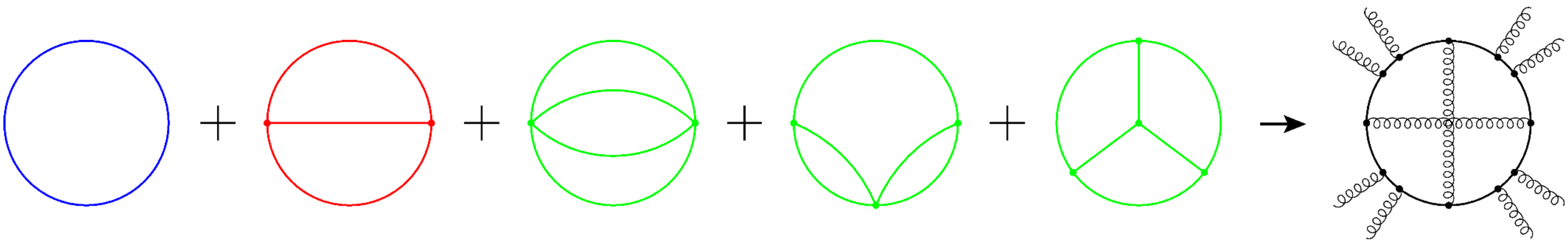
A general method for Feynman loop integrals calculation



马滟青

北京大学

微扰量子场论研讨会, 2021/05/15-16, 上海交通大学

Outline

I. Introduction

II. Auxiliary mass flow

III. Applications

IV. Outlook

The future of particle physics

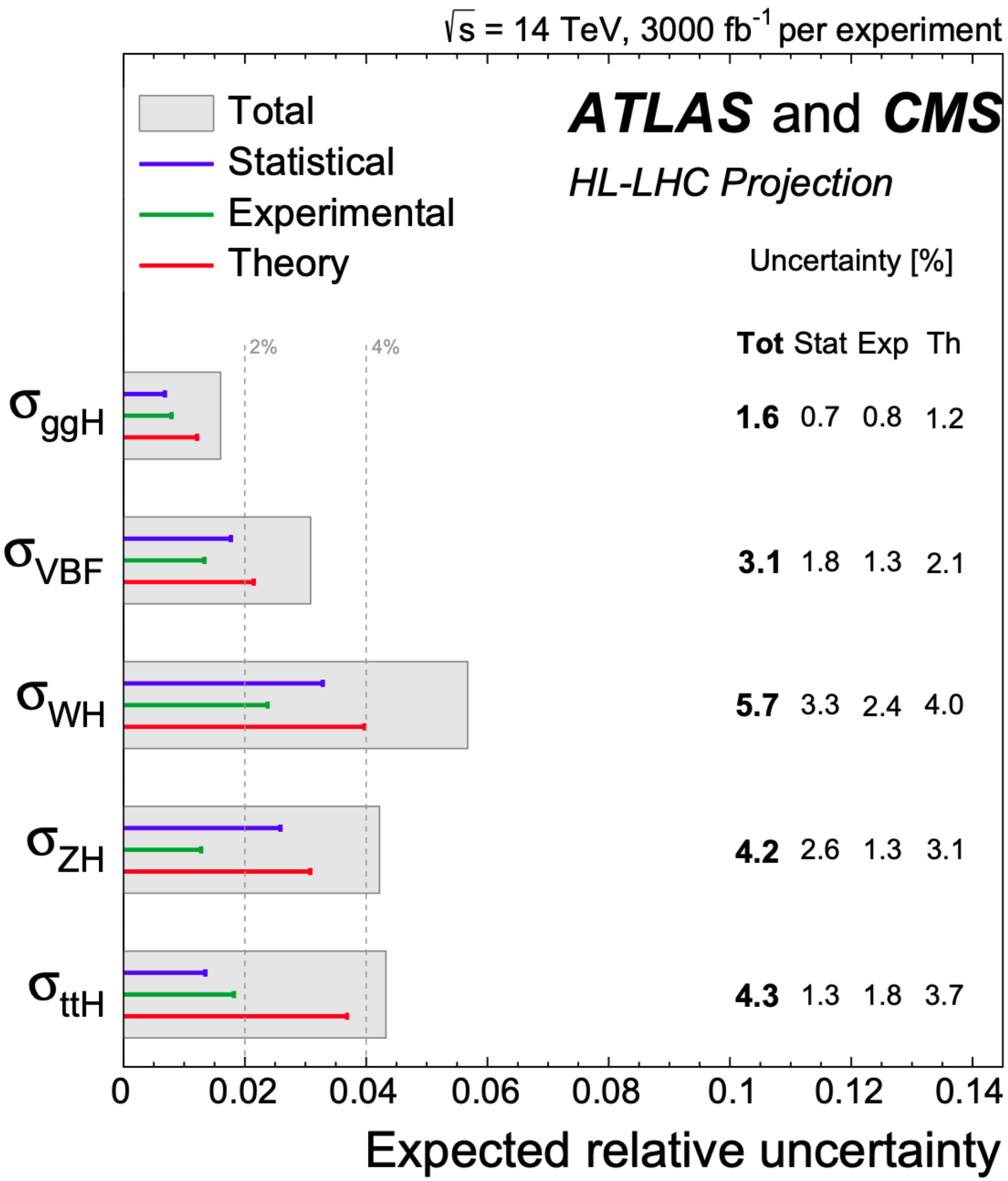
➢ Current status of LHC

- After 40 years test: SM is still very successful
- No clear signal of new physics

➢ To test SM or discover NP

- Three possible choices: precision/energy/cosmology
- Experiment: precision measurement!
- Theory: precision calculation!

High luminosity LHC projection



- **Theoretical uncertainty needs further reduced**
- **Perturbative calculation at high order!**

Perturbative QFT

1. Generate Feynman amplitudes

- Feynman diagrams and Feynman rules
- New developments: unitarity, recurrence relation

2. Calculate Feynman loop integrals

3. Calculate phase-space integrals

- Monte Carlo simulation with IR subtractions
- Mapping to loop integrals via reverse unitarity

$$\int \frac{d^D p}{(2\pi)^D}(2\pi)\delta_+(p^2) = \lim_{\eta\to 0^+}\int \frac{d^D p}{(2\pi)^D}\left(\frac{i}{p^2+i\eta}+\frac{-i}{p^2-i\eta}\right)$$

➢ **Feynman loop integrals**

- **Encoding the main nontrivial information of QFT**

$$\lim_{\eta\to 0^+}\int\prod_{i=1}^{L}\frac{\mathrm{d}^D\ell_i}{\mathrm{i}\pi^{D/2}}\prod_{\alpha=1}^{N}\frac{1}{(q_\alpha^2-m_\alpha^2+\mathrm{i}\eta)^{\nu_\alpha}}$$

- q_α: **linear combination of loop momenta and external momenta**
- **Taking** $\eta\to 0^+$ **before taking** $D\to 4$

Multi-loop: a challenge for intelligence

- One-loop integrals: systematical approach existed as early as 1970s

’t Hooft, Veltman, NPB (1979); Passarino, Veltman, NPB (1979); Oldenborgh, Vermaseren (1990)

- Further developments of unitarity-based method in the past decade

Britto, Cachazo, Feng, 0412103; Ossola, Papadopoulos, Pittau, 0609007; Giele, Kunszt, Melnikov, 0801.2237

See B. Feng’s talk for recent development
B.W. Xiao’s talk for non-covariant QFT

- About 40 years later, a satisfactory method for multi-loop calculation is still missing

Main strategy

1) Reduce loop integrals to basis (Master Integrals)

- Mainly integration-by-parts (IBP) reduction: Chetyrkin, Tkachov, NPB (1981) Laporta, 0102033

  the main bottleneck

  extremely time consuming for multi-scale problems

  unitarity-based reduction cannot give complete reduction beyond one-loop

2) Calculate MIs/original integrals

- Differential equations (depends on reduction and BCs) Kotikov, PLB (1991)
- Difference equations (depends on reduction and BCs) Laporta, 0102033
- Sector decomposition (extremely time-consuming) Binoth, Heinrich, 0004013
- Mellin-Barnes representation (nonplanar, time) Usyukina (1975) Smirnov, 9905323

IBP reduction

➢ A result of dimensional regularization

Chetyrkin, Tkachov, NPB (1981)

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_j^\mu} \left(v_k^\mu \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}} \right) = 0\,, \quad \forall j, k$$

$$\Downarrow$$

- **Linear equations:** $\sum_{i=1} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$
- M_i **scalar integrals,** Q_i **polynomials in** $D, \vec{s}, \eta$

➢ For each problem, the number of MIs is FINITE

Smirnov, Petukhov, 1004.4199

- **Feynman integrals form a finite dimensional linear space**
- **Reduce thousands of loop integrals to much less MIs**

Difficulty of IBP reduction

➢ Solve IBP equations

Laporta's algorithm, 0102033

$$\sum_{i=1} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$$

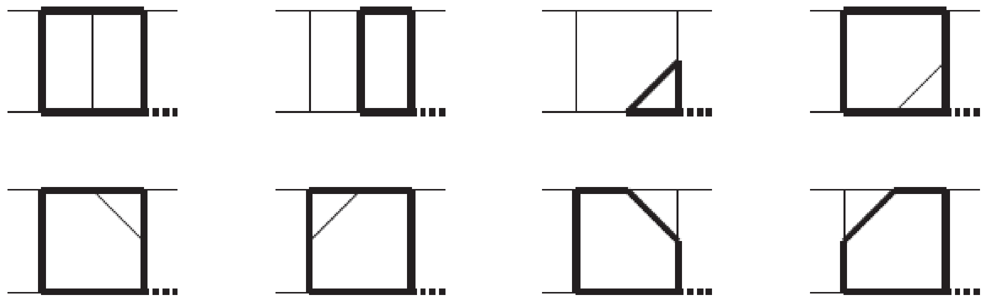
- Very large scale of linear equations (can be billions of) E.g., Laporta 1910.01248
- Equations are coupled
- × Explicit solution for multi-scale problem: hard to get, expression can be too large
- × Numerical solution at each phase space point : too slow

➢ Cutting-edge problems

- Hundreds GB RAM
- Months of runtime using super computer

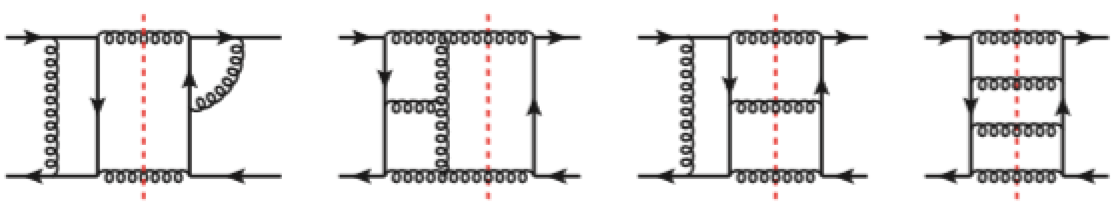
Difficulty of MIs calculation

- **Analytical (depending on reduction):** R. Bonciani, et.al 2016
  e.g. Higgs → 3 partons (Euclidean Region)



200MB, 10 min

- **Numerical (sector decomposition, independent of reduction): e.g. Quarkonium decay at NNLO** Feng, Jia, Sang, 1707.05758



NNLO (Virtual Squared) NNLO (Double Virtual) NNLO (Virtual – Real) NNLO (Double Real)

10^5 CPU core-hour

Recent developments

➢ Selected improvements for reduction

- **Finite field method** Manteuffel, Schabinger, 1406.4513
- **Direct solution** Kosower, 1804.00131
- **Syzygies method** Böhm, Georgoudis, Larsen, Schönemann, Zhang, 1805.01873, Bendle et.al., 1908.04301
- **Obtain one coefficient at each step** Chawdhry, Lim, Mitov, 1805.09182
- **Expansion of small parameters** Xu, Yang, 1810.12002; Mishima, 1812.04373
- **Intersection Numbers** Frellesvig, et. al., 1901.11510, 1907.02000

➢ Selected improvements for evaluation

- **Quasi-Monte Carlo method** Li, Wang, Yan, Zhao, 1508.02512
- **Finite basis** Manteuffel, Panzer, Schabinger, 1510.06758
- **Uniform-transcendental basis** Henn, 2013
- **Loop-tree duality** Capatti, Hirschi, Kermanschah, Ruijl, 1906.06138

State-of-the-art computation

- 2→2 process with massive particles at two-loop order: almost done $g+g\to t+\bar{t},\quad g+g\to H+H(g)$

- Frontier in the following decade:
  - 2→3 processes at two loops (3j/γ, V/H+2j $t\bar{t}$+j, $t\bar{t}H$,…)
  - 2→2 processes at three loops (2j/γ, V/H+j, $t\bar{t}$, HH, …)
  - 2→1 processes at four loops (j, V/H)
  - Two-loop EW corrections ($e^+e^-\to HZ$)

- Very challenging
  - Two-loop $g+g\to H+H\ (g)$: complete IBP reduction cannot be achieved within tolerable time

    Borowka et. al., 1604.06447
    Jones, Kerner, Luisoni, 1802.00349
  - Four-loop $g+g\to H$ (NNLP in HTL): 860 days (wall time!)

    Davies, Herren, Steinhauser, 1911.10214

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Modified FIs

Liu, YQM, Wang, 1711.09572
Liu, YQM, 1801.10523

- **Modify Feynman loop integral by keeping finite η**

$$\mathcal{M}(D,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha + \mathrm{i}\eta)^{\nu_\alpha}}$$

$$\mathcal{D}_\alpha \equiv q_\alpha^2 - m_\alpha^2$$

- Think it as an analytical function of η
- Physical result is defined by

$$\mathcal{M}(D,\vec{s},0) \equiv \lim_{\eta\to 0^+} \mathcal{M}(D,\vec{s},\eta)$$

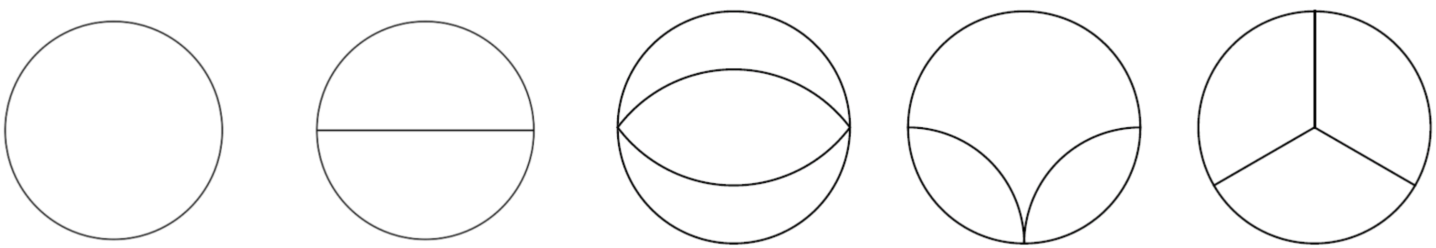
Expansion at infinity

➢ Expansion of propagators around $\eta = \infty$

$$\frac{1}{[(\ell+p)^2 - m^2 + \mathrm{i}\eta]^\nu} = \frac{1}{(\ell^2 + \mathrm{i}\eta)^\nu} \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \left(\frac{-2\ell \cdot p - p^2 + m^2}{\ell^2 + \mathrm{i}\eta} \right)^n$$

- Only one region in the method of region: $l^\mu \sim |\eta|^{1/2}$
- No external momenta in denominator, vacuum integrals
- Simple enough to deal with

➢ Vacuum MIs with equal internal masses



- Analytical results are known up to 3-loop
- Numerical results are known up to 5-loop

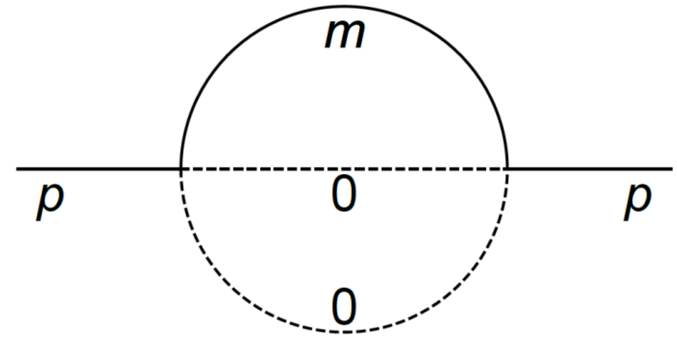
Davydychev,Tausk, NPB(1993)
Broadhurst, 9803091
Kniehl, Pikelner, Veretin, 1705.05136

Schroder, Vuorinen, 0503209
Luthe, PhD thesis (2015)
Luthe, Maier, Marquard, Ychroder, 1701.07068

Example

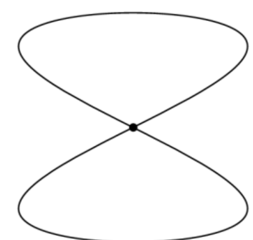
➢ Sunrise integral

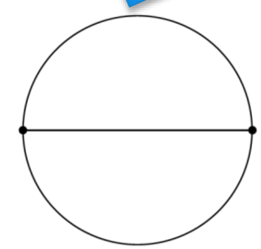
$$\hat{I}_{\nu_1\nu_2\nu_3} \equiv \int \prod_{i=1}^{2} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}}$$



$$\mathcal{D}_1 = (\ell_1+p)^2 - m^2,\ \mathcal{D}_2 = \ell_2^2,\ \mathcal{D}_3 = (\ell_1+\ell_2)^2$$

$$I_{111} = \eta^{D-3}\left\{\left[1 - \frac{D-3}{3}\frac{m^2}{\mathrm{i}\eta} + \frac{(D+4)(D-3)}{9D}\frac{p^2}{\mathrm{i}\eta}\right] I_{2,2}^{\mathrm{bub}} - \mathrm{i}\left[\frac{(D-2)^2}{3D}\frac{p^2}{\mathrm{i}\eta}\right] I_{2,1}^{\mathrm{bub}} + \mathcal{O}(\eta^{-2})\right\}$$





A new representation

➢ Asymptotic expansion: a convergent series

$$\mathcal{M}(D,\vec{s},\eta) = \eta^{LD/2-\sum_\alpha \nu_\alpha} \sum_{\mu_0=0}^{\infty} \eta^{-\mu_0} \mathcal{M}^{\mathrm{bub}}_{\mu_0}(D,\vec{s})$$

$$\mathcal{M}^{\mathrm{bub}}_{\mu_0}(D,\vec{s}) = \sum_{k=1}^{B_L} I^{\mathrm{bub}}_{L,k}(D) \sum_{\vec{\mu}\in\Omega^r_{\mu_0}} C_k^{\mu_0\cdots\mu_r}(D) s_1^{\mu_1}\cdots s_r^{\mu_r}$$

- $I^{\mathrm{bub}}_{L,k}(D)$: k-th master vacuum integral at L-loop order
- $C_k^{\mu_0\ldots\mu_r}(D)$: rational functions of D
- A convergent series, defines an analytical function around $\eta=\infty$

➢ A new representation

- Uniqueness theorem of analytical functions: physical FI is uniquely determined by this asymptotic series via analytical continuation
- A new series representation of FIs
- All FIs (therefore scattering amplitudes) are determined by equal-mass vacuum integrals

Find relations

$$\sum_{i=1}^{n} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta) = 0$$

- **Decomposition of $Q_i(D,\vec{s},\eta)$**

$$Q_i(D,\vec{s},\eta) = \sum_{(\lambda_0,\vec{\lambda})\in\Omega_{d_i}^{r+1}} Q_i^{\lambda_0\cdots\lambda_r}(D)\,\eta^{\lambda_0} s_1^{\lambda_1}\cdots s_r^{\lambda_r}$$

$$\Rightarrow \sum_{k,\rho_0,\vec{\rho}} f_k^{\rho_0\cdots\rho_r}\,\mathcal{I}_{L,k}^{\text{bub}}(D)\;\eta^{\rho_0} s_1^{\rho_1}\cdots s_r^{\rho_r} = 0$$

- **Linear equations:** $f_k^{\rho_0\cdots\rho_r} = 0$
  - With enough constraints $\Rightarrow Q_i^{\lambda_0\ldots\lambda_r}(D)$
  - With finite field technique, only integers in a finite field are involved, equations can be efficiently solved
- **Relations among $G \equiv \{M_1, M_2, \ldots, M_n\}$ can be determined**

Analytical continuation

- **Set up and solve DEs of MIs**

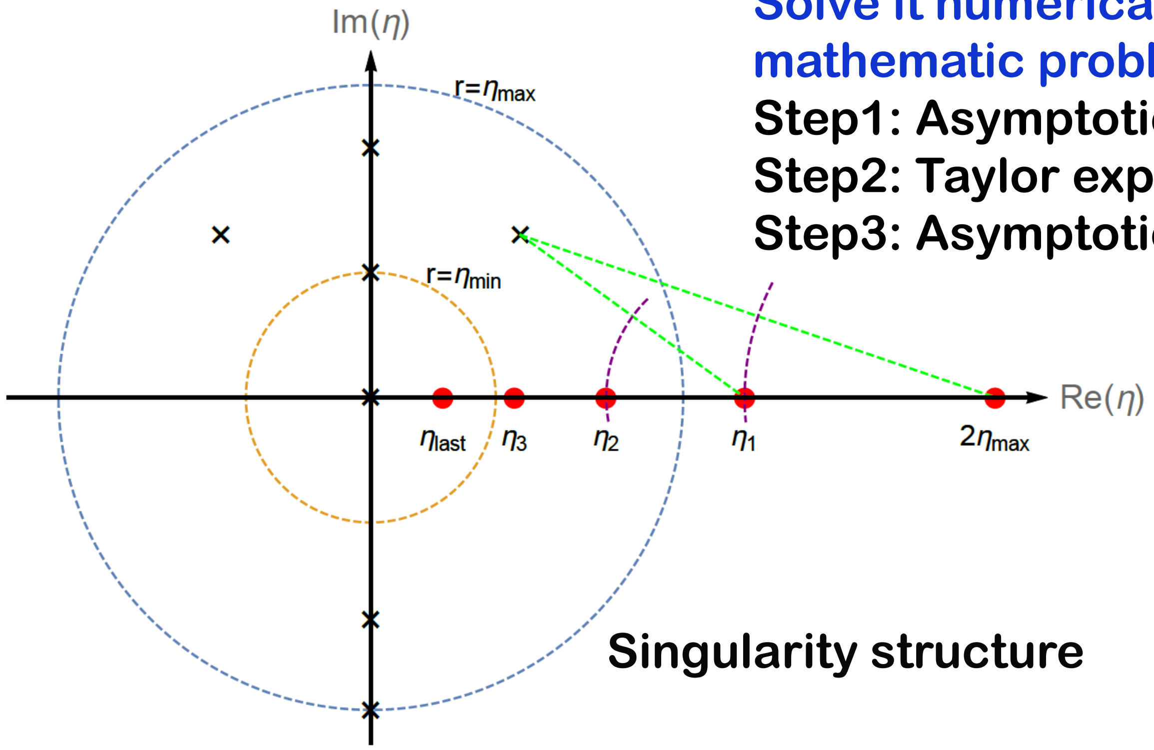
$$\frac{\partial}{\partial \eta}\vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta) \quad \text{with known } \vec{I}(D;\infty)$$

Solve it numerically: a well-studied mathematic problem

Step1: Asymptotic expansion at $\eta = \infty$

Step2: Taylor expansion at analytical points

Step3: Asymptotic expansion at $\eta = 0$



Singularity structure

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Reduction: a two-step strategy

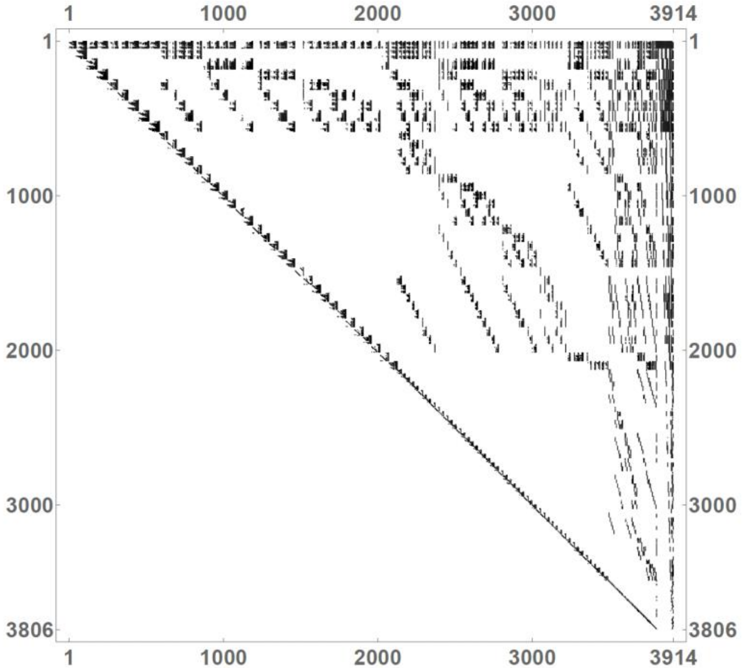
Guan, Liu, YQM, 1912.09294

➢ First-step: numerical

- Reduce to MIs at some (~100) phase space points, over finite field
- Fine if not efficient enough, can use AMF, traditional IBP, intersection number, …

➢ Second-step: all-purpose

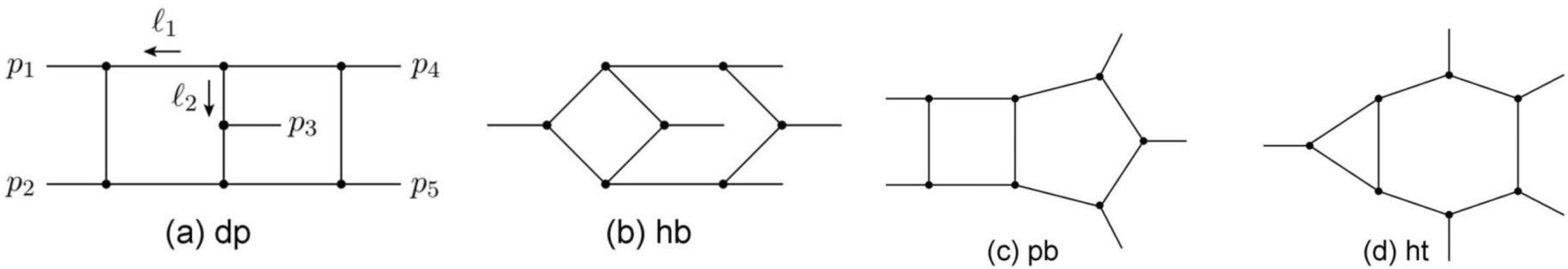
- Construct block-triangular reduction systems
- Very efficient to use



Reduction: example

➢ 5 gluon scattering

Guan, Liu, YQM, 1912.09294

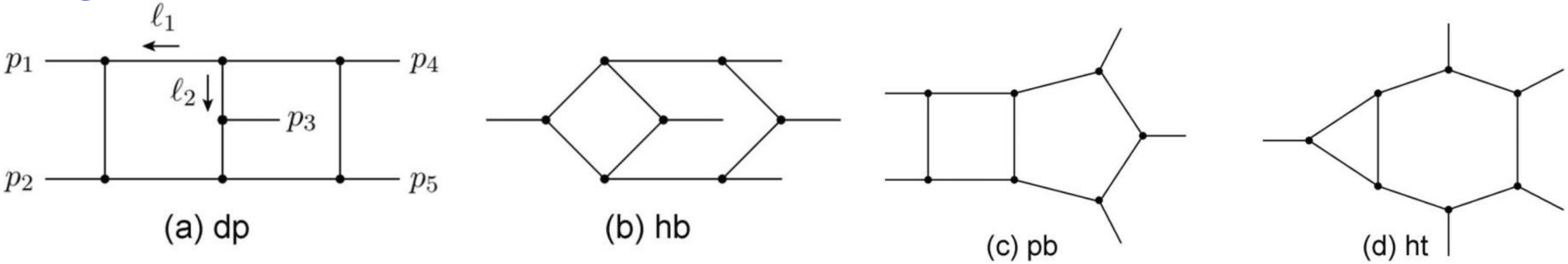


- For the first time, complete reduction of all two-loop 5-gluon FIs, size of results: 66MB, 40MB, 31MB, 11MB
- Easy to obtain, about 200 CPU core-hour
- Fast enough for numerical evaluation: <1s for each phase space point, fast by 100 times v.s. traditional IBP

Reduction: comparison

- **Compare with explicit solution**
  - Explicit solutions are very hard to obtain
  - The size of explicit solution can be too huge to be used
- **Efforts in literature**

- Complete reduction of (c) get a file ~ 20GB Chawdhry, Lim, Mitov, 1805.09182
- Reduction 26 out of 3000+ FIs of (a) (not the most complicated ones), get a file ~ 2GB Bendle et.al., 1908.04301



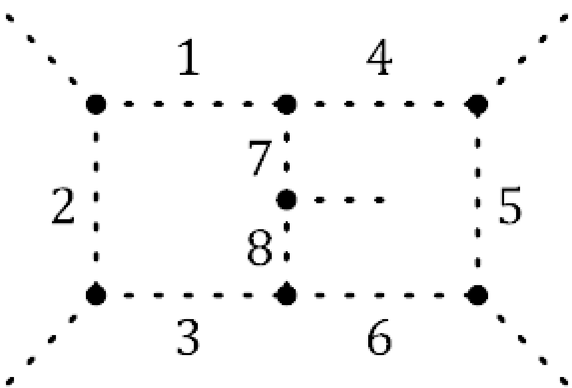
- UT basis, multivariate partial fraction: 186MB for (a)

Bendle et.al., 2104.06866

Evaluation: strategy to introduce η

Liu, YQM, in preparation

➢ Try to control #MIs: propagator mode

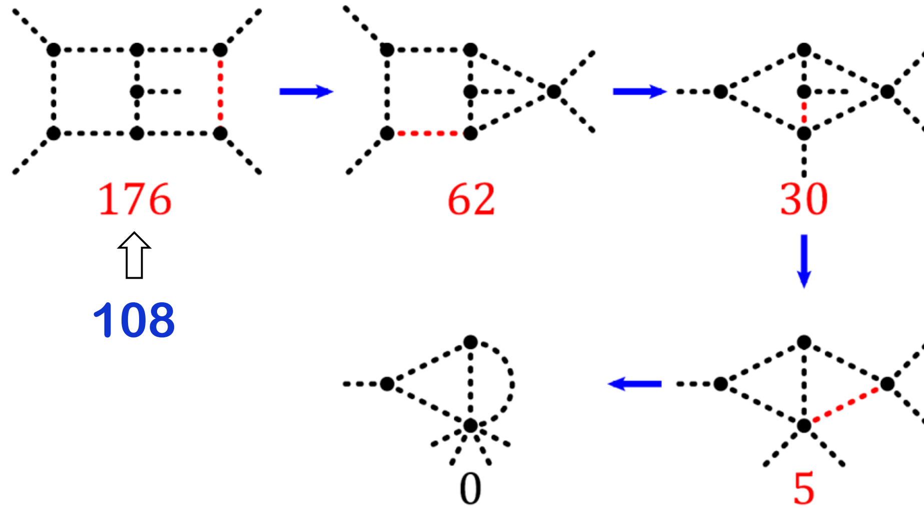


mode	propagators	#MIs
all	{1,2,3,4,5,6,7,8}	476
loop	{4,5,6,7,8}	305
	{1,2,3,4,5,6}	319
branch	{4,5,6}	233
	{7,8}	234
propagator	{4}	178
	{5}	176
	{7}	220

Evaluation: a two-loop example

➢ Two-loop double-pentagon

- Time =5h=(40*5s+3000*0.05s)*45+…
- Set DEs:90%; solve: 10%.
- New reduction strategy: 100× fast



$$I_{\rm phy}[1,1,1,1,1,1,1,1,0,0,0] =$$
$$-\,0.069435625172637 76\epsilon^{-4}$$
$$+\,(1.162256636711287 + 1.416359853446717\mathrm{i})\epsilon^{-3}$$
$$+\,(37.82474332116938 + 15.91912443581739\mathrm{i})\epsilon^{-2}$$
$$+\,(86.2861798369034 + 166.8971535711277\mathrm{i})\epsilon^{-1}$$
$$-\,(4.1435965578662 - 333.0996040071305\mathrm{i})$$
$$-\,(531.834114822928 - 1583.724672502141\mathrm{i})\epsilon$$
$$-\,(2482.24025323261 2 - 2567.398291724192\mathrm{i})\epsilon^2$$
$$-\,(8999.90369367113 - 19313.42643829926\mathrm{i})\epsilon^3$$
$$-\,(28906.9558269676 2 - 17366.82954322838\mathrm{i})\epsilon^4.$$

$$\vec{s} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$$

$$\vec{s}_0 = \left\{4, -\frac{113}{47}, \frac{281}{149}, \frac{349}{257}, -\frac{863}{541}\right\}$$

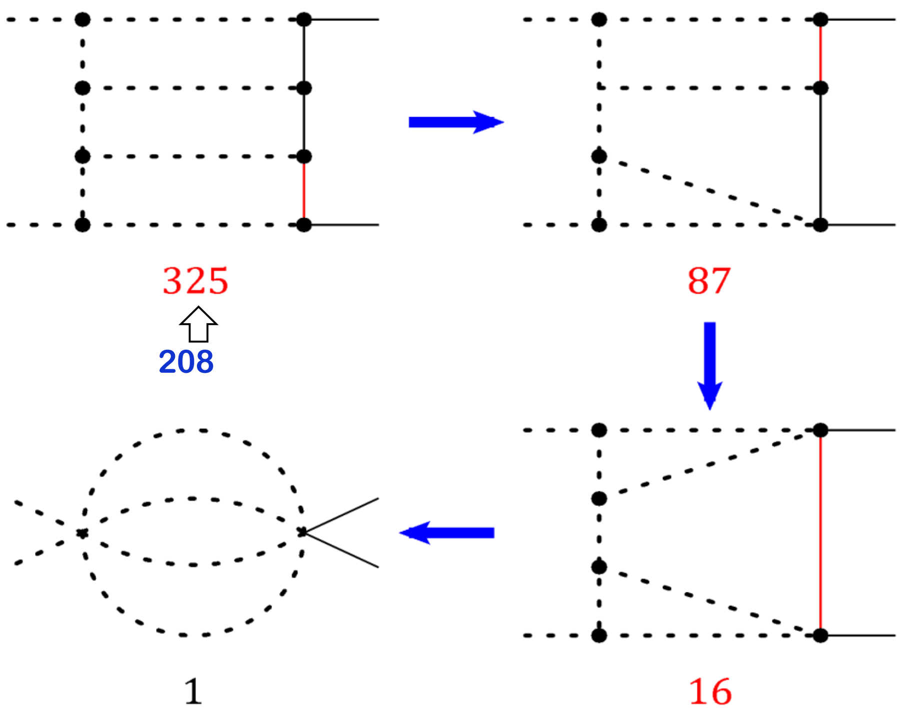
Consistent with literature:

Chicherin, et. al. 1901.05932

Chicherin, Sotnikov, 2009.07803

Evaluation: a three-loop example

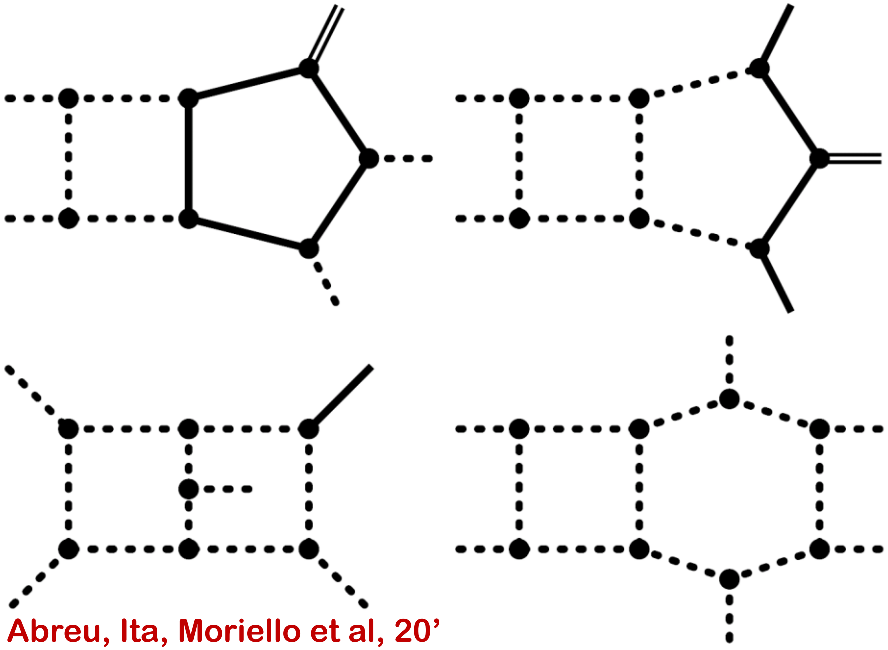
- $t\bar{t}$ **hadroproduction at three-loop order**
  - Time =15h=(40*50s+6000*0.6s)*8+…
  - Set DEs:90%; solve: 10%.
  - New reduction strategy: 100× fast



- New result, (highly nontrivial) consistence checked

Evaluation: other Examples

- **Two-loop: H/V+2j, $t\bar{t}$H, 4j**



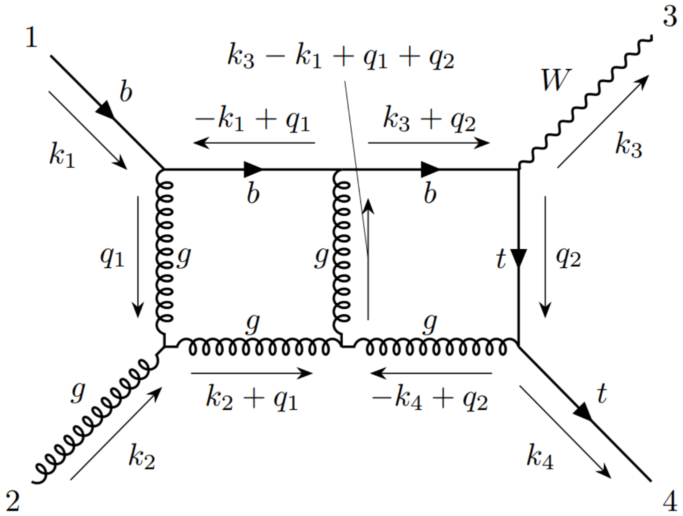
Abreu, Ita, Moriello et al, 20'
Canko, et al, 20'

- New results (except W+2j), highly nontrivial consistence checked

Other applications of AMF

- **Directly reduce amplitudes (avoid tensor reduction)**

Wang, Li, Basat, 1901.09390
Basat, Li, Wang, 2102.08225



- **Calculate two-loop MIs**

FFs of $g \to Q\bar{Q}({}^1S_0^{[1,8]}) + X$ — Zhang, et.al., 1810.07656

$e^+e^- \to H^\pm W^\mp$ — Yang, et.al., 2005.11010

$gg \to ZZ$ — Brønnum-Hansen, Wang, 2101.12095

Outlook

- Feynman integrals are completely determined by vacuum integrals
- General strategy to do reduction and evaluate MIs: correct, efficient, useful
- Ready for complete NNLO 3j/γ;
  Other interesting processes: stay tune

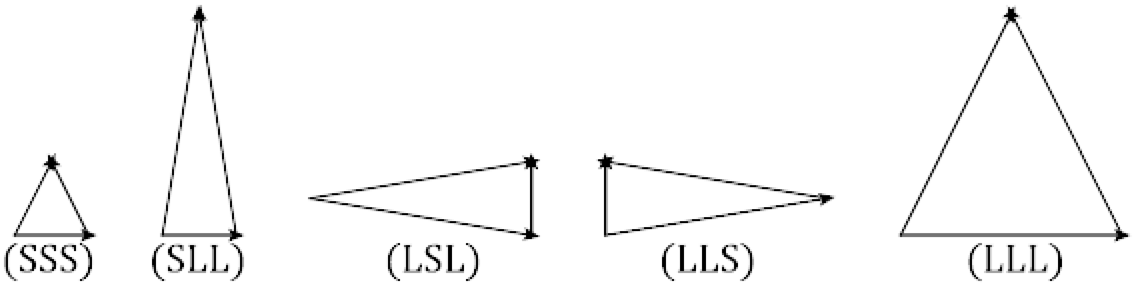
Thank you!

Integration regions

➢ General integration region

- loop momentum of each branch can be either $O(1)$ or $O(\sqrt{\eta})$
- regions for one-loop: (S), (L)

- regions for two-loop: (S,S,S), (S,L,L), (L,S,L), (L,L,S), (L,L,L)



- $R_1 = 2, R_2 = 5, R_3 = 15, R_4 = 47, \ldots$

Expansion

➢ Expansion in each region

- all large: single-mass vacuum integrals

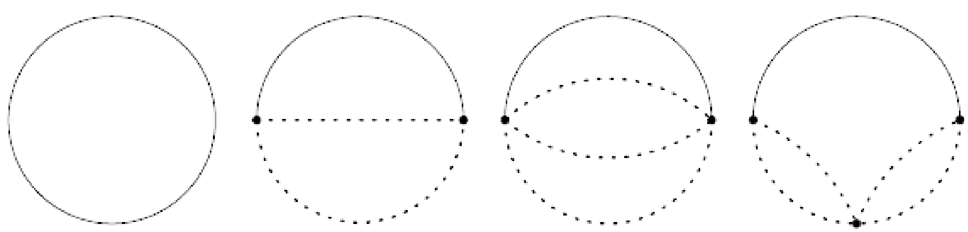
$$\frac{1}{((\ell+p)^2-m^2-k\eta)^\nu} \sim \frac{1}{(\ell^2-k\eta)^\nu},$$

- mixed: factorized integrals with a factor being vacuum integrals

$$\frac{1}{((\ell_S+\ell_L+p)^2-m^2-k\eta)^\nu} \sim \frac{1}{(\ell_L^2-k\eta)^\nu}.$$

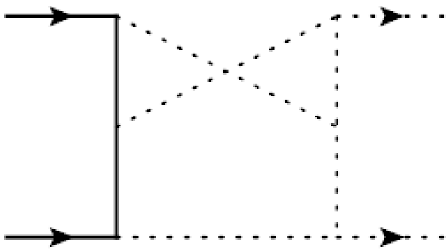
- all small: integrals with fewer propagators

$$\frac{1}{((\ell+p)^2-m^2-\eta)^\nu} \sim \frac{1}{(-\eta)^\nu}.$$



Example

➢ **2-loop non-planar sector for** $\mathrm{Q}+\bar{\mathrm{Q}}\rightarrow g+g$



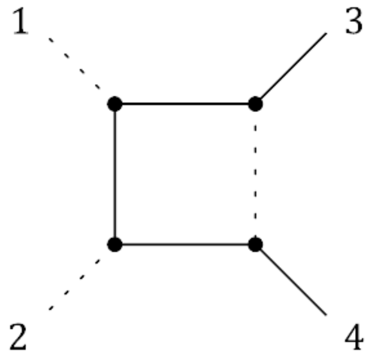
- 168 master integrals
- Traditional method sector decomposition: $O(10^4)$ CPU core-hour
- Our method: a few minutes

Feng, Jia, Sang, 1707.05758

MIs can be thought as special functions, and DEs tell us how to evaluate these special functions

Infrared Divergences

➢ Example: one-loop four-point integral



$s = 10, t = -3, m^2 = 1$

$$I[1,1,1,1] = \frac{0.0665971 - 0.101394i}{\epsilon} + (-0.0133705 + 0.287857i).$$

- **eta-reg:** $I[1,1,1,1](\eta) \sim (0.0665971 - 0.101394i)\log(\eta) + 0.0250704 + 0.22933i.$
- **dim-reg:** $I[1,1,1,1](\eta) \sim \eta^{-\epsilon} f_1 + f_2 + \eta^{1/2-\epsilon} f_3,$

$$f_1 = \frac{-0.0665971 + 0.101394i}{\epsilon} + (0.0384409 - 0.0585265i),$$
$$f_2 = \frac{0.0665971 - 0.101394i}{\epsilon} + (-0.0133705 + 0.287857i),$$
$$f_3 = 0.1309.$$

- **take $\eta \to 0$, only f_2 survives**

What is reduction

➢ Reduction

- Find relations between loop integrals
- Use them to express all loop integrals as linear combinations of MIs

➢ Relations among $G \equiv \{M_1, M_2, \ldots, M_n\}$

$$\sum_{i=1}^{n} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$$

- $Q_i(D, \vec{s}, \eta)$: homogeneous polynomials of $\vec{s}, \eta$ of degree d_i

➢ Constraints from mass dimension

$$2d_1 + \mathrm{Dim}(\mathcal{M}_1) = \cdots = 2d_n + \mathrm{Dim}(\mathcal{M}_n)$$

- Only 1 degree of freedom in $\{d_i\}$, chosen as $d_{\max} \equiv \mathrm{Max}\,\{d_i\}$

Reduction

- **With $G = G_1 \cup G_2$, satisfy**
  - G_1 is more complicated than G_2
  - G_1 can be reduced to G_2
- **Algorithm** ***Search for efficient relations***
  1. Set $d_{\max} = 0$
  2. Find out all reduction relations among G with fixed $d_{\max}$
  3. If obtained relations are enough to determine G_1 by G_2, stop;

     else, $d_{\max} = d_{\max} + 1$ and go to step 2
- **Conditions for G_1 and G_2**
  1. Relations among G_1 and G_2 are not too complicated: easy to find
  2. $\# G_1$ is not too large: numerically diagonalize relations easily

Reduction scheme with only dots

- FIs: $\vec{\nu} = (\nu_1, \ldots, \nu_N), \nu_i \geq 0$
  - $\mathbf{0}^{\pm} \equiv \mathbf{Identity}, \ \mathbf{m}^{\pm} \equiv (\mathbf{m}-\mathbf{1})^{\pm}\mathbf{1}^{\pm}$
  - $\mathbf{1}^{+}(5,1,0,3) = \{(6,1,0,3), (5,2,0,3), (5,1,0,4)\}$
  - $\mathbf{1}^{-}(5,1,0,3) = \{(4,1,0,3), (5,0,0,3), (5,1,0,2)\}$
- 1-loop: $G_1 = \mathbf{1}^{+}\vec{\nu}, G_2 = \mathbf{1}^{-}\mathbf{1}^{+}\vec{\nu}$ Duplancic and Nizic, 0303184
- Multi-loop:

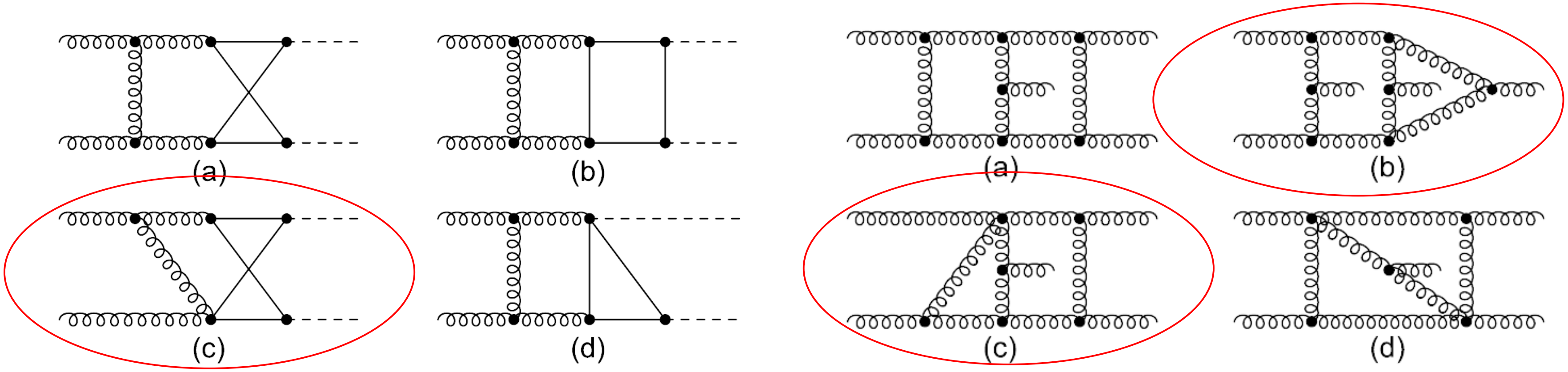
$$G_1 = \mathbf{m}^{+}\vec{\nu}, G_2 = \{\mathbf{1}^{-}\mathbf{m}^{+}, \mathbf{1}^{-}(\mathbf{m}-\mathbf{1})^{+}, \ldots, \mathbf{1}^{-}\mathbf{1}^{+}\}\vec{\nu}$$

  - $m = 2,3$ in examples, $\#G_1$ is not too large, include dozens of integrals
  - Relations among G_1 and G_2 are not too complicated, see examples

A step-by-step reduction is realized!

Examples

➢ 2-loop $g+g\to H+H$ and $g+g\to g+g+g$



$g+g\to H+H$				$g+g\to g+g+g$			
Sector	Type	$d_{\max}$	$\mathbf{m}^+$	Sector	Type	$d_{\max}$	$\mathbf{m}^+$
1(a)	7-NP	1	$\mathbf{3}^+$	2(a)	8-NP	1	$\mathbf{3}^+$
1(b)	7-P	1	$\mathbf{3}^+$	2(b)	8-NP	3	$\mathbf{3}^+$
1(c)	6-NP	5	$\mathbf{3}^+$	2(c)	7-NP	4	$\mathbf{3}^+$
1(d)	6-P	4	$\mathbf{2}^+$	2(d)	6-NP	2	$\mathbf{3}^+$

Difficulty:

- More legs > less legs
- Nonplanar > Planar
- $\mathbf{m}^+\vec{e} > \mathbf{m}^+\vec{\nu}$

- Relations can be obtained by a single-core laptop in a few hours
- Diagonalizing at each phase space point (floating number): 0.01 second
- Results checked numerically by FIRE