2000) ⟶ Two colourless (2004) ⟶ One or two Colourful (2014) ⟶ Three colourless (2020)
  - Inclusive (2000) ⟶ Differential (2006 -12) ⟶ Small pT (2018) ⟶ 3-D (2020)

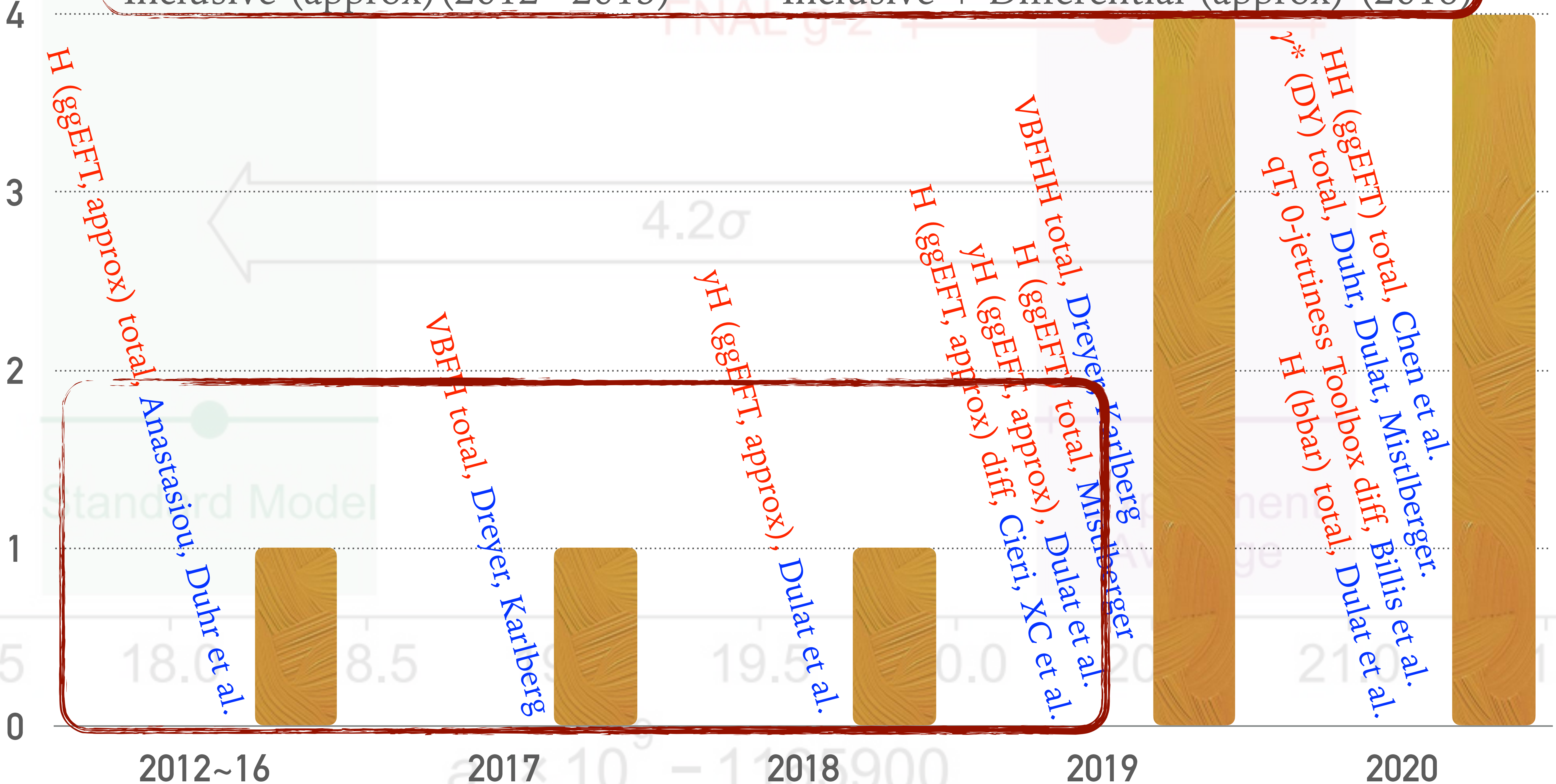


THE CUTTING-EDGE N3LO @ LHC

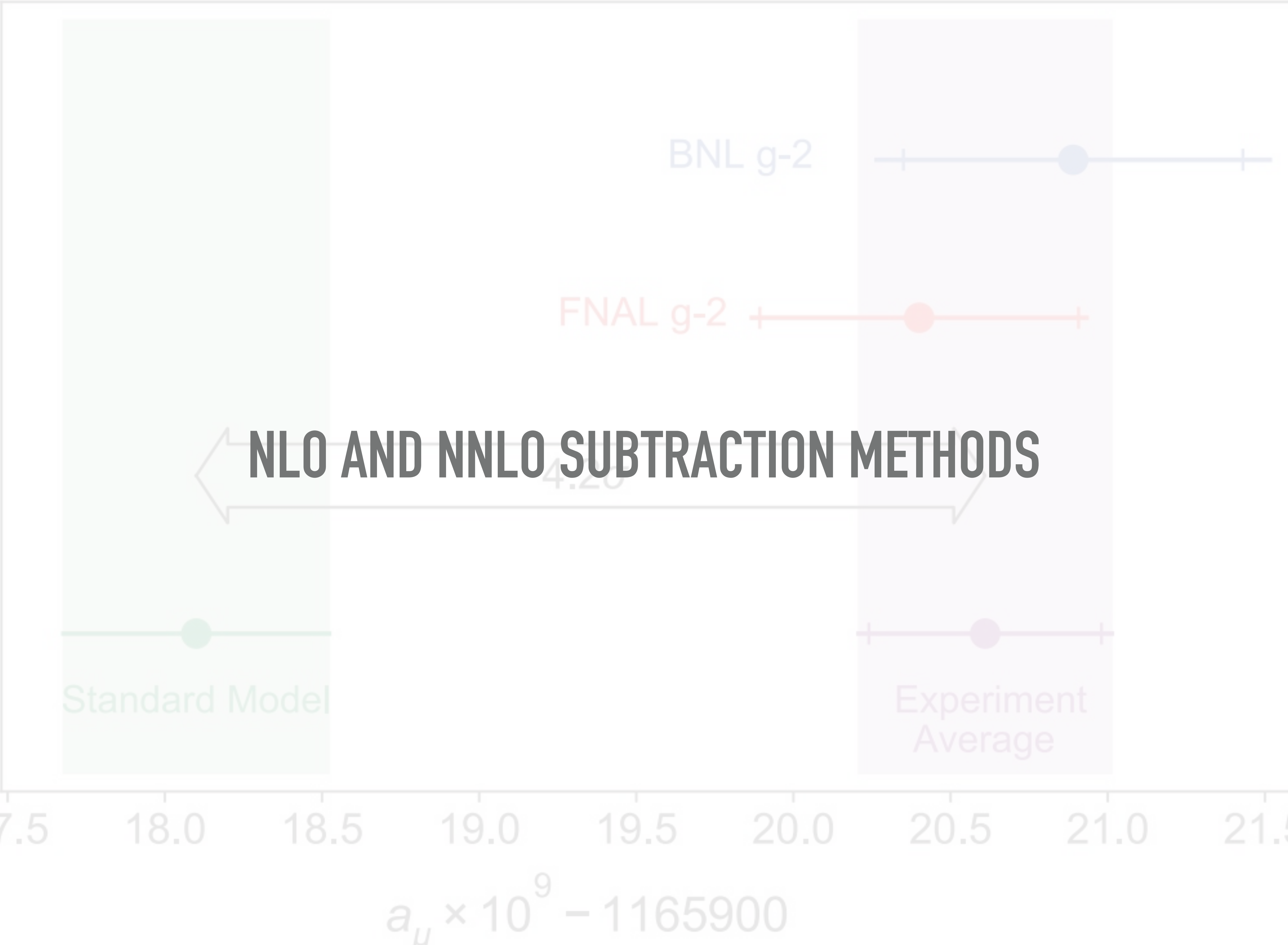
6 years for a "small" step

11 years for a "big" step

- LHC processes at N3LO accuracy (include secondary confirmation)
  - At the early stage like NNLO in 2000's (HNNLO @ 2004 to HN3LO @ 2015)
  - Inclusive (approx) (2012~2015) ⟶ Inclusive + Differential (approx) (2018)



NLO AND NNLO SUBTRACTION METHODS

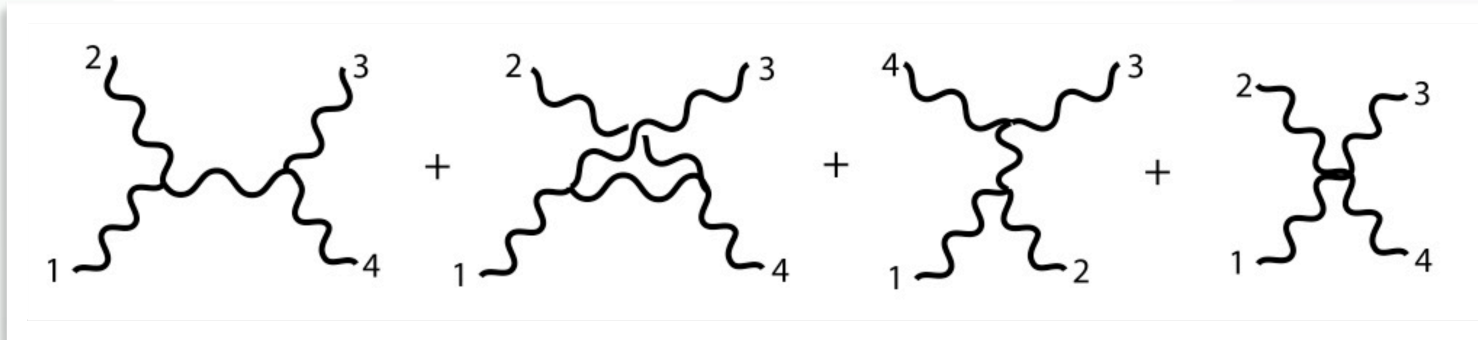


VARIOUS ESTABLISHED METHODS AND MORE ARE COMING

Slide from C. Williams at LoopFest 2019

BACKGROUND KNOWLEDGE: COLOUR-ORDERED AMPLITUDES

- **Basic concept from four gluon scattering:** L.Dixon hep-ph/9601359



- **Convert QCD couplings by SU(3) generators:**

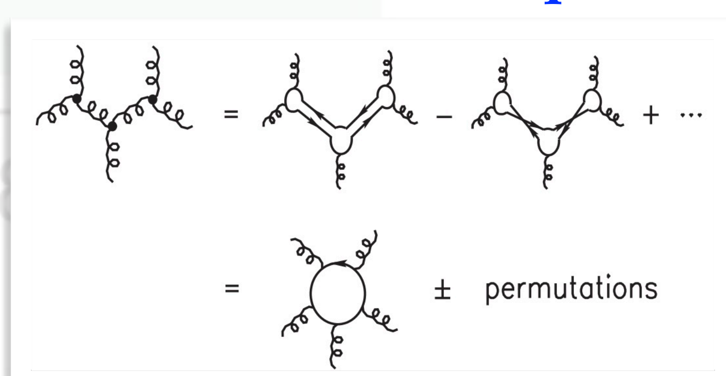
$$-gf^{abc} = \frac{ig}{\sqrt{2}}\mathrm{tr}[T^aT^bT^c - T^aT^cT^b]$$

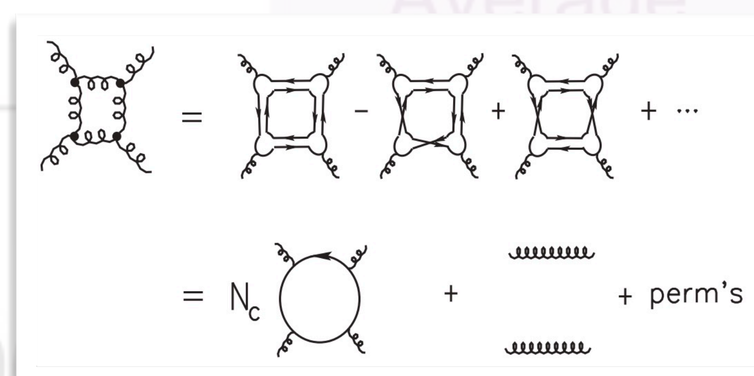
$$\begin{aligned} -ig^2 f^{abe} f^{cde} &= i\frac{g^2}{2}\mathrm{tr}([T^a,T^b][T^c,T^d]) \\ &= i\frac{g^2}{2}\mathrm{tr}(T^aT^bT^cT^d - T^aT^bT^dT^c - T^bT^aT^cT^d + T^bT^aT^dT^c) \end{aligned}$$

- **Group Feynman diagrams by trace of SU(3) generator:**

$$\begin{aligned} i\mathcal{M} = & \, i\mathbf{M}(1234)\cdot\mathrm{tr}[T^aT^bT^cT^d] + i\mathbf{M}(1243)\cdot\mathrm{tr}[T^aT^bT^dT^c] \\ & + i\mathbf{M}(1324)\cdot\mathrm{tr}[T^aT^cT^bT^d] + i\mathbf{M}(1342)\cdot\mathrm{tr}[T^aT^cT^dT^b] \\ & + i\mathbf{M}(1423)\cdot\mathrm{tr}[T^aT^dT^bT^c] + i\mathbf{M}(1432)\cdot\mathrm{tr}[T^aT^dT^cT^b] \end{aligned}$$

- **Generalise the idea for quarks and loop diagrams**





BACKGROUND KNOWLEDGE: DYNAMIC IR DIVERGENCES

- Modern tools for calculation: recursion relations like CSW, BCFW, CHY

$$\mathbf{M}(1_-2_-3_+4_+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \qquad \mathbf{M}(1_+\cdots i_-\cdots j_-\cdots n_+) = i\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

$s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j = \langle ij\rangle[ji]$ MHV amplitudes, S. J. Parke and T. R. Taylor (1986)

- Easy control of IR behaviour for both dynamic and explicit divergences: factorisation
- Dynamic IR divergence for tree matrix elements
  - Single soft gluon with momentum $p_j \to 0$:

$$|\mathcal{M}^0_{m+1}(\cdots,i,j,k,\cdots)|^2 \to s_{ijk}|\mathcal{M}^0_m(\cdots,i,k,\cdots)|^2 \quad \text{with Eikonal factor} \quad s_{ijk} = \frac{2s_{ik}}{s_{ij}s_{jk}}$$

  - Single collinear limit $p_j // p_k$ that $p_j = zp_{\tilde{K}}$, $p_k = (1-z)p_{\tilde{K}}$:

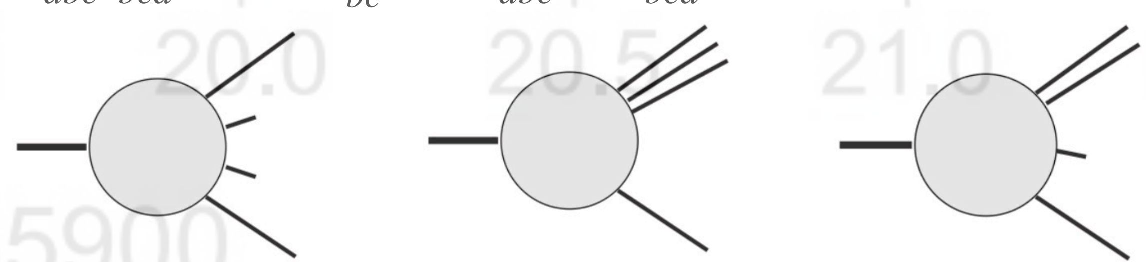
$$|\mathcal{M}^0_{m+1}(\cdots,i,j,k,l,\cdots)|^2 \to \frac{P_{jk\to\tilde{K}}}{s_{jk}}|\mathcal{M}^0_m(\cdots,i,\tilde{K},l,\cdots)|^2 \quad \text{with spin averaged splitting functions:}$$

$$P_{qg\to Q}(z) = \frac{1+(1-z)^2-\epsilon z^2}{z},\ P_{q\bar{q}\to G}(z) = \frac{z^2+(1-z)^2-\epsilon}{1-\epsilon},\ P_{gg\to G}(z) = \frac{2z}{1-z} + \frac{1-z}{z} + z(1-z)$$

  - Double soft gluon with momentum $p_b + p_c \to 0$:

$$s_{abcd} = \frac{2s_{ad}^2}{s_{ab}s_{bcd}s_{abc}s_{cd}} + \frac{2s_{ad}}{s_{bc}}\left(\frac{1}{s_{ab}s_{cd}} + \frac{1}{s_{ab}s_{dcd}} + \frac{1}{s_{cd}s_{abc}} - \frac{4}{s_{abc}s_{bcd}}\right) + \frac{2(1-\epsilon)}{s_{bc}^2}\left(\frac{s_{ab}}{s_{abc}} + \frac{s_{cd}}{s_{bcd}} - 1\right)^2$$

  - Various double unresolved limits:
    Double soft, triple collinear, soft and collinear:
    s_{abcd} $\quad s_{d,abc}$ $\quad P_{ijk\to\tilde{K}}$ $\quad \tilde{P}_{ijk\to\tilde{K}}$



BACKGROUND KNOWLEDGE: EXPLICIT IR DIVERGENCES

- Dynamic IR divergence for 1-loop matrix elements
  - Single soft gluon with momentum $p_j \to 0$:
    $$|\mathscr{M}^1_{m+1}(\cdots,i,j,k,\cdots)|^2 \to s^1_{ijk}|\mathscr{M}^0_m(\cdots,i,k,\cdots)|^2 + s_{ijk}|\mathscr{M}^1_m(\cdots,i,k,\cdots)|^2$$
  - Single collinear limit $p_j//p_k$ that $p_j = zp_{\tilde{K}}$, $p_k = (1-z)p_{\tilde{K}}$:
    $$|\mathscr{M}^1_{m+1}(\cdots,i,j,k,l,\cdots)|^2 \to \frac{P^1_{jk\to\tilde{K}}}{s_{jk}}|\mathscr{M}^0_m(\cdots,i,\tilde{K},l,\cdots)|^2 + \frac{P_{jk\to\tilde{K}}}{s_{jk}}|\mathscr{M}^1_m(\cdots,i,\tilde{K},l,\cdots)|^2$$
- Explicit IR divergence for 1-loop matrix elements S. Catani hep-ph/9802439

  $|\mathscr{M}^1_m(\cdots,i,j,k,\cdots)|^2 = 2\mathscr{Re}\langle\mathscr{M}_m|\mathscr{M}^1_m\rangle$ with 1-loop factorised as:

  $|\mathscr{M}^1_m\rangle = |\mathscr{M}^1_m(\mu^2;\{p\})\rangle = \mathbf{I}^{(1)}(\epsilon,\mu^2;\{p\})|\mathscr{M}_m(\{p\})\rangle + |\mathscr{M}^{1,fin}_m(\mu^2;\{p\})\rangle$

  The explicit 1-loop IR divergence is calculated in d-dimension, factorised in di-pole operator:
  $$\mathbf{I}^{(1)}(\epsilon,\mu^2;\{p\}) = \frac{e^{-\epsilon\psi(1)}}{2\Gamma(1-\epsilon)}\sum_{i\in\{p\}}\frac{\mathscr{V}_i(\epsilon)}{T_i^2}\sum_{j\neq i}T_i\cdot T_j\left(\frac{\mu^2 e^{-i\lambda_{ij}\pi}}{s_{ij}}\right)^{\epsilon} \quad \text{with } \mathscr{V}_i(\epsilon) = \frac{T_i^2}{\epsilon^2}+\frac{\gamma_i}{\epsilon}$$
- Explicit IR divergence for 2-loop matrix elements

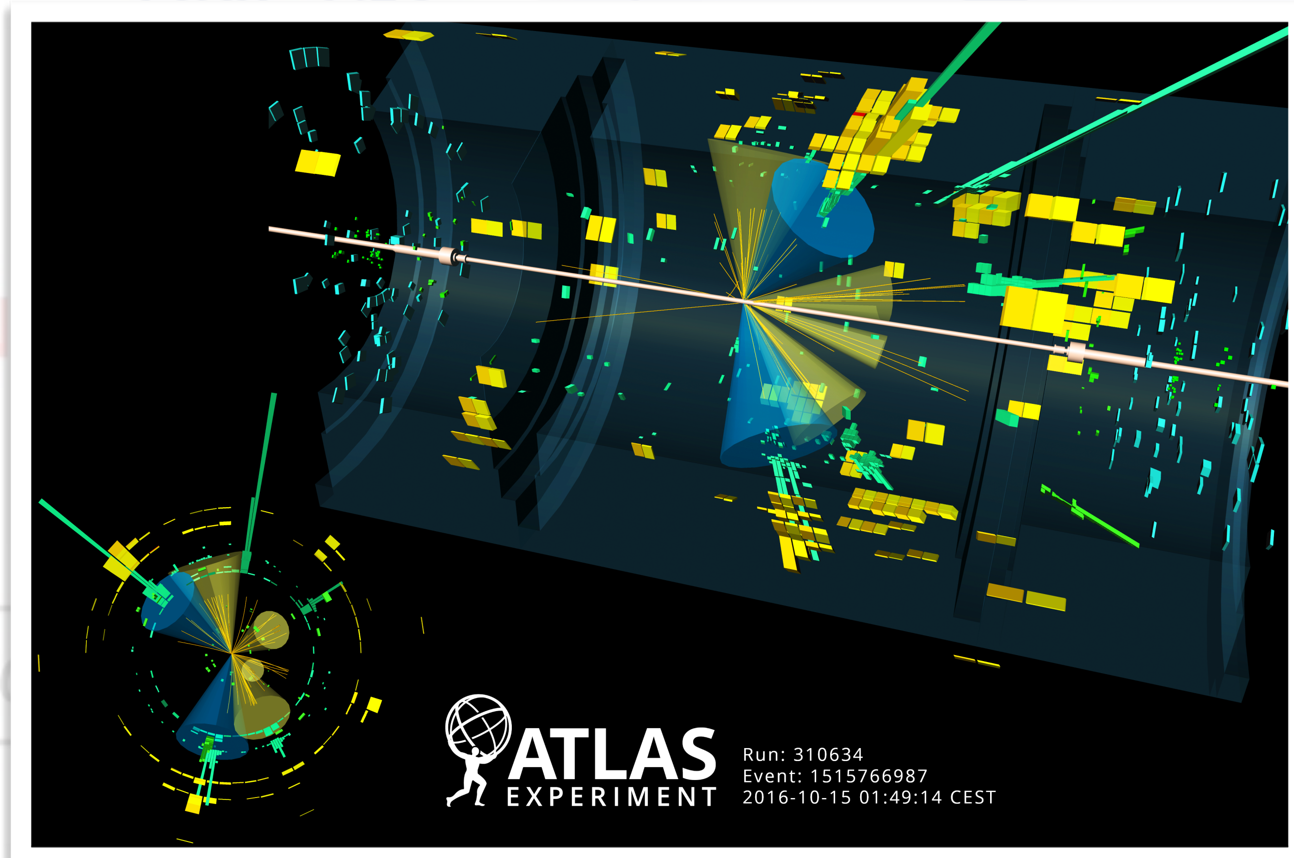
  $|\mathscr{M}^2_m(\cdots,i,j,k,\cdots)|^2 = 2\mathscr{Re}\langle\mathscr{M}_m|\mathscr{M}^2_m\rangle + \langle\mathscr{M}^1_m|\mathscr{M}^1_m\rangle$ with 2-loop factorised as:

  $|\mathscr{M}^2_m\rangle = |\mathscr{M}^2_m(\mu^2;\{p\})\rangle = \mathbf{I}^{(1)}(\epsilon,\mu^2;\{p\})|\mathscr{M}^1_m(\mu^2;\{p\})\rangle + \mathbf{I}^{(2)}(\epsilon,\mu^2;\{p\})|\mathscr{M}^0_m(\mu^2;\{p\})\rangle + |\mathscr{M}^{2,fin}_m(\mu^2;\{p\})\rangle$

  The new 2-loop di-pole operator is factorised as:
  $$\mathbf{I}^{(2)}(\epsilon,\mu^2;\{p\}) = -\frac{1}{2}\mathbf{I}^{(1)}(\epsilon,\mu^2;\{p\})\left(\mathbf{I}^{(1)}(\epsilon,\mu^2;\{p\})+\frac{4\pi\beta_0}{\epsilon}\right)+\frac{e^{+\epsilon\psi(1)}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{2\pi\beta_0}{\epsilon}+K\right)\mathbf{I}^{(1)}(2\epsilon,\mu^2;\{p\})+\mathbf{H}^{(2)}(\epsilon,\mu^2;\{p\})$$
- Kinoshita-Lee-Nauenberg theorem: all IR divergences cancel at each perturbation order of QFT (1964)

BACKGROUND KNOWLEDGE: JET ALGORITHM

- **How to describe complicated EXP events?**
  - Three shock waves in a snap shoot
  - Ten interaction pile up events
  - Tens to hundreds of particles hit detectors
  - Many displaced interactions
  - Soft, collinear, detector resolution
- **Theory limitations**
  - Each new particle carries 3 new D.O.F
    - Phase space integration is low
  - QFT is not perfect
    - IR divergence (this talk)
    - UV divergence (solved intrinsically)
    - Scale dependence (double edged)
- **Phenomenology is the bridge between EXP and THE**



- **How to predict complicated EXP events?**
  - Need an agreement between EXP and THE to abstract the core scattering process: use clustering algorithm to group final states particles from QCD radiations
  - Need to be simple: fast process by EXP
  - Need to be IR safe to avoid QFT limitation
  - Need to have sensitivity to new physics

SEQUENTIAL CLUSTERING ALGORITHM

- **Based on the following distance measures:**
  - Take all final state QCD particles in a **list**
  - Distance d_{ij} between two final state particles i and j:

$$d_{ij} = \min(k_{Ti}^{2p}, k_{Tj}^{2p})\frac{\Delta_{ij}^2}{R^2} \qquad \Delta_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2 \qquad k_T^2 = p_x^2 + p_y^2$$

  - Distance between initial beam (B) and final particle i:

$$d_{iB} = k_{Ti}^{2p}$$

- **Compute all distance d_{ij} and d_{iB} in the list, find the smallest**
  - If smallest is a d_{ij}, **combine** (sum four momenta) the two particles i and j, replace i and j in the list by the combined momentum as one particle
  - If smallest is a d_{iB}, **remove** particle i from the list, call it a **jet**
- **Repeat until all particles are clustered into jet (empty the list)**
- **Parameter R is called jet-cone size,** to control the distance between any pair of final state jets
- **Parameter p governs the relative power between energy and geometrical scales to distinguish the three algorithms:** 1 = kT, 0=C/A, -1=anti-kT
- **Beam direction is not IR safe and with low EXP sensitivity,** add fiducial cuts: $p_T^{\rm jet} > x$ GeV for any jet and usually with x=30 GeV, discard the event if fiducial cuts were not satisfied

NLO ANATOMY FOR 2 TO N PROCESS

- NLO cross section:

$$d\hat{\sigma}_{NLO} = \int_{d\Phi_{N+1}} \left(d\hat{\sigma}^{R}_{NLO} - d\hat{\sigma}^{S}_{NLO}\right) + \int_{d\Phi_N} \left(d\hat{\sigma}^{V}_{NLO} - d\hat{\sigma}^{T}_{NLO}\right)$$

$$0 = \int_{d\Phi_{N+1}} d\hat{\sigma}^{S}_{NLO} + \int_{d\Phi_N} d\hat{\sigma}^{T}_{NLO}$$

- $d\hat{\sigma}$ is the matrix elements at tree (R) or 1-loop (V) level
- Observables are based on at least N objects in

$$d\Phi_N = \delta\left(\sum_{i=1}^{N} p_i - \sqrt{s}\right)\prod_{i=1}^{N}\frac{d^4p_i}{(2\pi)^3}\delta(p_i^2 - m_i^2)\theta(E_i)$$

- Fulfilled by the requirement of at least N jets
  - Defined by jet algorithm with minimum p_T^{jet} requirement
  - Constrain $d\hat{\sigma}^{V}_{NLO}$ having only explicit IR divergence
  - Allow $d\hat{\sigma}^{R}_{NLO}$ becoming N or N+1 jets event
    N jets: with 1 particle being soft or collinear within one of the N jets
    N+1 jets: all particles are resolved without dynamic IR divergence
- Construct S and T subtraction terms to remove IR divergence

NNLO ANATOMY FOR 2 TO N PROCESS

- NNLO cross section:
  - Observables are still based on at least N objects in phase space Φ_N
  - Allow up to 2 emission at RR, 1 emission at RV and only explicit IR divergence for VV
  - Subtraction terms to remove IR divergence at each integration
  - No unphysical reminder:

$$d\hat{\sigma}_{NNLO} = \int_{d\Phi_{N+2}} \left(d\hat{\sigma}^{RR}_{NNLO} - d\hat{\sigma}^{S}_{NNLO}\right) + \int_{d\Phi_{N+1}} \left(d\hat{\sigma}^{RV}_{NNLO} - d\hat{\sigma}^{T}_{NNLO}\right) + \int_{d\Phi_{N}} \left(d\hat{\sigma}^{VV}_{NNLO} - d\hat{\sigma}^{U}_{NNLO}\right)$$

$$\int_{d\Phi_{N+2}} d\hat{\sigma}^{S}_{NNLO} + \int_{d\Phi_{N+1}} d\hat{\sigma}^{T}_{NNLO} + \int_{d\Phi_{N}} d\hat{\sigma}^{U}_{NNLO} = 0$$

- What is the perfect subtraction term?
  - $d\hat{\sigma}^{S}_{NNLO} = d\hat{\sigma}^{RR}_{NNLO}$ etc. that we need to integrate all D.O.F. analytically
  - Numerical integration? Computer can not handle ∞
  - Possible for QCD beta function (N3LO), DIS structure function (N3LO) (analytical integration on top of 1 or 2 D.O.F.)
  - Impossible for W+4 jets at NLO (integrate 3 of 20 D.O.F), ambiguity for parton identity

ANTENNA SUBTRACTION AT NLO

➤ Example for $pp \to H + Jet$

$$d\hat{\sigma}_{NLO} =$$
$$+ \int_{d\Phi_{H+2}} \left(d\hat{\sigma}^{R}_{NLO} - d\hat{\sigma}^{S}_{NLO}\right)$$
$$+ \int_{d\Phi_{H+1}} \left(d\hat{\sigma}^{V}_{NLO} - d\hat{\sigma}^{T}_{NLO}\right)$$

$$d\hat{\sigma}^{S}_{NLO} \sim X_3^0 d\hat{\sigma}^{B}_{LO}$$

$$d\hat{\sigma}^{T}_{NLO} = -\int_1 d\hat{\sigma}^{S}_{NLO}$$

- Subtraction terms constructed from Antenna functions
- Each antenna has two specified hard radiators + 1 unresolved patrons
- Momentum mappings $d\Phi_{H+2} \to d\Phi_{H+1}$ give the P.S. to reduced ME
- Integrated Antenna functions all known and contain explicit poles
- Explicit pole cancellation between integrated Antenna functions and loop calculations is analytical

➤ Now explain each parts in details

ANTENNA SUBTRACTION

- **Antenna Function X30:**

$$|\mathcal{M}^0_{m+1}(\cdots,i,j,k,\cdots)|^2 \xrightarrow{E_j\sim 0} s_{ijk}|\mathcal{M}^0_m(\cdots,i,k,\cdots)|^2$$

$$|\mathcal{M}^0_{m+1}(\cdots,i,j,k,l,\cdots)|^2 \xrightarrow{p_j//p_k} \frac{P_{jk\to\tilde{K}}}{s_{jk}}|\mathcal{M}^0_m(\cdots,i,\tilde{K},l,\cdots)|^2$$

$$|\mathcal{M}^0_3(i,j,k)|^2 = X^0_3(i,j,k)|\mathcal{M}^0_2(\widetilde{ij},\widetilde{jk})|^2$$

$$X^0_3(i,j,k) = \frac{|\mathcal{M}^0_3(i,j,k)|^2}{|\mathcal{M}^0_2(\tilde{I},\tilde{K})|^2}, \textbf{ with } |\mathcal{M}^0_2(\tilde{I},\tilde{K})|^2 \sim s = s_{ij}+s_{jk}+s_{ik}$$

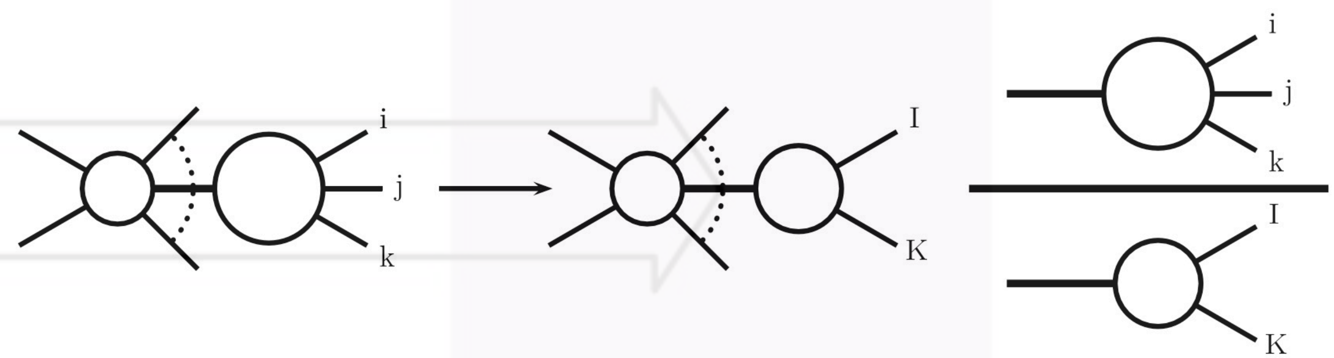
- **Generalise to X40 and X31:**

$$X^0_4(i,j,k,l)| = \frac{|\mathcal{M}^0_4(i,j,k,l)|^2}{|\mathcal{M}^0_2(\widetilde{ijk},\widetilde{jkl})|^2},\ X^1_3(i,j,k)| = \frac{|\mathcal{M}^1_3(i,j,k)|^2}{|\mathcal{M}^0_2(\widetilde{ij},\widetilde{jk})|^2} - X^0_3(i,j,k)\frac{|\mathcal{M}^1_2(\widetilde{ij},\widetilde{jk})|^2}{|\mathcal{M}^0_2(\widetilde{ij},\widetilde{jk})|^2}$$

- **No P.S. constrain on antenna functions with multiple divergence included**
- **Need to find complete set of all parton combinations**
- **Dynamics IR divergences are spin averaged from full matrix element**

ANTENNA SUBTRACTION

- **Momentum mapping**
  What is the momentum inside the reduced matrix elements? $p_{\widetilde{ij}}, \quad p_{\widetilde{ijk}}$
  Given that they belong to a set with less external parton than the full process
  - 3 to 2 final-final case: all momentum in final states $\{p_i, p_j, p_k\} \to \{p_{\widetilde{ij}}, p_{\widetilde{jk}}\}$
    - Momentum conservation and on-shell conditions:
      $p_I^2 = p_K^2 = 0, \quad p^\mu_{\widetilde{ij}} + p^\mu_{\widetilde{jk}} = p^\mu_i + p^\mu_j + p^\mu_k$
    - Initial parameterisation:
      $p^\mu_{\widetilde{ij}} = x p^\mu_i + r p^\mu_j + z p^\mu_k$
      $p^\mu_{\widetilde{jk}} = (1-x) p^\mu_i + (1-r) p^\mu_j + (1-z) p^\mu_k$



    - with three free parameters and two on-shell condition—one free choice:

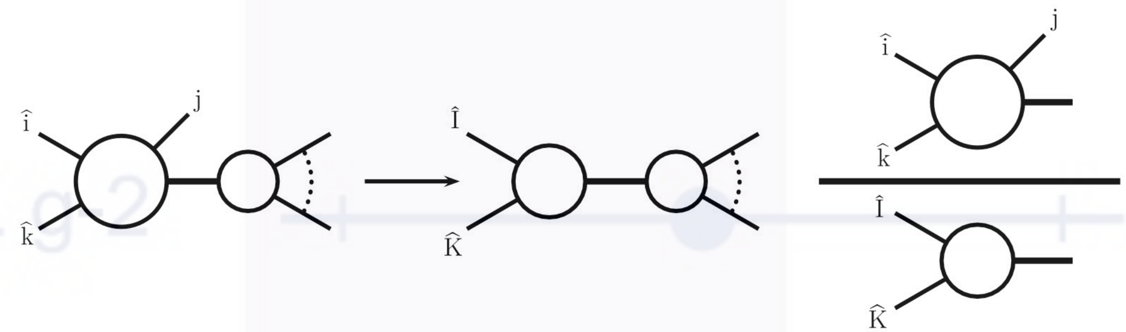
$$x = \frac{(1+\rho)s - 2rs_{jk}}{2(s_{ij}+s_{ik})}, \quad z = \frac{(1-\rho)s - 2rs_{ij}}{2(s_{jk}+s_{ik})}, \quad \rho^2 = 1 + \frac{4r(1-r)s_{ij}s_{jk}}{ss_{ik}}$$

$$\text{with} \quad r = \frac{s_{jk}}{s_{ij}+s_{jk}}, \quad s = s_{ij} + s_{jk} + s_{ki}$$

    - Check various IR limits of p_j:

$$\begin{aligned} p_{\widetilde{(ij)}} &\to p_i, & p_{\widetilde{(jk)}} &\to p_k & &\text{when } j \text{ is soft,} \\ p_{\widetilde{(ij)}} &\to p_i + p_j, & p_{\widetilde{(jk)}} &\to p_k & &\text{when } i \text{ becomes collinear with } j, \\ p_{\widetilde{(ij)}} &\to p_i, & p_{\widetilde{(jk)}} &\to p_j + p_k & &\text{when } j \text{ becomes collinear with } k. \end{aligned}$$

ANTENNA SUBTRACTION



- **Momentum mapping**
  - 3 to 2 inital-initial case: both hard radiator in initial state $\{p_{\hat{i}}, p_j, p_{\hat{k}}\} \rightarrow \{p_{\hat{I}}, p_{\hat{K}}\}$
    - How to absorb p_j (with $p_T \neq 0$) into initial states $p_{\hat{I}}$ and $p_{\hat{K}}$ (with $p_T = 0$)?

      Rescale initial state momentum: $p^\mu_{\hat{I}} = x_i p^\mu_{\hat{i}}, \quad p^\mu_{\hat{K}} = x_k p^\mu_{\hat{k}}$

      Apply Lorentz boost to all final states except p_j:

      $\Lambda^\mu_\nu(q, \tilde{q}) \quad$ with $\quad q^\mu = p^\mu_{\hat{i}} + p^\mu_{\hat{k}} - p^\mu_j, \quad \tilde{q}^\mu = x_i p^\mu_{\hat{i}} + x_k p^\mu_{\hat{k}}$
    - Such Lorentz boost is:

      $$\Lambda^\mu_\nu(q, \tilde{q}) = g^\mu_\nu - \frac{2(q+\tilde{q})^\mu (q+\tilde{q})_\nu}{(q+\tilde{q})^2} + \frac{2\tilde{q}^\mu q_\nu}{q^2}, \quad \tilde{p}^\mu_l = \Lambda^\mu_\nu(q, \tilde{q}) p_{l,\nu} \text{ for } l \neq j$$
    - By requiring $q^2 = \tilde{q}^2$ and $\Lambda^\mu_\nu(q, \tilde{q})$ is a transverse boost, we fix x_i and x_k:

      $$x_i = \sqrt{\frac{s_{ik}+s_{jk}}{s_{ik}+s_{ij}}}\sqrt{\frac{s_{ij}+s_{ik}+s_{jk}}{s_{ik}}}, \quad x_k = \sqrt{\frac{s_{ik}+s_{ij}}{s_{ik}+s_{jk}}}\sqrt{\frac{s_{ij}+s_{ik}+s_{jk}}{s_{ik}}}$$
    - Check in various IR limits of p_j:

      $$\begin{aligned} &\bar{p}_i \to p_i\,, && \bar{p}_k \to p_k && \text{when } j \text{ becomes soft}\,, \\ &\bar{p}_i \to (1-z_i)p_i\,, && \bar{p}_k \to p_k && \text{when } j \text{ becomes collinear with } \hat{i}\,, \\ &\bar{p}_k \to (1-z_k)p_k\,, && \bar{p}_i \to p_i && \text{when } j \text{ becomes collinear with } \hat{k}\,. \end{aligned}$$

ANTENNA SUBTRACTION

- **Integrated Antenna functions**
  - **P.S. factorisation (final-final example)**

$$d\Phi_{m+1}(p_3,\cdots,p_{m+3};p_1,p_2)=\delta(\sum_{j=3}^{m+3}p_j-p_1-p_2)\prod_{i=3}^{m+3}\frac{d^4p_i}{(2\pi)^3}\delta(p_i^2-m_i^2)\theta(E_i)$$

$$=d\Phi_m(p_3,\cdots,p_{\tilde{ij}},p_{\tilde{jk}},\cdots,p_{m+3};p_1,p_2)\cdot d\Phi_{X_{ijk}}(p_i,p_j,p_k;p_{\tilde{ij}}+p_{\tilde{kl}})$$

  - **Factorise with S subtraction term (example A30 for $d\hat{\sigma}^S_{NLO}$):**

$$A_3^0|\mathcal{M}_m^0|^2d\Phi_{m+1}(p_3,\cdots,p_{m+3};p_1,p_2)=A_3^0(i_q,j_g,k_{\bar{q}})d\Phi_{X_{ijk}}(p_i,p_j,p_k;p_{\tilde{ij}}+p_{\tilde{kl}})\cdot$$
$$|\mathcal{M}_m^0(1_g,2_g,\cdots,\tilde{ij}_q,\tilde{jk}_{\bar{q}},\cdots,m_g)|^2d\Phi_m(p_3,\cdots,p_{\tilde{ij}},p_{\tilde{jk}},\cdots,p_{m+3};p_1,p_2)$$

  - **Integrated antenna function $\mathcal{A}_3^0(s_{IK},\epsilon)$:**
    - Lorentz boost $A_3^0d\Phi_{X_{ijk}}$ to C.O.M of $p_{\tilde{ij}}+p_{\tilde{kl}}\rightarrow p_I+p_K$
    - Rewrite integral in d-dimension using $d^dp_i\delta(p_i^2)=E_i^{d-3}dE_id\Omega_i^{d-1}/2$
    - Lineup integral and integrand variables in s_{xy} with $\int d\Omega^d=2\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}$

$$d\Phi_{X_{ijk}}=\frac{(4\pi)^{\epsilon-2}}{\Gamma(1-\epsilon)}s^{2\epsilon-1}\int_0^s ds_{ik}\int_0^{s-s_{ik}}[s_{ik}s_{jk}(s-s_{ik}-s_{jk})]^{-\epsilon}ds_{jk}$$

ANTENNA SUBTRACTION

- Integrated antenna function $\mathcal{A}_3^0(s,\epsilon) = \int_j A_3^0 d\Phi_{X_{ijk}}$:

  Using the definition of A30:

$$A_3^0(i_q, j_g, k_{\bar{q}}) = \frac{|\mathcal{M}_3^0(i_q, j_g, k_{\bar{q}})|^2}{|\mathcal{M}_2^0(I,K)|^2} = A_3^0(s, s_{ik}, s_{jk}) =$$

$$\frac{1}{s}\left(\frac{s_{ik}}{s_{jk}} + \frac{s_{jk}}{s_{ik}} + \frac{2s(s - s_{ik} - s_{jk})}{s_{ik}s_{jk}}\right) - \frac{\epsilon}{s}\left(\frac{s_{ik}}{s_{jk}} + \frac{s_{jk}}{s_{ik}} + 2\right)$$

  Finally we could integrate all variables and get:

$$\mathcal{A}_3^0(s,\epsilon) = \frac{(4\pi^\epsilon)}{8\pi^2 e^{\epsilon\gamma_E} s^\epsilon}\left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{19}{4} - \frac{7}{12}\pi^2 + \epsilon\left(\frac{109}{8} - \frac{7}{8}\pi^2 - \frac{25}{3}\xi(3)\right) + \mathcal{O}(\epsilon^2)\right]$$

- Repeat initial-initial and final-initial P.S. integral to achieve a library of integrated antenna functions $J_2^{(1)}(s_{ij},\epsilon)$

- Cross check explicit IR divergence between $J_2^{(1)}(ij,\epsilon)$ and 1-loop ME:

$$pole\{J_2^{(1)}(s_{ij}/\mu^2,\epsilon)\} = -2I^{(1)}(\epsilon,\mu^2;s_{ij})$$

- Di-pole cancel with integrated antenna to have $d\hat{\sigma}^V_{NLO} - d\hat{\sigma}^T_{NLO}$ finite

ANTENNA SUBTRACTION

- **NLO H+J example subtraction terms**

$$d\hat{\sigma}_{NLO} = + \int_{d\Phi_{H+2}} \left(d\hat{\sigma}^{R}_{NLO} - d\hat{\sigma}^{S}_{NLO}\right) + \int_{d\Phi_{H+1}} \left(d\hat{\sigma}^{V}_{NLO} - d\hat{\sigma}^{T}_{NLO}\right)$$

$$d\hat{\sigma}^{S}_{NLO} \sim X^0_3 d\hat{\sigma}^{B}_{LO}$$

$$d\hat{\sigma}^{T}_{NLO} = -\int_1 d\hat{\sigma}^{S}_{NLO}$$

- **For $q\bar{q} \to ggH$ sub process from real,**

$$\hat{\sigma}^R_{NLO} = |\mathcal{M}^0_4(\hat{1}_q, 3_g, 4_g, \hat{2}_{\bar{q}})|^2$$

$$\hat{\sigma}^S_{NLO} = + D^0_3(\hat{1}_q, 3_g, 4_g)\,|\mathcal{M}^0_4(\hat{\tilde{1}}_q, \widetilde{34}_g, \hat{2}_{\bar{q}})|^2 + D^0_3(\hat{2}_q, 4_g, 3_g)\,|\mathcal{M}^0_4(\hat{1}_q, \widetilde{34}_g, \hat{\tilde{2}}_{\bar{q}})|^2$$

- **The corresponding virtual is**

$$\hat{\sigma}^V_{NLO} = |\mathcal{M}^1_3(\hat{1}_q, 3_g, \hat{2}_{\bar{q}})|^2$$

$$\hat{\sigma}^T_{NLO} = + \mathcal{D}^0_3(s_{\hat{1}3}/\mu^2, \epsilon)\,|\mathcal{M}^0_3(\hat{\tilde{1}}_q, 3_g, \hat{2}_{\bar{q}})|^2 + \mathcal{D}^0_3(s_{\hat{2}3}/\mu^2, \epsilon)\,|\mathcal{M}^0_3(\hat{1}_q, 3_g, \hat{\tilde{2}}_{\bar{q}})|^2$$

- **Repeat real for** $gg \to ggH, \quad gg \to q\bar{q}H, \quad q\bar{q} \to Q\bar{Q}H, \quad qg \to qgH$
- **Repeat virtual for** $gg \to gH, \quad qg \to qH$

(TAKE A BREAK)

SORRY THAT I DON’T HAVE TIME, SO I WRITE YOU A LONG LETTER

ANTENNA SUBTRACTION AT NNLO

- Example for $pp \to H + Jet$

$$d\hat{\sigma}_{NNLO} = \int_{d\Phi_{H+3}} (d\hat{\sigma}^{RR}_{NNLO} - d\hat{\sigma}^{S}_{NNLO}) + \int_{d\Phi_{H+2}} (d\hat{\sigma}^{RV}_{NNLO} - d\hat{\sigma}^{T}_{NNLO}) + \int_{d\Phi_{H+1}} (d\hat{\sigma}^{VV}_{NNLO} - d\hat{\sigma}^{U}_{NNLO})$$

$$d\hat{\sigma}^{S}_{NNLO} \sim X_3^0 |\mathcal{M}^0_{H+4}|^2 + X_4^0 |\mathcal{M}^0_{H+3}|^2$$

$$d\hat{\sigma}^{T}_{NNLO} \sim X_3^0 |\mathcal{M}^1_{H+3}|^2 + X_3^1 |\mathcal{M}^0_{H+3}|^2$$

$$d\hat{\sigma}^{U}_{NNLO} = -\int_2 d\hat{\sigma}^{S}_{NNLO} - \int_1 d\hat{\sigma}^{T}_{NNLO}$$

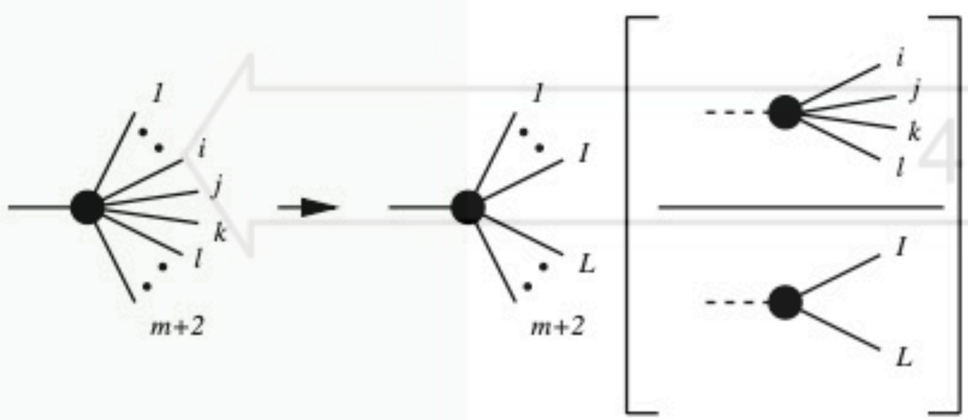
- Subtraction terms constructed from antenna functions
- Each antenna has two specified hard radiators + 1 or 2 unresolved patrons
- Momentum mappings
  $d\Phi_{H+3} \to d\Phi_{H+1(2)}$
  $d\Phi_{H+2} \to d\Phi_{H+1}$
  give the P.S. to reduced ME
- Integrated antenna functions all known and contain explicit poles
- Explicit pole cancellation between integrated antenna functions and loop calculations is analytical

- Now explain each parts in details

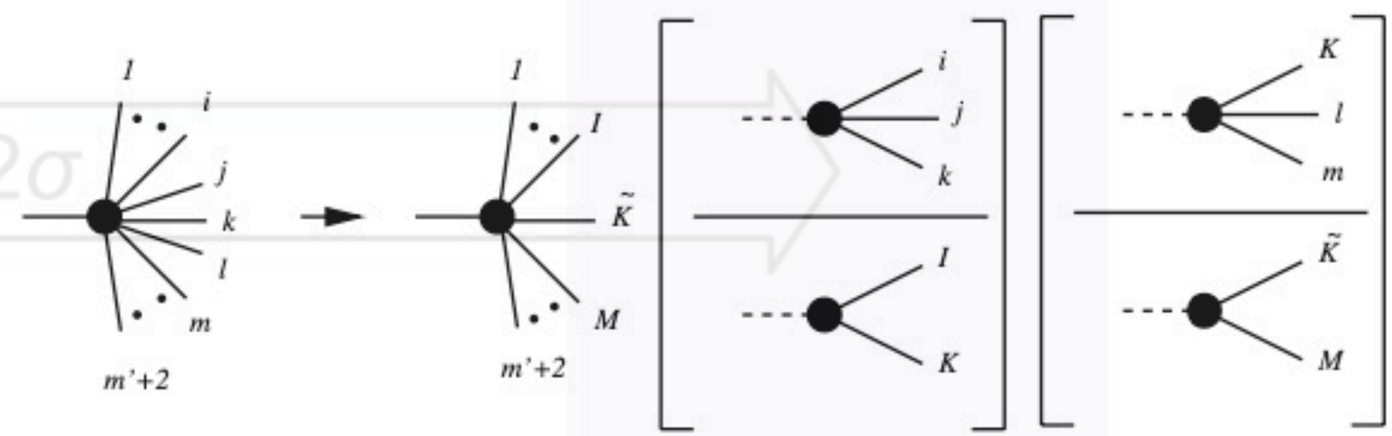
ANTENNA SUBTRACTION AT NNLO (RR LEVEL)

$$d\hat{\sigma}^{S}_{NNLO} = \boxed{d\hat{\sigma}^{S,a}} + \boxed{d\hat{\sigma}^{S,b_1}} + \boxed{d\hat{\sigma}^{S,b_2}} + \boxed{d\hat{\sigma}^{S,c}} + \boxed{d\hat{\sigma}^{S,d}}$$

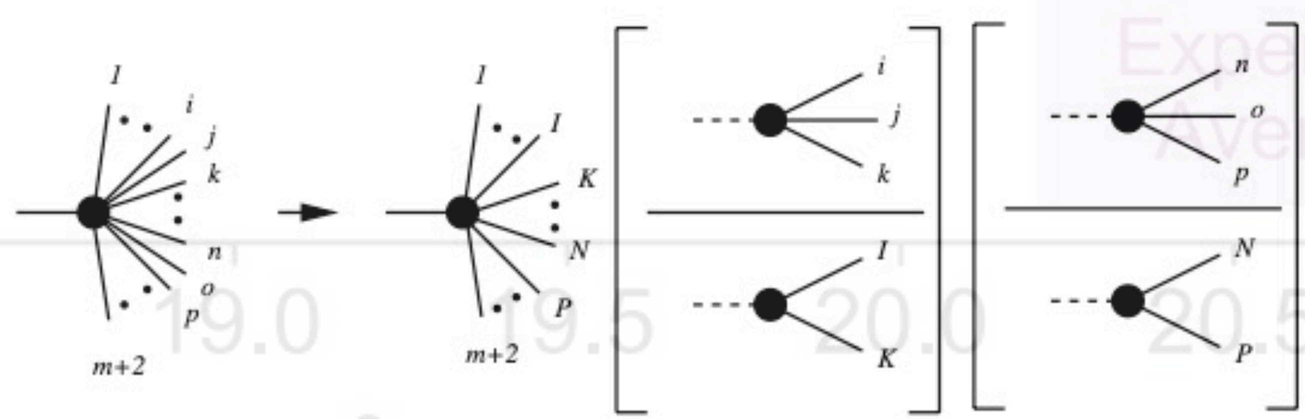
- First to remove single unresolved limits (NLO structure)
- Three possible colour ordering of double unresolved particles



(a) colour connected $X_4^0 \otimes |\mathcal{M}_n^0|^2$



(b) colour almost connected $X_3^0 \otimes X_3^0 \otimes |\mathcal{M}_n^0|^2$



(c) colour not connected $X_3^0 \otimes X_3^0 \otimes |\mathcal{M}_n^0|^2$

ANTENNA SUBTRACTION AT NNLO (RR LEVEL)

$$d\hat{\sigma}^{S}_{NNLO} = \boxed{d\hat{\sigma}^{S,a}} + \boxed{d\hat{\sigma}^{S,b_1}} + \boxed{d\hat{\sigma}^{S,b_2}} + \boxed{d\hat{\sigma}^{S,c}} + \boxed{d\hat{\sigma}^{S,d}}$$

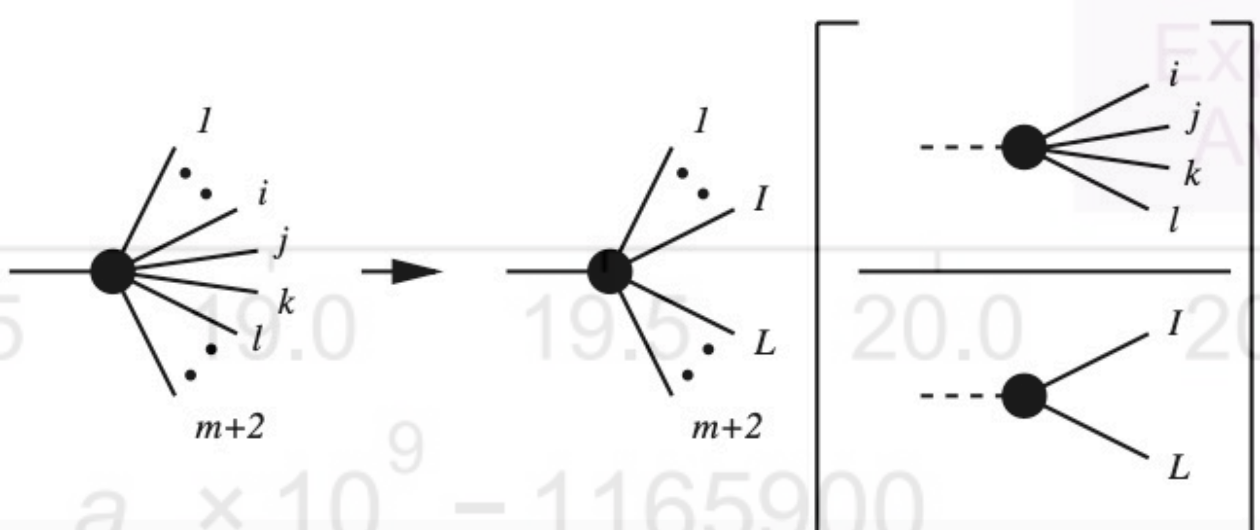
$$\boxed{d\hat{\sigma}^{S,a}} \sim X_3^0 |\mathcal{M}^0_{n+1}|^2$$

$$\boxed{d\hat{\sigma}^{S,b_1}} \sim X_4^0 |\mathcal{M}^0_{n}|^2$$

$$\boxed{d\hat{\sigma}^{S,b_2}} \; \boxed{d\hat{\sigma}^{S,d}} \sim X_3^0 X_3^0 |\mathcal{M}^0_{n}|^2$$

$$\boxed{d\hat{\sigma}^{S,c}} \sim X_3^0 X_3^0 |\mathcal{M}^0_{n}|^2 + \frac{s_{ik}}{s_{ij}s_{jk}} X_3^0 |\mathcal{M}^0_{n}|^2$$

$$|\mathcal{M}^0_{n+2}(\cdots, i, j, k, l, \cdots)|^2 - X_4^0(i,j,k,l)|\mathcal{M}^0_{n}(\cdots, I, L, \cdots)|^2$$



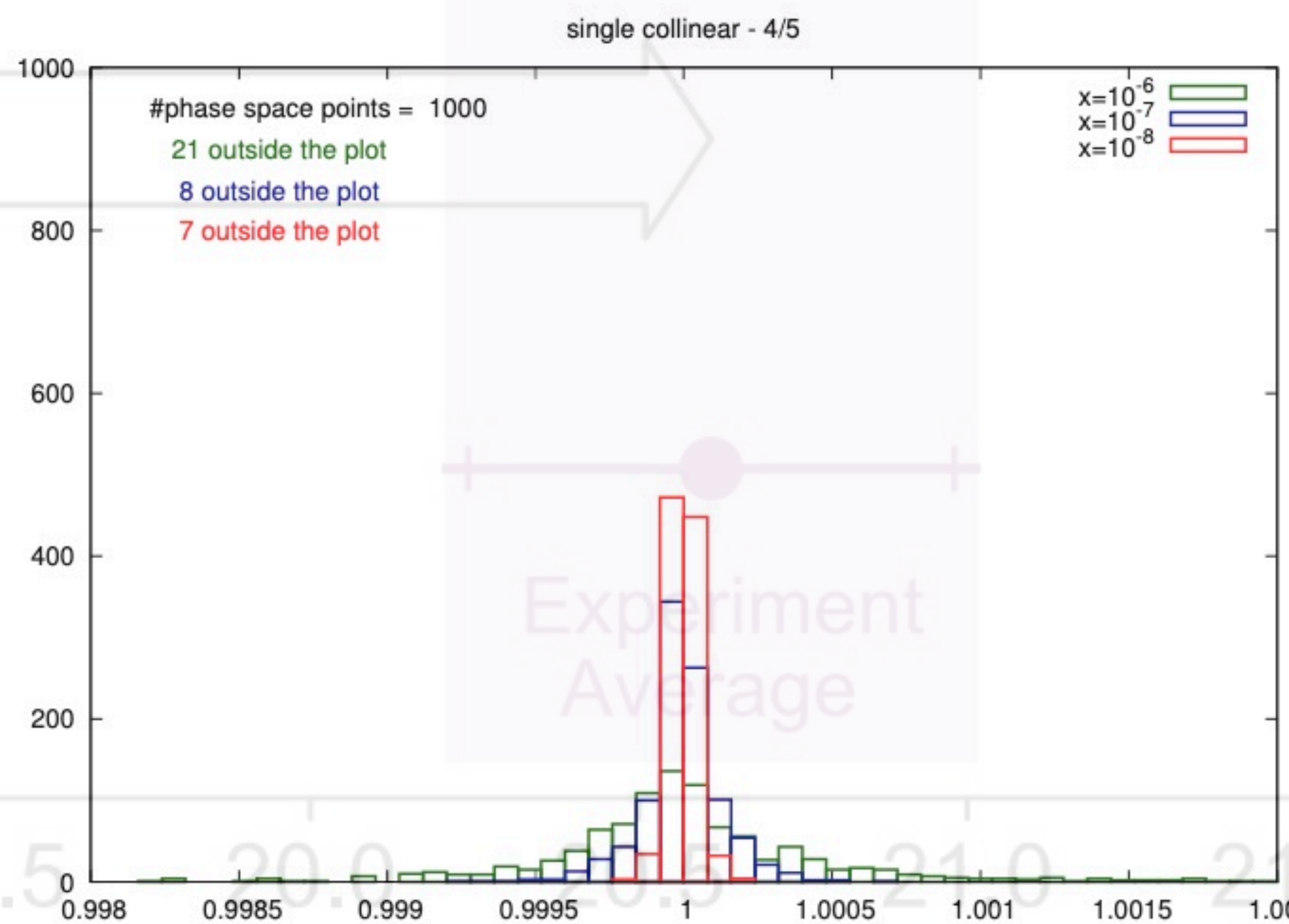
ANTENNA SUBTRACTION AT NNLO (RR LEVEL)

$$d\hat{\sigma}^{S}_{NNLO} = \boxed{d\hat{\sigma}^{S,a}} + \boxed{d\hat{\sigma}^{S,b_1}} + \boxed{d\hat{\sigma}^{S,b_2}} + \boxed{d\hat{\sigma}^{S,c}} + \boxed{d\hat{\sigma}^{S,d}}$$

- Test structure

$$R = \frac{d\hat{\sigma}^{RR}_{NNLO}}{d\hat{\sigma}^{S}_{NNLO}}$$

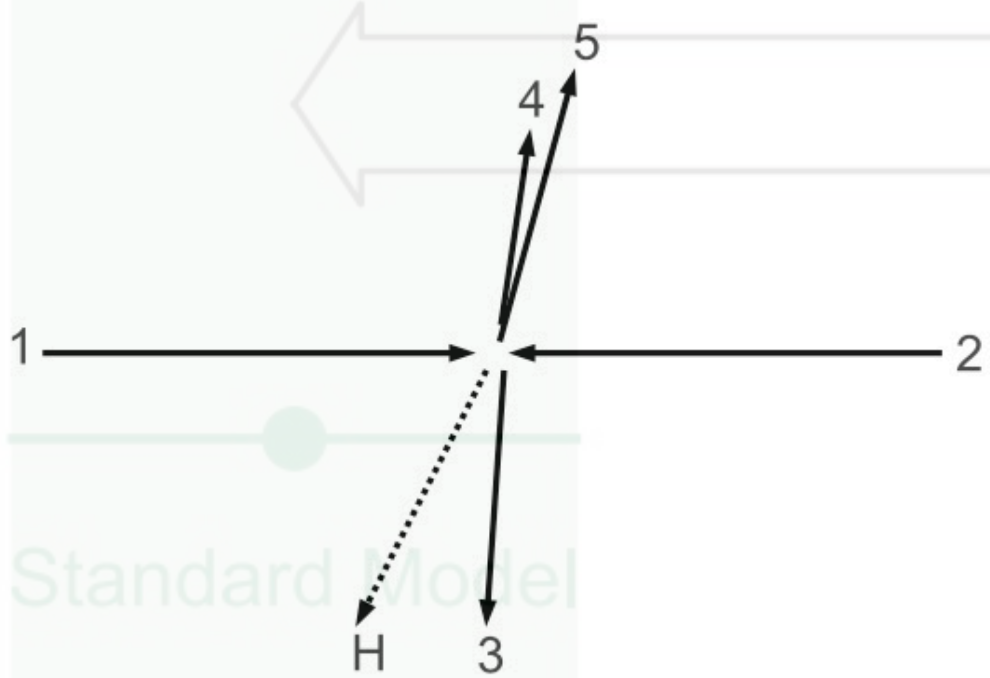
- R ~horizontal axis (centre at one near the unresolved region)
- Number of P.S. points in each bin $\sim$ vertical axis
- Control singular region to achieve spike plot

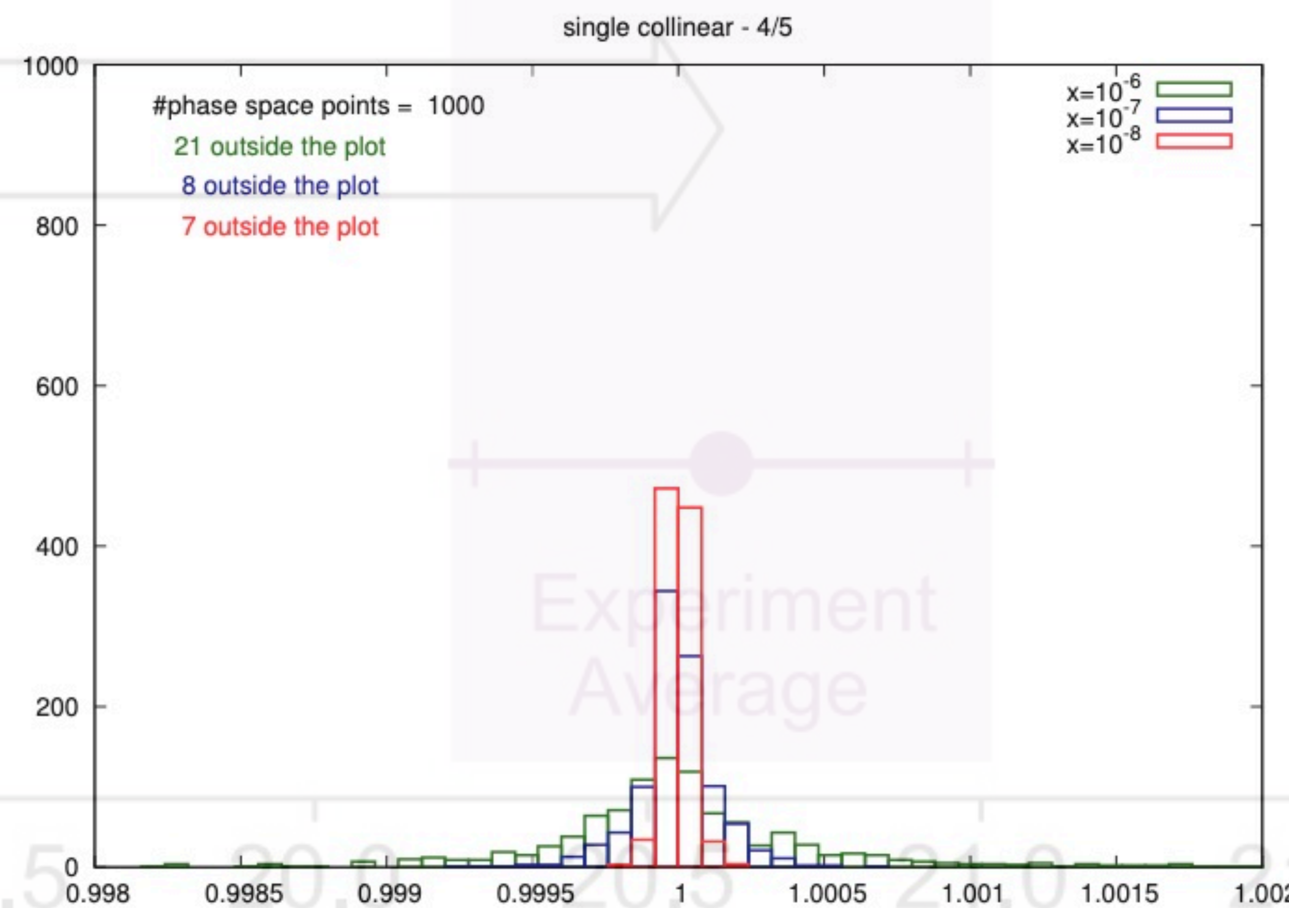


ANTENNA SUBTRACTION AT NNLO (RR LEVEL)

- Single collinear limit

$$d\hat{\sigma}^{S}_{NNLO} = \boxed{d\hat{\sigma}^{S,a}} + \boxed{d\hat{\sigma}^{S,b_1}} + \boxed{d\hat{\sigma}^{S,b_2}} + \boxed{d\hat{\sigma}^{S,c}} + \boxed{d\hat{\sigma}^{S,d}}$$



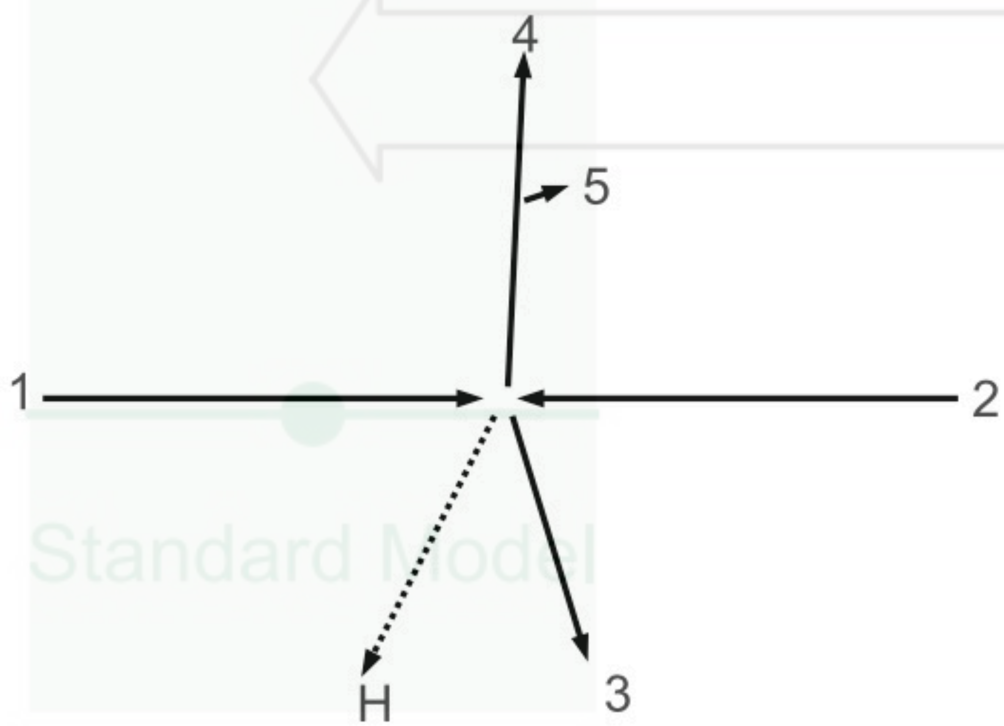


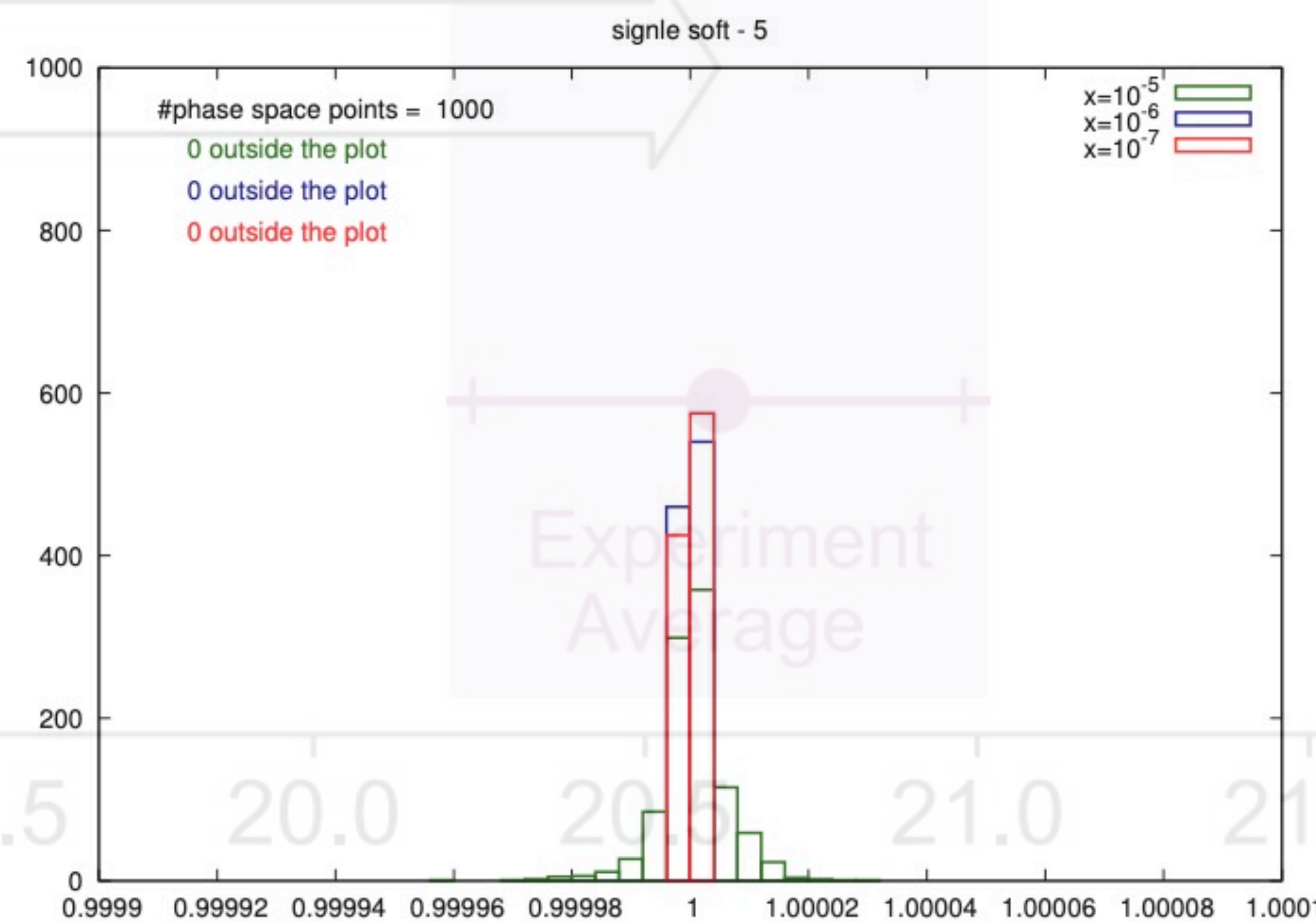
$$x = \frac{s_{45}}{s}$$

ANTENNA SUBTRACTION AT NNLO (RR LEVEL)

- Single soft limit

$$d\hat{\sigma}^{S}_{NNLO} = \boxed{d\hat{\sigma}^{S,a}} + \boxed{d\hat{\sigma}^{S,b_1}} + \boxed{d\hat{\sigma}^{S,b_2}} + \boxed{d\hat{\sigma}^{S,c}} + \boxed{d\hat{\sigma}^{S,d}}$$



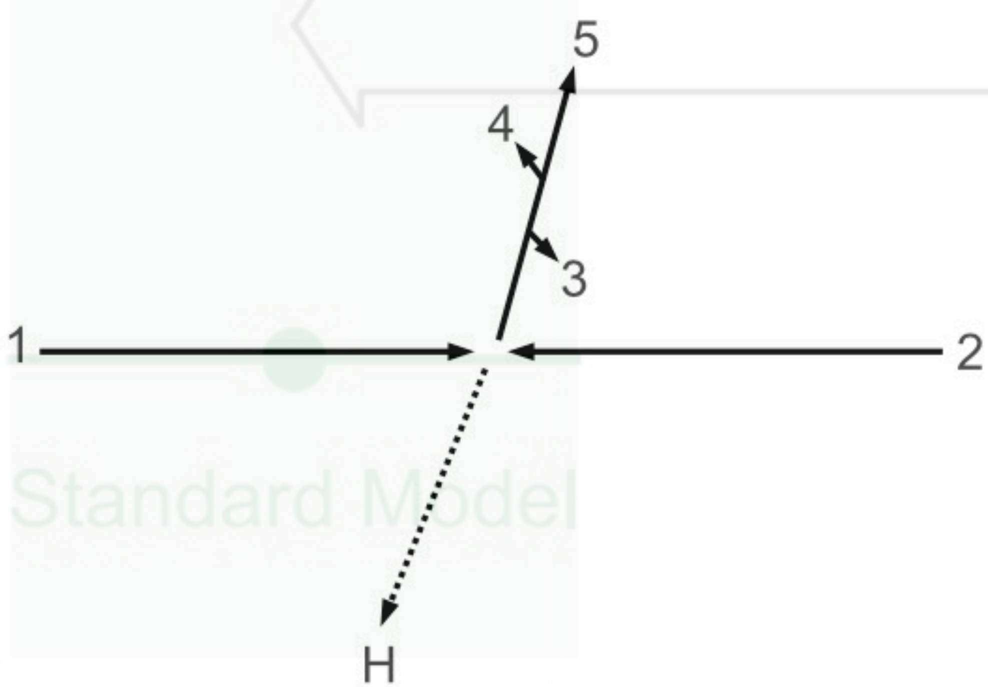


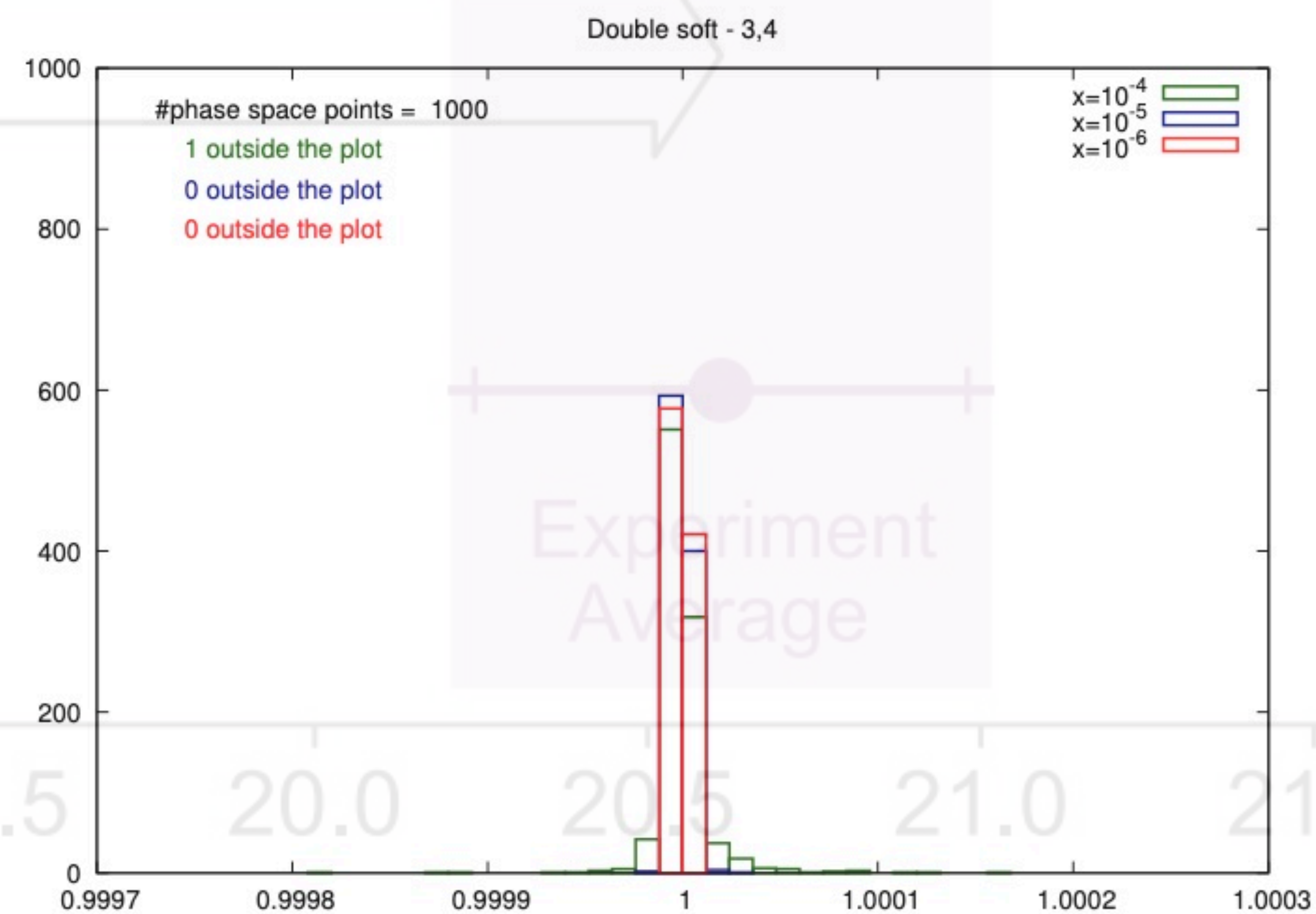
$$xs = s_{35} + s_{45} + s_{5H}$$

ANTENNA SUBTRACTION AT NNLO (RR LEVEL)

- Double soft limit

$$d\hat{\sigma}^{S}_{NNLO} = \boxed{d\hat{\sigma}^{S,a}} + \boxed{d\hat{\sigma}^{S,b_1}} + \boxed{d\hat{\sigma}^{S,b_2}} + \boxed{d\hat{\sigma}^{S,c}} + \boxed{d\hat{\sigma}^{S,d}}$$



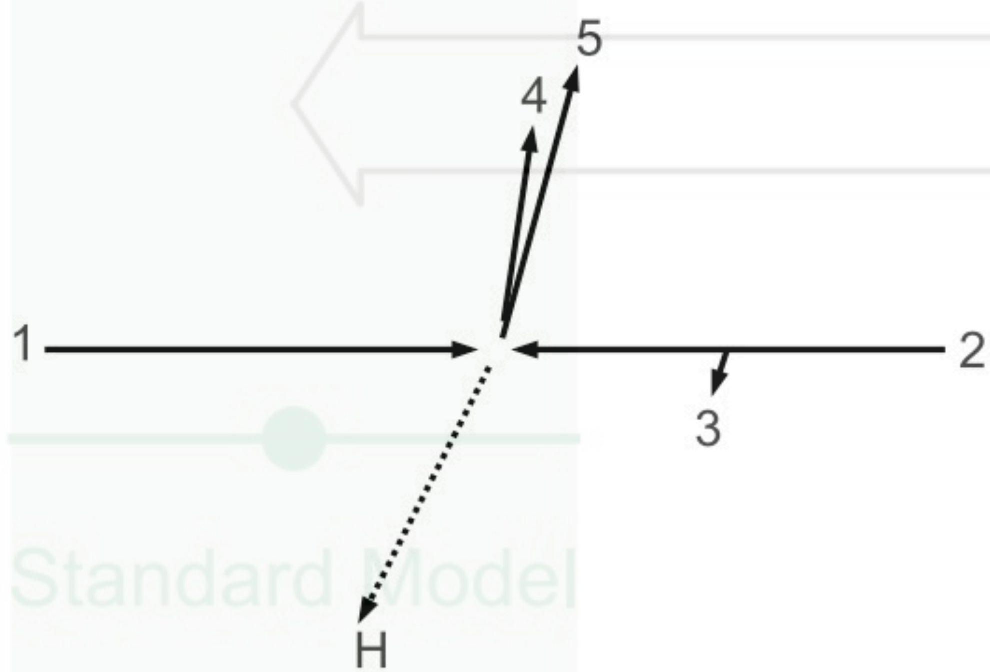


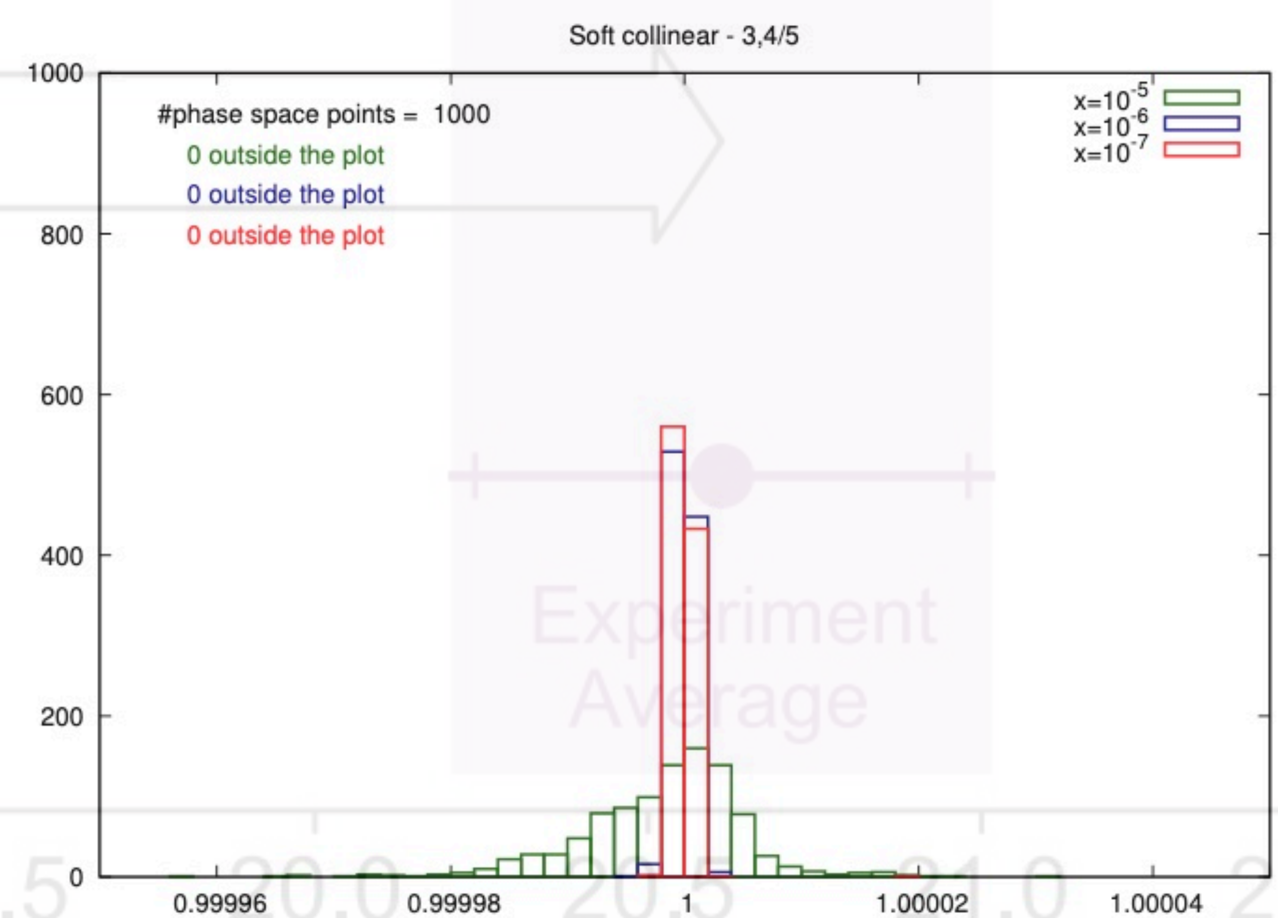
$$xs = s_{34} + s_{45} + s_{35} + s_{4H} + s_{3H}$$

ANTENNA SUBTRACTION AT NNLO (RR LEVEL)

- Soft collinear limit

$$d\hat{\sigma}^{S}_{NNLO} = \boxed{d\hat{\sigma}^{S,a}} + \boxed{d\hat{\sigma}^{S,b_1}} + \boxed{d\hat{\sigma}^{S,b_2}} + \boxed{d\hat{\sigma}^{S,c}} + \boxed{d\hat{\sigma}^{S,d}}$$





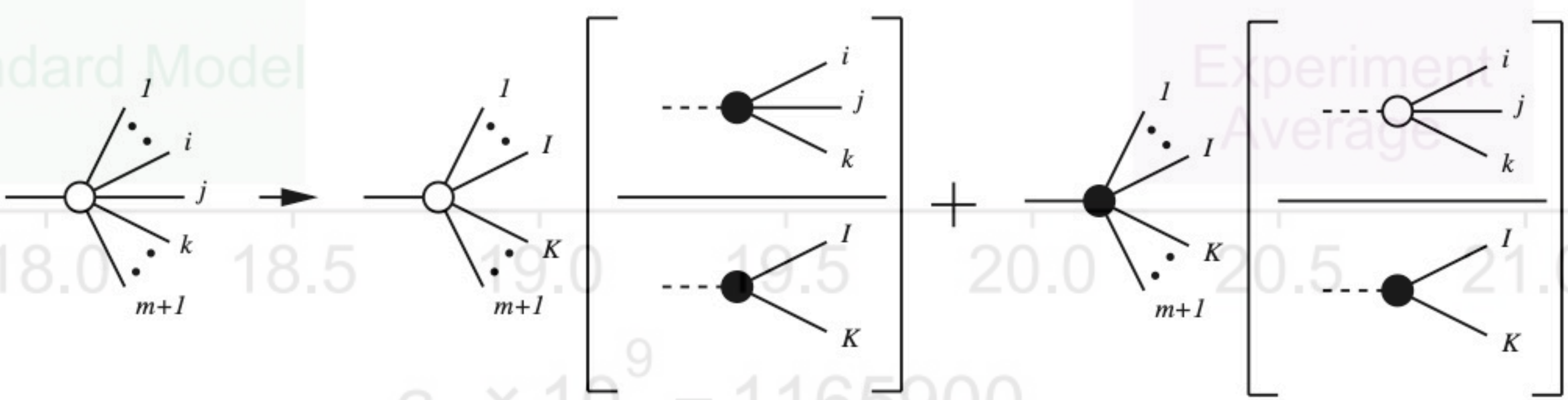
$$x = \frac{s_{45}}{s}, \quad xs = s_{34} + s_{35} + s_{3H}$$

ANTENNA SUBTRACTION AT NNLO (RV LEVEL)

$$d\hat{\sigma}^T_{NNLO} = \boxed{d\hat{\sigma}^{T,a}} + \boxed{d\hat{\sigma}^{T,b_1}} + \boxed{d\hat{\sigma}^{T,b_2}} + \boxed{d\hat{\sigma}^{T,c}}$$

$$\boxed{d\hat{\sigma}^{T,a}} = J^{(1)}_{n+1}|\mathcal{M}^0_{n+1}|^2 \sim \int_1 X^0_3 |\mathcal{M}^0_{n+1}|^2$$

$$\boxed{d\hat{\sigma}^{T,b}} : |\mathcal{M}^1_{n+1}|^2 \to X^0_3|\mathcal{M}^1_n|^2 + X^1_3|\mathcal{M}^0_n|^2$$



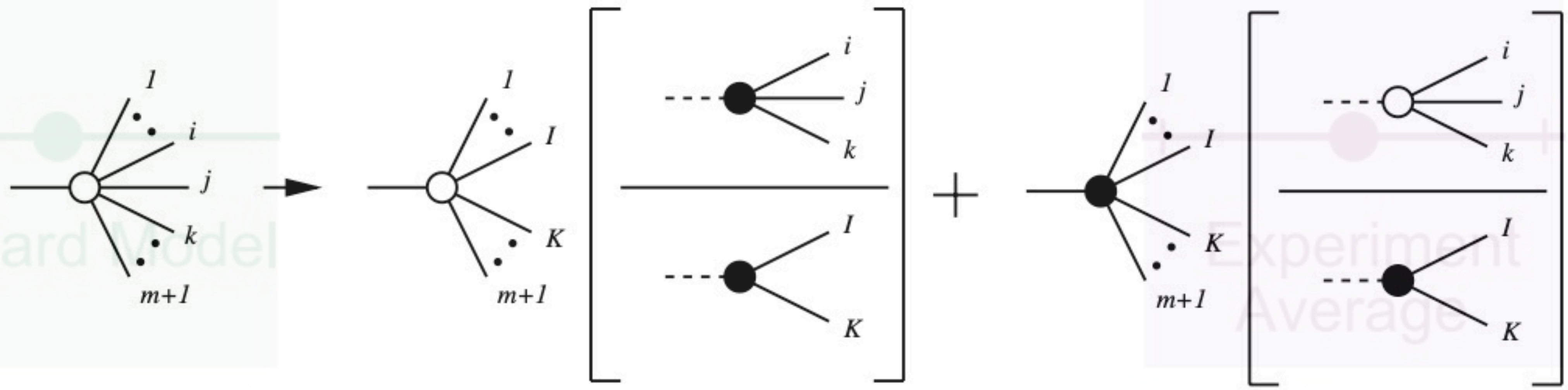
ANTENNA SUBTRACTION AT NNLO (RV LEVEL)

$$d\hat{\sigma}^{T}_{NNLO} = \boxed{d\hat{\sigma}^{T,a}} + \boxed{d\hat{\sigma}^{T,b_1}} + \boxed{d\hat{\sigma}^{T,b_2}} + \boxed{d\hat{\sigma}^{T,c}}$$

$$\boxed{d\hat{\sigma}^{T,a}} = J^{(1)}_{n+1}|\mathcal{M}^0_{n+1}|^2 \sim \int_1 X^0_3|\mathcal{M}^0_{n+1}|^2 \sim \mathcal{O}(\epsilon^{-2})$$

$$\boxed{d\hat{\sigma}^{T,b_1}} = X^0_3[|\mathcal{M}^1_n|^2 + J^{(1)}_n|\mathcal{M}^0_n|^2] \sim \mathcal{O}(\epsilon^0)$$

$$\boxed{d\hat{\sigma}^{T,b_2}} = X^1_3(\mu^2)|\mathcal{M}^0_n|^2 + J^{(1)}_X X^0_3|\mathcal{M}^0_n|^2 - M_X X^0_3 J^{(1)}_2|\mathcal{M}^0_n|^2 \sim \mathcal{O}(\epsilon^0)$$

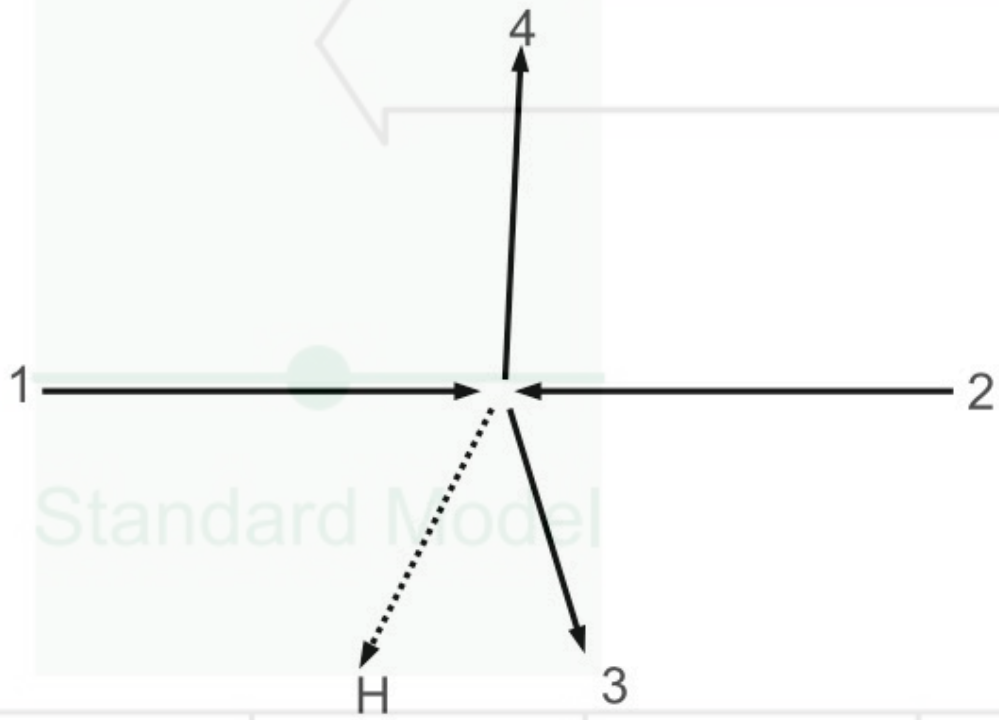


$$\boxed{d\hat{\sigma}^{T,c}} = -\int_1 d\hat{\sigma}^{S,c} + d\hat{\sigma}^{T,c_1} + d\hat{\sigma}^{T,c_2} \sim \mathcal{X}^0_3 X^0_3|\mathcal{M}^0_n|^2 \sim \mathcal{O}(\epsilon^0)$$

ANTENNA SUBTRACTION AT NNLO (RV LEVEL)

$$d\hat{\sigma}^{T}_{NNLO} = \boxed{d\hat{\sigma}^{T,a}} + \boxed{d\hat{\sigma}^{T,b_1}} + \boxed{d\hat{\sigma}^{T,b_2}} + \boxed{d\hat{\sigma}^{T,c}}$$

- Normal phase space preserve explicit pole cancellation



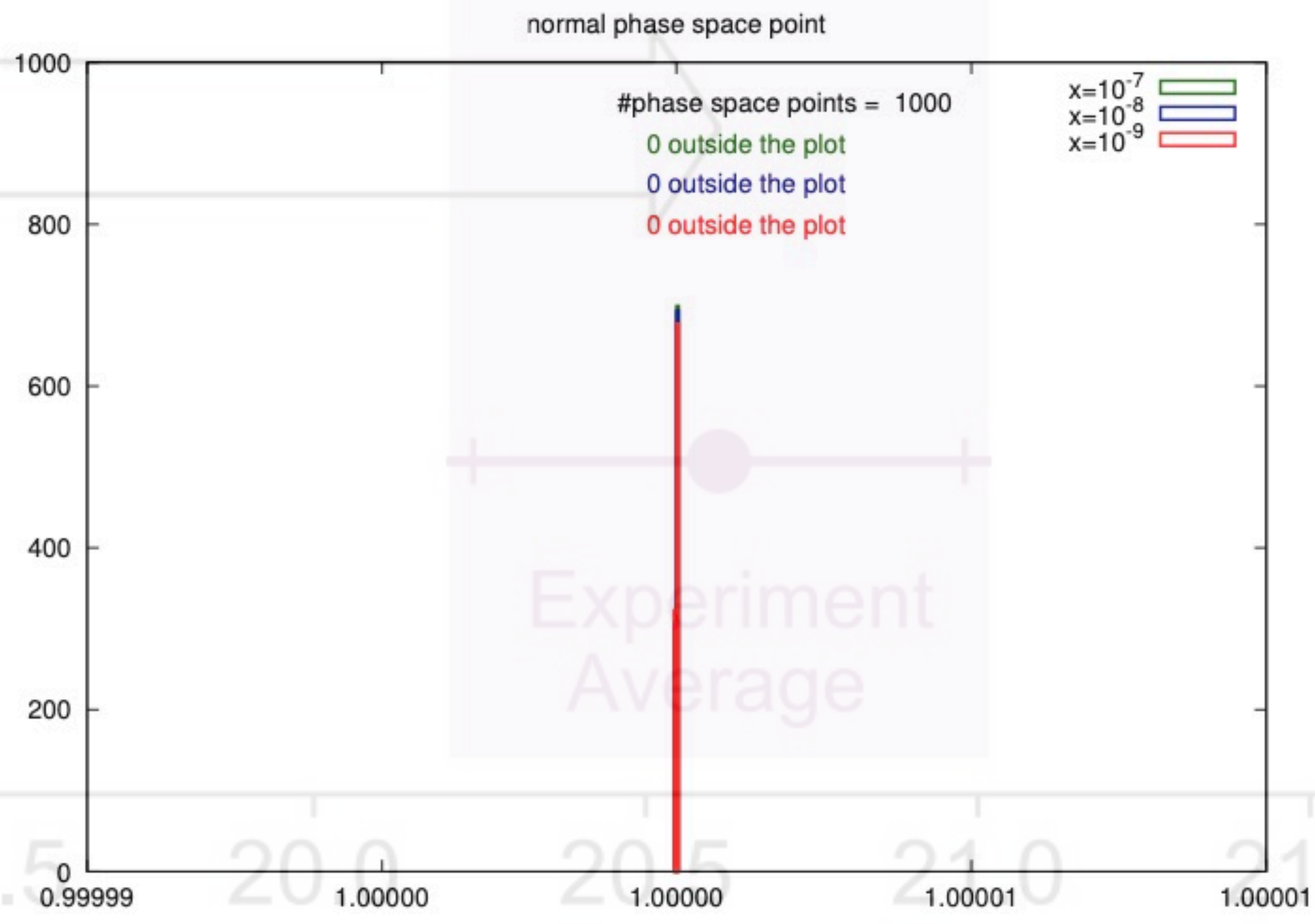


Figure: $\frac{1}{\epsilon^2}$ pole

ANTENNA SUBTRACTION AT NNLO (RV LEVEL)

$$d\hat{\sigma}^{T}_{NNLO} = \boxed{d\hat{\sigma}^{T,a}} + \boxed{d\hat{\sigma}^{T,b_1}} + \boxed{d\hat{\sigma}^{T,b_2}} + \boxed{d\hat{\sigma}^{T,c}}$$

- Normal phase space preserve explicit pole cancellation

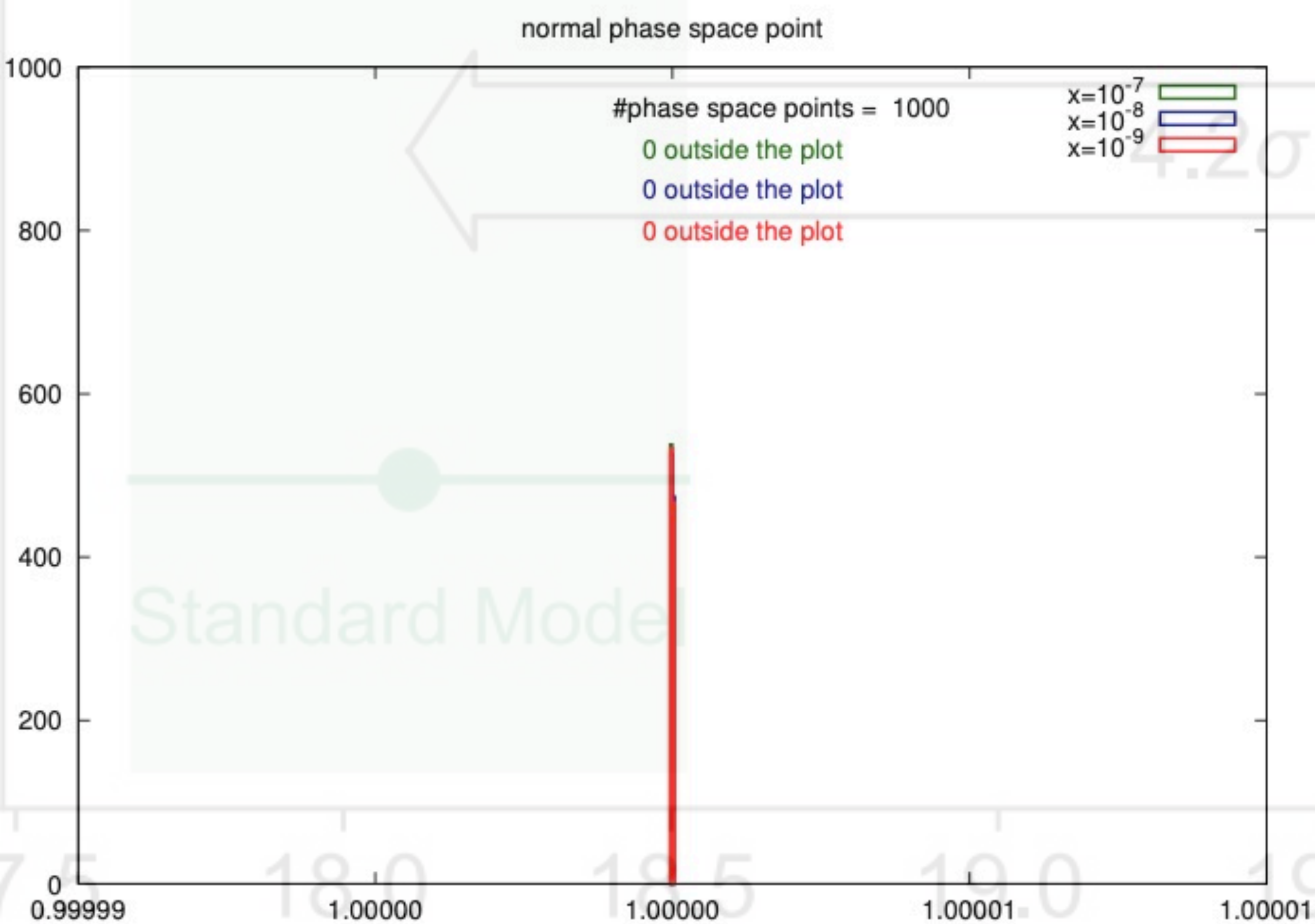


Figure: $\frac{1}{\epsilon}$ pole

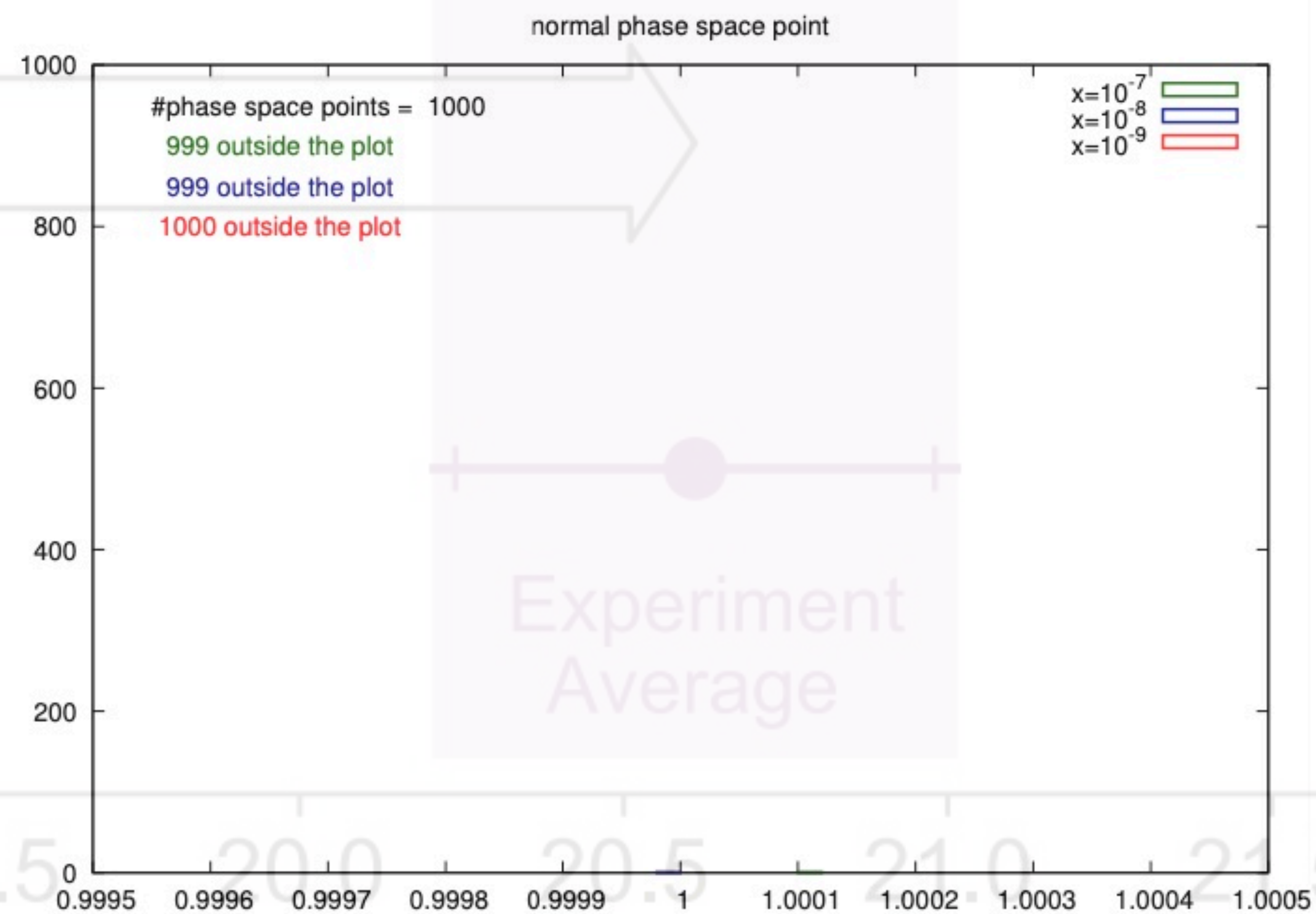
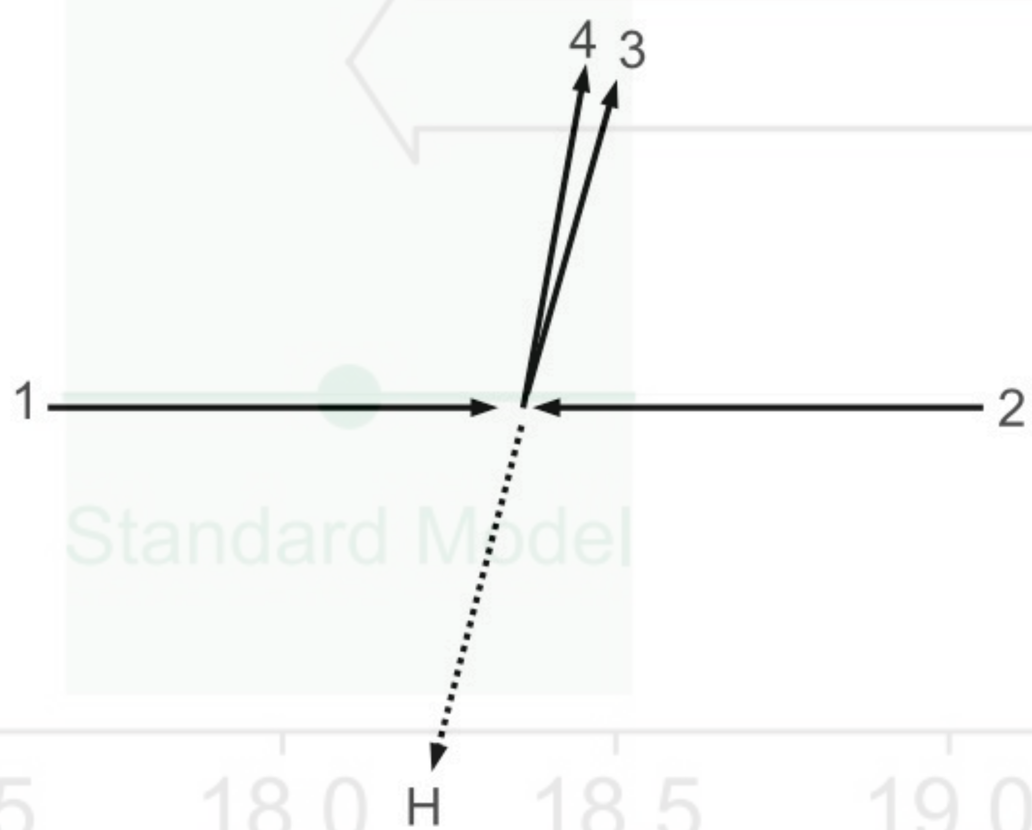


Figure: ϵ^0 pole

ANTENNA SUBTRACTION AT NNLO (RV LEVEL)

$$d\hat{\sigma}^{T}_{NNLO} = \boxed{d\hat{\sigma}^{T,a}} + \boxed{d\hat{\sigma}^{T,b_1}} + \boxed{d\hat{\sigma}^{T,b_2}} + \boxed{d\hat{\sigma}^{T,c}}$$

- Single collinear limit preserve both explicit and dynamic pole cancellation



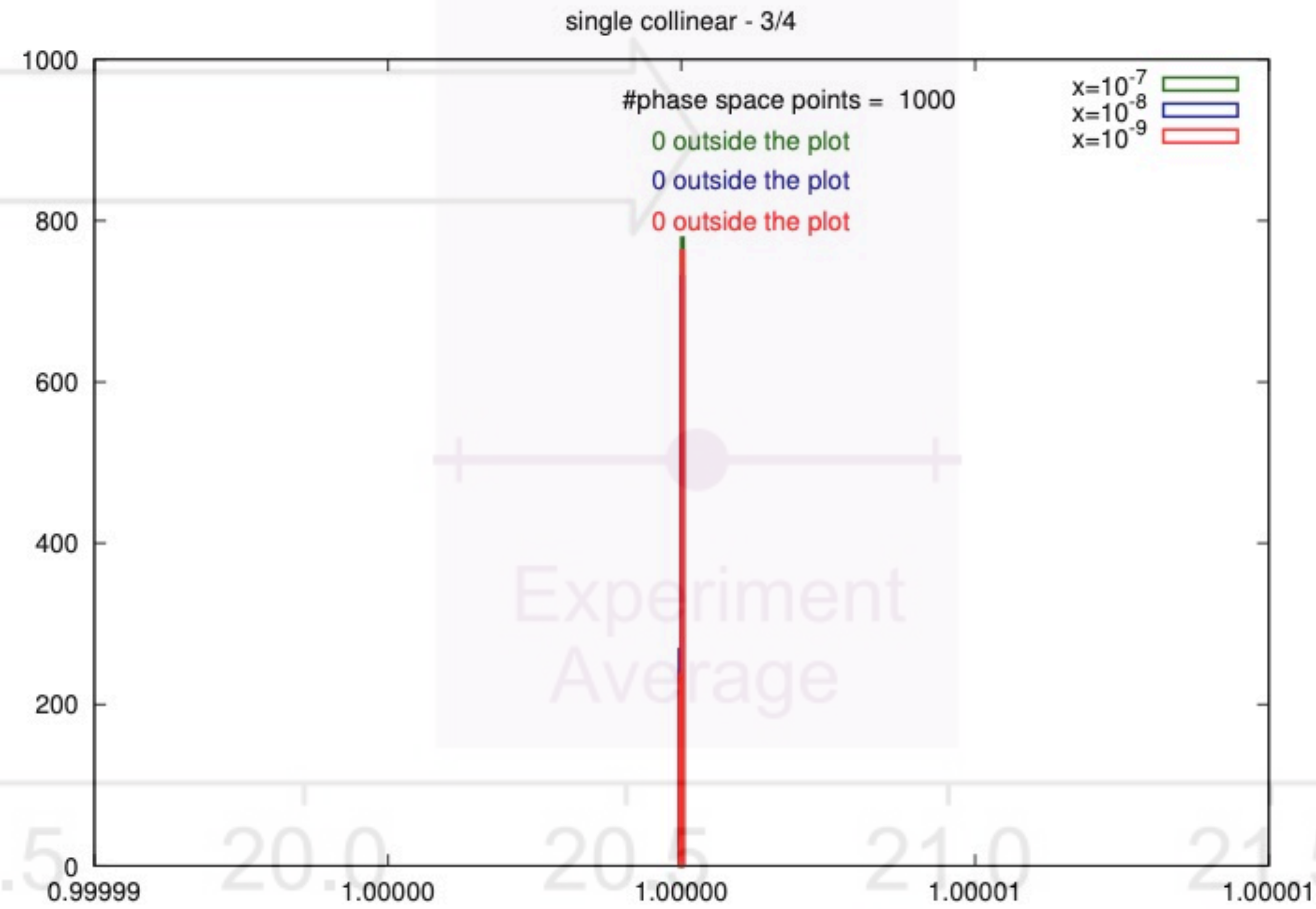


Figure: $\frac{1}{\epsilon^2}$ pole

ANTENNA SUBTRACTION AT NNLO (RV LEVEL)

$$d\hat{\sigma}^{T}_{NNLO} = \boxed{d\hat{\sigma}^{T,a}} + \boxed{d\hat{\sigma}^{T,b_1}} + \boxed{d\hat{\sigma}^{T,b_2}} + \boxed{d\hat{\sigma}^{T,c}}$$

- Single collinear limit preserve both explicit and dynamic pole cancellation

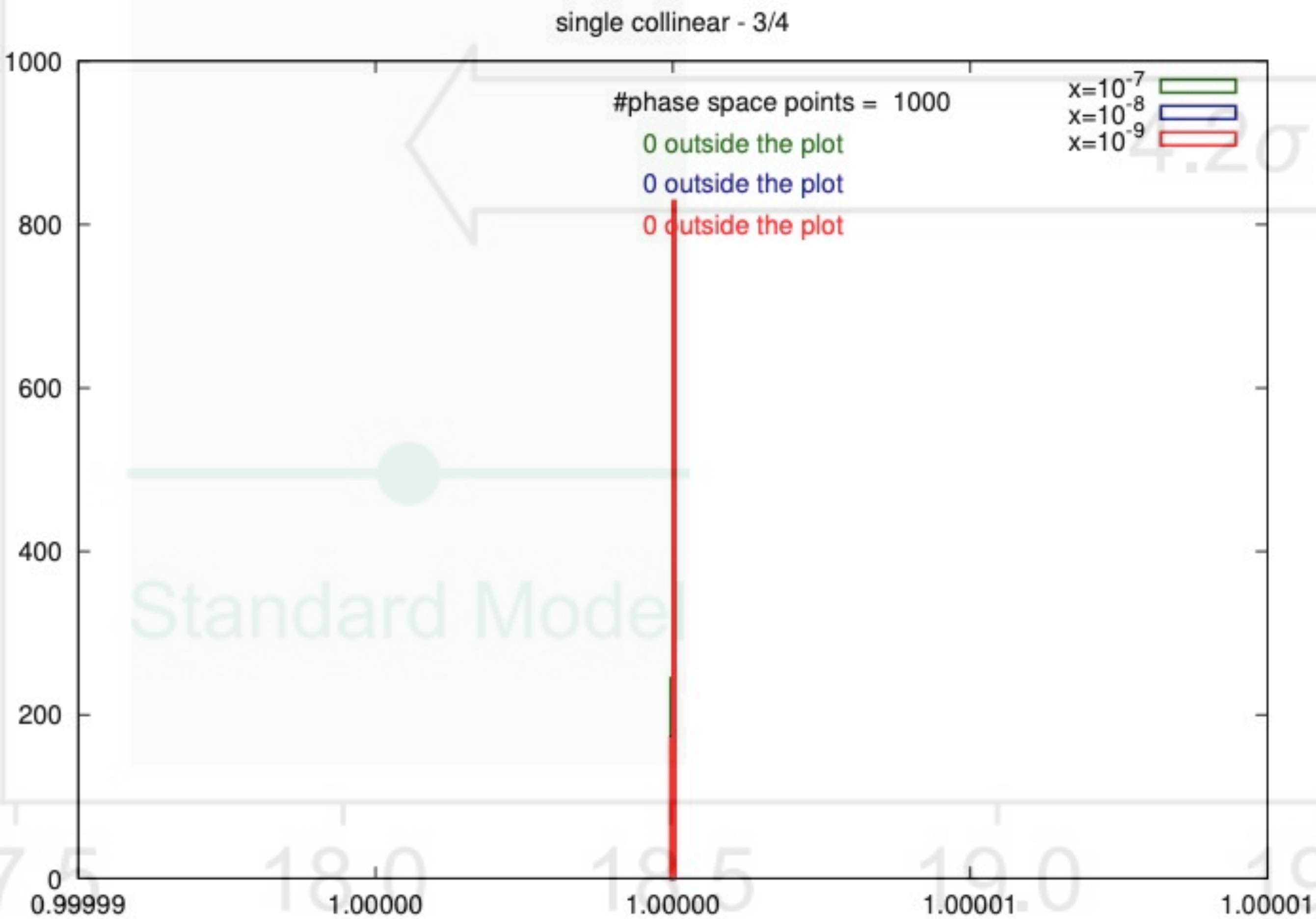


Figure: $\frac{1}{\epsilon}$ pole

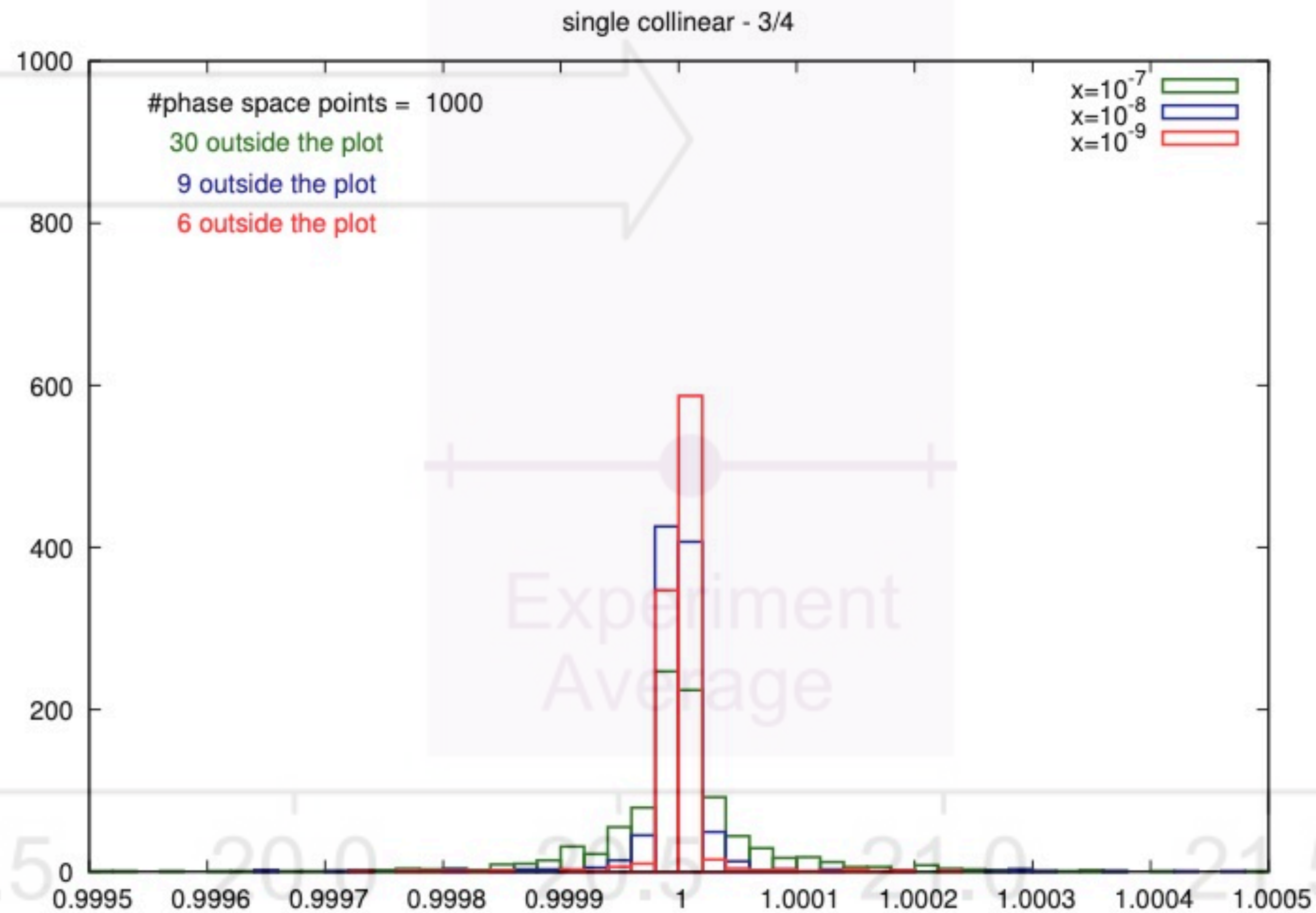
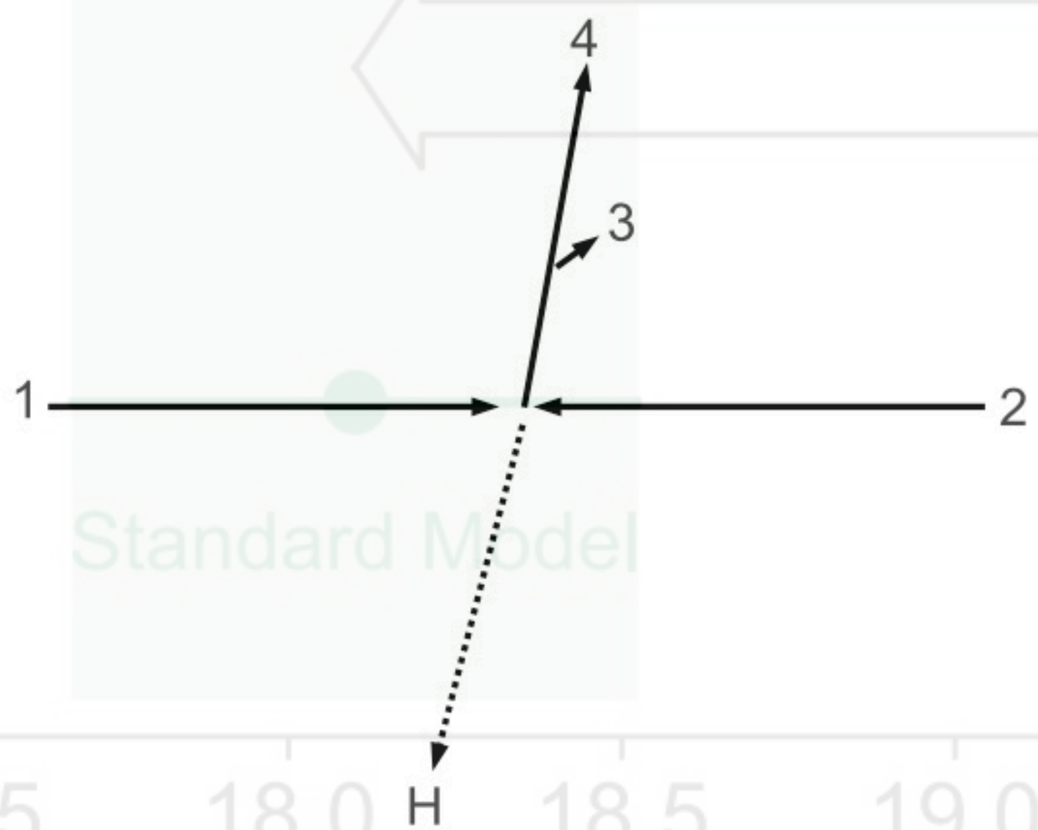


Figure: ϵ^0 pole

ANTENNA SUBTRACTION AT NNLO (RV LEVEL)

$$d\hat{\sigma}^{T}_{NNLO} = \boxed{d\hat{\sigma}^{T,a}} + \boxed{d\hat{\sigma}^{T,b_1}} + \boxed{d\hat{\sigma}^{T,b_2}} + \boxed{d\hat{\sigma}^{T,c}}$$

- Single soft limit preserve both explicit and dynamic pole cancellation



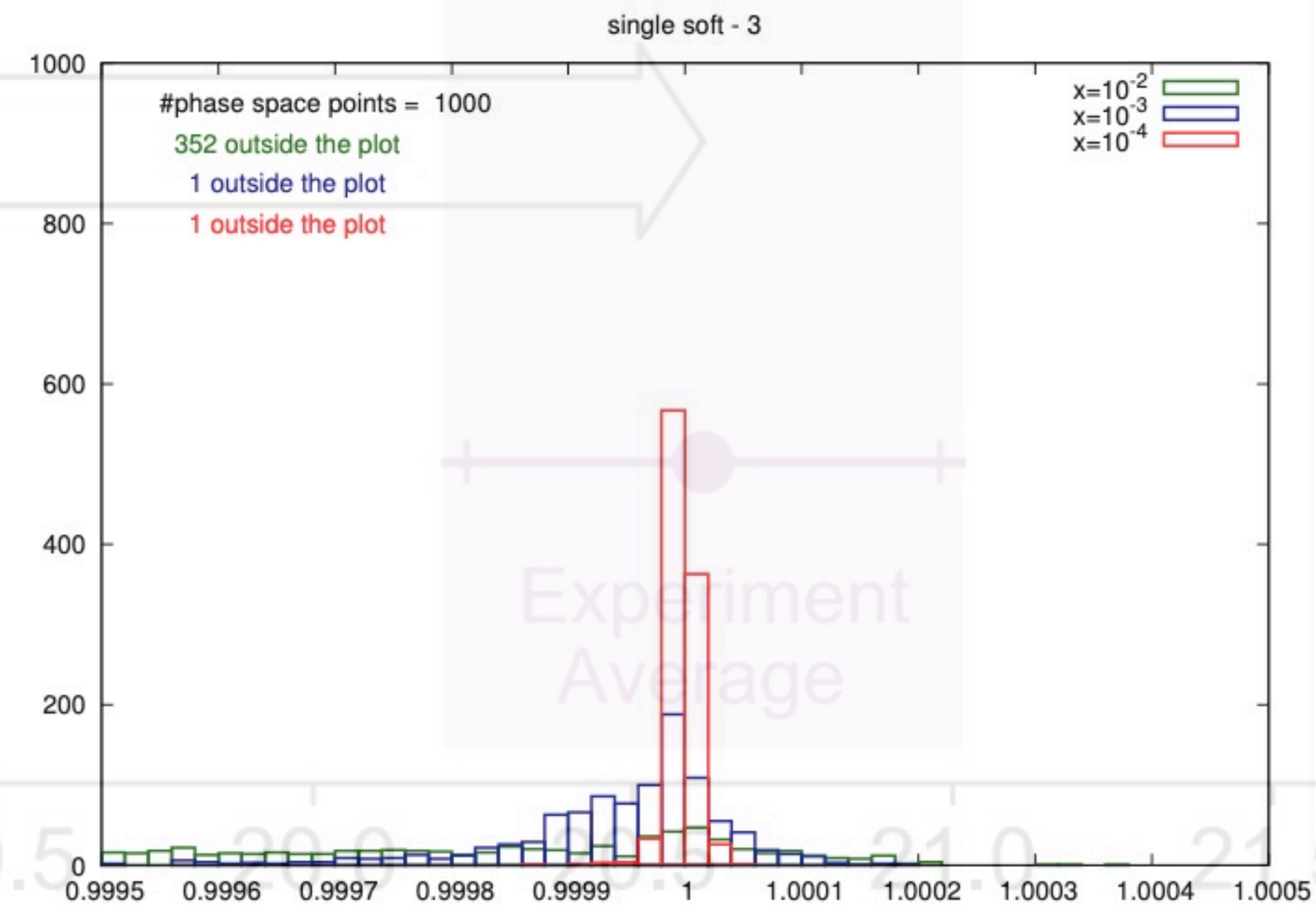
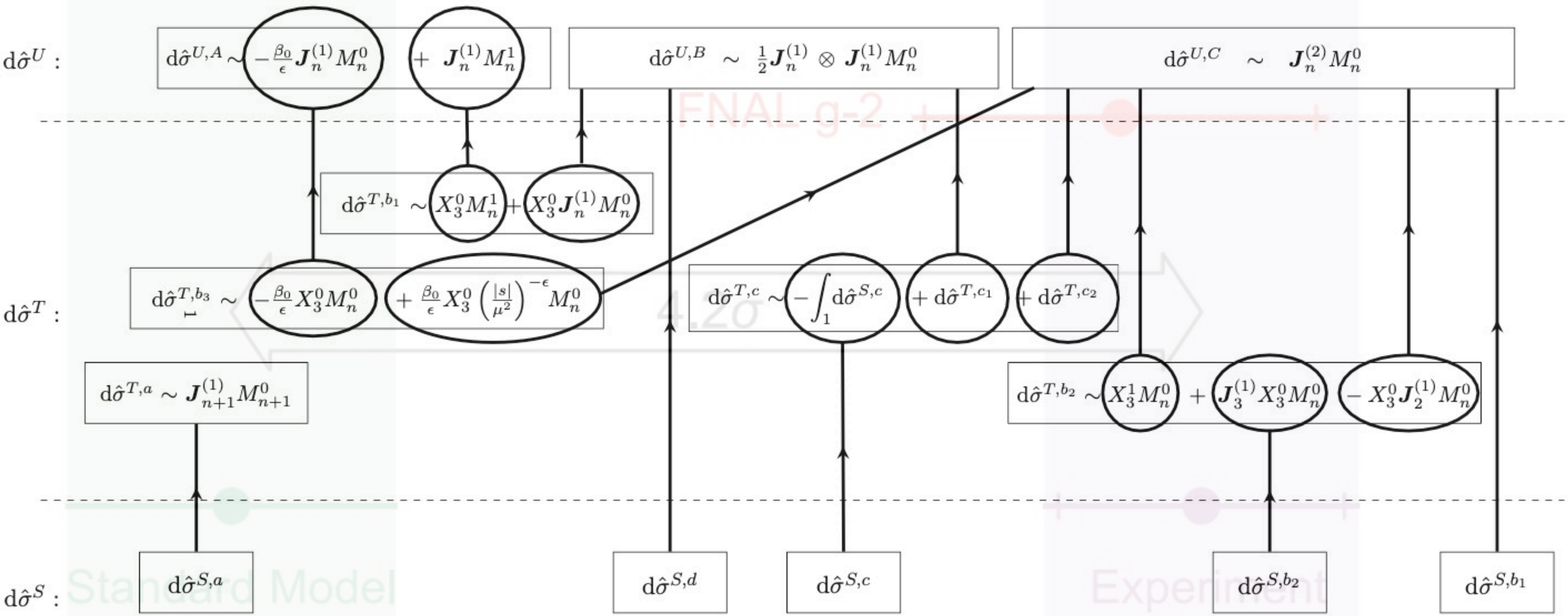


Figure: ϵ^0 pole

ANTENNA SUBTRACTION AT NNLO (VV LEVEL)



- Double virtual level only have explicitly poles and no parton become unresolved
- Collect all subtraction terms (integrated) and add back in $d\hat{\sigma}^{U}_{NNLO}$

ANTENNA SUBTRACTION AT NNLO (VV LEVEL)

$$d\hat{\sigma}^{U}_{NNLO} = \boxed{d\hat{\sigma}^{U,a}} + \boxed{d\hat{\sigma}^{U,b}} + \boxed{d\hat{\sigma}^{U,c}}$$

$$\boxed{d\hat{\sigma}^{U,a}} = J_n^{(1)}\left(|\mathcal{M}_n^1|^2 - \frac{\beta_0}{\epsilon}|\mathcal{M}_n^0|^2\right)$$

$$\boxed{d\hat{\sigma}^{U,b}} = -\frac{1}{2}J_n^{(1)} \otimes J_n^{(1)}|\mathcal{M}_n^0|^2$$

$$\boxed{d\hat{\sigma}^{U,c}} = J_n^{(2)}|\mathcal{M}_n^0|^2$$

$$pole\{d\hat{\sigma}^{VV}_{NNLO}\} = pole\left\{J_n^{(1)} \otimes |\mathcal{M}_n^1|^2 - \left(\frac{1}{2}J_n^{(1)} \otimes J_n^{(1)} + \frac{\beta_0}{\epsilon} - J_n^{(2)}\right)|\mathcal{M}_n^0|^2\right\}$$

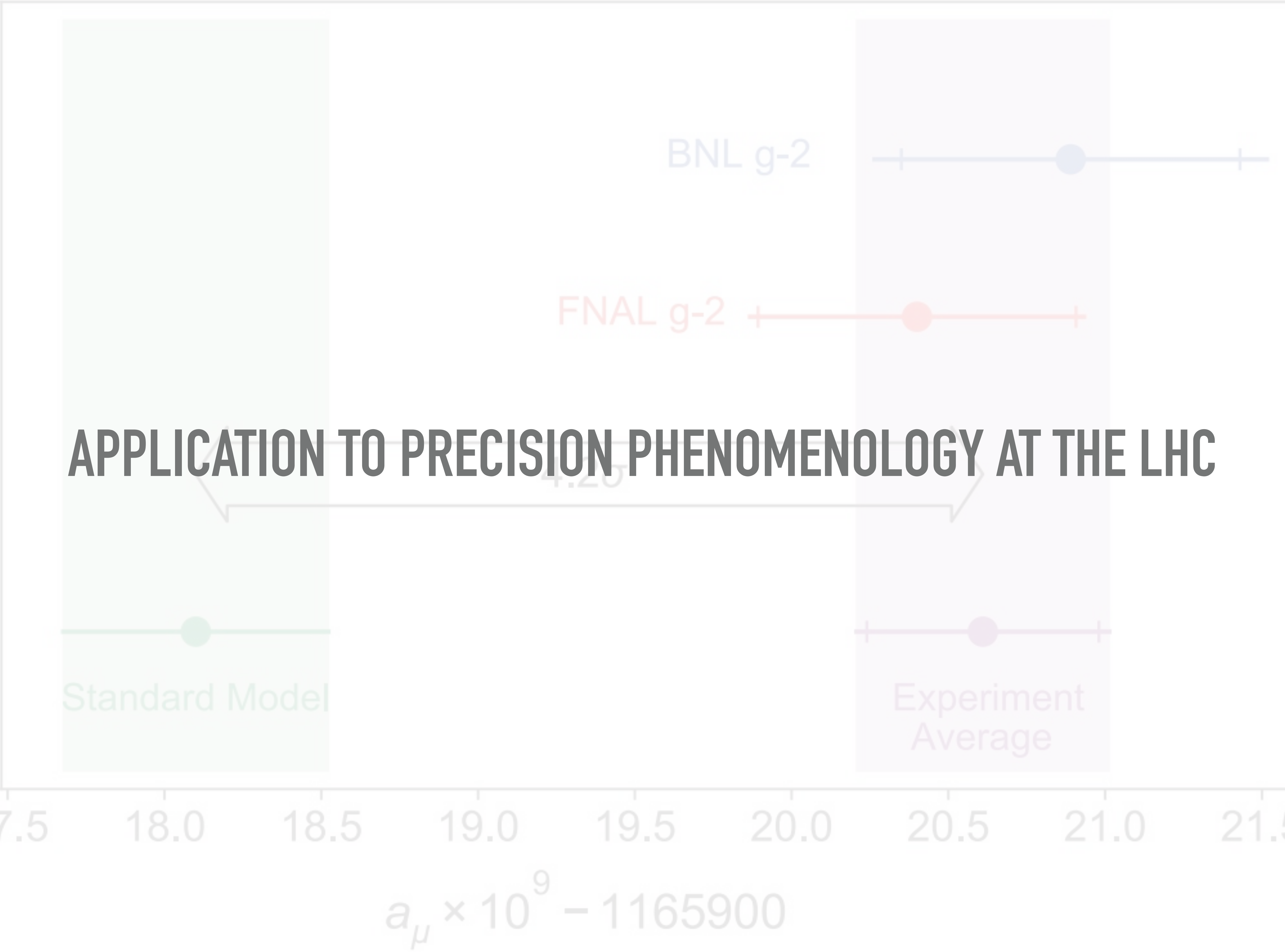
- Explicit IR divergence can be arranged in di-pole structure
- Analytically check the cancellation against VV ME:

$$|\mathcal{M}_m^2\rangle = |\mathcal{M}_m^2(\mu^2;\{p\})\rangle = \mathbf{I}^{(1)}(\epsilon,\mu^2;\{p\})|\mathcal{M}_m^1(\mu^2;\{p\})\rangle + \mathbf{I}^{(2)}(\epsilon,\mu^2;\{p\})|\mathcal{M}_m^0(\mu^2;\{p\})\rangle + |\mathcal{M}_m^{2,fin}(\mu^2;\{p\})\rangle$$

$$\mathbf{I}^{(2)}(\epsilon,\mu^2;\{p\}) = -\frac{1}{2}\mathbf{I}^{(1)}(\epsilon,\mu^2;\{p\})\left(\mathbf{I}^{(1)}(\epsilon,\mu^2;\{p\}) + \frac{4\pi\beta_0}{\epsilon}\right) + \frac{e^{+\epsilon\psi(1)}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{2\pi\beta_0}{\epsilon} + K\right)\mathbf{I}^{(1)}(2\epsilon,\mu^2;\{p\}) + \mathbf{H}^{(2)}(\epsilon,\mu^2;\{p\})$$

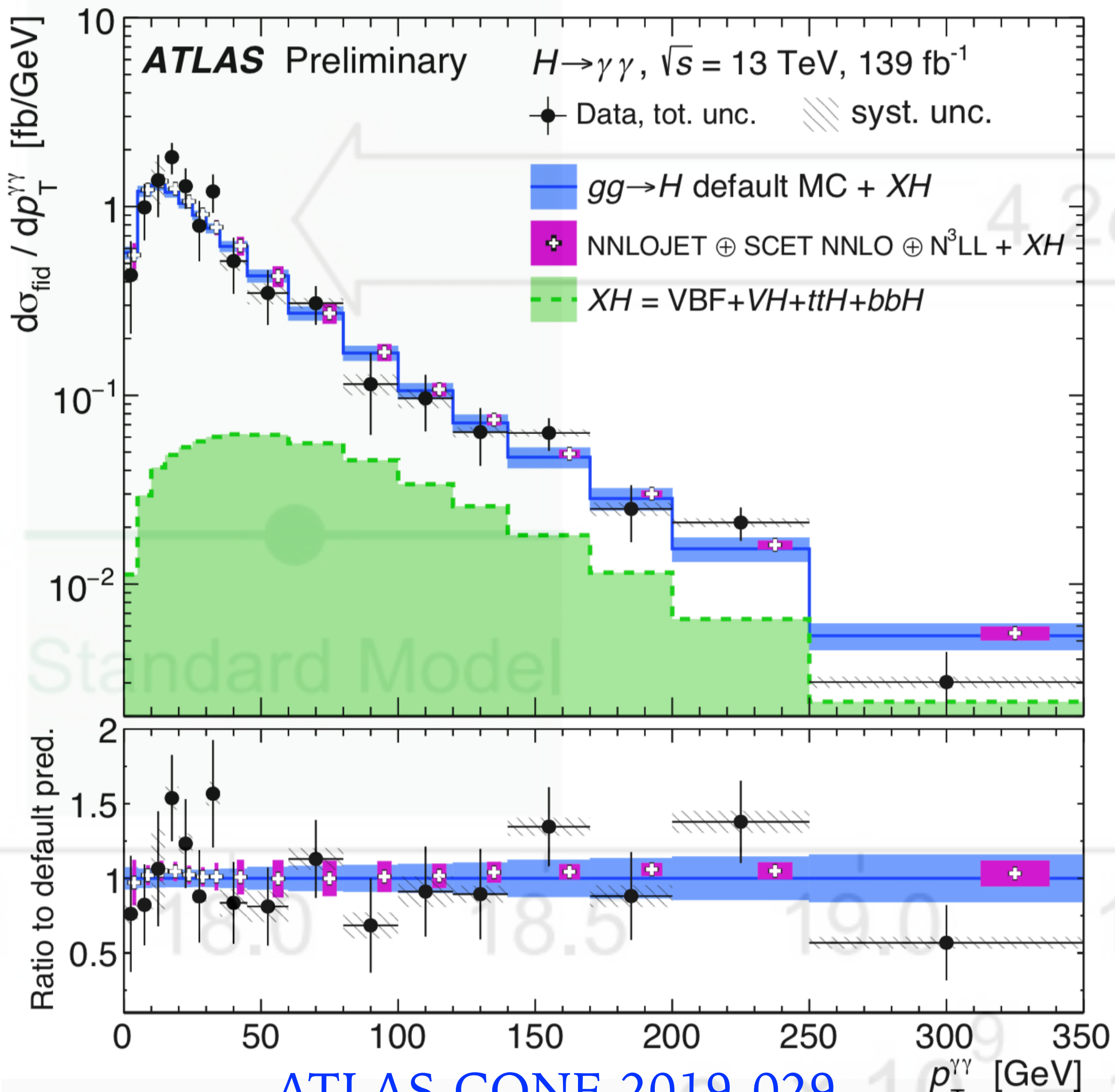
- Complete NNLO calculation with all IR divergence regulated

APPLICATION TO PRECISION PHENOMENOLOGY AT THE LHC

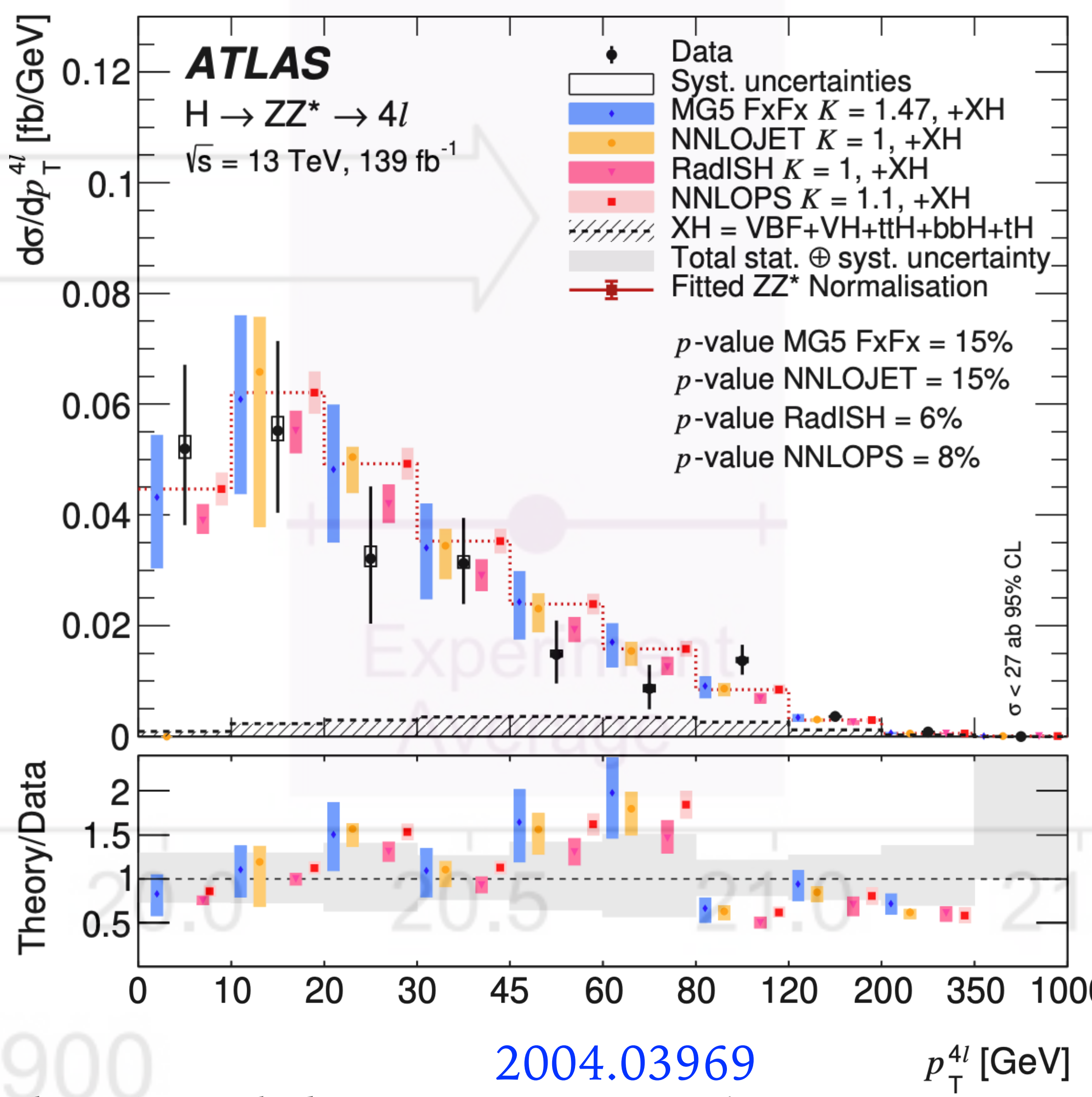


APPLICATION TO PRECISION PHENOMENOLOGY AT THE LHC

- H+J production at NNLO (using NNLOJET package + SCET/RadISH):
  - Join Higgs production with decay channels: $H \to \gamma\gamma$, $2l2\nu$, $,4l$ etc
  - Predictions of HpT at NNLO compare with ATLAS results
  - Theory uncertainty is currently ahead of EXP error 10% vs. 25%



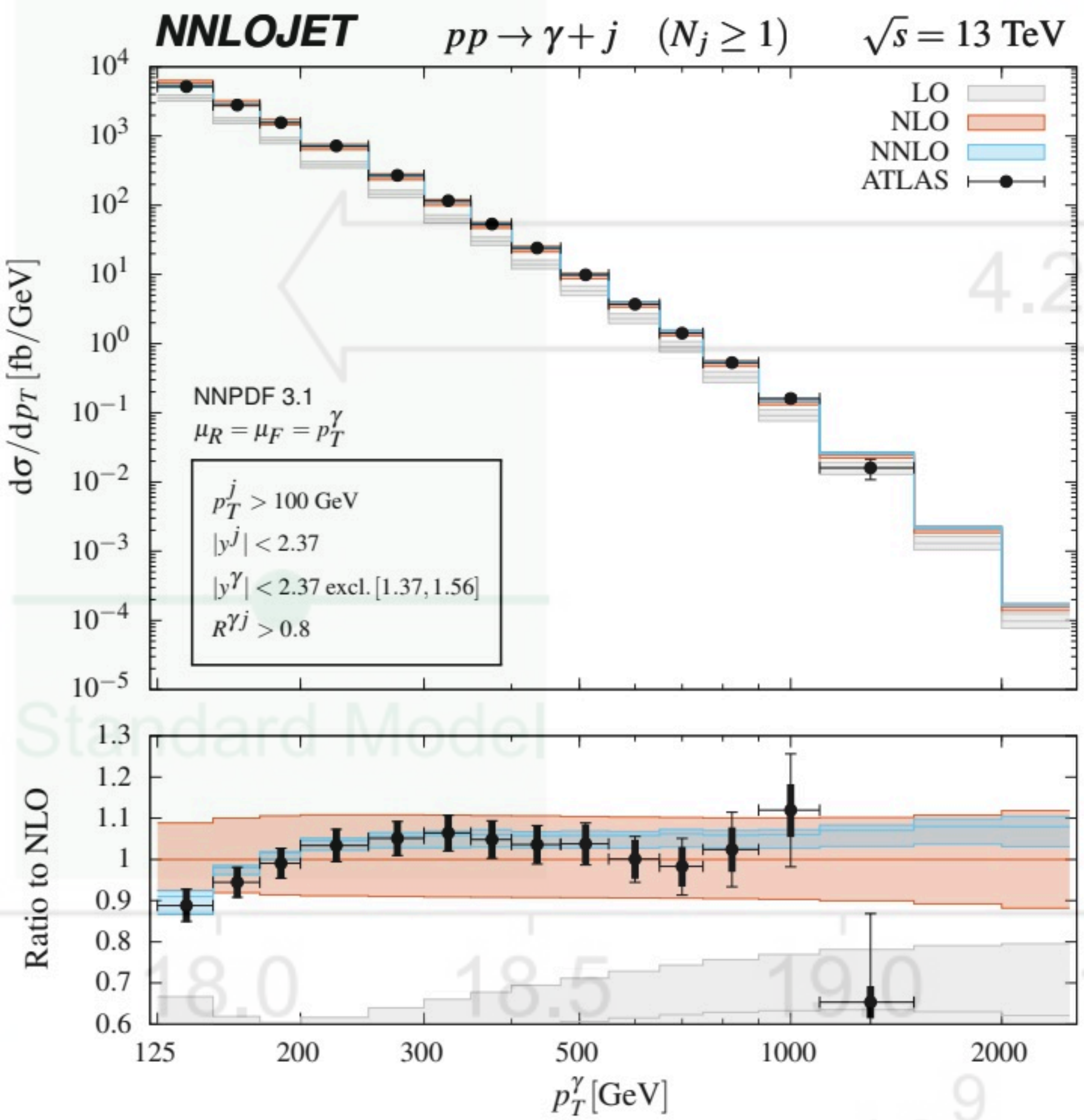
ATLAS-CONF-2019-029



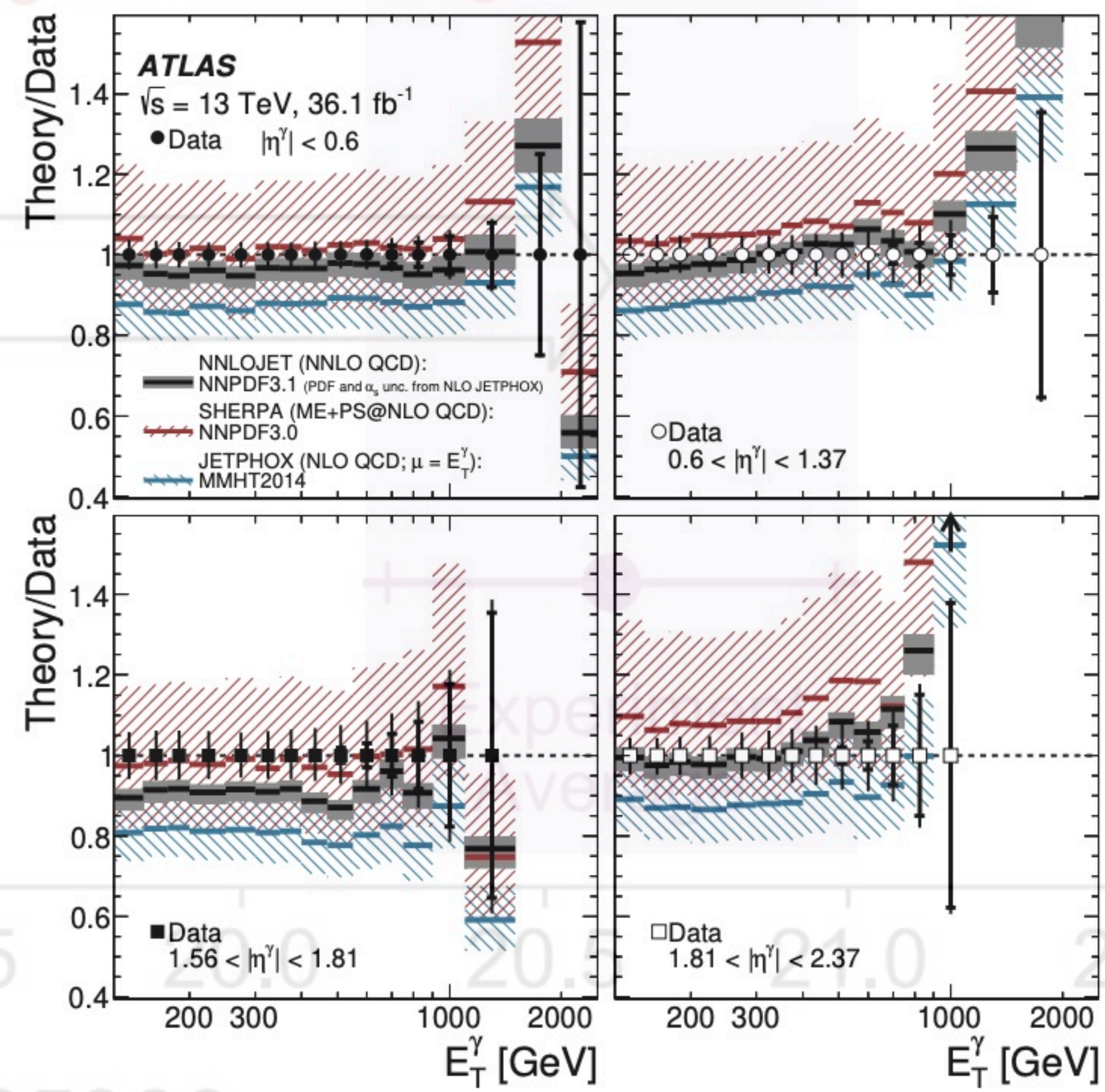
2004.03969

APPLICATION TO PRECISION PHENOMENOLOGY AT THE LHC

- γ+J production at NNLO (using NNLOJET package):
  - Common event at LHC but huge background to interference with signal
  - Theory uncertainty is currently comparable with EXP error, both at 3~5%



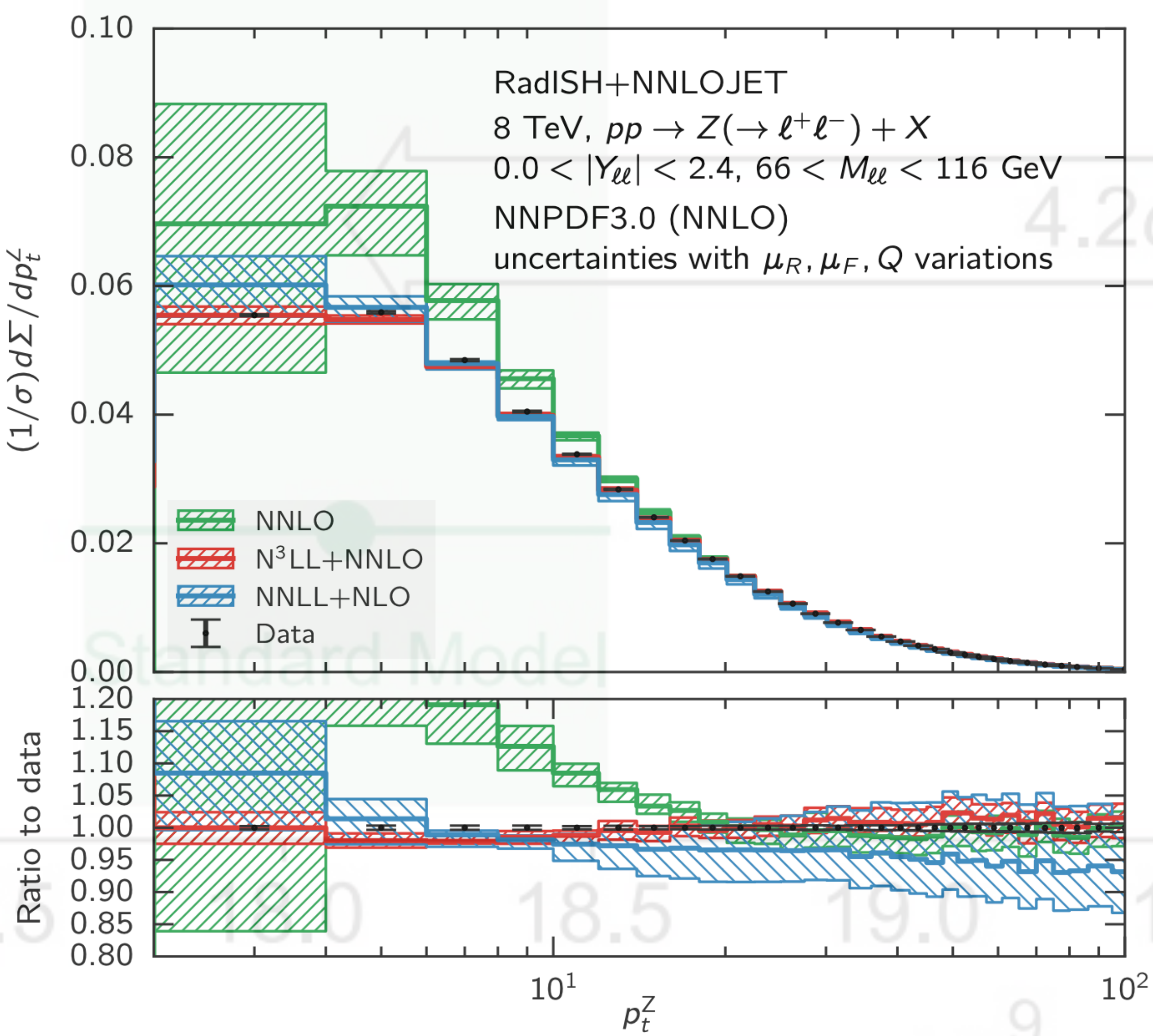
1904.01044



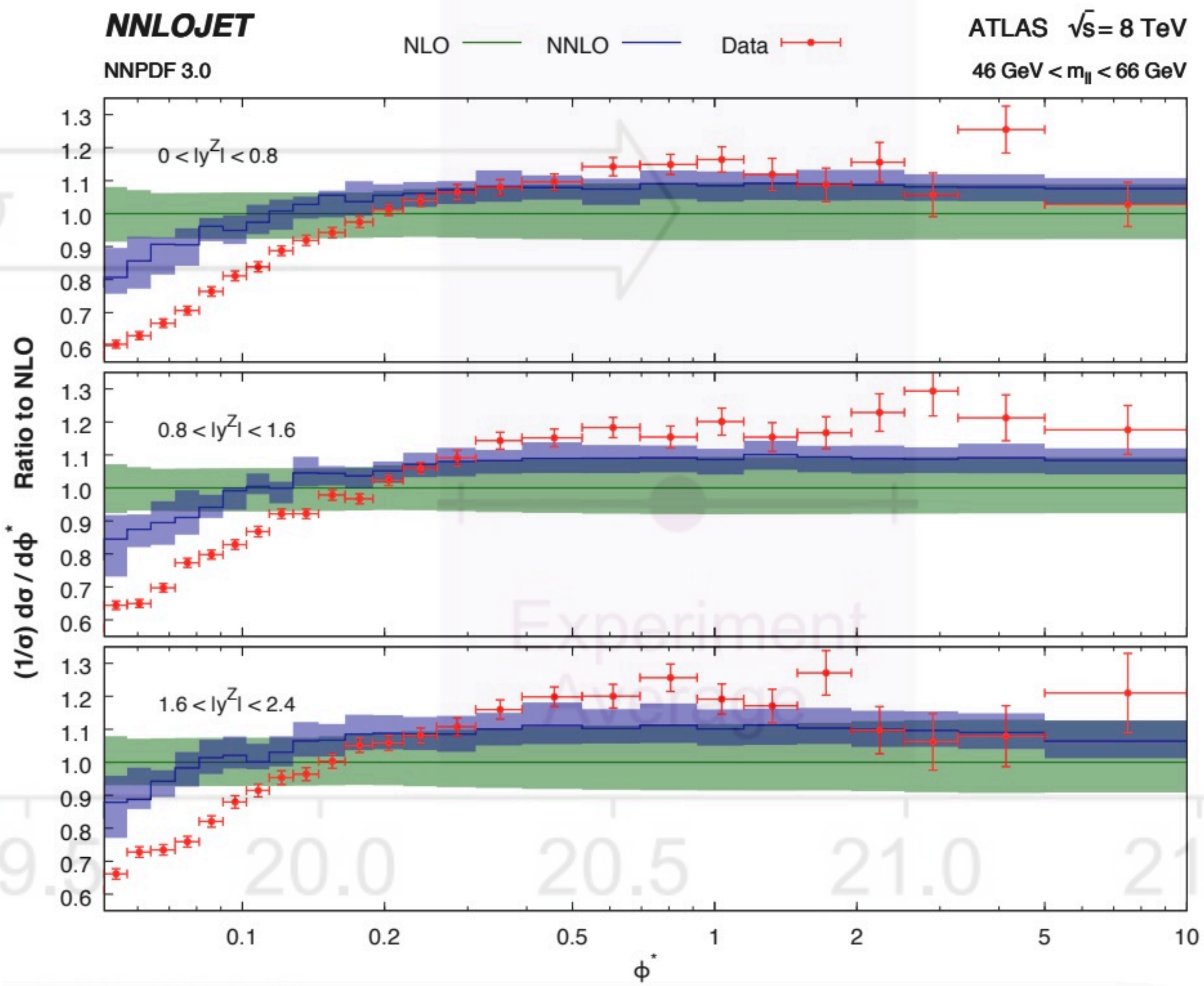
2004.03969

APPLICATION TO PRECISION PHENOMENOLOGY AT THE LHC

- Z+J production at NNLO (using NNLOJET package):
  - The "standard candle" of the Standard Model
  - Theory uncertainty is currently behind EXP error, 1% vs. 0.2%
  - Accurate data are used to abstract PDF and affect other phenomenology studies

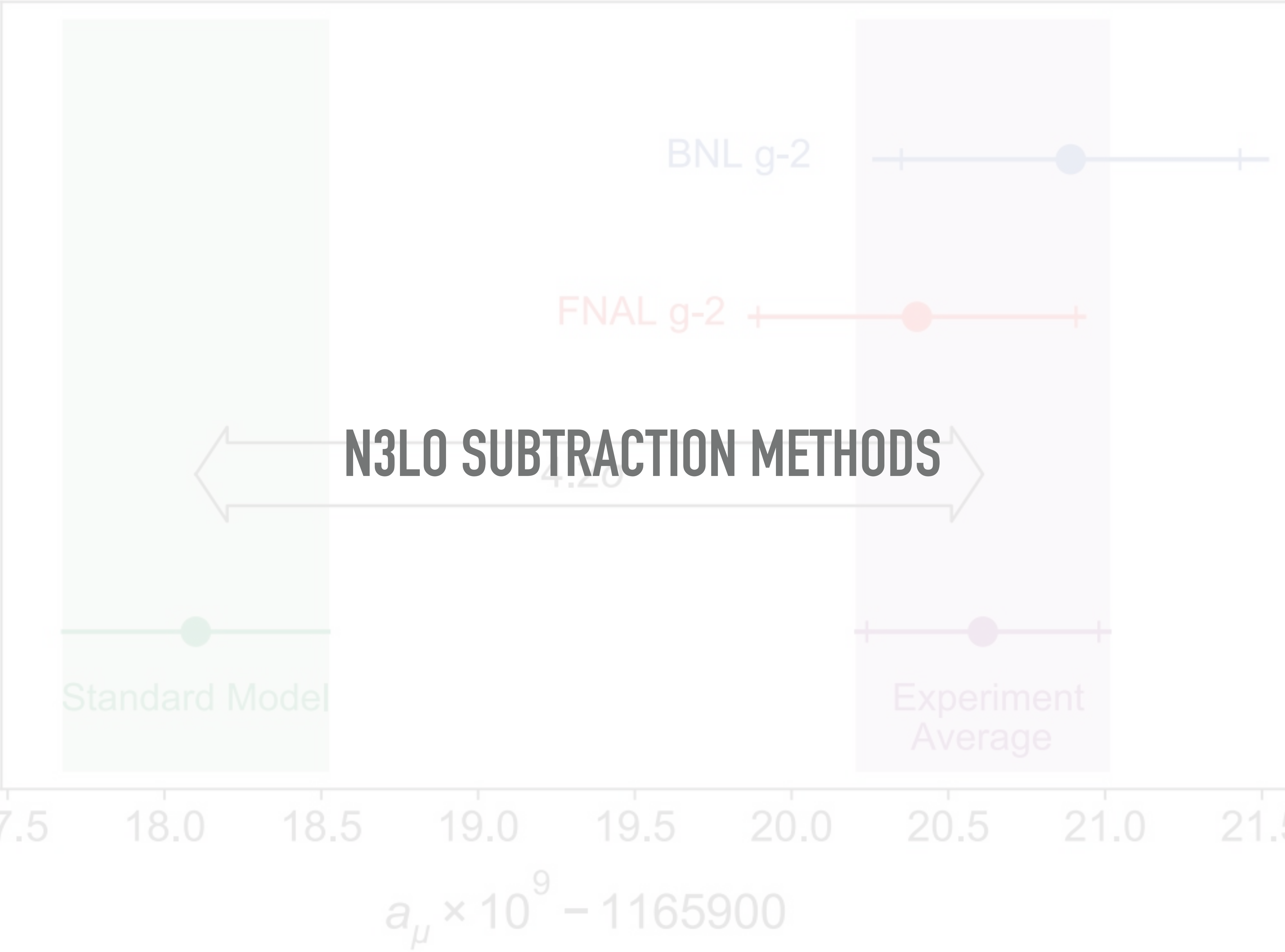


1805.05916



1610.01843

N3LO SUBTRACTION METHODS



QT SUBTRACTION AT N3LO (APPROXIMATED)

- Extend qT-subtraction method to N3LO (Cieri, XC et al. 1807.11501).
  In qT (CSS) factorisation to Higgs production at N3LO:

$$\frac{d\sigma}{dp_T^2 dy} = \frac{m_H^2}{s}\sigma_{LO}^H \int_0^{+\infty} db \frac{b}{2} J_0(bp_T) S_g(m_H, b) \sum_{a_1,a_2} \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} \left[HC_1C_2\right]_{gg:a_1a_2} \prod_{i=1,2} f_{a_i/h_i}(x_i/z_i, b_0^2/b^2)$$

$$S_c(M,b) = \exp\left[-\int_{b_0^2/b^2}^{M^2} \frac{dq^2}{q^2}\left(A_c\big(\alpha_s(q^2)\big)\ln\frac{M^2}{q^2} + B_c\big(\alpha_s(q^2)\big)\right)\right]$$

- Apply q_T^{cut} to factorise full N3LO into two parts.

$$d\sigma^H_{N^3LO} = \mathcal{H}^H_{N^3LO} \otimes d\sigma^H_{LO}\Big|_{\delta(p_T)} + \left[d\sigma^{H+jet}_{NNLO} - d\sigma^{H,CT}_{N^3LO}\right]_{p_T > q_T^{cut}}$$

- Above q_T^{cut}, recycle H+jet at NNLO from NNLOJET with qT counter terms (CT) to regulate IR divergence.
- Below q_T^{cut}, factorise real radiations from hard coefficient functions at $\delta(p_T)$ in HN3LO package.
- Most of the factorised components of $\delta(p_T)$ contribution are known analytically at N3LO.
- We use a constant $C_{N3}\delta_{ga}\delta_{gb}(1-z)$ to approximate the unknown pieces.
- Numerically abstract the C_{N3} coefficient using exact N3LO total cross section (1802.00833, 1802.00827).

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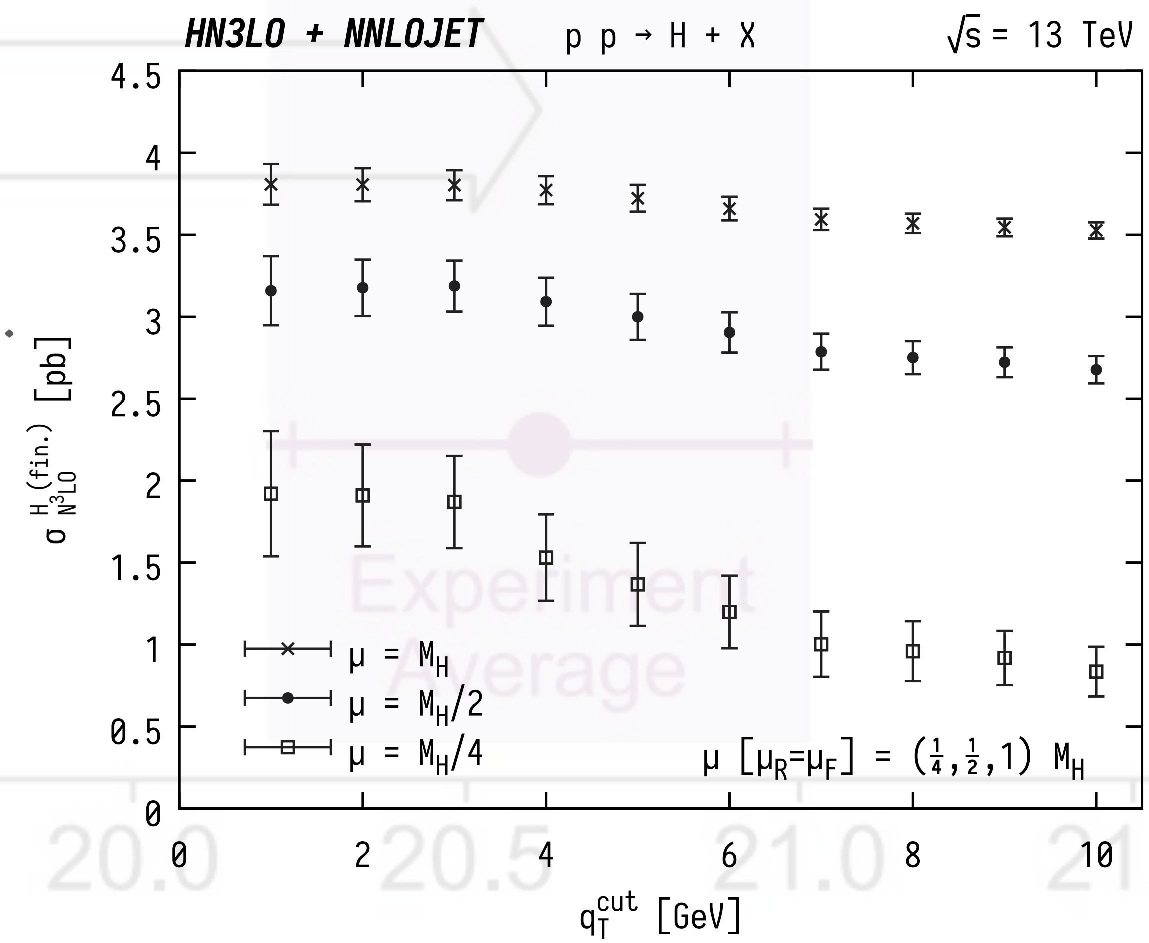

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$$S_c(M,b) = \exp\left[- \int_{b_0^2/b^2}^{M^2} \frac{dq^2}{q^2} \left(A_c(\alpha_s(q^2)) \ln \frac{M^2}{q^2} + B_c(\alpha_s(q^2)) \right) \right]$$

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➤ Above q_T^{cut}, recycle H+jet at NNLO from NNLOJET with qT counter terms (CT) to regulate IR divergence.

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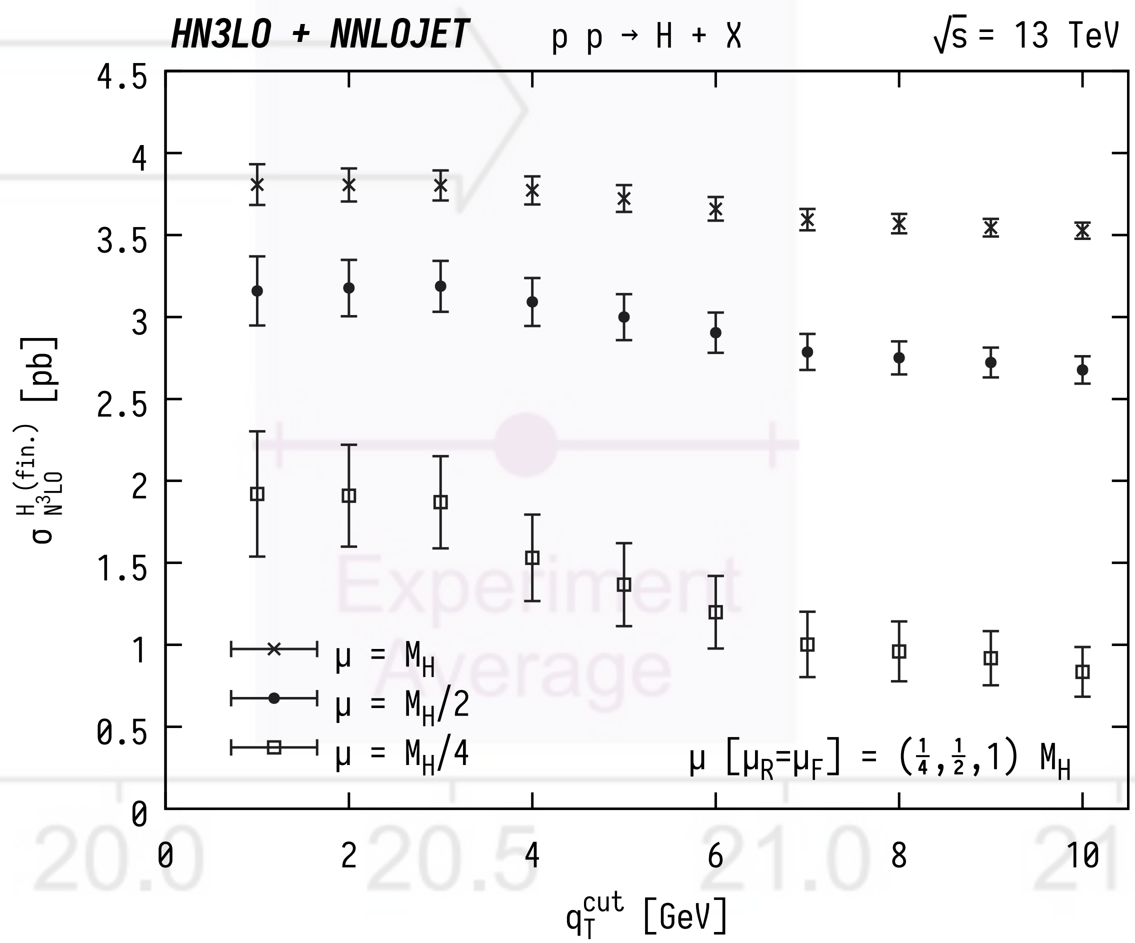
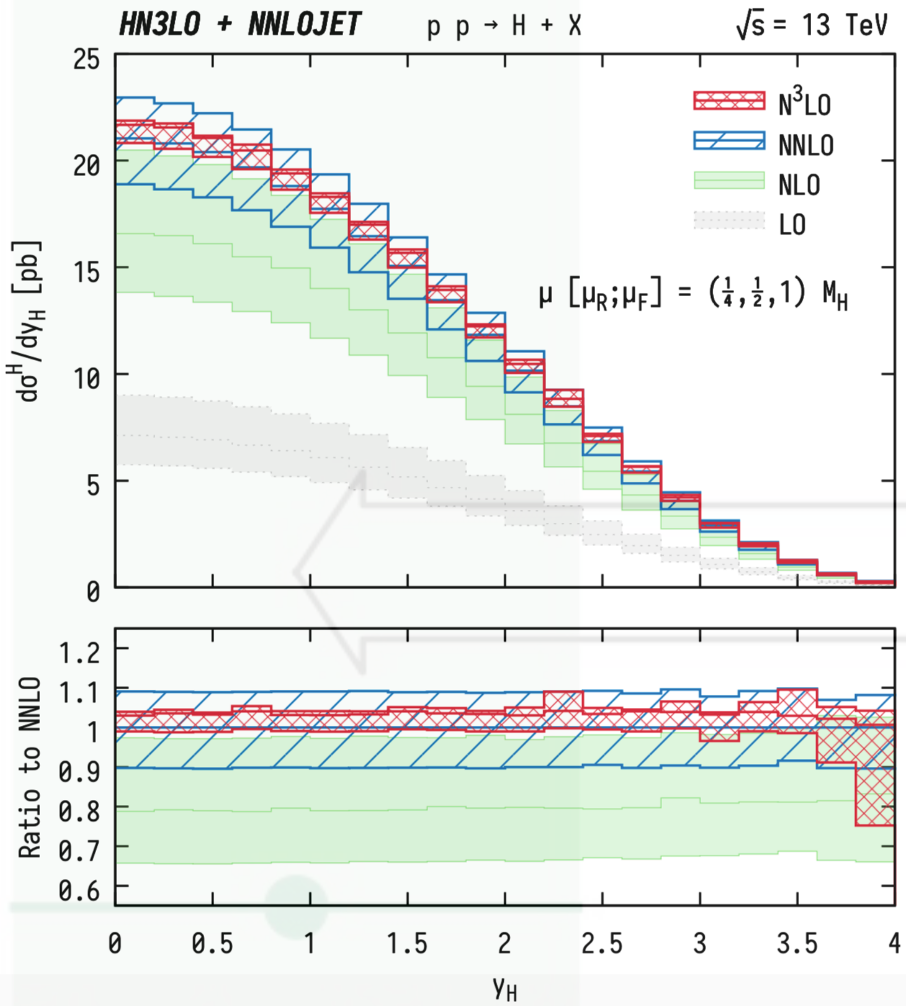


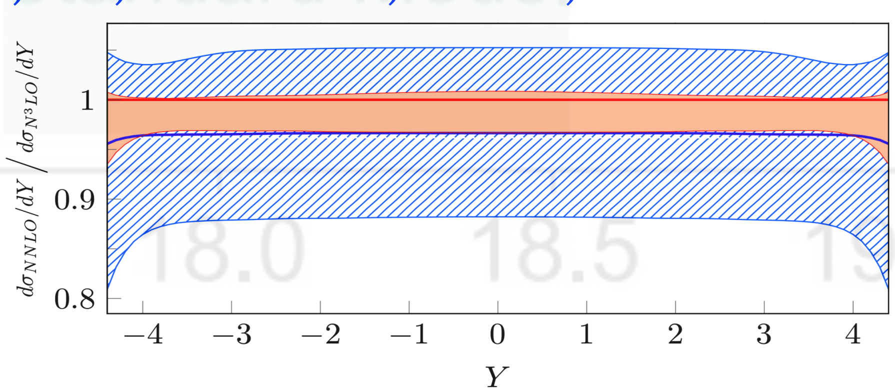
➤ Numerically abstract the C_{N3} coefficient using exact N3LO total cross section (1802.00833, 1802.00827).

HIGGS RAPIDITY DISTRIBUTIONS AT N3LO (APPROXIMATED)

- N3LO differential observables at the LHC from qT-subtraction and threshold expansion

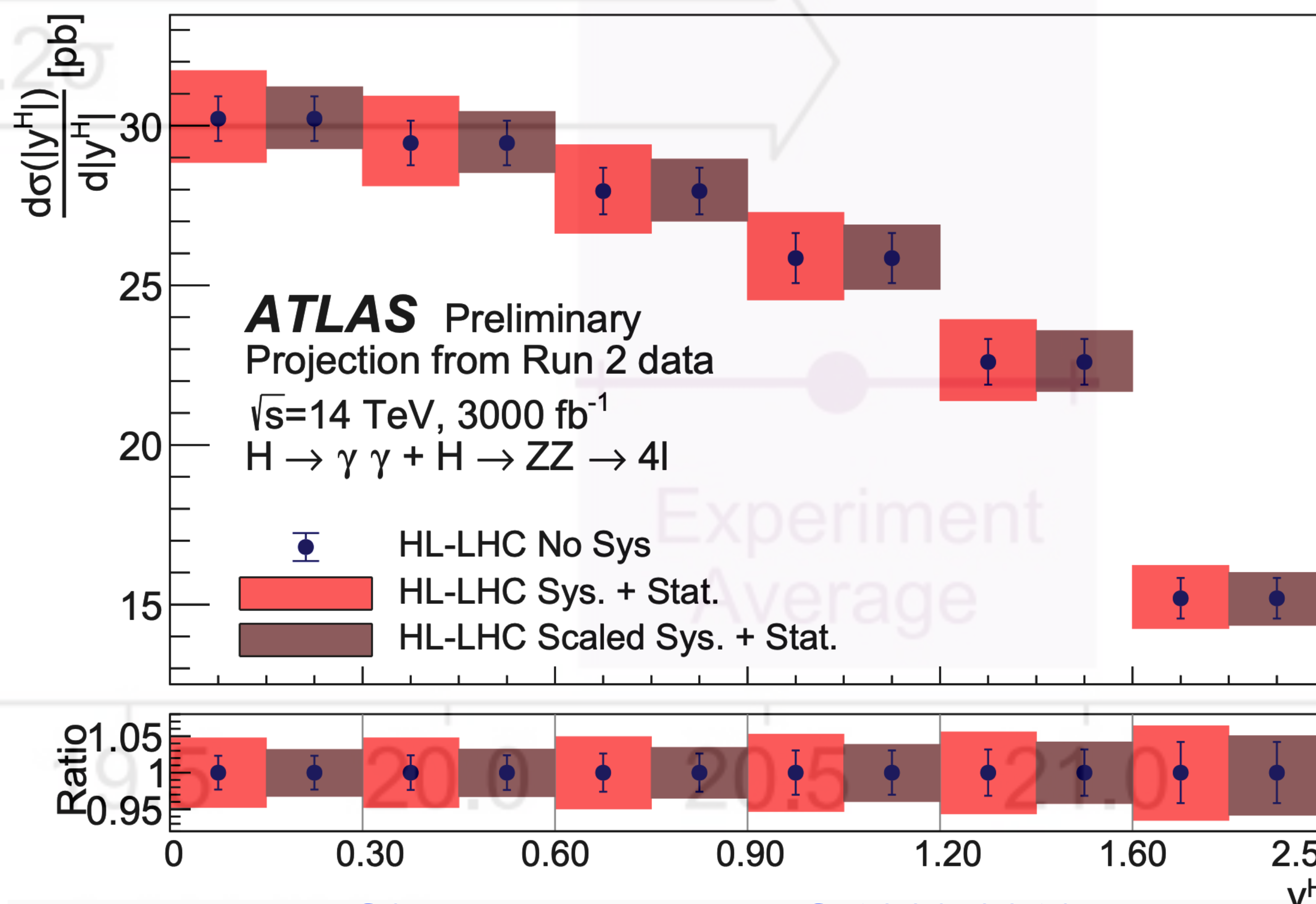


Cieri, XC, Gehrmann, Glover, Huss 1807.11501



Dulat, Mistlberger, Pelloni 1810.09462

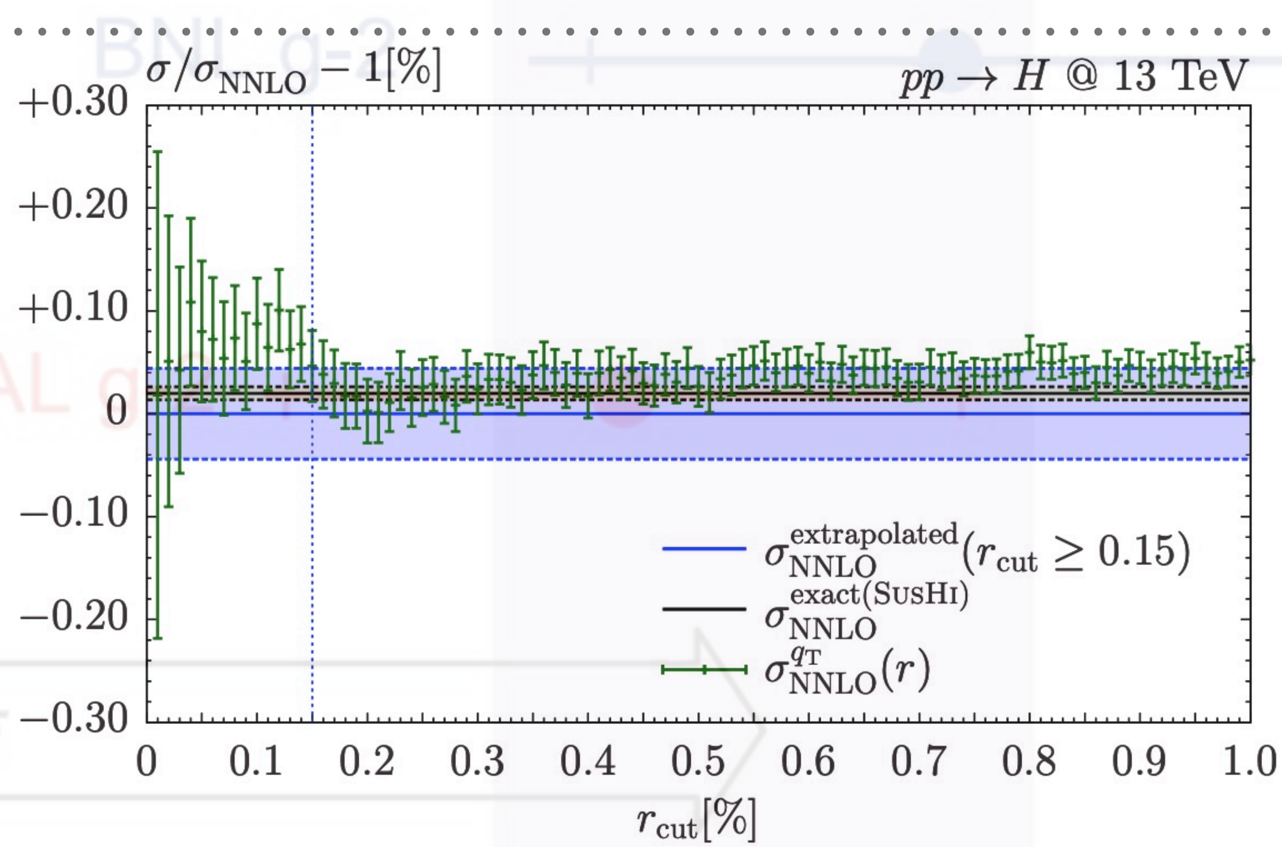
- Remarkably flat K-factor (as expected)
- QCD scale uncertainty reduced to $^{+1\%}_{-3\%}$
- Comparable to (S2) HL-LHC projections $\pm$ 3%
- Future upgrade to reduce PDF and α_s uncertainties

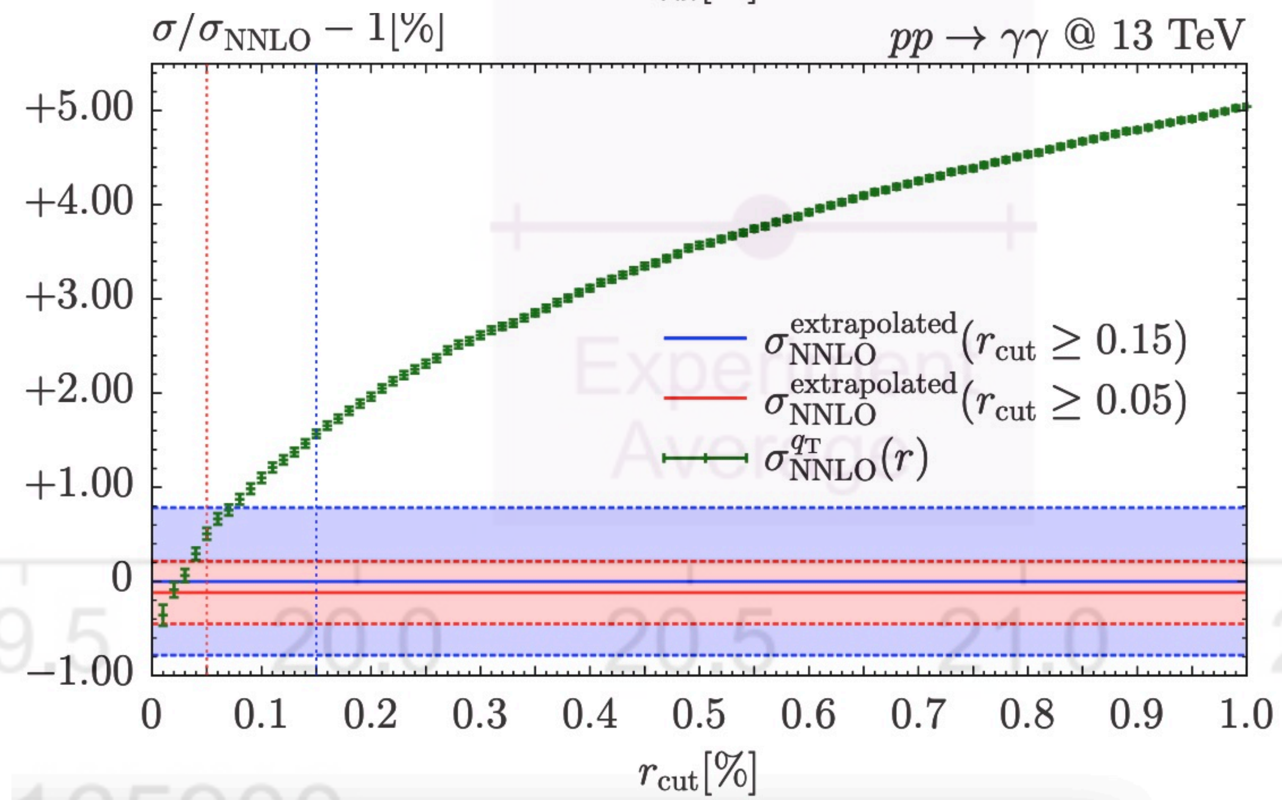


WG2 report on HL-LHC 1902.00134

LIMITATION OF QT SUBTRACTION AT N3LO

- EXP never measure directly the Higgs Boson but its decay products
  - Various fiducial cuts are needed to identify final state decay products (Photon-isolation, jet algorithm, lepton isolation, energy veto for neutrinos)
  - qT subtraction at NNLO already shown it limitation with fiducial cuts
  - More general NNLO subtraction methods are not yet ready for N3LO
- However, remember what is perfect subtraction method?
  - Use the scattering ME itself
  - Need integrated ME which is possible only for simple process
  - Did I mention the analytical calculation?



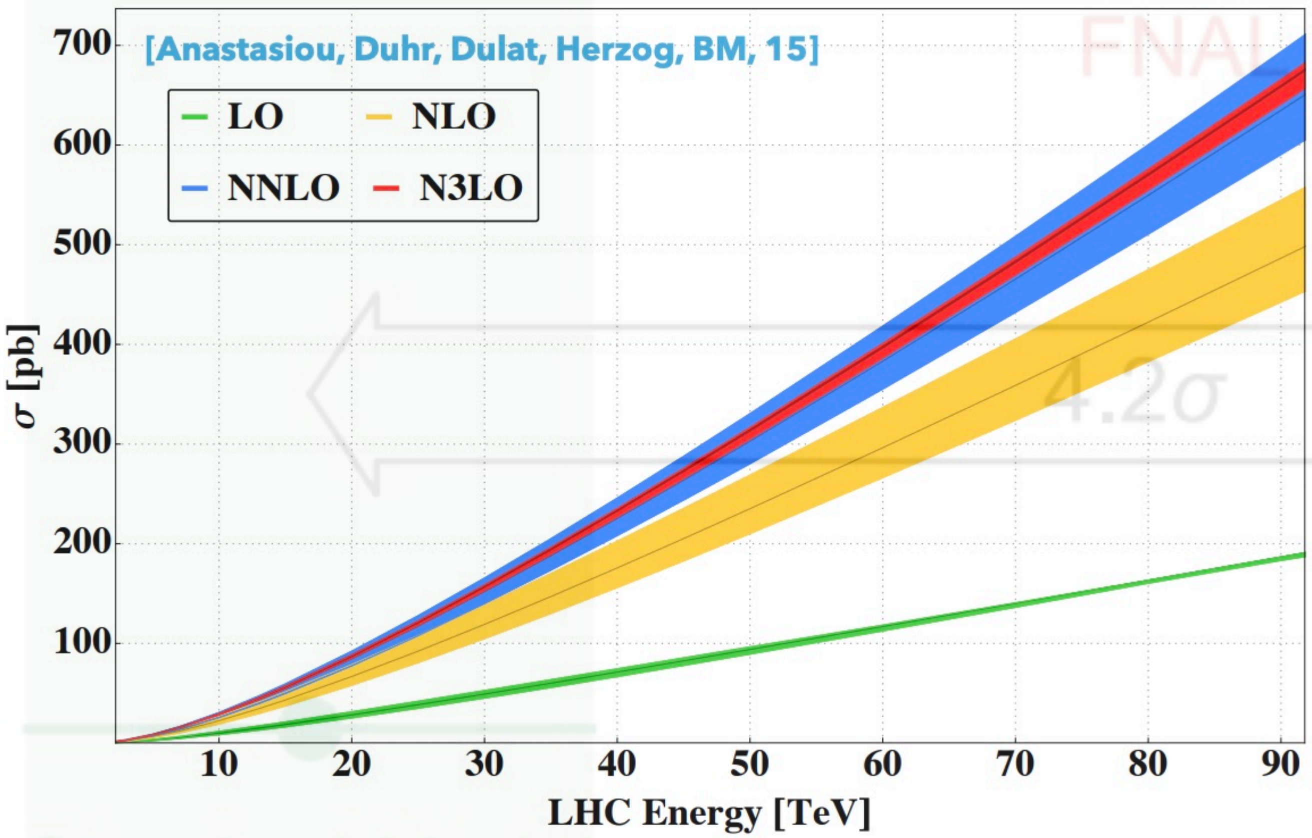


1711.06631

P2B SUBTRACTION AT N3LO

- N3LO Higgs total cross section was known in 2015
  - Complete integrate all real emissions, loop momentums to achieve total cross section

1503.06056



- Going from inclusive to differential in yH took 4 years
  - Keeping the rapidity information (PDF convolution fraction) and integrate all real emission kinematics
  - With x_1 and x_2 degree of freedom kept, we have all information for Born kinematic of Higgs production

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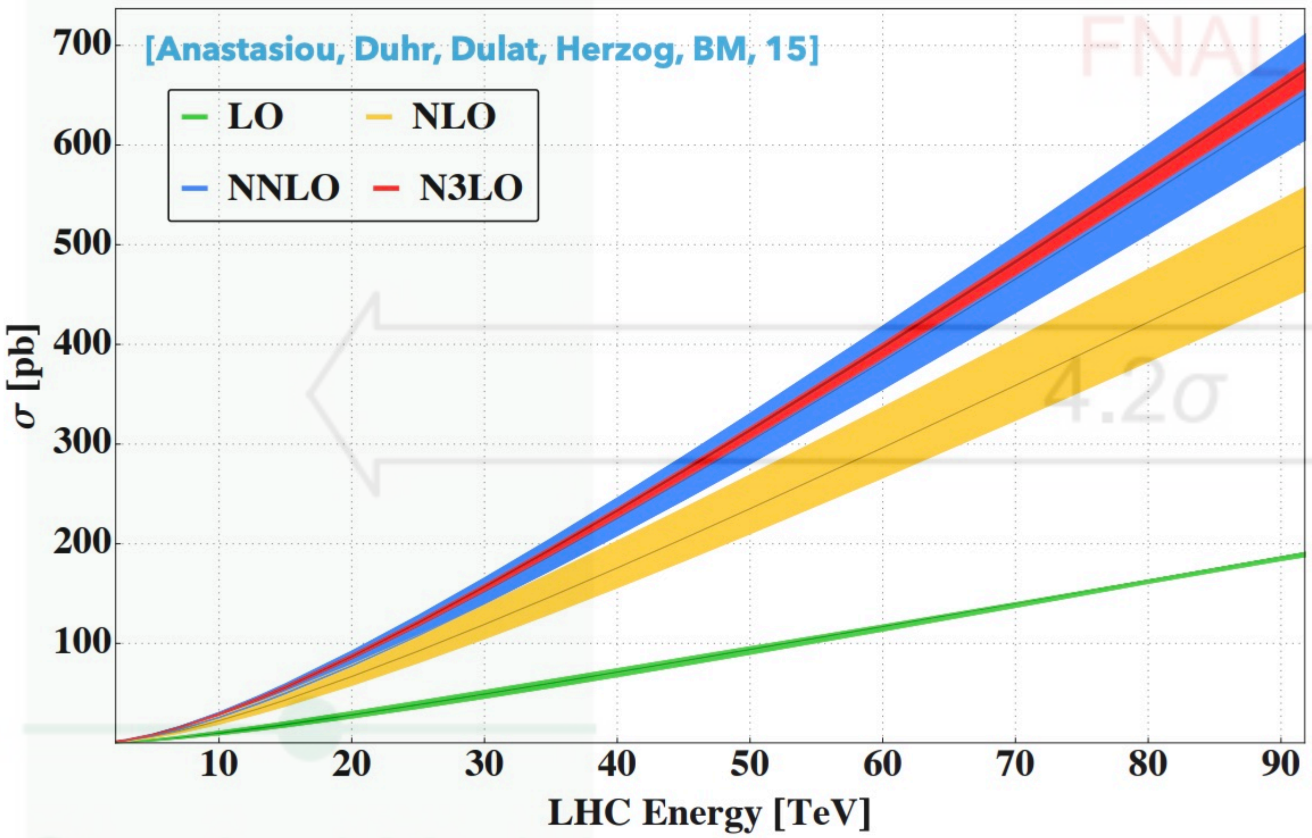
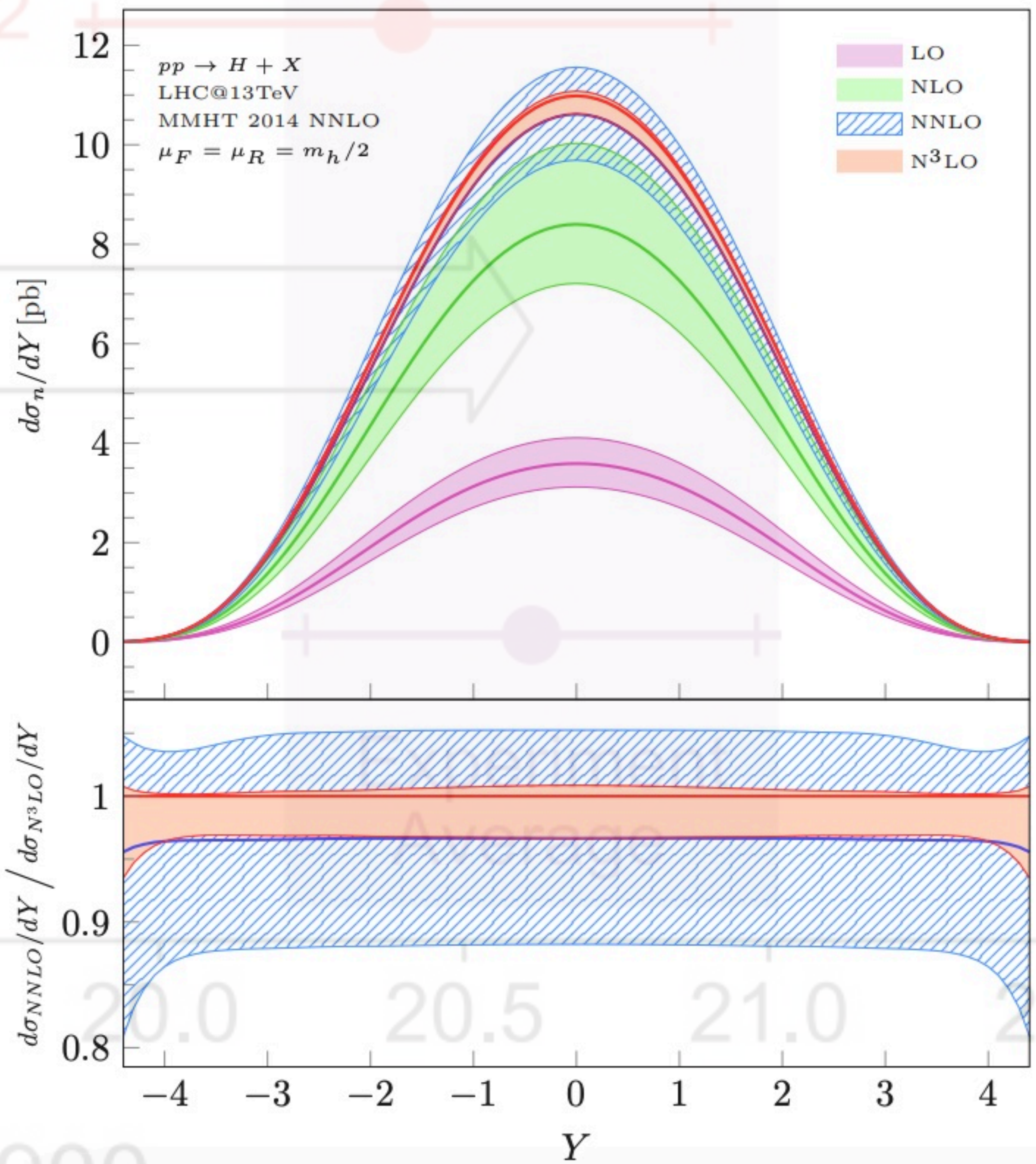


$$\frac{d\sigma_{P\,P\to H+X}}{dY} = \hat{\sigma}_0 \sum_{i,j} \int_0^1 dx_1 dx_2 dy_1 dy_2 f_i(y_1) f_j(y_2)$$

$$\times \delta(\tau - x_1 x_2 y_1 y_2)\delta\left(Y - \frac{1}{2}\log\left(\frac{x_1 y_1}{x_2 y_2}\right)\right)\eta_{ij}(x_1, x_2)$$



1810.09462

- Going from inclusive to differential in yH took 4 years
  - Keeping the rapidity information (PDF convolution fraction) and integrate all real emission kinematics
  - With x_1 and x_2 degree of freedom kept, we have all information for Born kinematic of Higgs production

P2B SUBTRACTION AT N3LO

- All ingredients ready for "Projection to Born" subtraction at N3LO
  - Higgs rapidity distribution retain all Born level differential information of Higgs production
  - Combine with H+J at NNLO with the perfect subtraction term for beyond Born kinematic

$$\frac{\mathrm{d}\sigma_F^{\mathrm{N}^k\mathrm{LO}}}{\mathrm{d}\mathcal{O}} = \left(\frac{\mathrm{d}\sigma_{F+\mathrm{jet}}^{\mathrm{N}^{(k-1)}\mathrm{LO}}}{\mathrm{d}\mathcal{O}} - \frac{\mathrm{d}\sigma_{F+\mathrm{jet}}^{\mathrm{N}^{(k-1)}\mathrm{LO}}}{\mathrm{d}\tilde{\mathcal{O}}} \right) + \frac{\mathrm{d}\sigma_F^{\mathrm{N}^k\mathrm{LO}}}{\mathrm{d}\tilde{\mathcal{O}}}$$

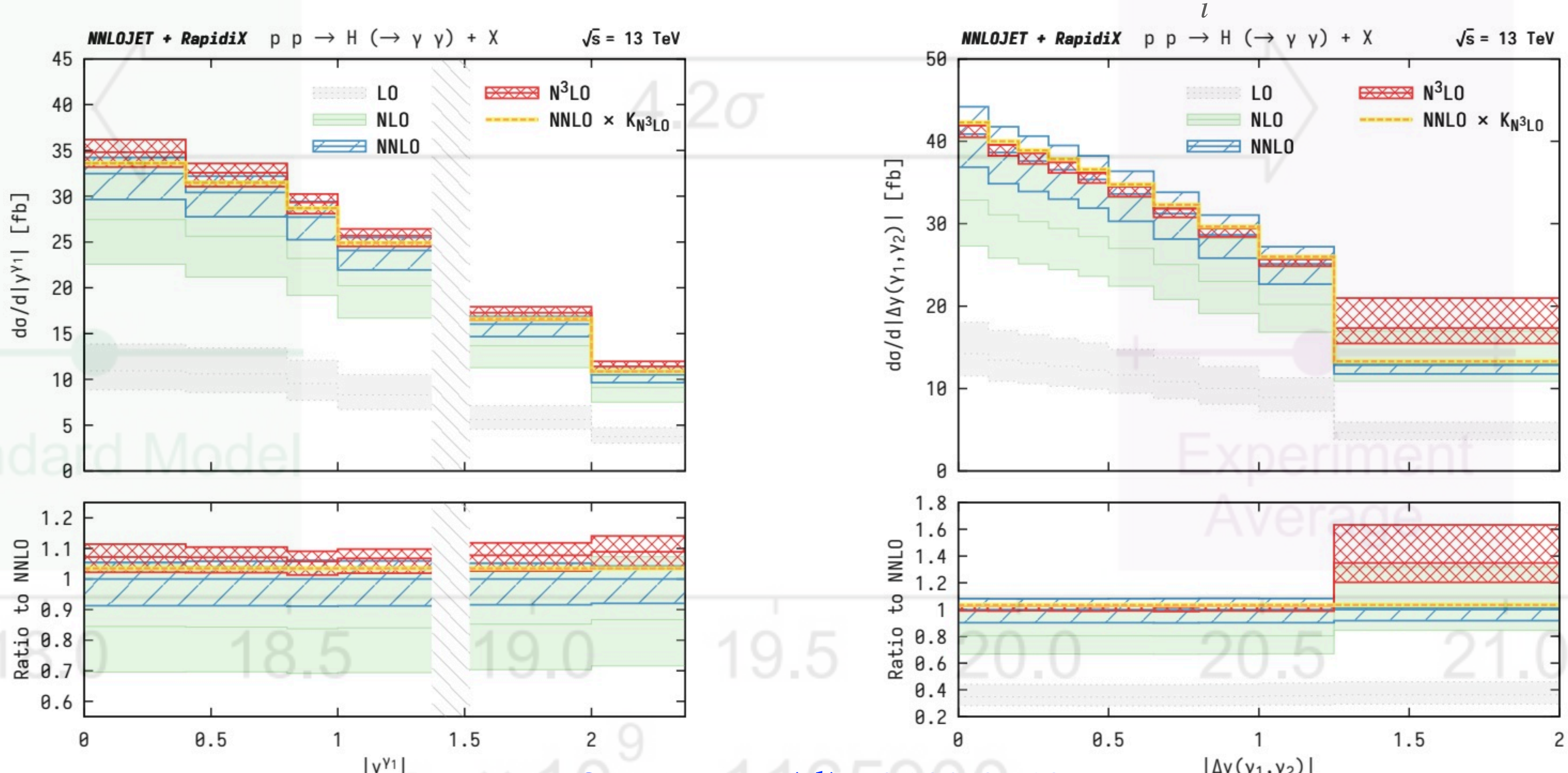
- General idea of "Projection to Born"
  - Use H+J at NNLO to subtract the IR divergence of H+J at NNLO
  - Define momentum mapping to map H+J, JJ, JJJ kinematic to H Born kinematic $\mathcal{O} \xrightarrow{P2B} \tilde{\mathcal{O}}$:
    - Perfect subtraction of IR divergence at Born kinematic $\mathcal{O} = \tilde{\mathcal{O}}$
    - Differential information of real radiations kept in normal P.S. $\mathcal{O} \neq \tilde{\mathcal{O}}$
    - A mapping retain Higgs rapidity: Initial-Initial Antenna mapping!
  - What has been subtracted in H+J at NNLO are already integrated and added in $d\sigma_{H+X}^{N3LO}/dY$
    - $d\sigma_{H+X}^{N3LO}/dY$ lives in Born kinematic phase space $\tilde{\mathcal{O}}$

P2B SUBTRACTION AT N3LO

- Joint effort for NNLOJET and RapidiX:
  - Projet H+J@NNLO onto inclusive Higgs rapidity distribution from RapidiX
  - In Higgs to di-photon decay channel, apply LHC experiment fiducial cuts:

$$p_T^{\gamma_1} > 0.35 \times m_H, \quad p_T^{\gamma_1} > 0.25 \times m_H, \quad |\eta^\gamma| < 2.37 \text{ excluding } 1.37 < |\eta^\gamma| < 1.52$$

$$\text{For } p_i \in \Delta R_{i,\gamma} < 0.2 \text{ and } E_T^i > 1 \text{ GeV, only keep event with } \sum_i E_T^i < 5\% \times E_T^\gamma$$



NNLOJET + RapidiX 2102.07607

SUMMARY

- With limited theoretical tools to predict hadron collision, we could explain experimental results and test the Standard Model
- High Energy Physics is advancing to precision study at a steady speed, new breakthrough is expected for every decade
- NNLO QCD is the new standard for precision study, more consistent update to PDF and α_s will be available in the future
- NNLO+N3LL and N3LO predictions are available for limited observables. With realistic projection of theory progress, we can expect promising precisions at HL-LHC accuracy.
- More flexible subtraction methods are needed for 2 to 3 scattering at NNLO and 2 to 2 scattering at N3LO.
- New physics are already there, we need better tools to find them.

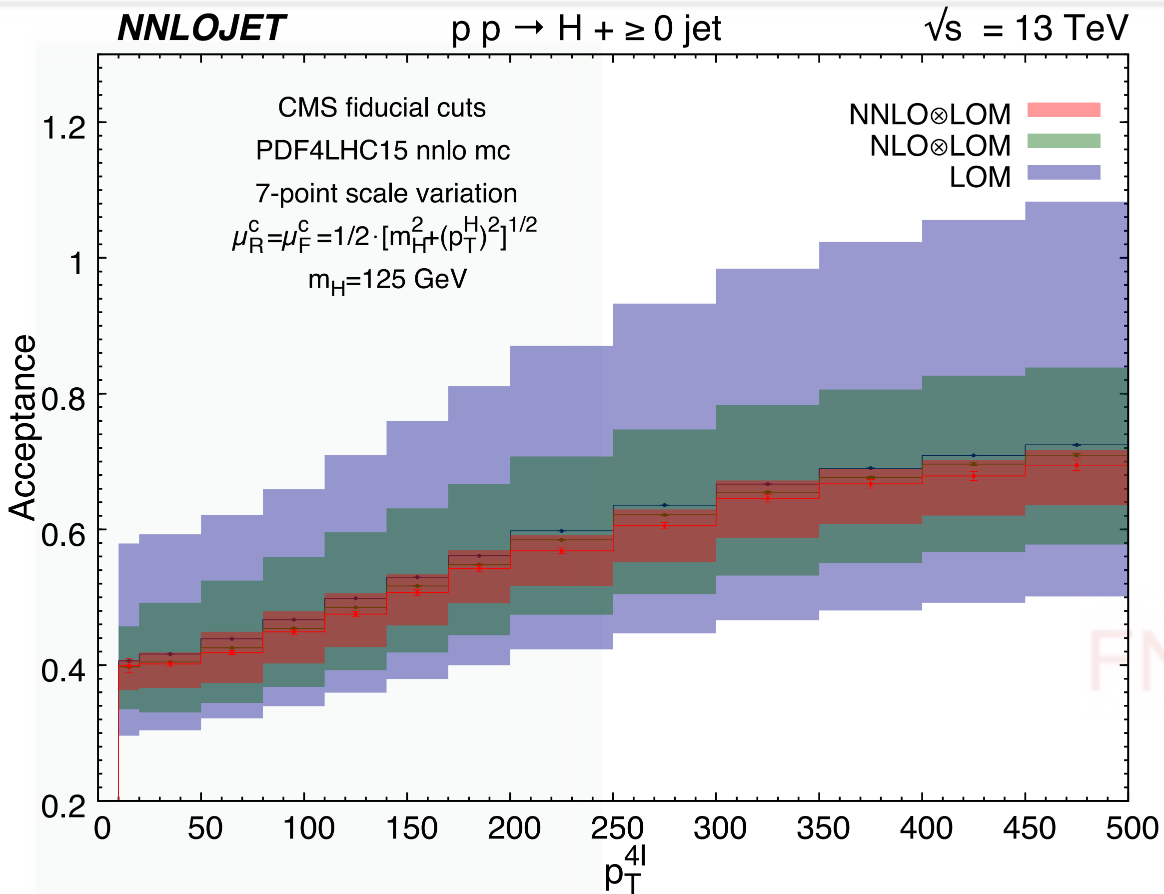
Thank You for your Attention

Total time (int. dimension Of the tree level)	LO	NLO	NNLO
H	1 min (3)	30 min (6)	300h (9)
H—>di-photon	1 min (3)	40 min (6)	400h (9)
H—>4l (2e2mu, 4e, 4mu require at least two separate runs)	2~3 min (9)	2h (12)	1000h (15)
H+j	3 min (6)	1.5h (9)	70000h (12)
H—>di-photon + jet	4 min (6)	2h (9)	90000h (12)
H—>4l (2e2mu, 4e, 4mu require at least two separate runs)+jet	20 min (12)	10h (15)	600000h (18)
H_qT	20 min (6)	5h (9)	7000000h (12)

ACCEPTANCE STUDY

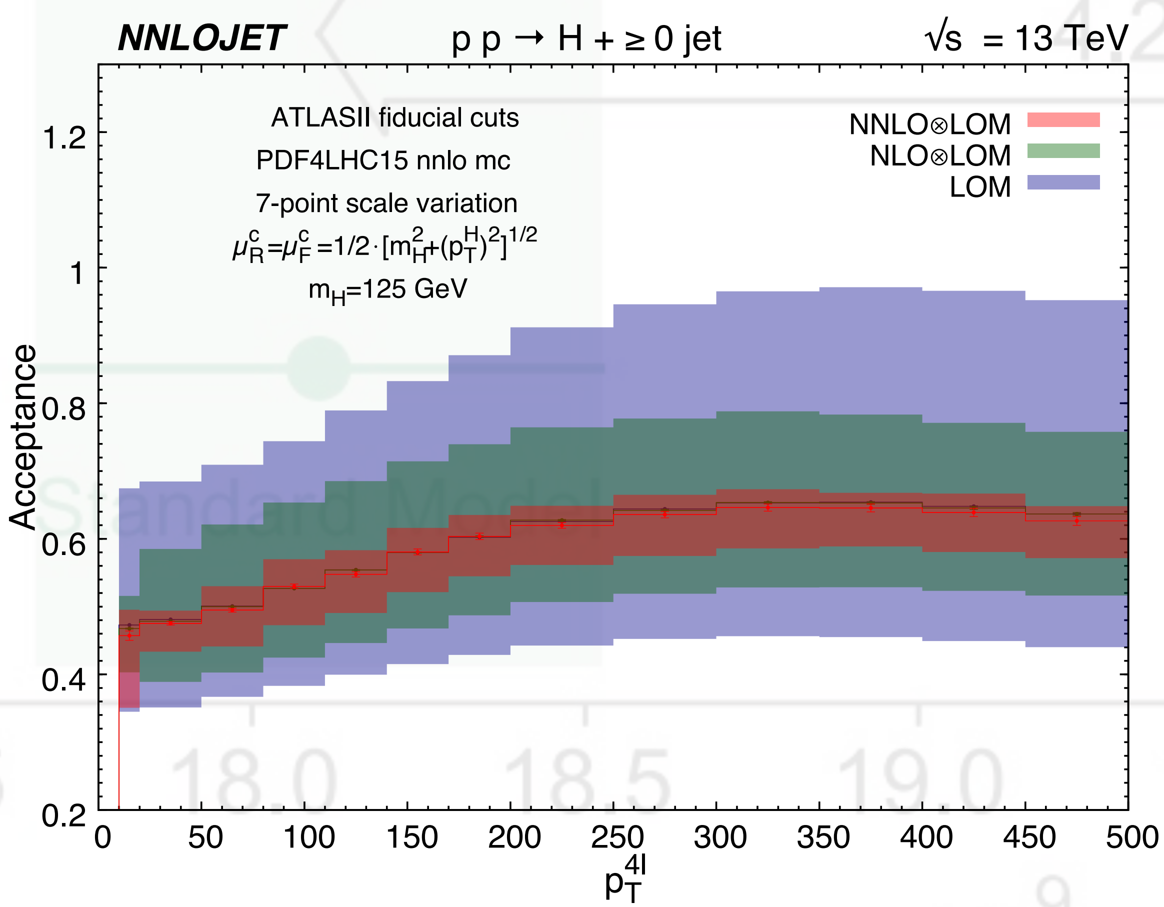
$H \to ZZ^* \to 4l$

CMS cuts



Acceptance deviate from each FO

ATLAS cuts



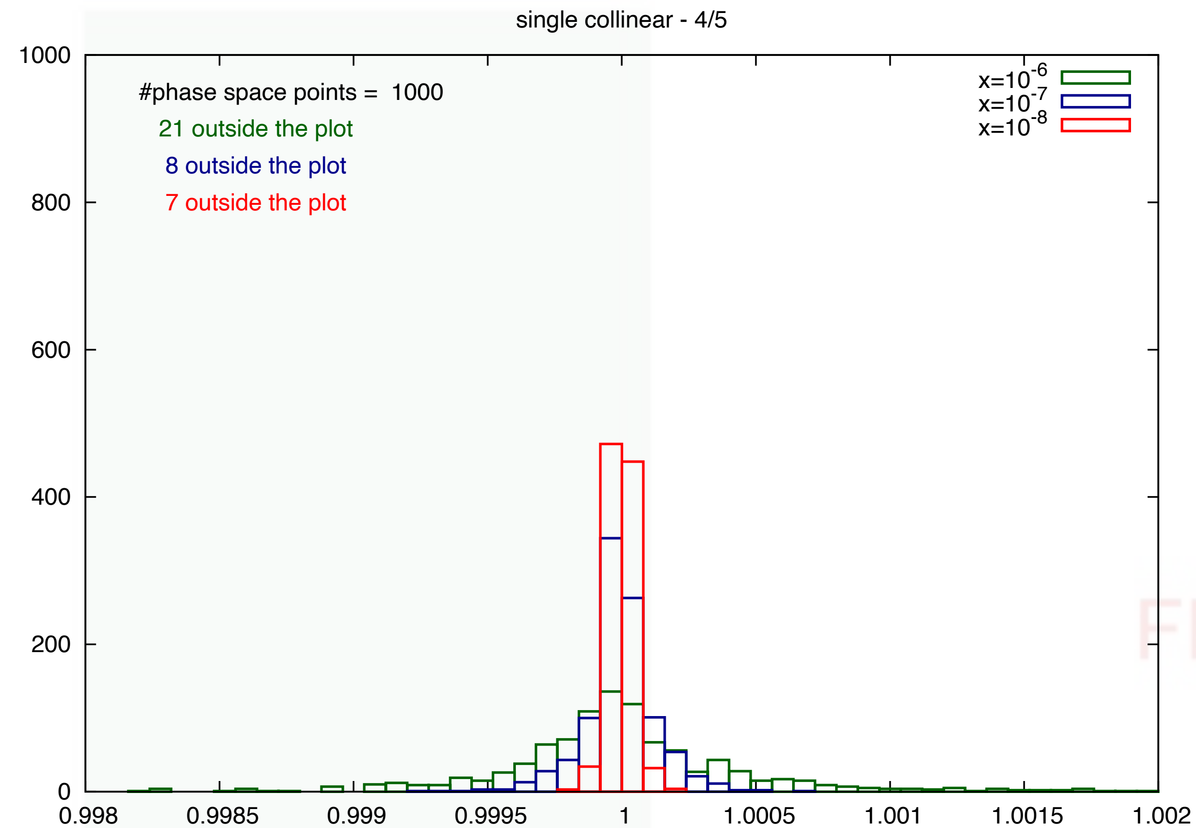
Acceptance consistent for each FO

- CMS (1706.09936) and ATLAS (1708.02810) use different lepton isolation algorithm in $ZZ^* \to 4l$

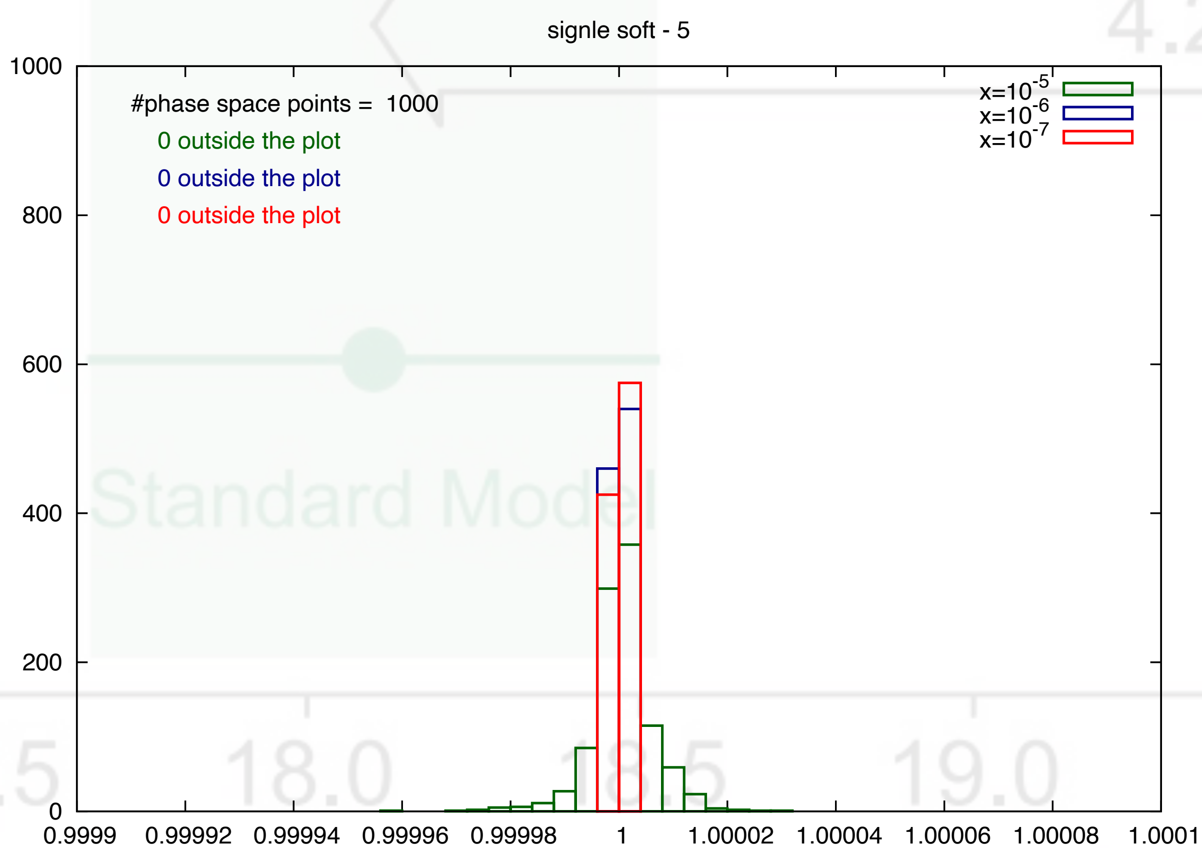
Fiducial Cuts	CMS	ATLAS
Lepton Isolation		
Cone size R^l	0.3	—
$\sum p_T^i / p_T^l \ (i \in R^l)$	< 35%	—
$\Delta R^{SF(DF)}(l_i, l_j)$	> 0.02	> 0.1(0.2)
Jet Definition (anti-kT with R=0.4)		
p_T^{jets} (GeV)	> 30	> 30
$\lvert y^{jets} \rvert$	< 2.5	< 4.4
$\Delta R(jet, e(\mu))$	—	> 0.2(0.1)

- Fixed order study of acceptance reveals detailed structures

$$A_{FO}(\mathcal{O}) = \frac{d\sigma_{FO}^{H(\to ZZ^* \to 4l)+jet}/d\mathcal{O}}{d\sigma_{FO}^{H+jet}/d\mathcal{O} \times (BR_{2e2\mu} + BR_{4\mu} + BR_{4e})}$$



Single collinear limit 4//5 with $x \sim 10^{-8}$



Single soft limit $5 \to 0$ with $x \sim 10^{-7}$

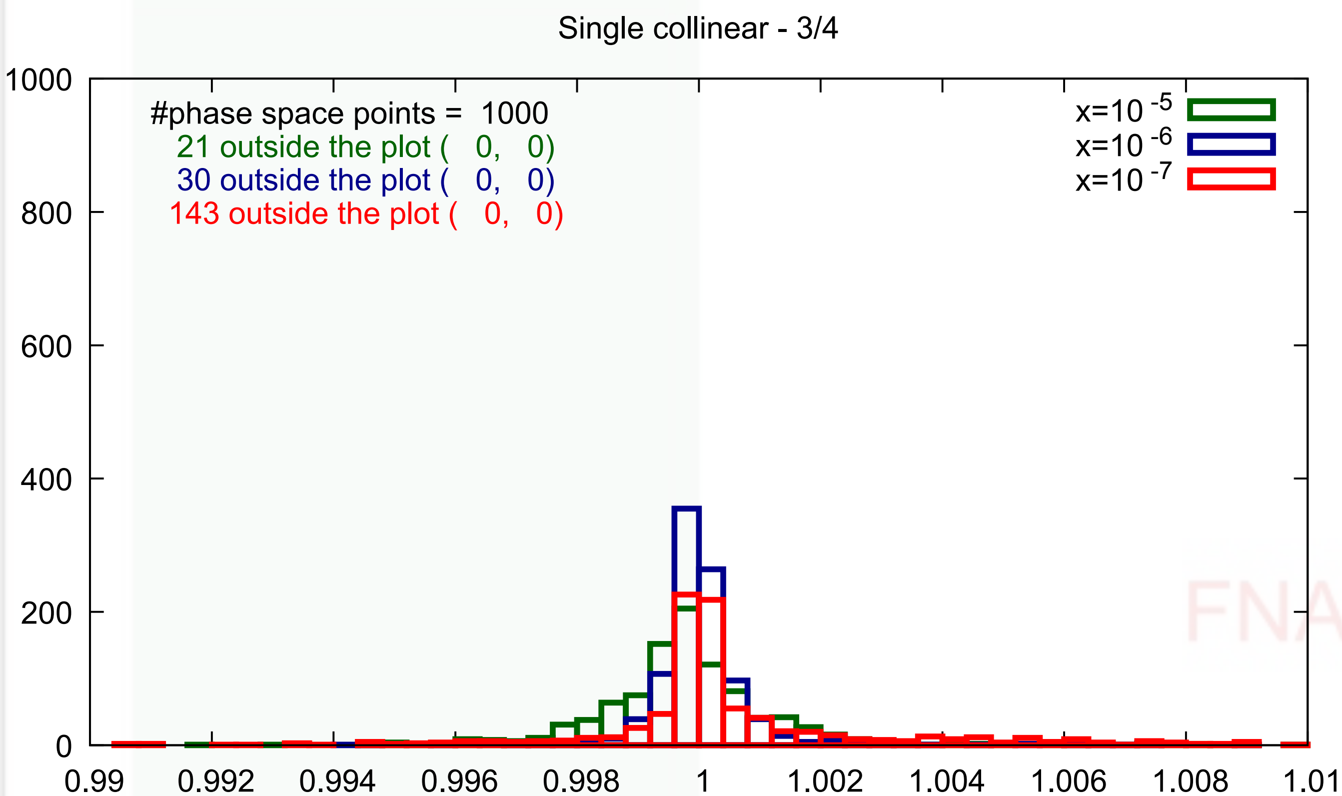
TEST NUMERICAL STABILITY OF MATRIX ELEMENTS

- Construct antenna subtraction terms (ATS) to mimic unresolved limits of matrix elements (ME)
- Test function (tree level): $R = \dfrac{ME^0}{AST^0}$
- R ~ the horizontal axis (centre at one near the unresolved region)
- Number of P.S. points in each bin ~ the vertical axis
- Controlling singular region correctly will achieve spike plots
- For example: $p_1 + p_2 \to p_3 + p_4 + p_5$

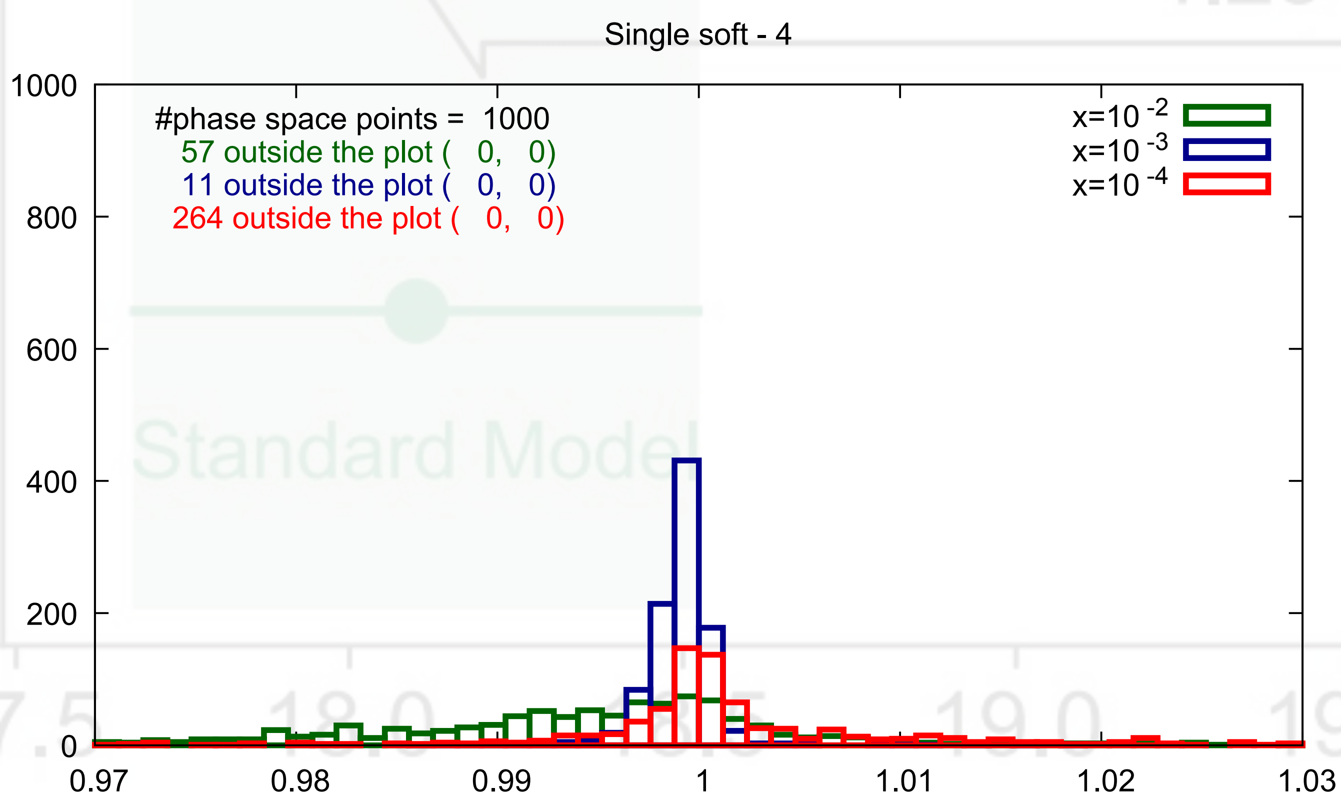
Single collinear limit: $x = \dfrac{s_{45}}{s}, \quad x \sim 10^{-8}$

Single soft limit: $xs = s_{35} + s_{45}, \quad x \sim 10^{-7}$

TEST NUMERICAL STABILITY OF MATRIX ELEMENTS



Single collinear limit 3//4 with $x \sim 10^{-6}$



Single soft limit $4 \to 0$ **with** $x \sim 10^{-3}$

- Ideally we would like to use ME from automated tools
- However, not many of them are numerical stable in IR singular regions
- OpenLoops2 is one of the best auto-tools optimised in IR singular regions
- However for a loop-induced process:

$$g_1 + g_2 \to \gamma + \gamma + g_3 + g_4$$

- Test function (loop induced):

$$R = \frac{ME^1}{AST^1}$$

- We observe spikes break down at
  single collinear limit: $x \sim 10^{-7}$
  single soft limit: $x \sim 10^{-4}$