Technique in AMF

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Loop is important…

Auxiliary Mass Flow (AMF)

Xiao Liu, Yan-Qing Ma, Chen-Yu Wang, Phys.Lett.B 779 (2018) 353-357
Xiao Liu, Yan-Qing Ma, Phys.Rev.D 99 (2019) 7, 071501

$$\mathcal{M}(D,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha + \mathrm{i}\eta)^{\nu_\alpha}},$$

$$\mathcal{M}(D,\vec{s},0^+) \equiv \lim_{\eta\to 0^+} \mathcal{M}(D,\vec{s},\eta),$$

$\eta \to \infty$ **Taylor expansion, i.e. series representation.**

Direct Taylor expansion (rescale)

$$q^\mu \to \sqrt{\eta}\, q^\mu$$

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{\left(\mathscr{D}_\alpha + \mathrm{i}\eta\right)^{\nu_\alpha}} = \eta^{DL/2-N_\nu} \sum_{k=0}^{\infty} \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{(\ell \cdot k)^m}{\left(\ell_\alpha^2 + \mathrm{i}\right)^{\nu_\alpha}}$$

$$= \eta^{DL/2-N_\nu} \sum_{k=0}^{\infty} \left(C_{k,1} I_1^{vac} + C_{k,2} I_2^{vac}\right) \eta^{-k}$$

Expansion based on Feynman parameterization

$$\widetilde{G}^{\mu_1\ldots\mu_R}_{\ell_1\ldots\ell_R} \equiv \int \mathbb{D}^L q \, \frac{q^{\mu_1}_{\ell_1}\ldots q^{\mu_R}_{\ell_R}}{\prod_{i=1}^{n}[(Q_i+K_i)^2 - m_i^2 + \imath\eta]^{\nu_i}} .$$

$$\widetilde{G}^{\mu_1\ldots\mu_R}_{\ell_1\ldots\ell_R} = \frac{(-1)^{N_\nu}}{\prod_{j=1}^{n}\Gamma(\nu_j)} \int \prod_{j=1}^{n} \mathrm{d}x_j \; x_j^{\nu_j-1} \delta(1-\sum_{l=1}^{n} x_l)$$
$$\times \sum_{m=0}^{[R/2]} \frac{\Gamma(N_\nu^{(m)})}{(-2)^m} \left[(\tilde{M}^{-1}\otimes g)^{(m)} \widetilde{\ell}^{(R-2m)}\right]^{\Gamma_1,\ldots,\Gamma_R}$$
$$\times U^{-D/2+m-R}\left(\frac{F}{U} - \imath\eta\right)^{-N_\nu^{(m)}} , \qquad (6)$$

Expansion based on Feynman parameterization

$$\widetilde{G}^{\mu_1\ldots\mu_R}_{\ell_1\ldots\ell_R} = \frac{(-1)^{N_\nu}}{\prod_{j=1}^n \Gamma(\nu_j)} \int \prod_{j=1}^{n} \mathrm{d}x_j \; x_j^{\nu_j-1} \delta(1-\sum_{l=1}^{n} x_l)$$
$$\times \sum_{m=0}^{[R/2]} \frac{\Gamma(N_\nu^{(m)})}{(-2)^m} \left[(\tilde{M}^{-1} \otimes g)^{(m)} \tilde{\ell}^{(R-2m)} \right]^{\Gamma_1,\ldots,\Gamma_R}$$
$$\times U^{-D/2+m-R} \left(\frac{F}{U} - \imath\eta \right)^{-N_\nu^{(m)}}, \quad (6)$$

$$v_l = \sum_{i=1}^{L} M^{-1}_{li} Q_i \,.$$

$$\mathcal{F}(\mathbf{x}) = \det(M) \left[\sum_{j,l=1}^{L} Q_j M^{-1}_{jl} Q_l - J - i\delta \right],$$

$$\mathcal{U}(\mathbf{x}) = \det(M)\,, \quad \tilde{M}^{-1} = \mathcal{U} M^{-1}\,, \quad \tilde{l} = \mathcal{U} v$$

$$G^{\mu_1\cdots\mu_R}_{l_1\cdots l_R} = \frac{\Gamma(N_\nu)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^{N} dx_j \, x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \int d^D\kappa_1 \cdots d^D\kappa_L$$
$$\times k_{l_1}^{\mu_1} \cdots k_{l_R}^{\mu_R} \left[\sum_{i,j=1}^{L} k_i^{\mathrm{T}} M_{ij} k_j - 2 \sum_{j=1}^{L} k_j^{\mathrm{T}} \cdot Q_j + J + i\delta \right]^{-N\nu},$$

Expansion based on Feynman parameterization

$$\widetilde{G}^{\mu_1\ldots\mu_R}_{\ell_1\ldots\ell_R} = \frac{(-1)^{N_\nu}}{\prod_{j=1}^{n}\Gamma(\nu_j)} \int \prod_{j=1}^{n} \mathrm{d}x_j \; x_j^{\nu_j-1} \delta(1-\sum_{l=1}^{n} x_l)$$
$$\times \sum_{m=0}^{[R/2]} \frac{\Gamma(N_\nu^{(m)})}{(-2)^m} \left[(\tilde{M}^{-1} \otimes g)^{(m)} \tilde{\ell}^{(R-2m)}\right]^{\Gamma_1,\ldots,\Gamma_R}$$
$$\times U^{-D/2+m-R} \left(\frac{F}{U} - \imath\eta\right)^{-N_\nu^{(m)}}, \tag{6}$$

$$\widetilde{\mathcal{M}}(\eta) = \sum_i \mathcal{C}_i \mathcal{F}_i,$$

$$\left(\frac{F}{U} - \imath\eta\right)^{-N_\nu^{(m)}} = (-\imath\eta)^{-N_\nu^{(m)}} \sum_{p=0}^{\infty} \binom{-N_\nu^{(m)}}{p} \times \frac{F^p}{U^p(-\imath\eta)^p}.$$

Expansion based on Feynman parameterization

$$\int \prod_{j=1}^{n} \mathrm{d}x_j \; x_j^{n_j-1} \delta(1 - \sum_{l=1}^{n} x_l) U^{-\tilde{D}/2},$$

$$\frac{1}{(q^2 + 2q \cdot k + k^2 - m^2 + \imath\eta)^n} \Rightarrow \frac{N}{(q^2 + \imath\eta)^n}$$

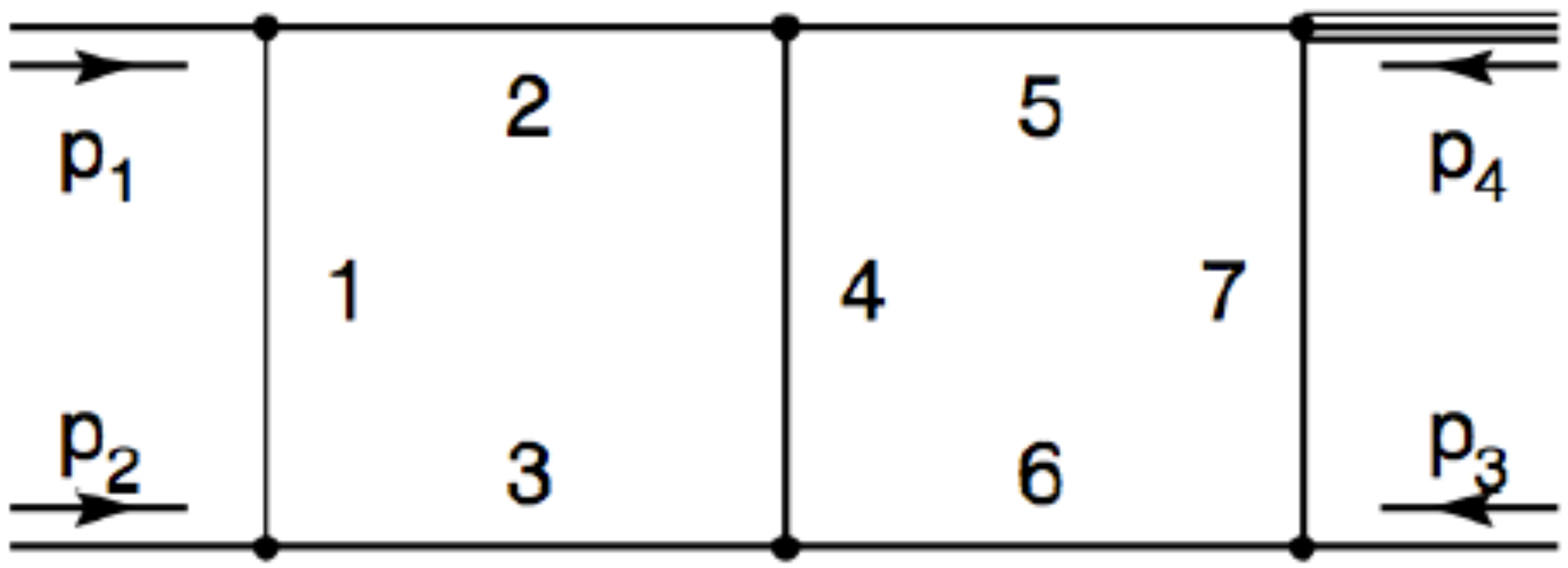
$$\mathcal{D}_1 = (q_1 - k_1)^2 \to x_1, \qquad \mathcal{D}_2 = (q_1 + k_2)^2 \to x_2,$$

$$\mathcal{D}_3 = (q_2 + k_1 + k_2)^2 \to x_3, \qquad \mathcal{D}_4 = q_2^2 \to x_4,$$

$$\mathcal{D}_5 = q_1^2 \to x_5, \qquad \mathcal{D}_6 = (q_1 - q_2 - k_1)^2 \to x_6,$$

Expansion based on Feynman parameterization

U and F can
be determined
geometrically



$$\mathcal{U}(\mathbf{x}) = \sum_{T \in \mathcal{T}_1} \Big[\prod_{j \in \mathcal{C}(T)} x_j \Big] ,$$

$$\mathcal{F}_0(\mathbf{x}) = \sum_{\hat{T} \in \mathcal{T}_2} \Big[\prod_{j \in \mathcal{C}(\hat{T})} x_j \Big] (-s_{\hat{T}}) ,$$

$$\mathcal{F}(\mathbf{x}) = \mathcal{F}_0(\mathbf{x}) + \mathcal{U}(\mathbf{x}) \sum_{j=1}^{N} x_j m_j^2 .$$

$$\mathcal{U} = x_{123} x_{567} + x_4 x_{123567} ,$$

$$\mathcal{F} = (-s_{12})(x_2 x_3 x_{4567} + x_5 x_6 x_{1234} + x_2 x_4 x_6 + x_3 x_4 x_5) + (-s_{23}) x_1 x_4 x_7 + (-p_4^2) x_7 (x_2 x_4 + x_5 x_{1234}) ,$$

where $x_{ijk\dots} = x_i + x_j + x_k + \cdots$ and $s_{ij} = (p_i + p_j)^2$.

Expansion based on Feynman parameterization

$$\int \mathrm{d}y_1 \delta \left(y_1 - \sum_{j \in [i_1]} x_j \right) = 1.$$

$$\int \mathrm{d}y_1\, \delta\left(y_1 - x_1 - x_2 - x_5\right) = 1,$$

$$\int \mathrm{d}y_2\, \delta\left(y_2 - x_3 - x_4\right) = 1,$$

$$\int \mathrm{d}y_3\, \delta\left(y_3 - x_6\right) = 1.$$

Expansion based on Feynman parameterization

$$
\begin{aligned}
&\int \prod_{k=1}^{n} \mathrm{d}x_k \; x_k^{n_k-1} \delta\left(1-\sum_{l=1}^{n} x_l\right) U^{-\widetilde{D}/2} \\
=&\int \prod_{m=1}^{X} \left(\prod_{j\in[i_m]} \left(\mathrm{d}x_j \; x_j^{n_j-1}\right)\right) \delta\left(1-\sum_{l=1}^{n} x_l\right) U^{-\widetilde{D}/2} \\
=&\int \prod_{m=1}^{X} \left(\prod_{j\in[i_m]} \left(\mathrm{d}x_j \; x_j^{n_j-1}\right) \mathrm{d}y_m \; \delta\left(y_m - \sum_{p\in[i_m]} x_p\right)\right) \\
&\times \delta\left(1-\sum_{l=1}^{n} x_l\right) U^{-\widetilde{D}/2} \\
=&\int \prod_{m=1}^{X} \left(\mathrm{d}y_m \; \frac{\prod_{j\in[i_m]} \Gamma(n_j)}{\Gamma\left(\sum_{j\in[i_m]} n_j\right)} y_m^{\left(\sum_{j\in[i_m]} n_j\right)-1}\right) \\
&\times \delta\left(1-\sum_{l=1}^{X} y_l\right) U^{-\widetilde{D}/2}. \qquad (11)
\end{aligned}
$$

Expansion based on Feynman parameterization

$$\int \prod_{j=1}^{6} \mathrm{d}x_j \, x_1 x_2^2 x_3^2 x_6^2 \delta(1 - \sum_{l=1}^{6} x_l) U^{-\widetilde{D}/2}$$

$$= \int (\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_5 \mathrm{d}y_1 \, \delta\,(y_1 - x_1 - x_2 - x_5) x_1 x_2^2) \, (\mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}y_2 \, \delta\,(y_2 - x_3 - x_4) x_3^2)$$

$$\times \, (\mathrm{d}x_6 \mathrm{d}y_3 \delta\,(y_3 - x_6) x_6^2) \; \delta(1 - \sum_{l=1}^{6} x_l) U^{-\widetilde{D}/2}$$

$$= \frac{1}{180} \int \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \, y_1^5 y_2^3 y_3^2 \, \delta(1 - y_1 - y_2 - y_3) U^{-\widetilde{D}/2},$$

Expansion based on Feynman parameterization

$$I^{(vac),\widetilde{D}}_{\nu_1,\nu_2,\nu_3} \equiv \int \frac{\mathrm{d}^{\widetilde{D}}q_1 \mathrm{d}^{\widetilde{D}}q_2}{[q_1^2+\imath]^{\nu_1}[q_2^2+\imath]^{\nu_2}[(q_1+q_2)^2+\imath]^{\nu_3}}$$
$$= (-i)^{\widetilde{D}+N_\nu} \int \frac{y_1^{\nu_1-1}\mathrm{d}y_1}{\Gamma(\nu_1)} \frac{y_2^{\nu_2-1}\mathrm{d}y_2}{\Gamma(\nu_2)} \frac{y_3^{\nu_3-1}\mathrm{d}y_3}{\Gamma(\nu_3)}$$
$$\times\delta(1-y_1-y_2-y_3)\Gamma(N_\nu-\widetilde{D})U^{-\widetilde{D}/2}. \quad (12)$$

$$\int \prod_{j=1}^{6} \mathrm{d}x_j\, x_1 x_2^2 x_3^2 x_6^2 \delta(1-\sum_{l=1}^{6} x_l) U^{-\widetilde{D}/2}$$
$$= \frac{1}{180}\int \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3\, y_1^5 y_2^3 y_3^2\, \delta(1-y_1-y_2-y_3)U^{-\widetilde{D}/2}$$
$$= 4 I^{(vac),D+6}_{\{6,4,3\}}.$$

$$I^{(vac),\widetilde{D}}_{\{\nu_1,\nu_2,\nu_3\}} \to I^{(vac),D}_{\{1,1,1\}}, I^{(vac),D}_{\{1,1,0\}}$$

$$I^{(vac),D+4}_{\{2,1,1\}} = \frac{3\imath(D-3)}{(D-1)D(D+1)(D+2)} I^{(vac),D}_{\{1,1,1\}} - \frac{(D-3)(7D-4)}{(D-1)D^2(D+1)(D+2)} I^{(vac),D}_{\{1,1,0\}}$$

Expansion based on Feynman parameterization

$$\widetilde{G}^{\mu_1\ldots\mu_R}_{\ell_1\ldots\ell_R} = \frac{(-1)^{N_\nu}}{\prod_{j=1}^{n}\Gamma(\nu_j)} \int \prod_{j=1}^{n} \mathrm{d}x_j \; x_j^{\nu_j-1} \delta(1-\sum_{l=1}^{n} x_l)$$
$$\times \sum_{m=0}^{[R/2]} \frac{\Gamma(N_\nu^{(m)})}{(-2)^m} \left[(\tilde{M}^{-1}\otimes g)^{(m)} \widetilde{\ell}^{(R-2m)}\right]^{\Gamma_1,\ldots,\Gamma_R}$$
$$\times U^{-D/2+m-R} \left(\frac{F}{U} - \imath\eta\right)^{-N_\nu^{(m)}}, \qquad (6)$$

$$\widetilde{\mathcal{M}}(\eta) = \sum_i \mathcal{C}_i \mathcal{F}_i,$$

$$\mathcal{C}_i = \eta^{LD/2-N_\nu+m_i^{\max}} \sum_{p=0}^{\infty} \sum_j \mathcal{A}_{0pj} \eta^{-p} I_{L,j}^{(vac),D},$$

Matching for the Reduction

$$\mathcal{C}_i = \eta^{\dim(\mathcal{C}_i)/2 - w_0} \left(\sum_{p=0}^{p_0} \sum_j \sum_{\substack{\alpha_1,\ldots,\alpha_t \\ |\alpha| = w_0 + p}} \left(a_{0p\alpha j}(D) \eta^{-p} \times s^\alpha I_{L,j}^{(vac),D} \right) + \mathcal{O}(\eta^{-p_0-1}) \right). \quad (20)$$

$$\widetilde{I}_k = \eta^{\dim(\widetilde{I}_k)/2 - w_k} \left(\sum_{p=0}^{p_0} \sum_j \sum_{\substack{\alpha_1,\ldots,\alpha_t \\ |\alpha| = w_k + p}} \left(a_{kp\alpha j}(D) \eta^{-p} \times s^\alpha I_{L,j}^{(vac),D} \right) + \mathcal{O}(\eta^{-p_0-1}) \right). \quad (25)$$

$$Z_{i0}\mathcal{C}_i + Z_{i1}\widetilde{I}_1 + \cdots + Z_{iS}\widetilde{I}_S = 0,$$

$$Z_k = \sum_{\substack{\lambda_1,\ldots,\lambda_t \\ |\lambda| \leqslant d_k/2}} z_{k\lambda_0\lambda}(D) \eta^{\lambda_0} s^\lambda,$$

Equation numbers (rows): different scale combo & different powers of η

To be solved (cols): z's

Matching for the Reduction

$$\begin{pmatrix} \mathbb{M}_{1,1} & \cdots & \mathbb{M}_{1,n_c} \\ \vdots & & \vdots \\ \mathbb{M}_{n_e,1} & \cdots & \mathbb{M}_{n_e,n_c} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_{n_c} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbb{M}(D) \cdot \mathbb{X}(D) = 0,$$

Elements of $\mathbb{M}$ may contain $D^{\mathcal{O}(10^3)}$

Do we really need all of them?

Goal: the reduction of integral/amplitude

$$\mathscr{M} = c_1 I_1 + c_2 I_2 + \cdots = a_{-4}\epsilon^{-4} + a_{-3}\epsilon^{-3} + a_{-2}\epsilon^{-2} + a_{-1}\epsilon^{-1} + a_0 + \mathcal{O}(\epsilon)$$

Matching for the Reduction

$$\mathbb{M}(\epsilon) = \mathbb{M}_0 + \mathbb{M}_1\epsilon + \cdots + \mathbb{M}_m\epsilon^m + \mathcal{O}(\epsilon^{m+1}),$$

$$\mathbb{X}(\epsilon) = \mathbb{X}_0 + \mathbb{X}_1\epsilon + \cdots + \mathbb{X}_m\epsilon^m + \mathcal{O}(\epsilon^{m+1}).$$

$$\begin{aligned}
&\mathbb{M}_0 \cdot \mathbb{X}_0 = 0,\\
&\mathbb{M}_0 \cdot \mathbb{X}_1 + \mathbb{M}_1 \cdot \mathbb{X}_0 = 0,\\
&\mathbb{M}_0 \cdot \mathbb{X}_2 + \mathbb{M}_1 \cdot \mathbb{X}_1 + \mathbb{M}_2 \cdot \mathbb{X}_0 = 0,\\
&\qquad \cdots\\
&\mathbb{M}_0 \cdot \mathbb{X}_m + \mathbb{M}_1 \cdot \mathbb{X}_{m-1} + \cdots + \mathbb{M}_m \cdot \mathbb{X}_0 = 0.
\end{aligned}$$

Matching for the Reduction

$$\mathbb{M}_0 \cdot \mathbb{X}_0 = 0$$ **with r_0 solutions**

$$\mathbb{M}_0 \cdot \mathbb{X}_0^{(r_0)} = 0,$$

$$\mathbb{X}_0^{(r_0)} \equiv (\mathbb{X}_0^1, \mathbb{X}_0^2, \cdots, \mathbb{X}_0^{r_0}).$$

However, $c_0\mathbb{X}_0^1 + \cdots + c_{r_0}\mathbb{X}_0^{r_0}$ **is also one solution**

$$\mathbb{M}_0 \cdot \mathbb{X}_1 + \mathbb{M}_1 \cdot \mathbb{X}_0 = 0,$$

$\Rightarrow$

$$\left(\mathbb{M}_1 \cdot \mathbb{X}_0^{(r_0)}, \mathbb{M}_0\right)\begin{pmatrix}\mathbb{C}_0^{(r_0,r_1)} \\ \mathbb{X}_1^{(r_1)}\end{pmatrix} = 0,$$

with r_1 solutions

$$\mathbb{X}_1^{(r_1)} \equiv (\mathbb{X}_1^1, \mathbb{X}_1^2, \cdots, \mathbb{X}_1^{r_1})$$

$\Leftarrow$

$$\mathbb{X}_0^{(r_1)} + \mathbb{X}_1^{(r_1)}\epsilon, \quad \mathbb{X}_0^{(r_1)} \equiv \mathbb{X}_0^{(r_0)} \cdot \mathbb{C}_0^{(r_0,r_1)}.$$

Matching for the Reduction

Suppose that up to ϵ^p we have the solutions

$$\mathbb{X}_0^{(r_p)} + \mathbb{X}_1^{(r_p)}\epsilon + \cdots + \mathbb{X}_p^{(r_p)}\epsilon^p.$$

Then at ϵ^{p+1} we can have

$$\left(\mathbb{M}_{p+1}\cdot\mathbb{X}_0^{(r_p)} + \cdots + \mathbb{M}_1\cdot\mathbb{X}_p^{(r_p)}, \mathbb{M}_0\right)\cdot\begin{pmatrix}\mathbb{C}_p^{(r_p,r_{p+1})}\\ \mathbb{X}_{p+1}^{(r_{p+1})}\end{pmatrix} = 0. \tag{26}$$

Matching for the Reduction

Then up to the next order ϵ^{p+1} we can obtain the solution,

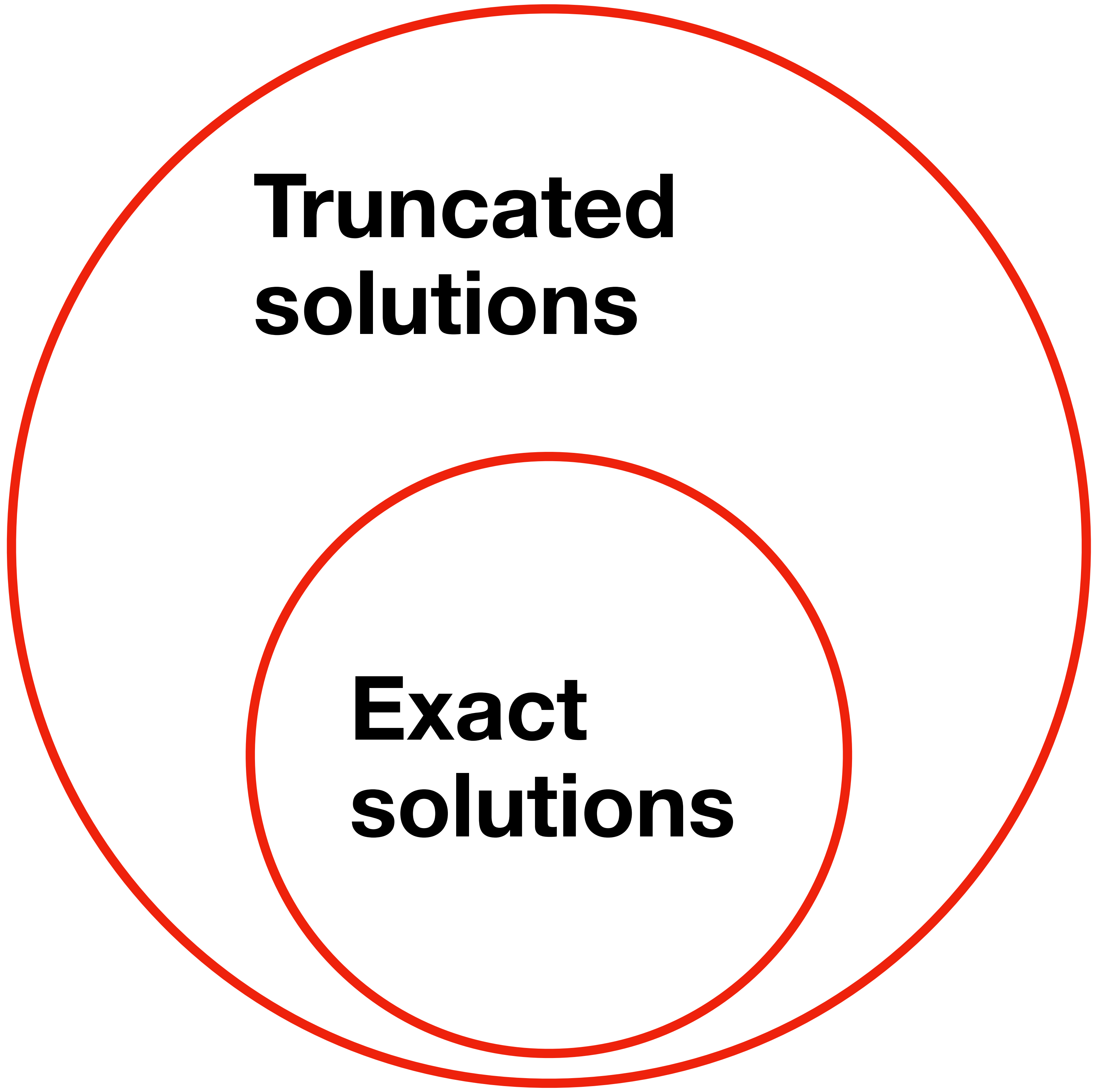
$$\mathbb{X}_0^{(r_{p+1})} + \mathbb{X}_1^{(r_{p+1})}\epsilon + \cdots + \mathbb{X}_{p+1}^{(r_{p+1})}\epsilon^{p+1}, \tag{27}$$

where

$$\begin{aligned} \mathbb{X}_0^{(r_{p+1})} &\equiv \mathbb{X}_0^{(r_p)} \cdot \mathbb{C}_p^{(r_p, r_{p+1})}, \\ &\cdots \\ \mathbb{X}_p^{(r_p+1)} &\equiv \mathbb{X}_p^{(r_p)} \cdot \mathbb{C}_p^{(r_p, r_{p+1})}. \end{aligned} \tag{28}$$

Therefore, by the iteration relations we can obtain the approximate soulutions $\mathbb{X}^{approx}(\epsilon)$,

$$\mathbb{X}^{approx}(\epsilon) = \mathbb{X}_0^{(r_m)} + \mathbb{X}_1^{(r_m)}\epsilon + \cdots + \mathbb{X}_m^{(r_m)}\epsilon^m. \tag{29}$$



Toolkits

Julia-lang code AMF.jl

- Direct Taylor expansion (as cross check for now) ✅
- Expansion based on formula of Feynman parameterization (FORM, Fermat, Julia) ✅

1. Expand for given order

$$\left(\frac{F}{U} - \imath\eta\right)^{-N_\nu^{(m)}} = (-\imath\eta)^{-N_\nu^{(m)}} \sum_{p=0}^{\infty} \binom{-N_\nu^{(m)}}{p} \frac{F^p}{U^p(-\imath\eta)^p}$$

obtain $c_1 s m_t^2 x_1^{n_1} x_2^{n_2} x_3^{n_3} U^m + \cdots$

2. Distribute (TH2) jobs for different scale combos (may further split terms according to the expansion order). Then recover I_{n_1,n_2,n_3}
3. Implement IBP and dimension shift, and then organize the coefficient constants (polynomial of D).
4. Summarize the final results of the coefficients into JLD files.

Julia-lang code MatchSeries.jl

- First problem is the order of η expansion. (Enough equations for the z's solution & enough equations for stablizing the solutions.)
  The truncation $\mathbb{M}(\epsilon) = \mathbb{M}_0 + \mathbb{M}_1\epsilon + \cdots + \mathbb{M}_m\epsilon^m + \mathcal{O}\left(\epsilon^{m+1}\right)$ cannot directly give the answer, since further more order of η expansion will change $\mathbb{M}_0$.

- Then we go back to the original equation $\mathbb{M}(D) \cdot \mathbb{X}(D) = 0$, and solve the nullity by choosing random numbers of D in several different (prime) finite fields.

- Found the mostly appeared number of nullities as the “estimated nullity”.
  Number of independent solutions should be no less than number of LHS.
  NOTE: the coefficient of reduction LHS should not be zero.

Chinese Remainder Theorem (CRT)

$$x = a_1 \pmod{m_1}$$

$$\dots$$

$$x = a_n \pmod{m_n}$$

$$x = \sum_{i=1}^{n} a_i b_i b_i' \pmod{M}$$

$b_i = M/m_i$ (the product of all the moduli except for m_i)

$$b_i' = b_i^{-1} \pmod{m_i}$$

If p is a prime number and a is a natural number, then

$$a^p \equiv a \pmod{p}.$$

Rational reconstruction

$$a \equiv n/d \pmod{M}.$$

$$da \equiv n \pmod{M}$$

Extended Euclidean algorithm gives sequences:

$$Ms_i + at_i = r_i. \qquad M > 2ND$$

Julia-lang code MatchSeries.jl

- Modularize the coefficient matrix into several finite fields
- Do the nullspace calculation in finite fields.
- Combine the nullspace matrices using CRT, by keeping the first element of nullspace as 1. Otherwise the solutions from different finite fields may differ by a factor, so that the reconstruction will fail.
- Rational reconstruction for the true solution.
- Check if the solution satisfy the nullspace matrix equation.

Thank you!