Feynman Integrals and Scattering Amplitudes from Wilson Loops

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based on work w. Song He, Qingling Yang, Chi Zhang [2012.15042]

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Outline

1. Motivation
2. One loop Chiral pentagon & Two-loop Double pentagon
3. Summary & Outlook

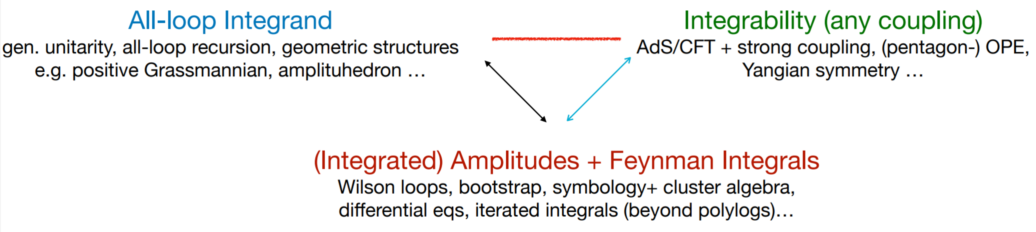
Section 1

Motivation

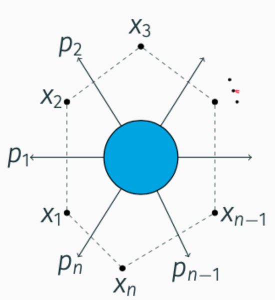
Amplitudes & Integrals

Scattering Amplitudes: QCD@LHC + Gravity@LIGO + strings, math, cosmology ...

hidden simplicity + structures in QFT: most remarkable $N = 4$ SYM (planar limit)



$\mathcal{N} = 4$ SYM & Wilson-loop Duality



MHV amplitude = null polygonal Wilson loops [Alday, Maldacena], [Brandhuber, Heslop, Travaglini], [Drummond, Henn, Korchemski, Sokatchev], [...]

$$A_n(p_1, \ldots, p_n) \leftrightarrow W_n(x_1, \ldots, x_n) \sim \langle \mathrm{Tr}\mathscr{P} \exp(i \int A \cdot dx) \rangle$$

Supersymmetric generalization: super-amplitude = super-Wilson loop [Mason, Skinner], [Caron-Huot]

$$\mathscr{A}_n(p_1, \ldots, p_n) \leftrightarrow \mathscr{W}_n(x_1, \ldots, x_n)$$

dual space:

$$x_{i+1} - x_i = p_i, \quad (\theta_{i+1} - \theta_i)^{\alpha A} = \lambda_i^\alpha \eta_i^A$$

Symmetry

Superconformal (amps) $+$ dual superconformal symmetry (Wilson loop, in dual space)
$\rightarrow$ Yangian symmetry

Symmetry broken by IR/UV divergence at loop-level.

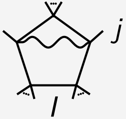
BDS ansatz [Bern, Dixon, Smirnov]: $A_n^{\mathrm{BDS}} \sim \exp(\Gamma_{\mathrm{cusp}} F_n^{1-\mathrm{loop}}/4)$
BDS-normalized amps:

$$R_{n,k} = \mathscr{A}_{n,k}/A_n^{\mathrm{BDS}}$$

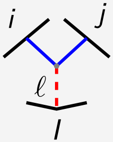
which breaks Yangian symmetry but leaves dual conformal symmetry (DCI).

Amplitudes from integral with unit leading singularities [Arkani-Hamed et al]

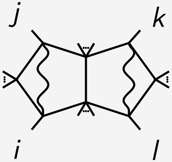
1-loop MHV amplitude $= \sum_{i<j<i}$



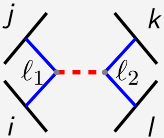
Wilson loop:



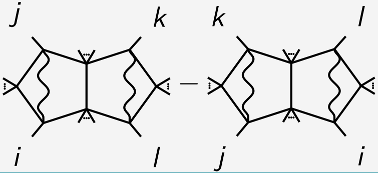
2-loop MHV amplitude $= \sum_{i<j<k<l<i}$



Wilson loop:

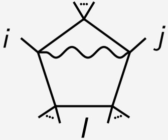


2-loop NMHV amplitude $|_{\chi_i \chi_j \chi_k \chi_l} =$

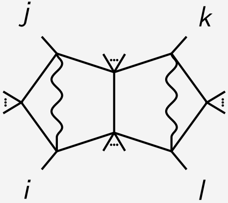


$(i, j, k, l$ non-adjacent$)$

One loop Chiral pentagon & Two-loop Double pentagon

One loop chiral pentagon 

: well-knwon result (can be obtained by Feynman parametrization/Differential equation).

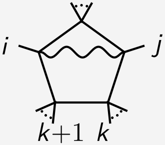
Two loop double pentagon 

: computed up to at most 7-point/at a special kinematic point [Bourjaily et al] of 8-point. We filled this gap in this work by WL dlog representation.

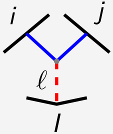
Section 2

One loop Chiral pentagon & Two-loop Double pentagon

WL dlog form: one loop chiral pentagon



$$= \int \mathrm{d}^4\ell \frac{(\ell - x_*)^2 (x_* - x_k)^2}{(\ell - x_i)^2 (\ell - x_{i+1})^2 (\ell - x_j)^2 (\ell - x_{j+1})^2 (\ell - x_k)^2} =$$



partial Feynman parametrization (to line integral):

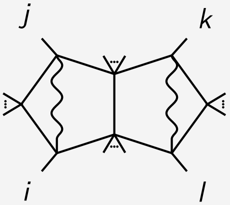
$$\frac{1}{(\ell - x_i)^2 (\ell - x_{i+1})^2} = \int_0^\infty \frac{\mathrm{d}\tau_X}{\left(\frac{1}{1+\tau_X}(\ell - x_i)^2 + \frac{\tau_X}{1+\tau_X}(\ell - x_{i+1})^2\right)^2} =: \int_0^\infty \frac{\mathrm{d}\tau_X}{(\ell - X(\tau_X))^4}$$

Swap the order of integration to integrate $\mathrm{d}^4\ell$ (star-triangle identity)

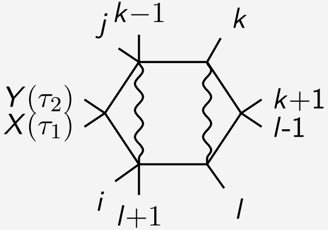
$$\int \mathrm{d}^2\tau \int \mathrm{d}^4\ell \frac{(\ell - x_*)^2 (x_* - x_k)^2}{(\ell - X(\tau_X))^4 (\ell - Y(\tau_Y))^4 (\ell - x_k)^2} \propto \int \frac{\mathrm{d}^2\tau}{(X - Y)^2 (X - x_k)^2 (Y - x_k)^2}$$

WL integral representation of double-pentagon integral

By Wilson loop duality, we have the following representation of the two loop double pentagon:



$$= \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \frac{\langle ijkl\rangle}{\langle iXjY\rangle}$$



It's easier to calculate these integrals than usual Feynman parameterization integrals.

Result: Algebraic words

algebraic letters (and its cyclic images):

$$\chi^{j,k}_{a,b,c,d} := \frac{\frac{\langle x_a x_b\rangle\langle x_d\, jk\rangle}{\langle x_d x_b\rangle\langle x_a\, jk\rangle} - z_{a,b,c,d}}{\frac{\langle x_a x_b\rangle\langle x_d\, jk\rangle}{\langle x_d x_b\rangle\langle x_a\, jk\rangle} - \bar{z}_{a,b,c,d}}$$

$$\sum_{\sigma_a\in\{0,1\}} (-1)^{\sigma_1+\cdots+\sigma_4} F(i+\sigma_1, j+\sigma_2, k+\sigma_3, l+\sigma_4) \otimes W^{i,j,k,l}_{a-i,\cdots,d-l}$$

where

$$W^{i,j,k,l}_{a-i,\cdots,d-l} = \chi^{j,k}_{a,b,c,d} \otimes \frac{\langle x_a\, jk\rangle\langle x_b\, il\rangle}{\langle x_a\, jl\rangle\langle x_b\, ik\rangle} + \text{cyclic} + \frac{1}{2}\left(\frac{\bar{z}(1-z)}{z(1-\bar{z})}\prod \chi\right) \otimes \frac{\langle x_a\, jl\rangle\langle x_b\, ik\rangle\langle x_c\, jl\rangle\langle x_d\, ik\rangle}{\langle x_a\, kl\rangle\langle x_b\, il\rangle\langle x_c\, ij\rangle\langle x_d\, jk\rangle}$$

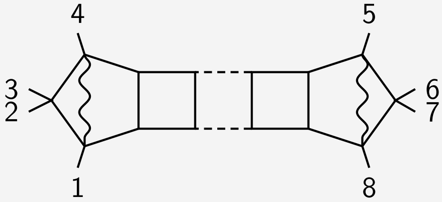
where the first four terms are given by cyclic rotation in i, j, k, l (thus also in a, b, c, d); Symmmetry $W^{i,j,k,l}_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} = W^{j,k,l,i}_{\sigma_2,\sigma_3,\sigma_4,\sigma_1}$ guarantees that all square roots cancel in $I_{\mathrm{dp}}(i,j,k,l) - I_{\mathrm{dp}}(j,k,l,i)$ and MHV amplitude.

Section 3

Summary & Outlook

Summary & Outlook

1. We found a new Wilson loop dlog representation for some Feynman integrals which helps us to calculate them, especially the double-pentagon integral which is nearly impossible to compute by old methods.
2. We directly see the algebraic words cancels in MHV amplitudes and the simplest NHHV components.
3. How to get compute the function with square roots?
4. Further explore the structures of integrals and amplitudes in WL representation, e.g. penta-ladders and its all loop structures.



Thank you!

Appendix

Structure of BDS-normalized amplitude

BDS-normalized amps:

$$R_{n,k} := \mathscr{A}_{n,k}/A_n^{\mathrm{BDS}} \sim \text{(Yangian invariants)} \times \text{(Transcendental functions)}$$

Yangian invariants are classified clearly [Arkani-Hamed et al].

Transcendental functions are DCI, and "uniform weight"=2L.

MHV and NMHV expected to be simplest: generalized polylogarithms [Goncharov]

$$G(a_1, \ldots, a_n; t_0) = \int_0^{t_0} d\log(t_1 - a_1) \int_0^{t_1} d\log(t_2 - a_2) \cdots \int_0^{t_{n-1}} d\log(t_n - a_n)$$

Result: General discussion

Total differential $\mathrm{d}I_{\mathrm{dp}}(i,j,k,l)$ reads

$$\frac{1}{2}R^{\bar{i}}_{j-1j}\mathrm{d}\log\frac{\langle i(i{-}1i{+}1)(j{-}1j)(kl)\rangle}{\langle\bar{i}j\rangle\langle j{-}1jkl\rangle}+M^{ikl}_{j-1j}\mathrm{d}\log\frac{\langle ij{-}1jk\rangle}{\langle j{-}1jkl\rangle}$$
$$-(j{-}1j\leftrightarrow jj{+}1)+(\bar{i}\leftrightarrow\bar{j})+(\bar{k}\leftrightarrow\bar{l})+(i\,j\leftrightarrow k\,l) \quad (1)$$

where R and M are symbols of two weight 3 functions; first one is rational while second one contains algebraic part.

Result: Rational Alphabet

Notation $a = i, i+1,\ b = j, j+1,\ c = k, k+1,\ d = l, l+1$

1. First entries: $\langle p{-}1pq{-}1q\rangle$, with $p, q = a, b, c, d$ (24);
2. Second entries: $\langle i\bar{j}\rangle$ (12); $\langle i(i{-}1\,i{+}1)(p{-}1\,p)(q{-}1\,q)\rangle$ with $p, q = b, c, d$ plus cyclic in i, j, k, l (48); all letters in first entries (24);
3. Last entries: $\langle a{-}1ajk\rangle$, $\langle a{-}1akl\rangle$, $\langle a{-}1ajl\rangle$ plus cyclic in i, j, k, l (24) ; $\langle i(i{-}1i{+}1)(b{-}1b)(kl)\rangle$ plus cyclic in i, j, k, l (8); $\langle i\bar{j}\rangle, \langle j\bar{i}\rangle, \langle k\bar{l}\rangle, \langle l\bar{k}\rangle$ (4);
4. Rest letters(appear only in third entries):$\langle i(b{-}1\,b)(c{-}1\,c)(d{-}1\,d)\rangle$ plus cyclic in i, j, k, l (32); $\langle(\bar{i}) \cap (ib{-}1b) \cap (\bar{k}) \cap (kd{-}1d)\rangle$ plus cyclic in i, j, k, l (16)

four-mass box function:

$$F(u,v) := \mathrm{Li}_2(1-z) - \mathrm{Li}_2(1-\bar{z}) + \tfrac{1}{2}\log(z/\bar{z})\log(v)$$

$$u = u_{a,b,c,d} = z\bar{z}\,,\ v = u_{b,c,d,a} = (1-z)(1-\bar{z}),\ u_{a,b,c,d} := \frac{x_{a,b}^2 x_{c,d}^2}{x_{a,c}^2 x_{b,d}^2}$$

Rationalization

The result of the one-loop hexagon involves the integral of "γ"-deformed four-mass box function:

$$\int \frac{\mathrm{d}^2\tau \langle ijkl\rangle}{\langle iXjY\rangle}\gamma F(u,v).$$

where γ is given by corresponding leading singularities evaluated at two solutions of the four-mass Schubert problem, and it's irrational. By the change of variable from τ to $z(\tau)$, the prefactor γ, together with $\mathrm{d}\log$ forms depending on τ, becomes a beautiful $\mathrm{d}\log$ of a rational function of $z(\tau)$! Then we can perform the integral:

$$\int_{z(0)}^{z(\infty)} \mathrm{d}\log\frac{z-w}{\bar{z}-w} F(z,\bar{z}) = (\text{rational symbol}) + S\left(F(z,\bar{z})\right) \otimes \left.\frac{z-w}{\bar{z}-w}\right|_{\tau=0}^{\tau=\infty}.$$