Multi-loop Feynman integrals with masses

Li Lin Yang
Zhejiang University

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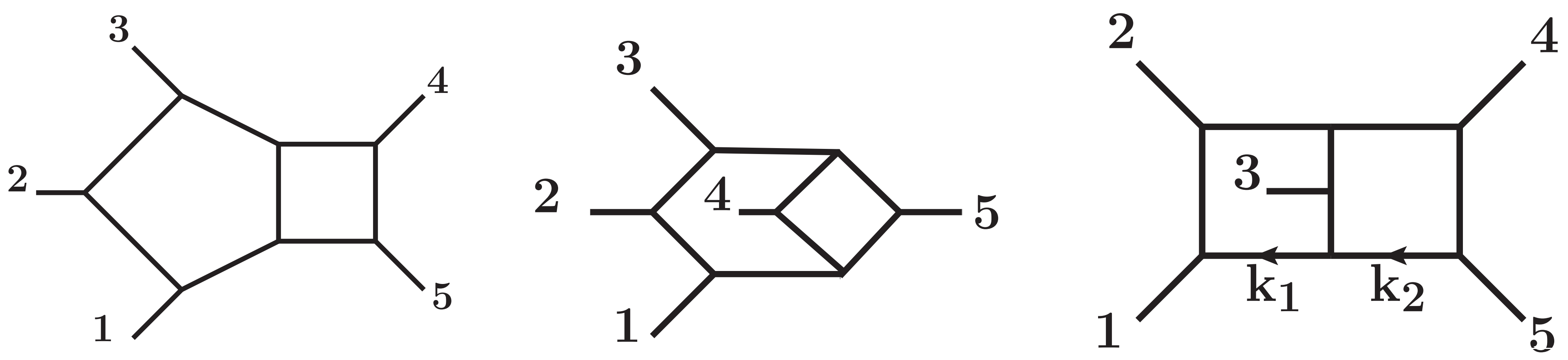

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Two-loop five-point integrals and amplitudes

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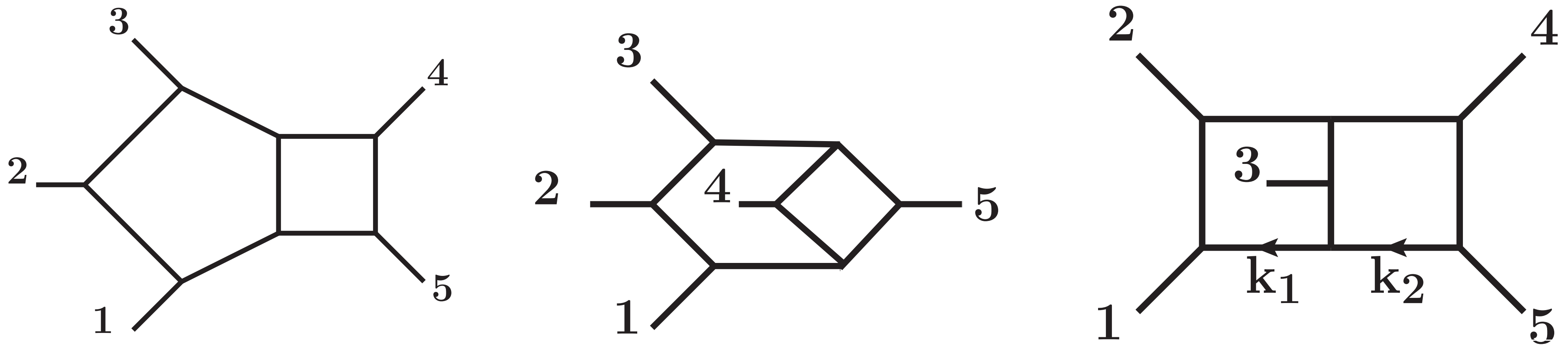

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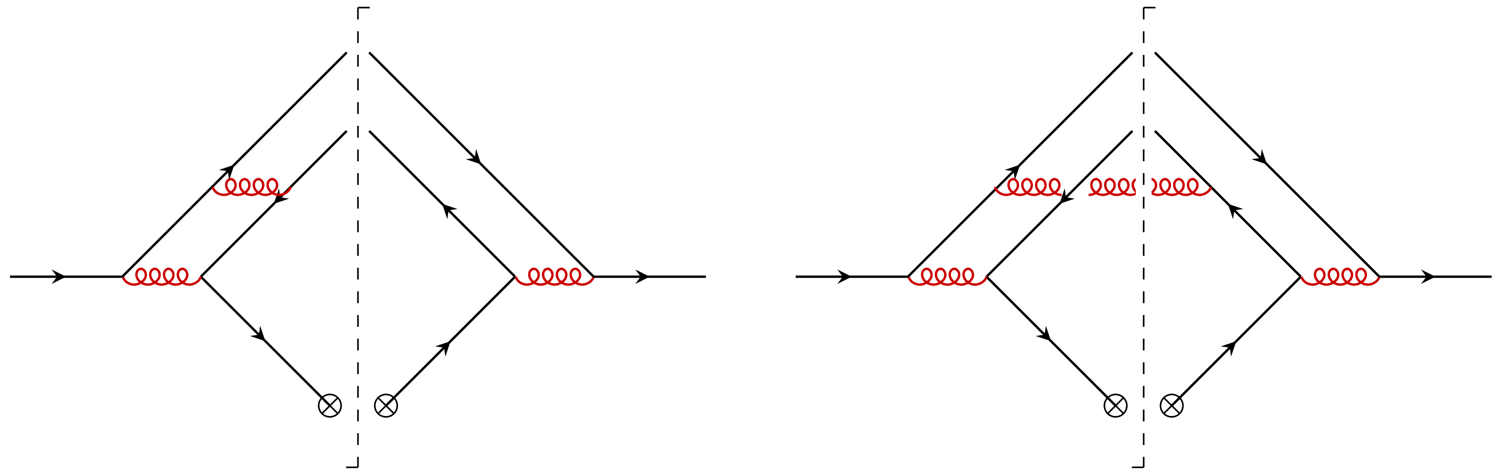

Luo et al.: 1912.05778; Hua Xing Zhu, …

Three-loop TMD PDFs

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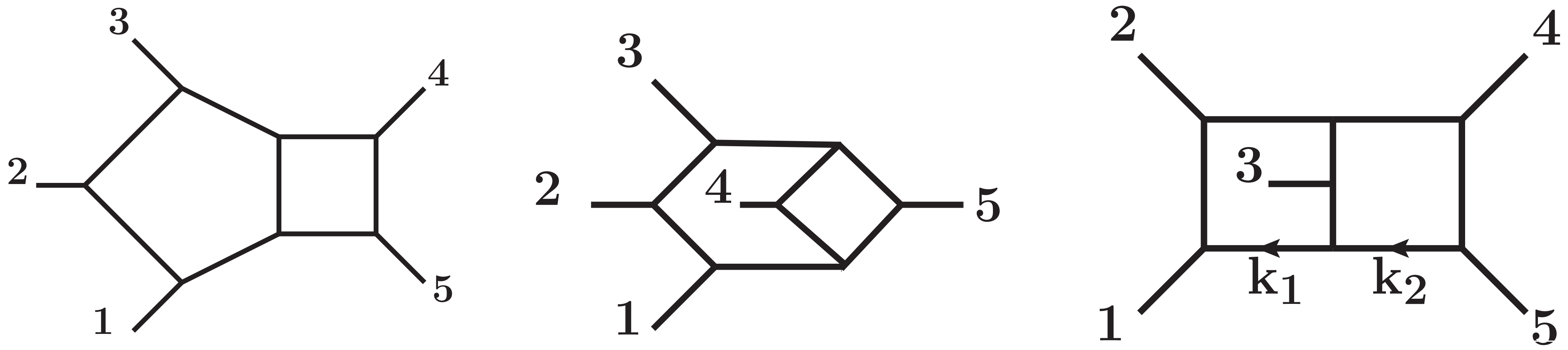

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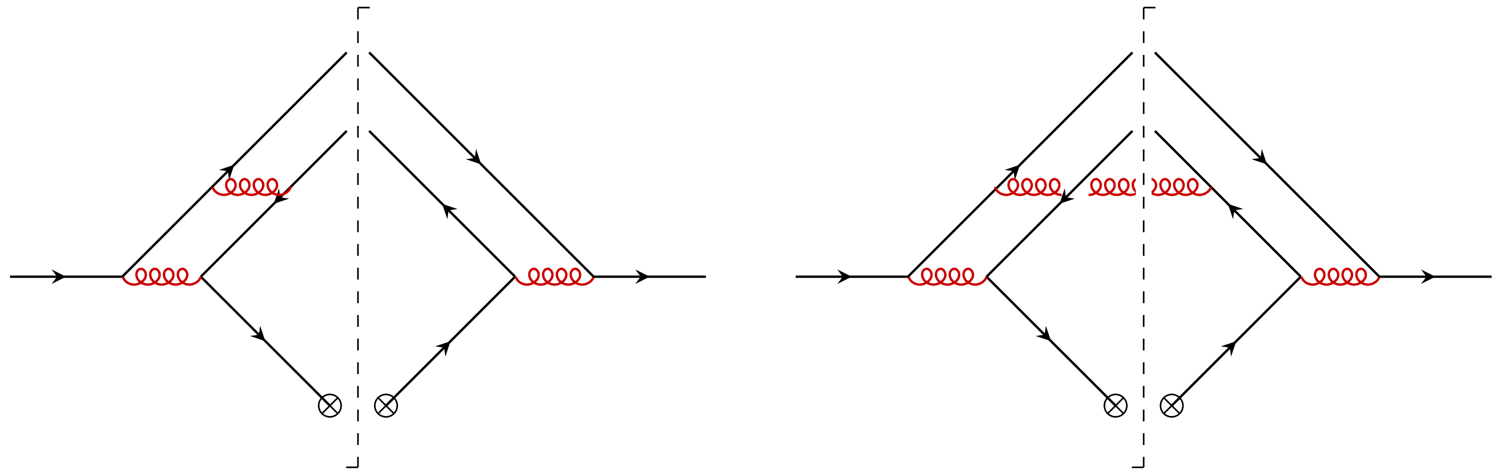

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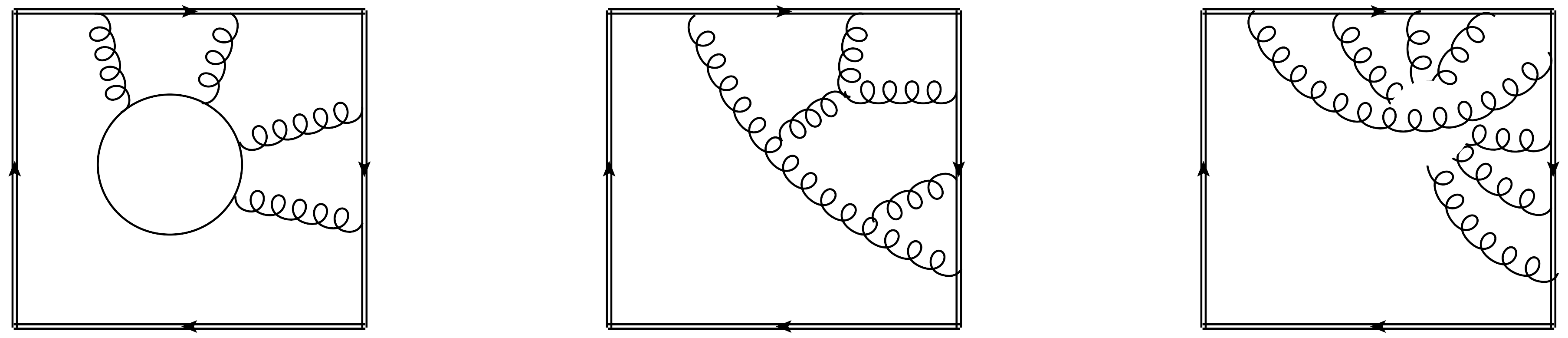
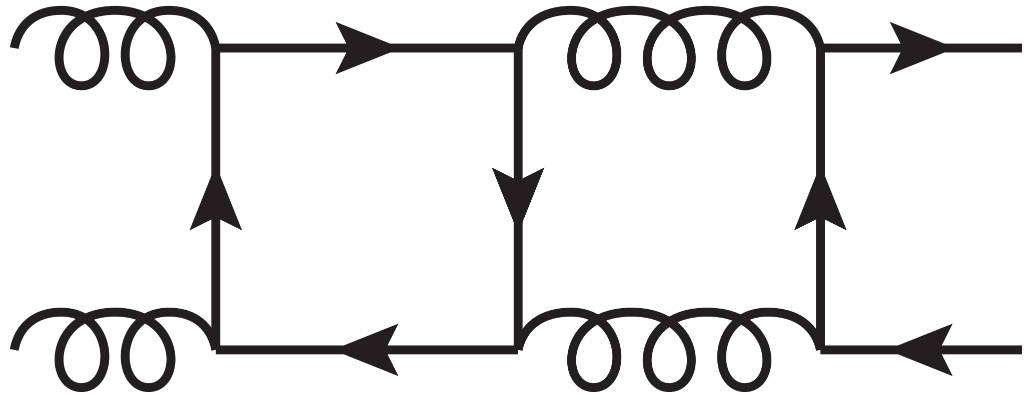


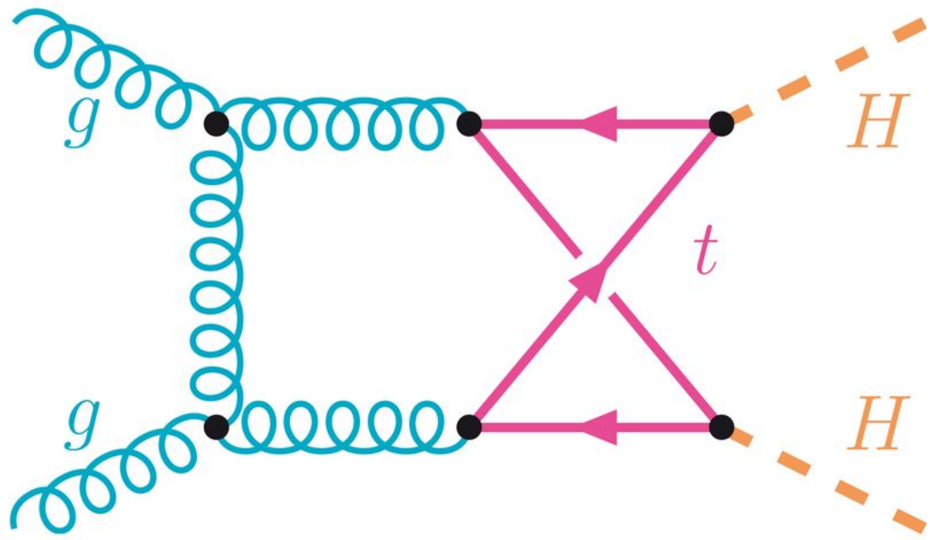
Henn et al.: 1911.10174; Gang Yang, …

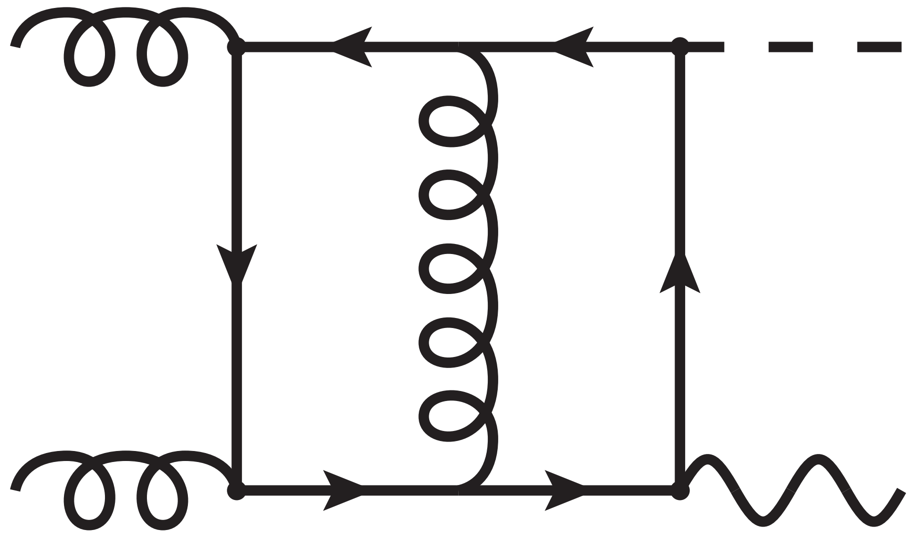
Four-loop cusp anomalous dimensions

Loop integrals with massive particles

We are also interested in loop integrals with **massive** particles, especially in electroweak physics







Loop integrals with massive particles

The analytic structure quickly becomes complicated when mass scales are introduced

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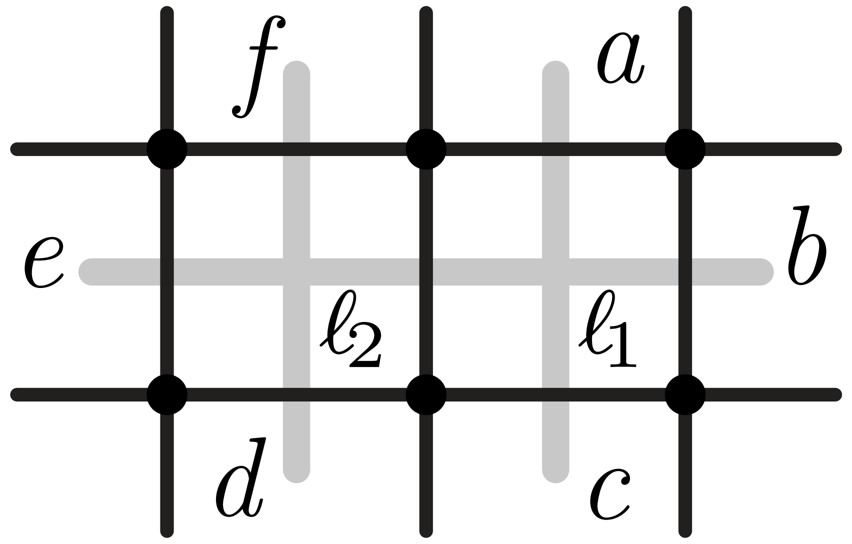

Simplest non-polylogarithmic massless integral

Bourjaily et al.: 1712.02785

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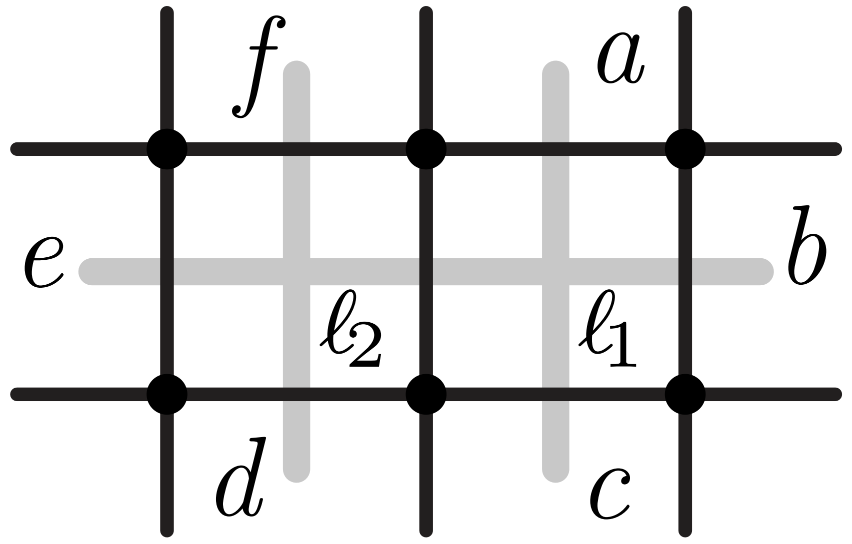

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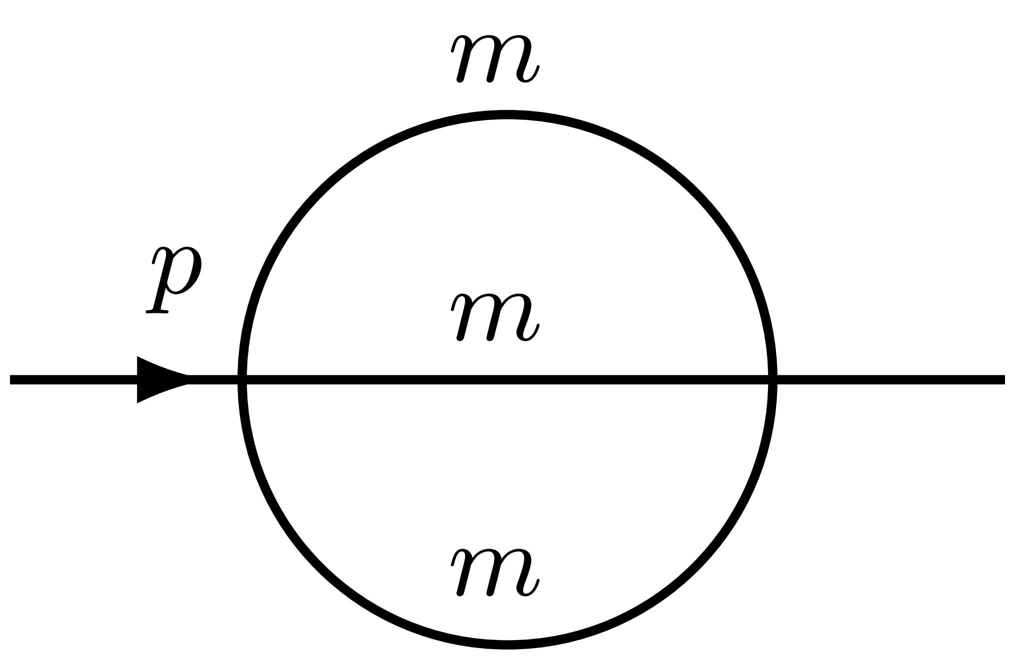

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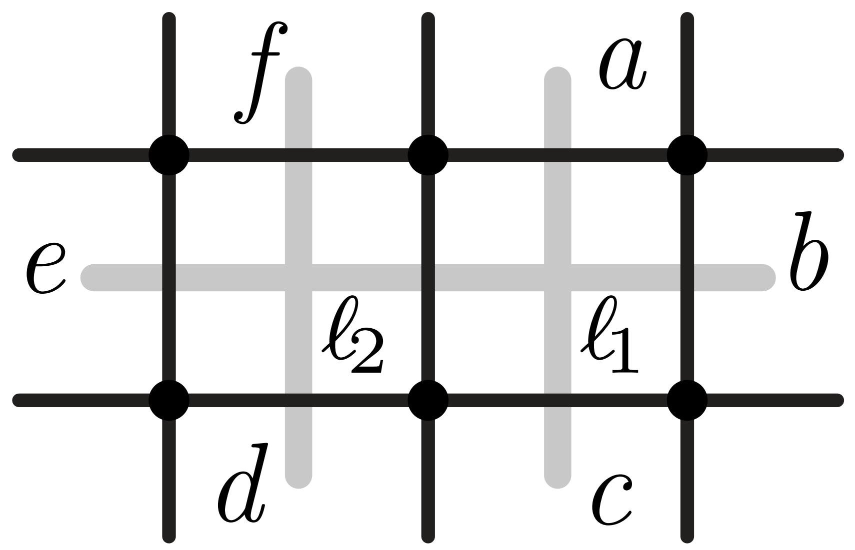

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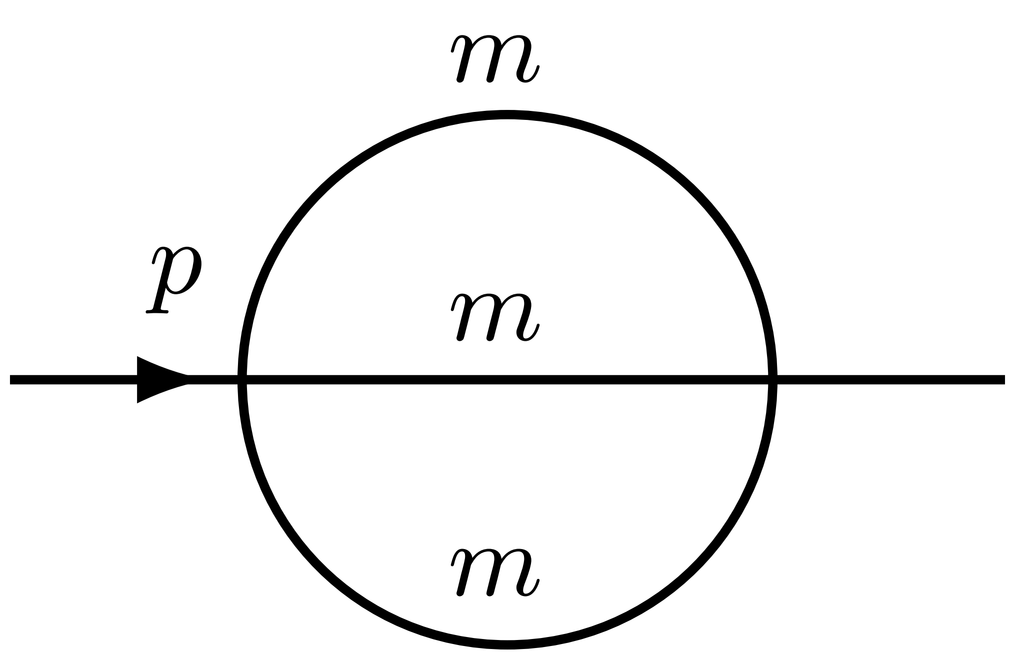

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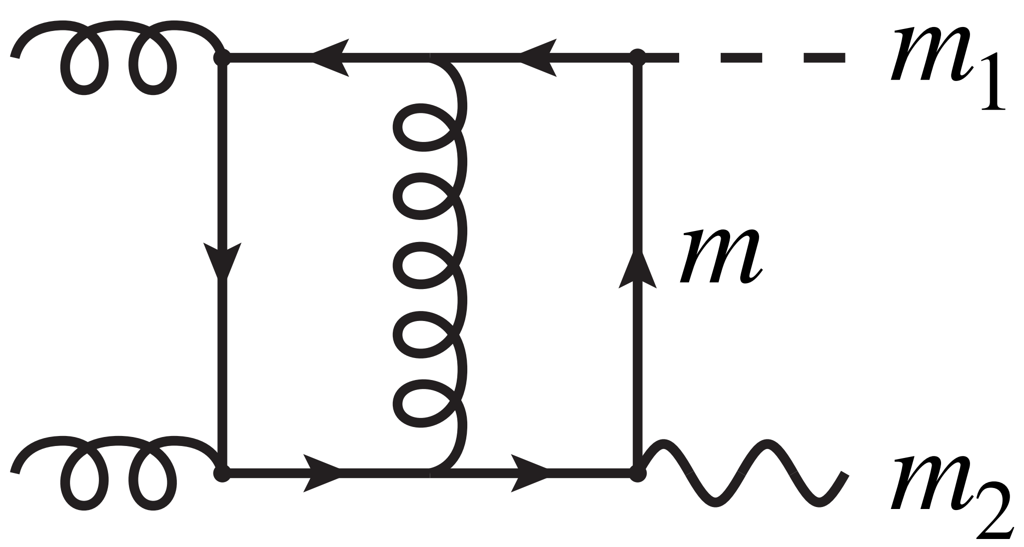

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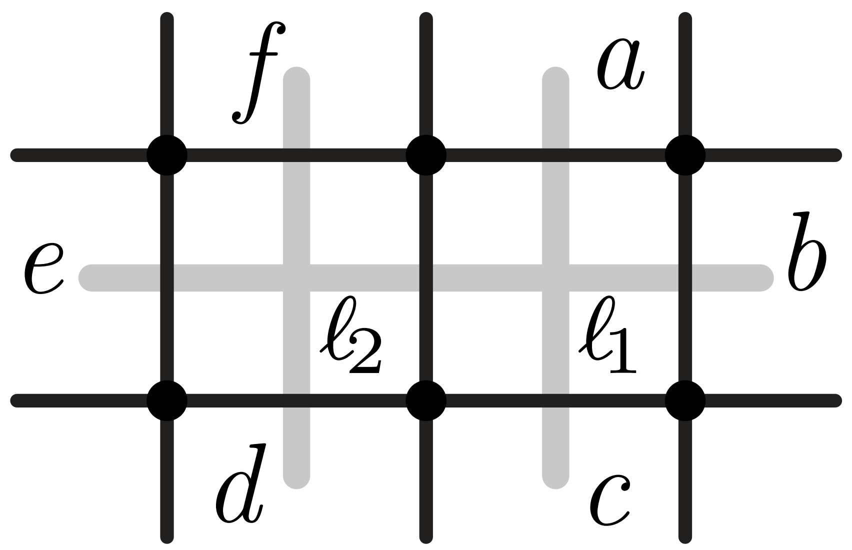

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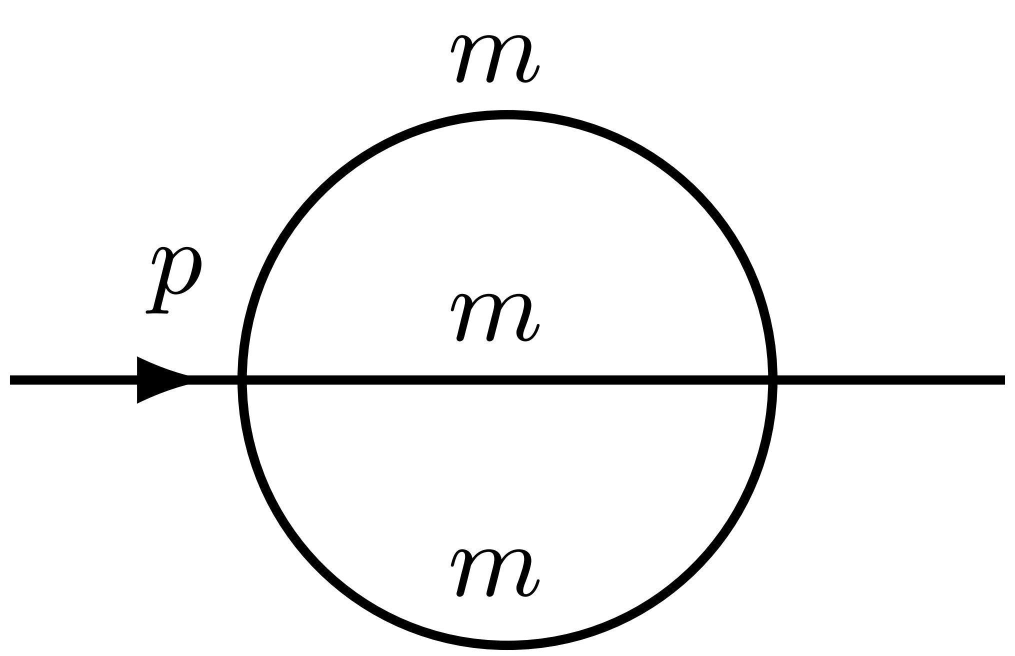

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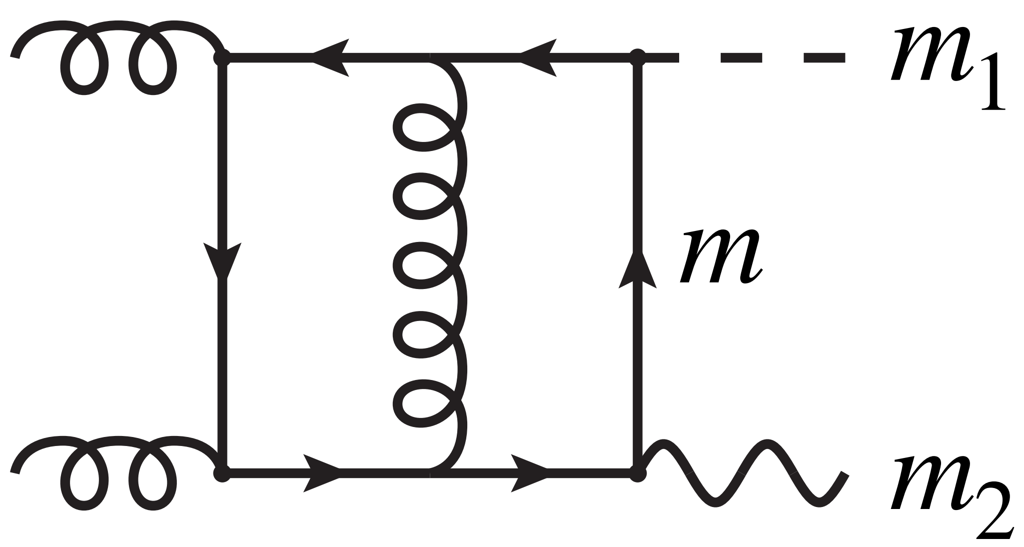

$m_1 = m_2 = 0$: polylogarithmic

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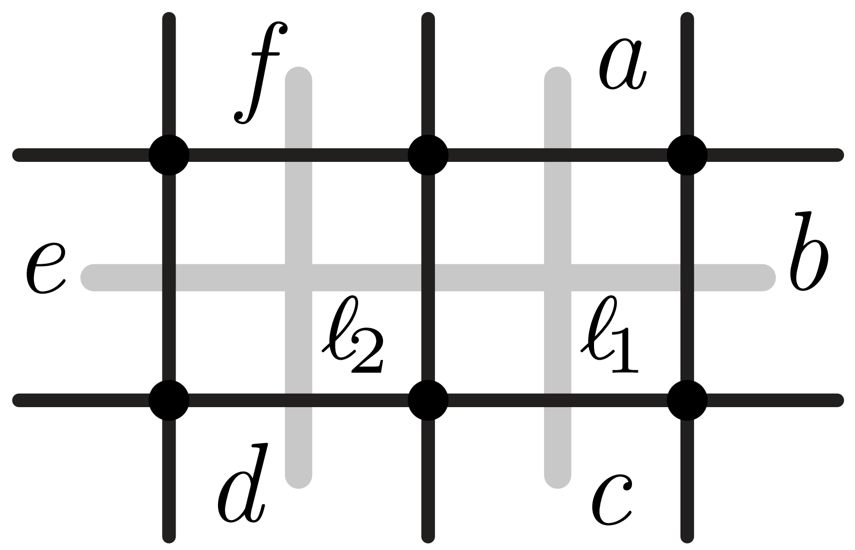

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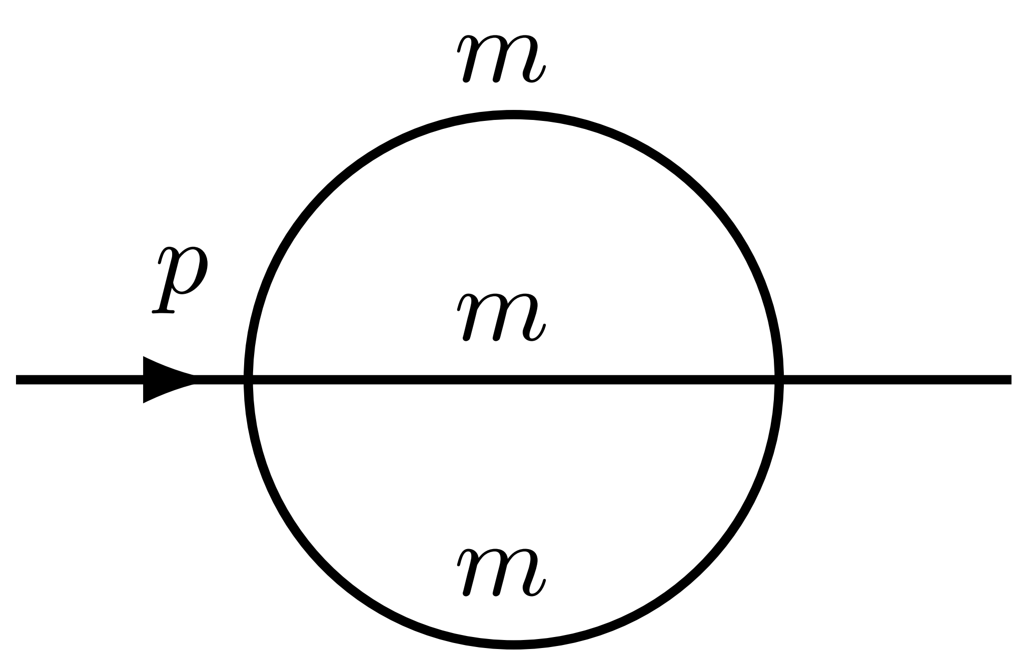

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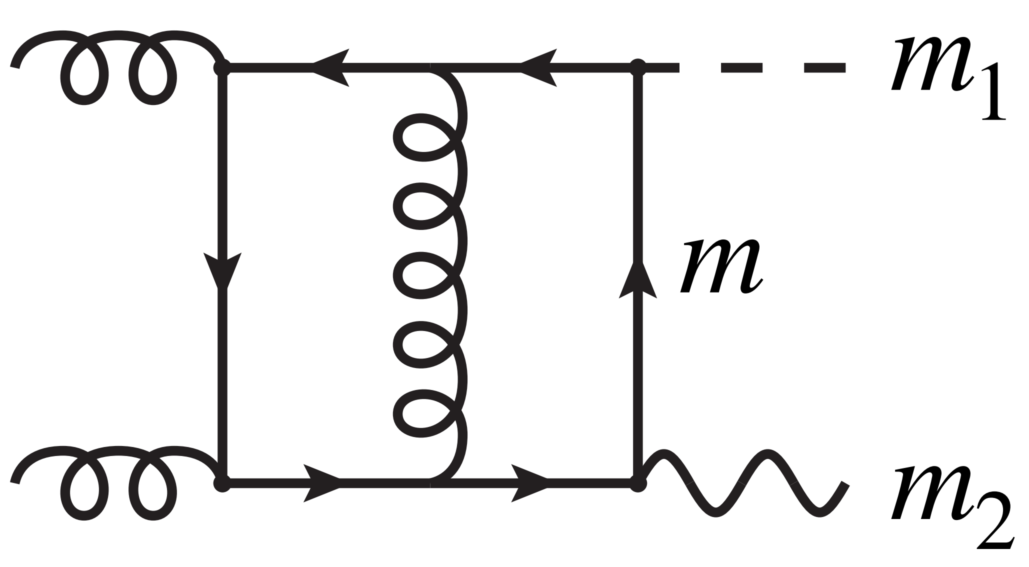

$$m_1 = m_2 = 0: \text{ polylogarithmic}$$

$$m_1 \neq 0, m_2 = 0: \text{ elliptic}$$

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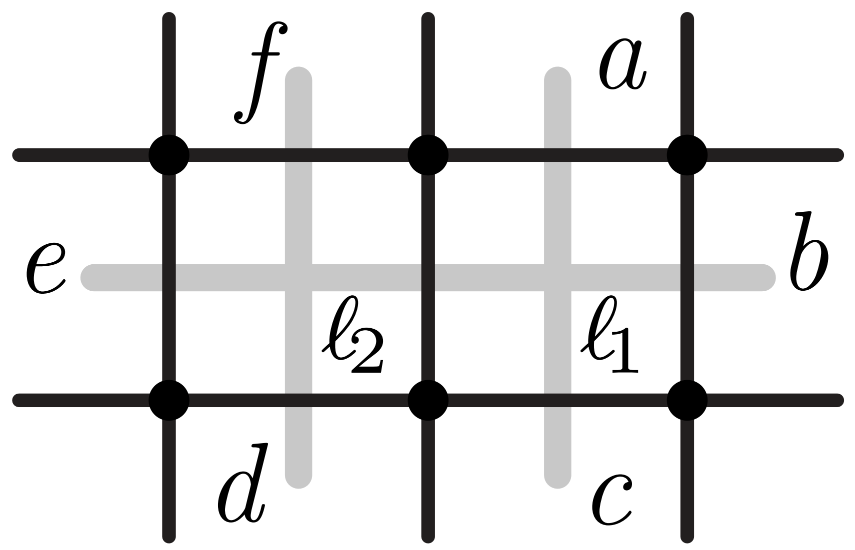

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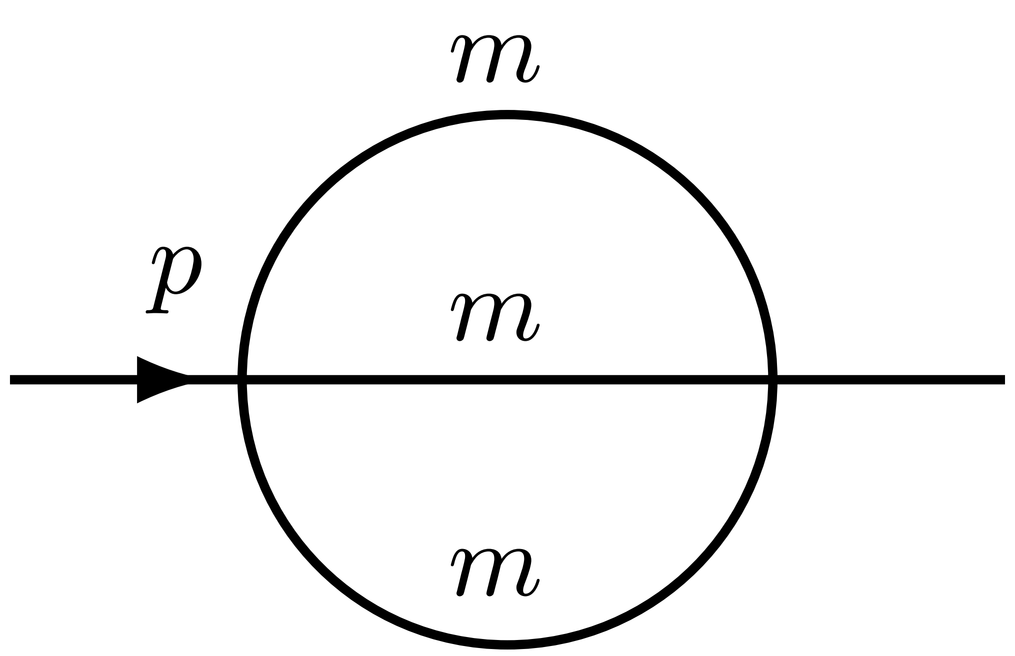

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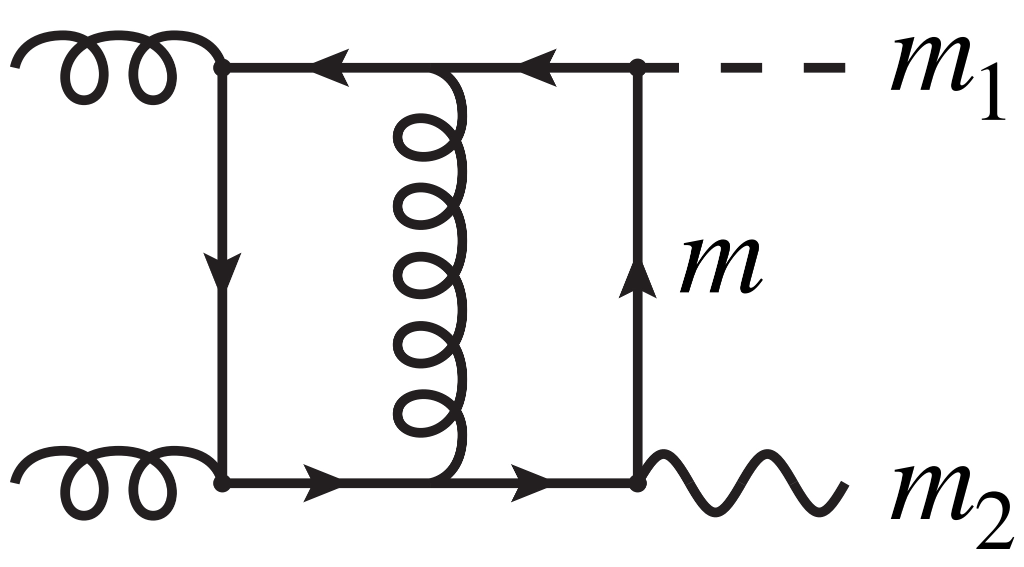

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May be computed numerically (sector decomposition, auxiliary mass flow, …)

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But we'd like to push the analytic method to its limit. And there are many questions:

- Given an integral family, how do we know which classes of functions will appear in the results?
- What is a good functional basis (with nice analytic/algebraic/geometric/numeric properties) to represent the results?
- How to organize the calculation procedure to naturally reflect these properties?
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Let's look at the simplest example: MPLs

Multiple polylogarithms (MPLs)

Generalization of logarithms and polylogarithms

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

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Pure functions:

$$dG(a_1, \ldots, a_n; z) = \sum_{i=1}^{n} G(a_1, \ldots, \hat{a}_i, \ldots, a_n; z)\, d\log \frac{a_{i-1} - a_i}{a_{i+1} - a_i}$$

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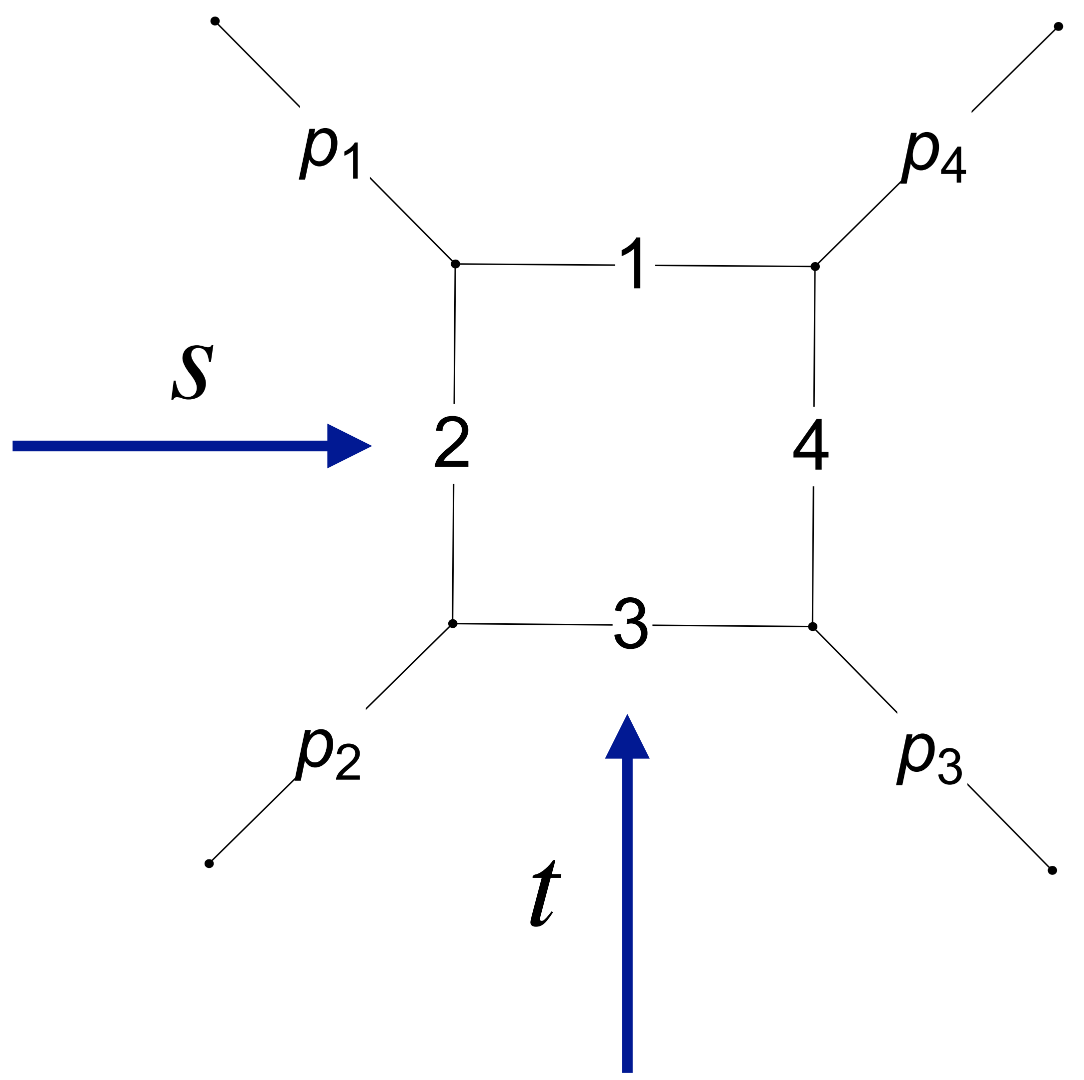
- Well-understood analytic structure (branch cuts): helps the analytic continuation to different kinematic regions
- Many algebraic properties (shuffle, stuffle, symbols, Hopf, …): helps to find the most compact representation See talk by Gang Yang
- Efficient numeric evaluation: helps to make phenomenological predictions

MPLs from canonical differential equations

Henn: 1304.1806, 1412.2296



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"canonical basis"

$x = t/s$

$$\partial_x \vec{g}(x;\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] \vec{g}(x,\epsilon)$$

"ϵ-form" simple poles

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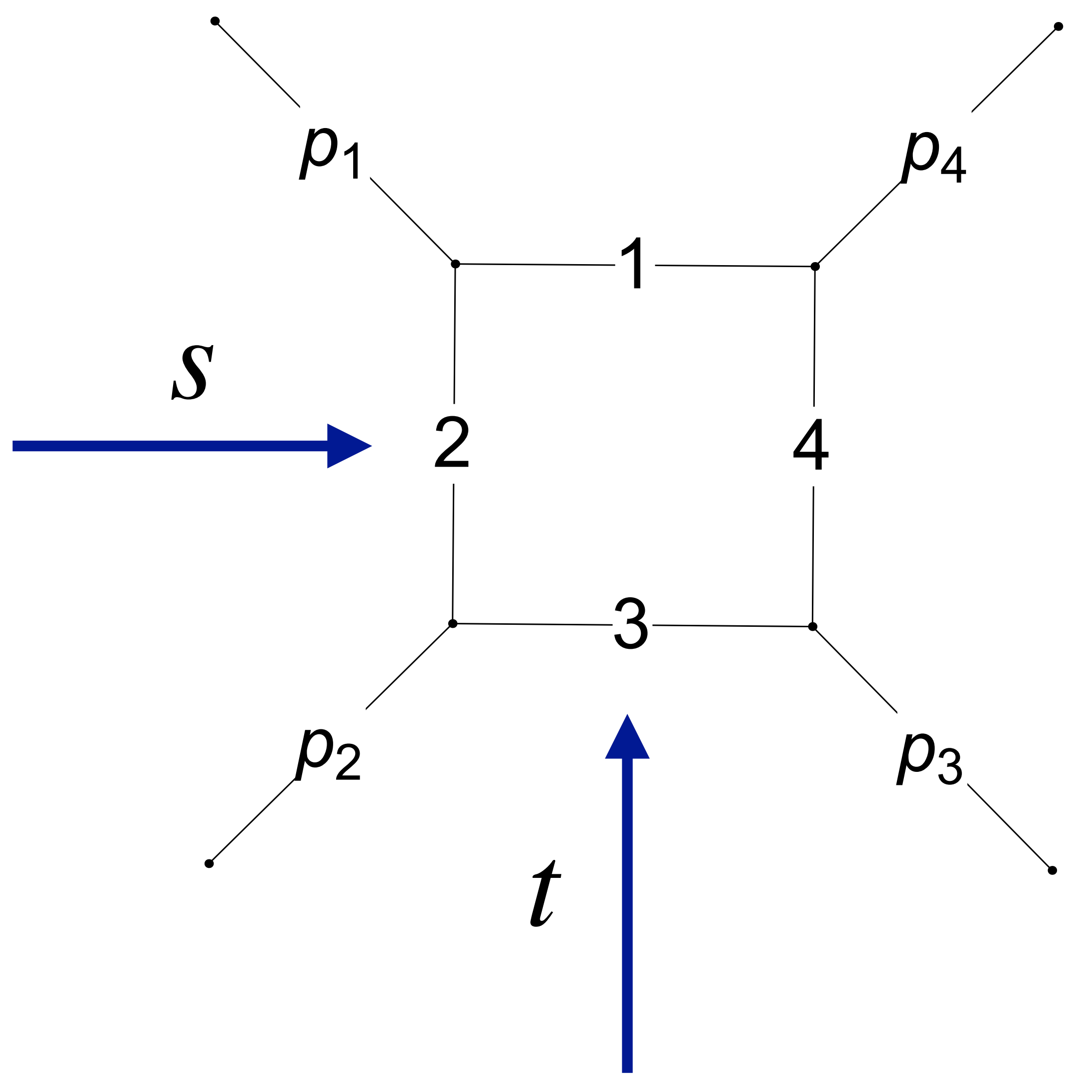

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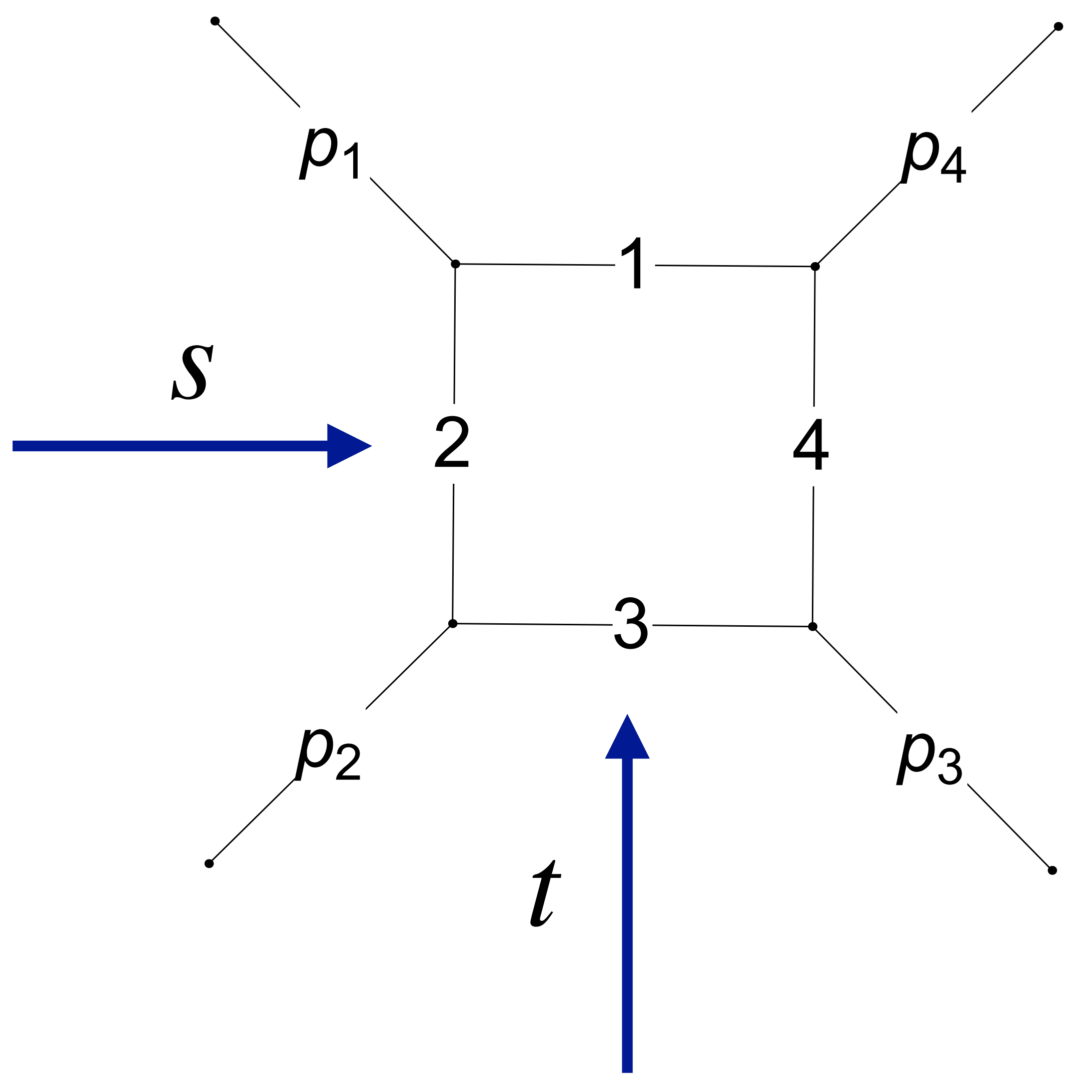

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$$\vec{f}(z,\epsilon) = \sum_n \epsilon^n \vec{f}^{(n)}(z)$$

MPLs from canonical DEs

Expansion coefficients in ϵ consist of iterated integrals

$$\vec{f}^{(n)}(z) \supset \int_{z_0}^{z} d\log(\alpha_n(z_n)) \cdots \int_{z_0}^{z_3} d\log(\alpha_2(z_2)) \int_{z_0}^{z_2} d\log(\alpha_1(z_1))$$

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In many cases can be converted to combinations of MPLs
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Other benefits of having a UT basis:

- Even if no explicit form as MPLs, can be efficiently evaluated numerically

Hidding: 2006.05510

- Amplitude coefficients often take a simpler form

Boehm et al.: 2008.13194

How to find a canonical basis?

The traditional way: starting from the DEs

$$\frac{\partial}{\partial x}\vec{f}(x,\epsilon) = A(x,\epsilon)\,\vec{f}(x,\epsilon)$$

Try to find a transformation matrix $T(x,\epsilon)$ such that $\vec{f}'(x,\epsilon) = T\vec{f}$ is a canonical basis, which means

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Algorithmic approach: Lee: 1411.0911

Program packages: Prausa: 1701.00725; Gituliar, Magerya: 1701.04269; Meyer: 1705.06252

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However, does not work well for irrational systems (e.g., with algebraic extensions like square-roots)

UT integrals and d-log integrands

It is believed that integrals with d-log integrands are canonical

$$\int_{\mathscr{C}} \left[G(\mathbf{z})\right]^{\epsilon} \bigwedge_{j=1}^{n} d\log f_j(\mathbf{z})$$

Studied a lot in the context of $\mathcal{N} = 4$ SYM

Arkani-Hamed et al., Bern et al., Drummond et al., Gehrmann et al., …

An example:

weight 0

weight 1

$$4\epsilon^2 \int_0^x \frac{dz_2}{z_2} \int_0^{z_2} \frac{dz_1}{z_2 - z_1} \left[\frac{z_2(x - z_2)(1 + z_1)}{z_1(z_1 - z_2)^2}\right]^{\epsilon} = 1 - \epsilon \log x + \epsilon^2 \left(\frac{1}{2}\log^2 x + 2\mathrm{Li}_2(-x)\right) + \cdots$$

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We'd like to find d-log integrands which can be interpreted as Feynman integrals

Need to choose a concrete representation…

Baikov representation

Baikov: hep-ph/9611449

Change of integration variables from momenta to propagators

$$\int \left[\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}}\right] \frac{1}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}} \propto \int_{\mathscr{C}} \left[\prod_{n=1}^{N} dz_n\right] \frac{\left[G(z_1, \ldots, z_N)\right]^{(d-L-E-1)/2}}{z_1^{a_1} \cdots z_N^{a_N}}$$

propagators

Gram determinant

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Other uses of Baikov rep.: IBP reduction, deriving differential equations, …

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We need a more generic form…

Generalized loop-by-loop Baikov representation

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irreducible scalar product (ISP)

If some z_i does not appear in the denominator, we can integrate over it to obtain a new representation, e.g.:

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$$\int_{z_-}^{z_+} (az^2 + bz + c)^{-1-\epsilon}\, dz \sim a^{\epsilon} \left(b^2 - 4ac\right)^{-1/2-\epsilon}$$

$\longrightarrow$ Equivalent to the loop-by-loop (LBL) Baikov representation

Generalized loop-by-loop Baikov representation

$$\int \left[\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}}\right] \frac{1}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}} \propto \int_{\mathscr{C}} \left[\prod_{n=1}^{N} dz_n\right] \frac{\left[G(z_1, \ldots, z_N)\right]^{(d-L-E-1)/2}}{z_1^{a_1} \cdots z_N^{a_N}}$$

irreducible scalar product (ISP)

If some z_i does not appear in the denominator, we can integrate over it to obtain a new representation, e.g.:

$$\int_{z_-}^{z_+} (az^2 + bz + c)^{-1-\epsilon}\, dz \sim a^{\epsilon} \left(b^2 - 4ac\right)^{-1/2-\epsilon} \longrightarrow$$

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Generalized (LBL) representation, new polynomial appears in denominator!

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Generalized (LBL) representation, new polynomial appears in denominator!

We will use this generalized form to look for d-log integrands, but how can we convert them back to Feynman integrals?

Geometric formulation of IBP equivalence

Frellesvig et al.: 1901.11510, 1907.02000, 2008.04823

Baikov integrals are special cases of generalized hypergeometric integrals

$$I = \int_{\mathscr{C}} u(z)\, \varphi(z)$$

multivalued function vanishing on the boundary $\partial\mathscr{C}$

single-valued differential n-form

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$$0 = \int_{\mathscr{C}} d(u(z)\xi(z)) = \int_{\mathscr{C}} u(z)\, \nabla_\omega \xi(z)$$

$(n-1)$-form

$\nabla_\omega \equiv d + \omega\, \wedge$ covariant derivative

$\omega \equiv d \log u$ connection

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$(n-1)$-form

$\nabla_\omega \equiv d + \omega\,\wedge$ covariant derivative

$\omega \equiv d\log u$ connection

$\varphi(z)$ and $\varphi(z) + \nabla_\omega \xi(z)$ are equivalent (in the sense of integration)

Twisted cohomology

The covariant derivative ∇_ω creates a cochain complex of differential forms

$$\Omega^0(M) \overset{\nabla_\omega}{\to} \Omega^1(M) \overset{\nabla_\omega}{\to} \cdots \Omega^{n-1}(M) \overset{\nabla_\omega}{\to} \Omega^n(M) \overset{\nabla_\omega}{\to} 0$$

Exact forms φ: $\varphi = \nabla_\omega \xi$ Closed forms φ: $\nabla_\omega \varphi = 0$

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The k-th twisted cohomology group

$$H^k_\omega = \frac{\ker(\nabla_\omega : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}(\nabla_\omega : \Omega^{k-1}(M) \to \Omega^k(M))} = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}}$$

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H^k_ω is a vector space whose elements (cocycles) are equivalence classes

$$\langle \varphi | : \varphi \sim \varphi + \nabla_\omega \xi$$

The same as IBP!

Decomposition in cohomology group

The dimension of the vector space H^n_ω is

$$\dim(H^n_\omega) = \nu = \#\text{ of master integrals with a given } \omega$$

We may find a basis with ν vectors $\quad \{\langle e_1|, \langle e_2|, \ldots, \langle e_\nu|\}$

All vectors can be written as a linear combination

$$\langle \varphi| = \sum_{i=1}^{\nu} c_i \langle e_i|$$

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How to project out these coefficients?

We need something like an "inner-product"

Intersection numbers

To define an inner-product, introduce a dual vector space H_{ω}^{n*} with elements $|\varphi_R\rangle$

Equivalence classes $\quad |\varphi_R\rangle \, : \, \varphi_R \sim \varphi_R + \nabla_{-\omega}\xi_R$

The intersection numbers Cho, Matsumoto (1995)

$$\langle \varphi_L | \varphi_R \rangle_{\omega} = \frac{1}{(2\pi i)^n} \int \iota_{\omega}(\varphi_L) \wedge \varphi_R = \frac{1}{(2\pi i)^n} \int \varphi_L \wedge \iota_{-\omega}(\varphi_R)$$

map to an equivalent, but compactly supported form

Decomposition via intersection numbers

Frellesvig et al.: 1901.11510, 1907.02000, 2008.04823

$$\langle \varphi | = \sum_{i=1}^{\nu} c_i \langle e_i |$$

Construct a dual basis of $\langle e_i |$

$$\langle e_i | d_j \rangle = \delta_{ij} \longrightarrow c_i = \langle \varphi | d_i \rangle$$

or $$\langle \varphi | = \sum_{i=1}^{\nu} \langle \varphi | d_i \rangle \langle e_i |$$

Just like the decomposition in a usual vector space…

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Just like the decomposition in a usual vector space…

Computing these inner-products is still a non-trivial task (skipped in this talk)

Application to generalized Baikov integrals

We identify

$$\omega = d\log u$$

$$u(z) = \prod_i \left[G_i(z)\right]^{-\gamma_i - \beta_i \epsilon}$$

polynomials resulting from integrating over some ISPs

$$\varphi(z) = \frac{f(z)}{z_1^{a_1} \ldots z_n^{a_n} G_1^{b_1} \ldots G_m^{b_m}} \bigwedge_{j=1}^{n} dz_j$$

candidates for d-log integrands

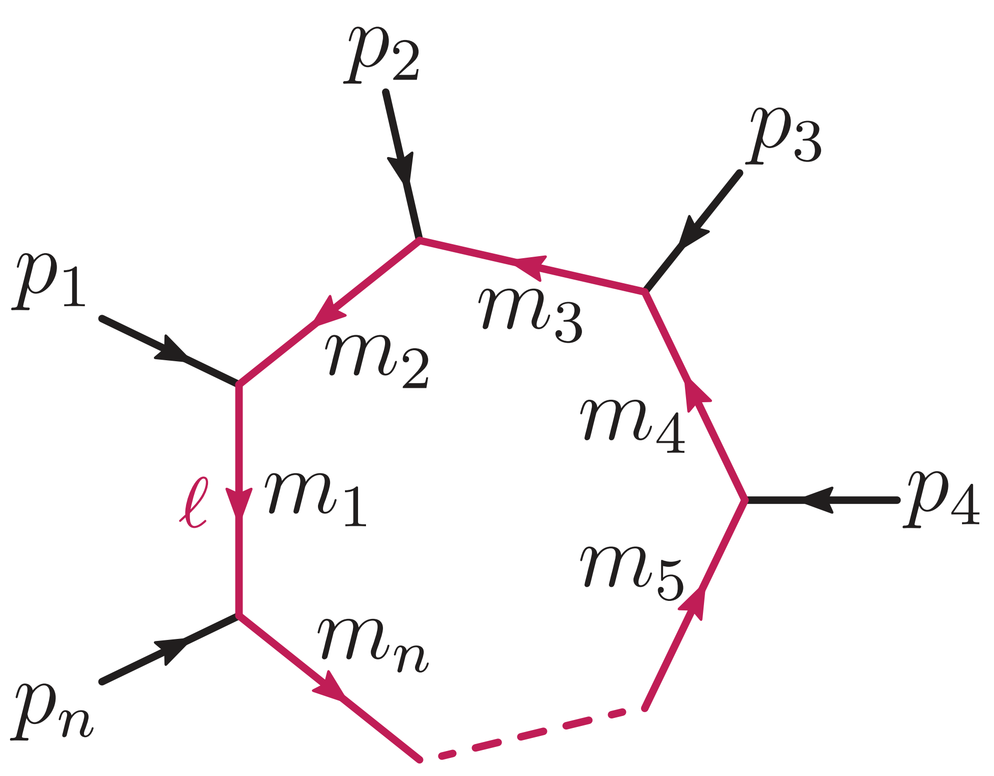
Given a candidate $\varphi(z)$, we can convert it to Feynman integrals using intersection theory, or by generalized IBP relations

Dlapa, Li, Zhang: 2103.04638

The remaining task is to construct enough n-forms belonging to H^n_ω which lead to d-log integrals…

One-loop integrals

Chen, Jiang, Xu, **LLY**: 2008.03045



An arbitrary one-loop topology

# of independent external momenta: $E = n - 1$

$$u(z) \sim [G(z)]^{(2-E)/2-\epsilon}$$

E even: $\quad \varphi(z) \sim \left[G(z)\right]^{(E-2)/2} \bigwedge_{i=1}^{E+1} d\log(z_i)$

E odd: $\quad \varphi(z) \sim \sqrt{G(\mathbf{0})}\left[G(z)\right]^{(E-3)/2} \bigwedge_{i=1}^{E+1} \frac{dz_i}{z_i}$

A useful relation:

$$\frac{\partial}{\partial x} \log \frac{1-\sqrt{\frac{x_2(x_1-x)}{x_1(x_2-x)}}}{1+\sqrt{\frac{x_2(x_1-x)}{x_1(x_2-x)}}} = \frac{\sqrt{x_1 x_2}}{x\sqrt{(x_1-x)(x_2-x)}}$$

Easy to verify that $u(z)\varphi(z)$ takes d-log form

Multi-loop integrals with maximal cuts

We now turn to multi-loop Baikov integrals

$$F_{a_1,\ldots,a_m,0,\ldots,0} = \int_{\mathcal{C}} \left[\prod_i \left[G_i(\boldsymbol{z})\right]^{-\gamma_i - \beta_i \epsilon} \right] \left[\prod_{j=1}^{m} \frac{dz_j}{z_j^{a_j}} \right] \prod_k dz_k$$

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For simplicity, we first consider maximal cuts

$$F^{m\text{-cut}}_{a_1,\ldots,a_m,0,\ldots,0} = \int_{\mathcal{C}'} \left[\prod_k dz_k\right]\left[\prod_{j=1}^{m}\oint_{z_j=0}\frac{dz_j}{z_j^{a_j}}\right]\times\prod_i \left[G_i(\boldsymbol{z})\right]^{-\gamma_i-\beta_i\epsilon}$$

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cut = integrate out using residues

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Good things about maximal cuts:

- Determines the differential equations up to sub-topologies
- Helps to identify whether the integral family involves elliptic integrals

Univariate construction

Chen, Jiang, Xu, **LLY**: 2008.03045

In the simplest cases, only one variable remains under maximal cuts

$$u(z) = \prod_i \left[G_i(z)\right]^{-\gamma_i - \beta_i \epsilon}$$

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If all γ_i's are integers, we can always factorize

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…with distinguished roots c_j

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$\dim(H_\omega) = \nu \longrightarrow$ needs to construct ν d-log integrals

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We choose: $$\hat{\phi}_i(z) = \frac{\mathcal{K}_0}{z - c_i} \prod_{j=0}^{\nu} (z - c_j)^{\gamma'_j}, \ (i = 1, \ldots, \nu)$$

Univariate construction

Chen, Jiang, Xu, **LLY**: 2008.03045

If one of the γ_i's is a half-integer

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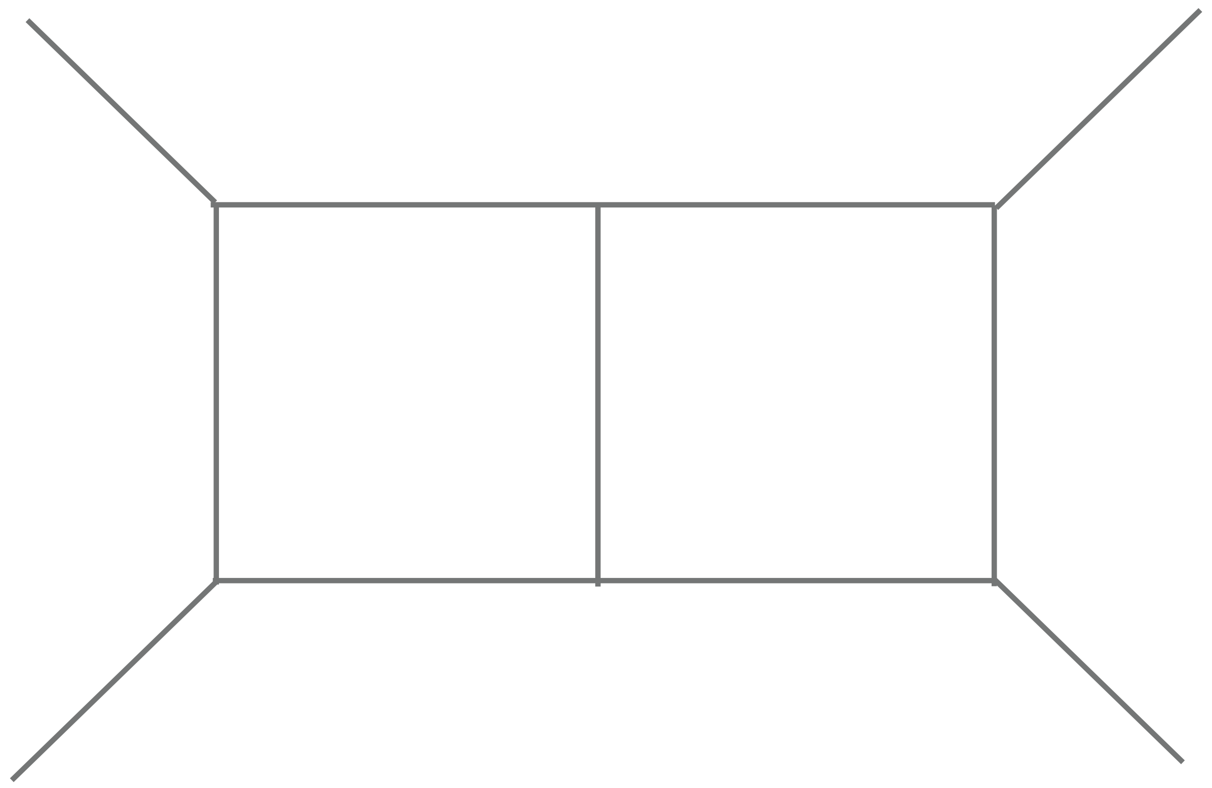
Beyond these two cases, one expect appearance of elliptic integrals!

Massless double box on maximal cuts

Chen, Jiang, Xu, **LLY**: 2008.03045



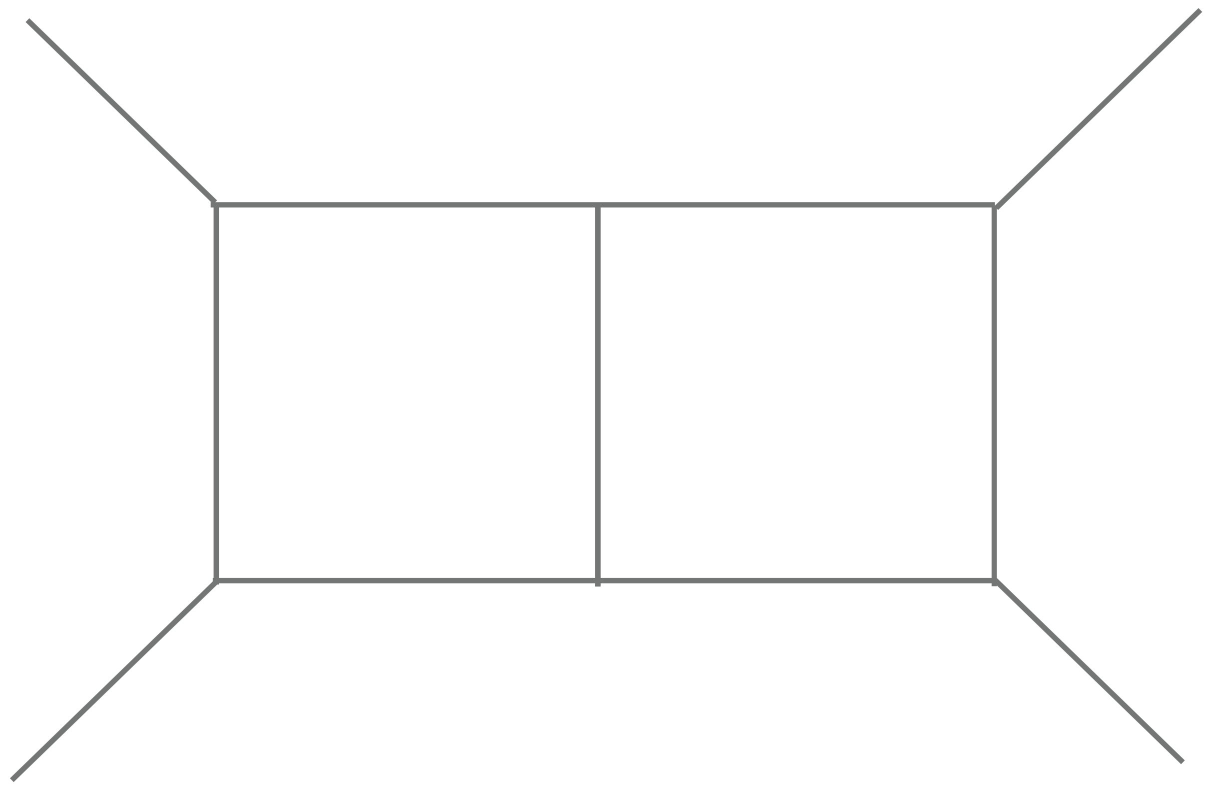
$$u(z) = \frac{1}{s^2}\left(\frac{t(s+t)}{s^2}\right)^{\epsilon} z^{-1-\epsilon}(s+z)^{\epsilon}(t-z)^{-1-2\epsilon}$$

$$\omega = d\log(u) = \left(\frac{\epsilon}{s+z} + \frac{1+2\epsilon}{t-z} - \frac{1+\epsilon}{z}\right) dz$$

$$\dim(H_\omega) = 2$$

Massless double box on maximal cuts

Chen, Jiang, Xu, **LLY**: 2008.03045



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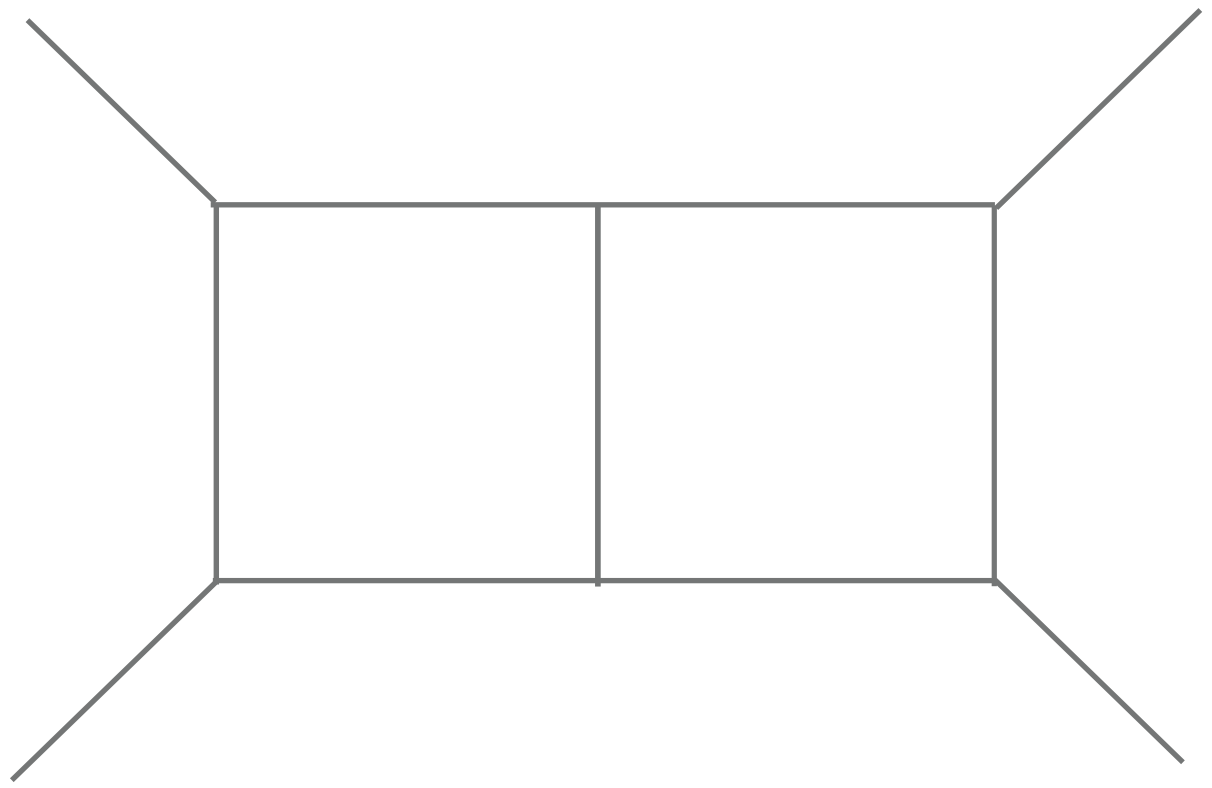
No half-integer powers: $\phi_1 = s^2 z dz\,, \quad \phi_2 = s^2(t-z)dz$

Massless double box on maximal cuts

Chen, Jiang, Xu, **LLY**: 2008.03045



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No half-integer powers: $\phi_1 = s^2 z dz\,,\quad \phi_2 = s^2(t-z)dz$

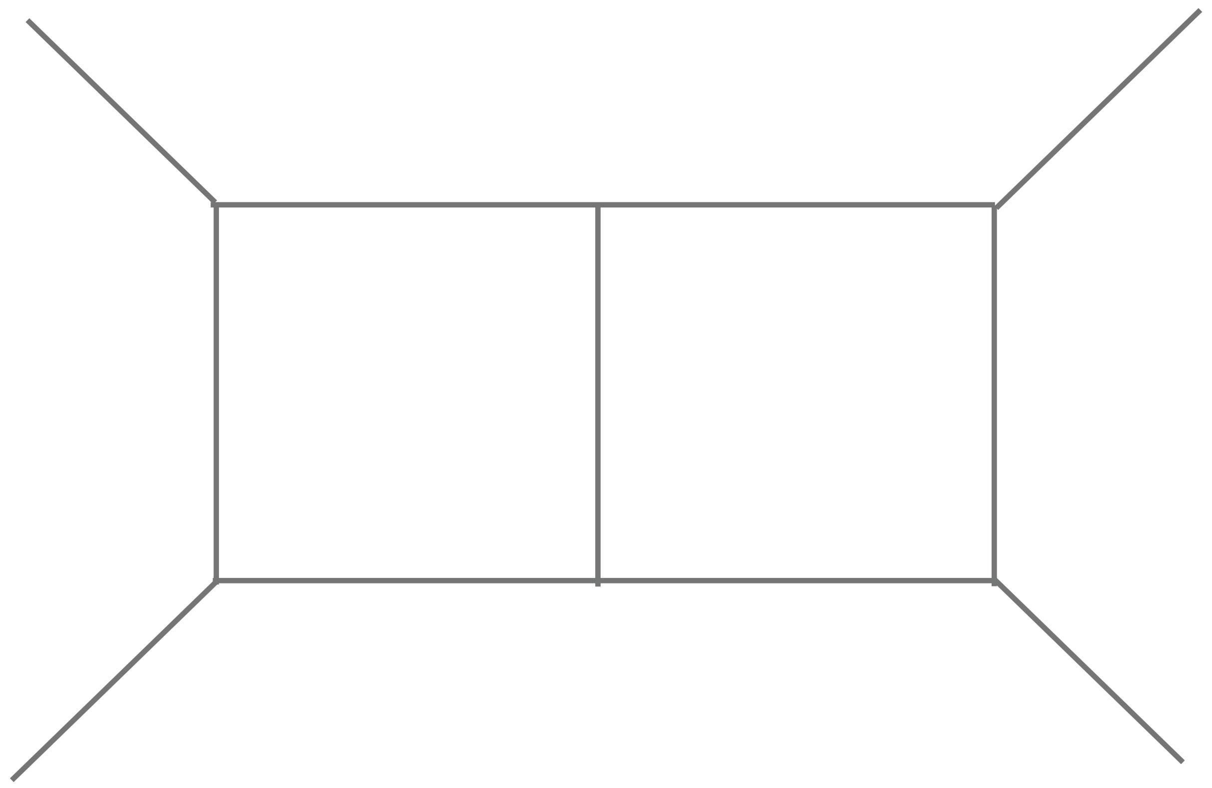
Pick two arbitrary master integrals:

$$E_1 = F_{1,1,1,1,1,1,1,0,0}\,,\quad E_2 = F_{1,2,1,1,1,1,1,0,0}$$

$$\longrightarrow \langle e_1| = dz\,,\quad \langle e_2| = \frac{1+2\epsilon}{z}dz$$

Massless double box on maximal cuts

Chen, Jiang, Xu, **LLY**: 2008.03045



Performing the decomposition gives

$$I_1 = -\frac{s(1+3\epsilon)}{2\epsilon}E_1 + \frac{st(1+\epsilon)}{2\epsilon(1+2\epsilon)}E_2$$

$$I_2 = \frac{s(1+3\epsilon)+2\epsilon t}{2\epsilon}E_1 - \frac{st(1+\epsilon)}{2\epsilon(1+2\epsilon)}E_2$$

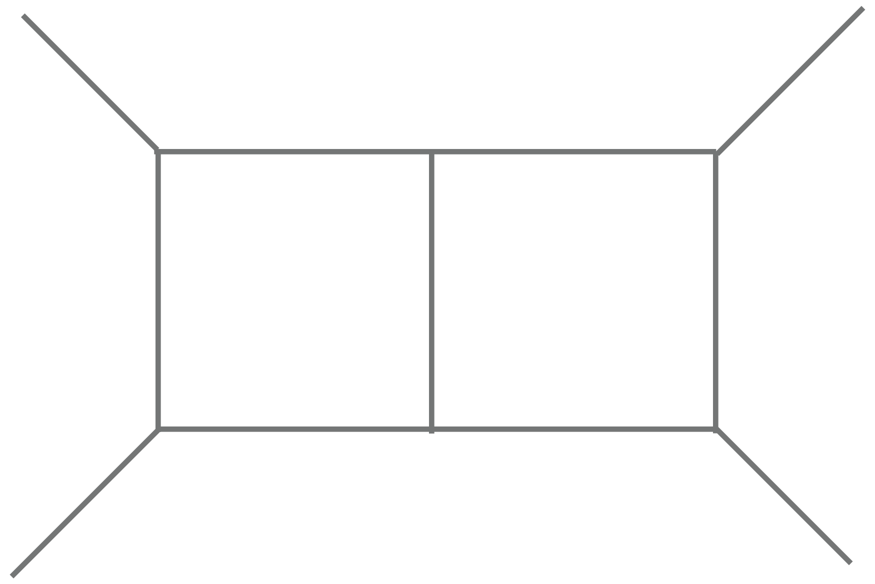
Check the DEs (on maximal cuts)

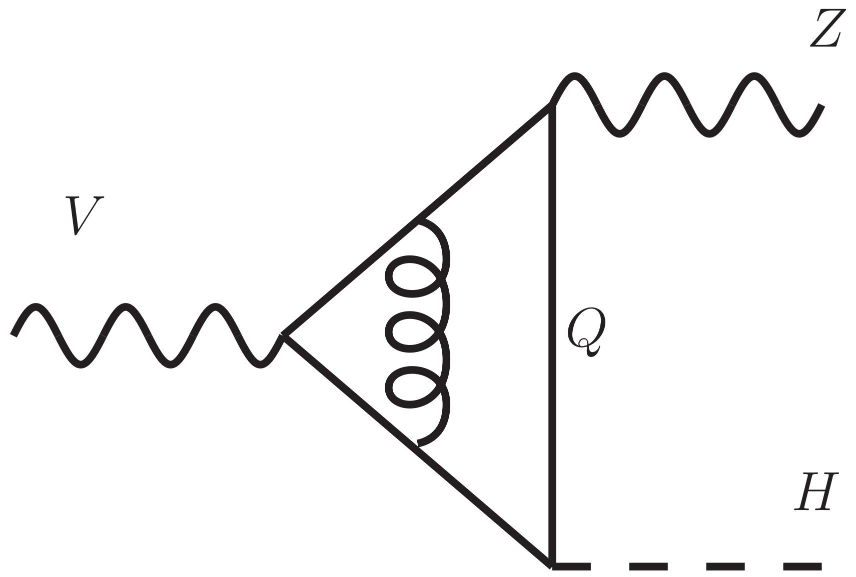
$$\frac{\partial}{\partial s}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \epsilon \begin{pmatrix} -\frac{2}{s} & \frac{1}{s+t} \\ \frac{2}{s} & -\frac{s+2t}{s(s+t)} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \quad \frac{\partial}{\partial t}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \epsilon \begin{pmatrix} 0 & -\frac{s}{t(s+t)} \\ -\frac{2}{t} & -\frac{s}{t(s+t)} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

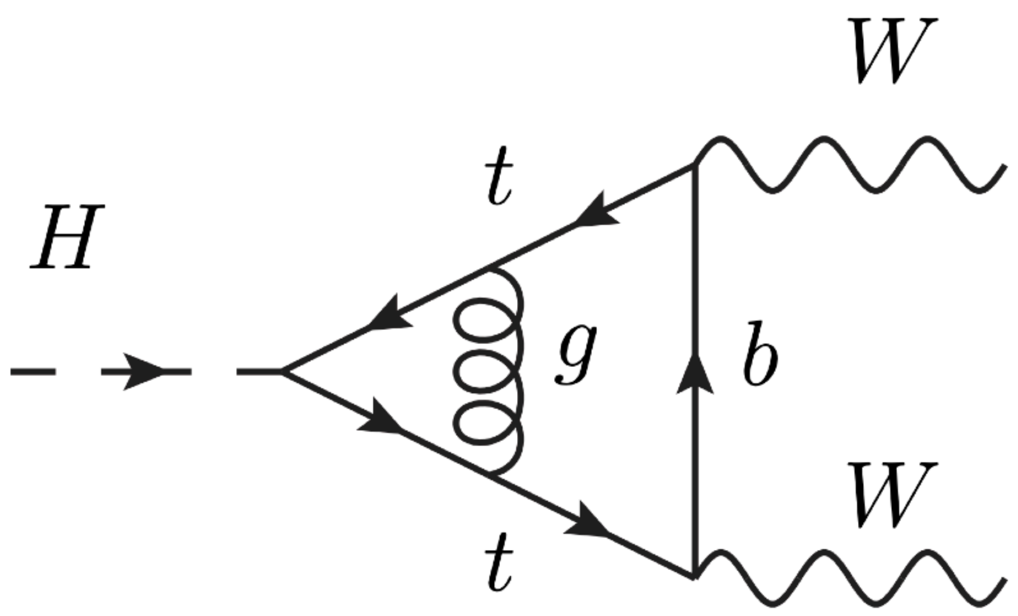
Multivariate construction

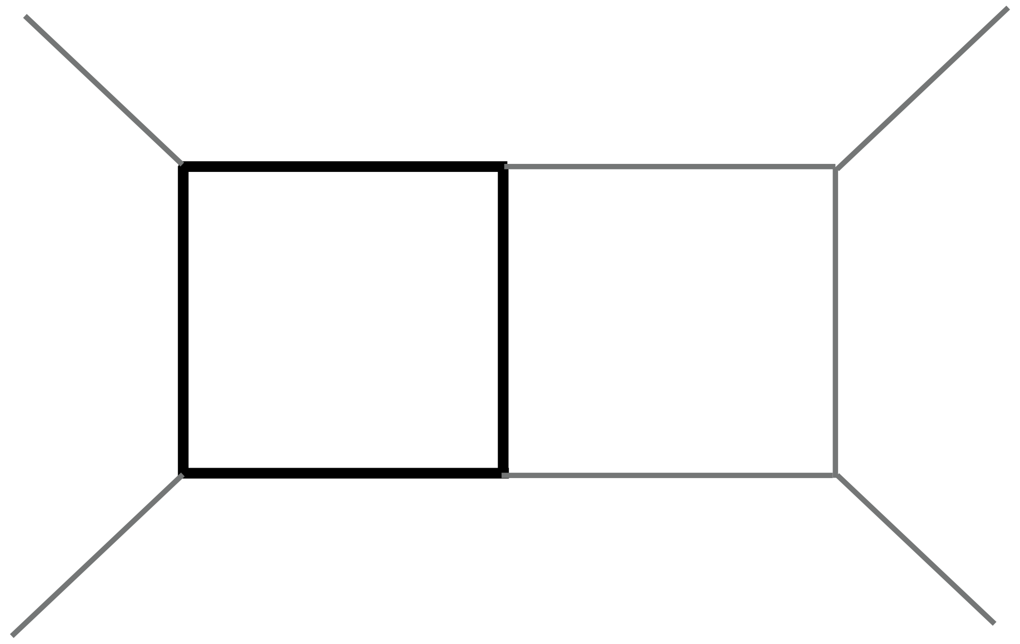
In the generic multivariate case, we perform the construction variable-by-variable

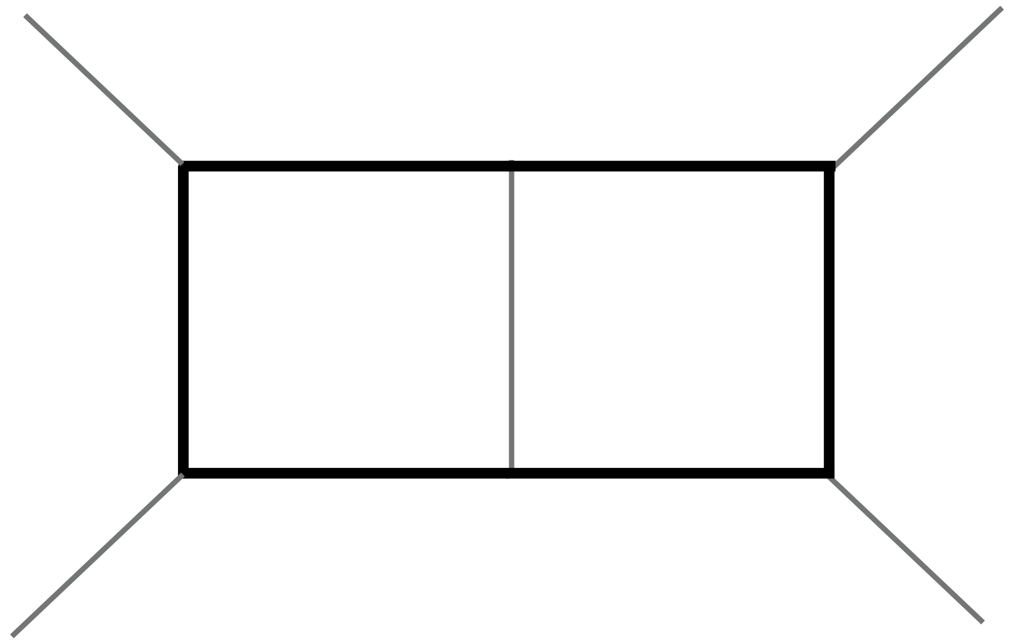
Many examples worked out:











Chen, Jiang, Xu, **LLY**: 2008.03045; Ma, Wang, Xu, **LLY**, Zhou: 2105.06316; and work to appear

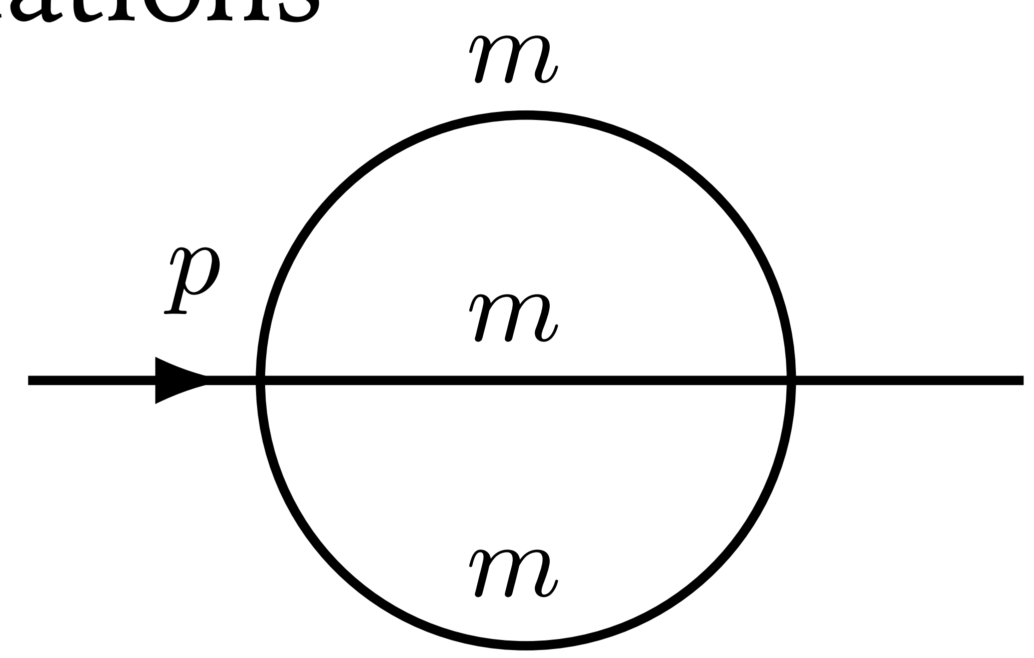
But there are still questions to be answered…

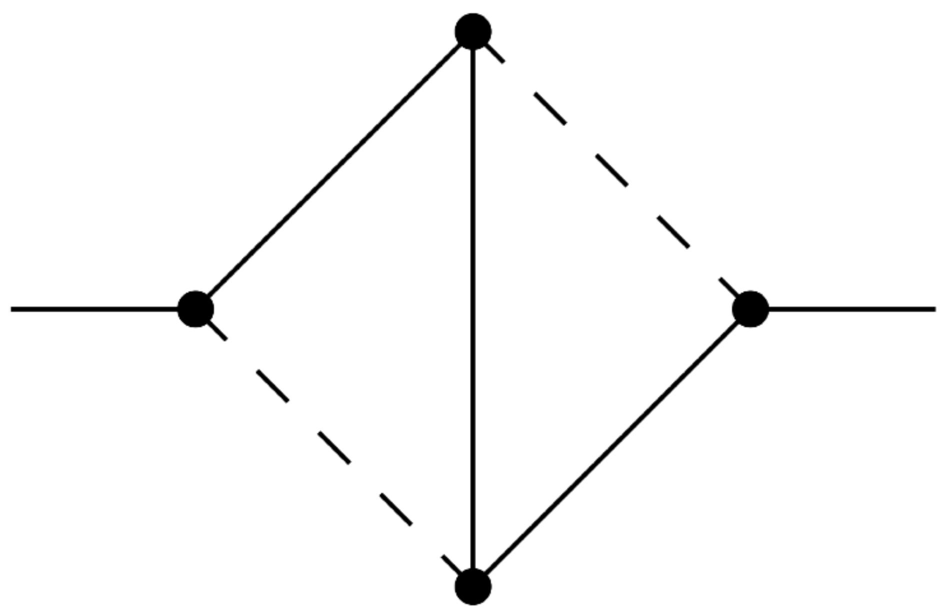
Beyond MPLs

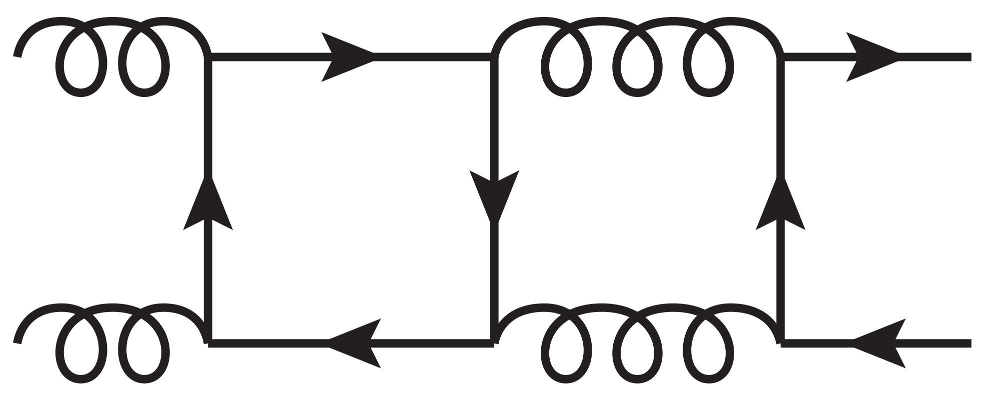
It is well-known that we need functions beyond MPLs, e.g.,
(integrals of) elliptic integrals in many realistic situations

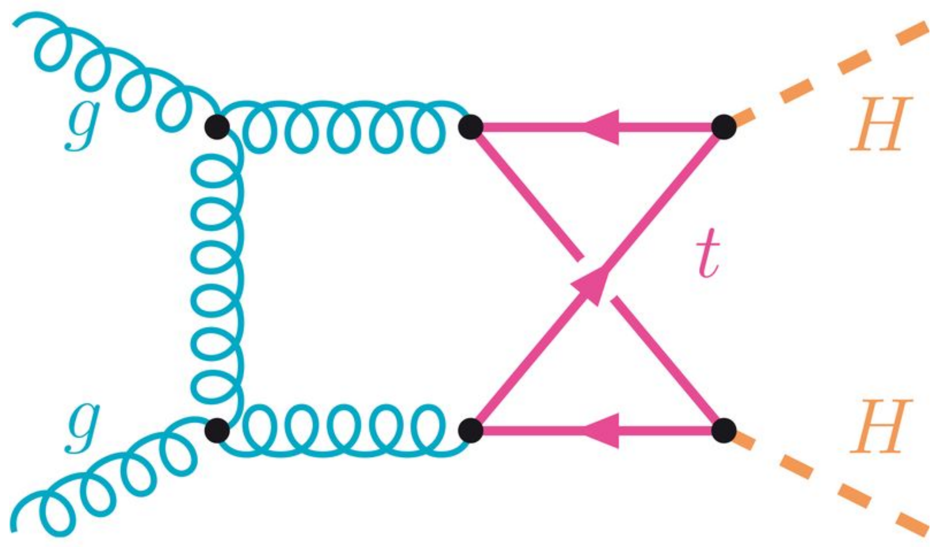
$$F(x;k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$E(x;k) = \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$









What is a good functional basis and how do we organize the calculation?

Elliptic MPLs

Broedel et al.: 1809.10698; and many more references

A class of functions containing (integrals) of elliptic integrals, and sharing many important features of ordinary MPLs

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

Pure functions

$$d\mathscr{E}_4 \sim \sum \text{(functions with lower weights)} \times \text{(one-forms)}$$

Shuffle algebra

$$\mathcal{E}_4(A_1 \cdots A_k; x, \vec{a})\, \mathcal{E}_4(A_{k+1} \cdots A_{k+l}; x, \vec{a}) = \sum_{\sigma \in \Sigma(k,l)} \mathcal{E}_4(A_{\sigma(1)} \cdots A_{\sigma(k+l)}; x, \vec{a})$$

And many more…

How do these functions arise from differential equations?

Elliptic canonical basis

In the elliptic cases, one may still find a basis which satisfy

$$d\vec{f}(z,\epsilon) = \epsilon\, dA(z)\, \vec{f}(z,\epsilon)$$

See, e.g., Weinzierl: 1912.02578 and references therein

where $\vec{f}(z,\epsilon)$ are non-algebraic combinations of Feynman integrals

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How can we construct such a basis? Our attempt: elliptic d-log forms

Look for combinations of $d\mathrm{K}(f_1(z)) \wedge d\log(f_2(z)) \wedge \cdots$ and $d\mathrm{E}(f_1(z)) \wedge d\log(f_2(z)) \wedge \cdots$

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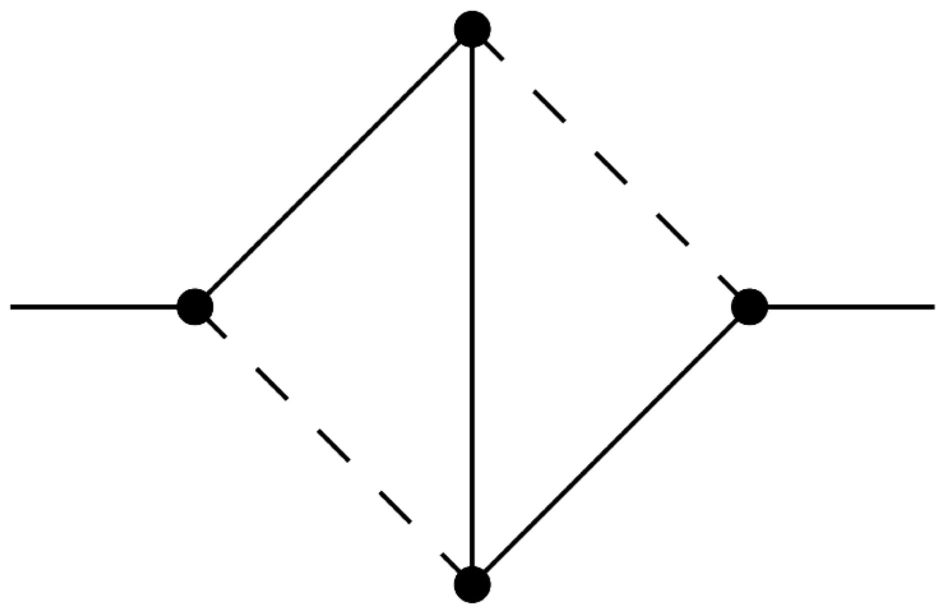
Preliminary results for the kite integral



Stay tuned…

Summary and outlook

- Constructive approach to find canonical bases for massive integrals (with many square-roots)
- Future generalization to elliptic sectors

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Thank you!