Scattering Amplitudes, Feynman Integrals and Wilson Loops

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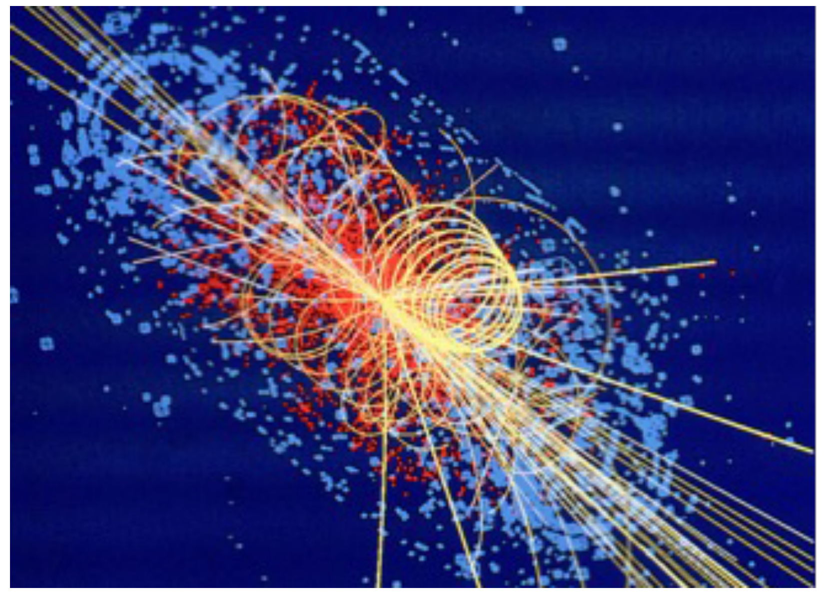
Based on works with (Simon Caron-Huot 1112.1060 …)
Zhenjie Li, Chi Zhang 1911.01290, 2009.11471
Zhenjie Li, Yichao Tang, Qinglin Yang 2012.13094
Zhenjie Li, Qinglin Yang, Chi Zhang 2012.15092

微扰量子场论研讨会 (上海)

May 2021

S-matrix in QFT

- Colliders at high energies need amplitudes of gluons/quarks



$gg \to gg \dots g$

- Fundamental level understanding of QFT & gravity incomplete: strong coupling, dualities, hidden symmetries & relations, quantum gravity & cosmology…

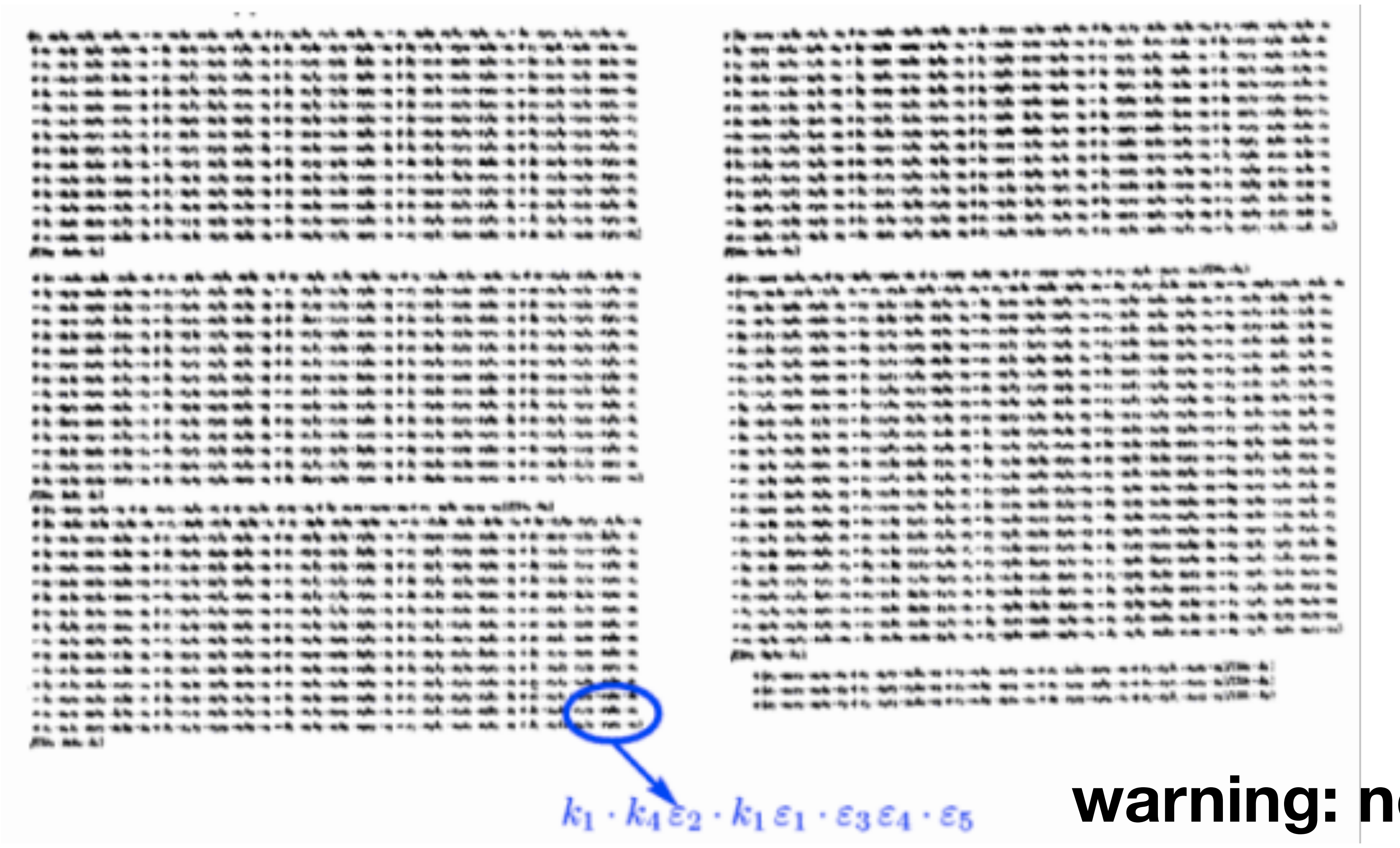
new structures & simplicity seen in perturbative gluon/graviton scattering (even trees)!

- Goal: new ideas & new pictures for QFT (gravity, strings, math…) from studying the S-matrix

Impossible computations?

Feynman diagrams manifest locality & unitarity, but usually no manifest symmetry

Challenging for more legs/loops: many diagrams, lots of terms, huge redundancy



Process	N_{FG}
$gg \to 2g$	4
$gg \to 3g$	25
$gg \to 4g$	220
$gg \to 5g$	2485
$gg \to 6g$	34300
$gg \to 7g$	559405
$gg \to 8g$	10525900
$gg \to 9g$	224449225
$gg \to 10g$	5348843500

warning: not with your bare hands!

Gluons: 2 states $h = \pm$, but manifest locality requires 4 states (huge redundancies)
Much worse for graviton scattering: redundancies from diff invariance

A prior no reason to expect any simplicity or structures in the S-matrix

Parke-Taylor formula


1985: heroic calculation of tree amp gg → gggg (results ~10 pages)

GLUONIC TWO GOES TO FOUR

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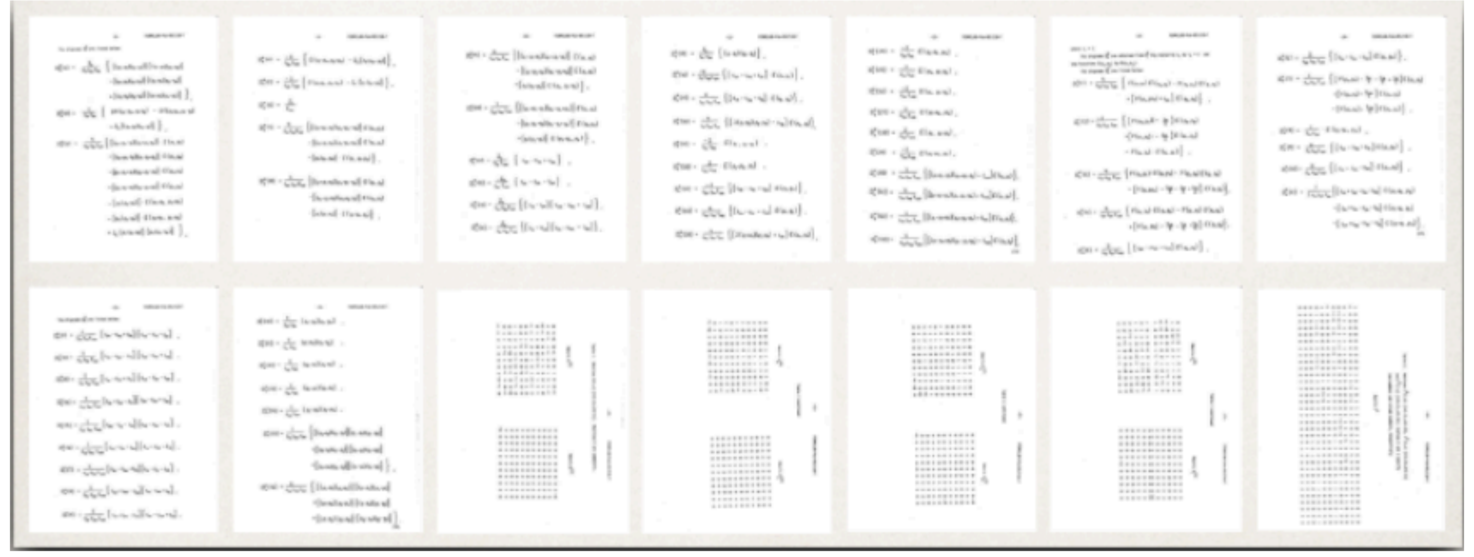
U.S.A.

ABSTRACT

The cross section for two gluon to four gluon scattering is given in a form suitable for fast numerical calculations.



Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

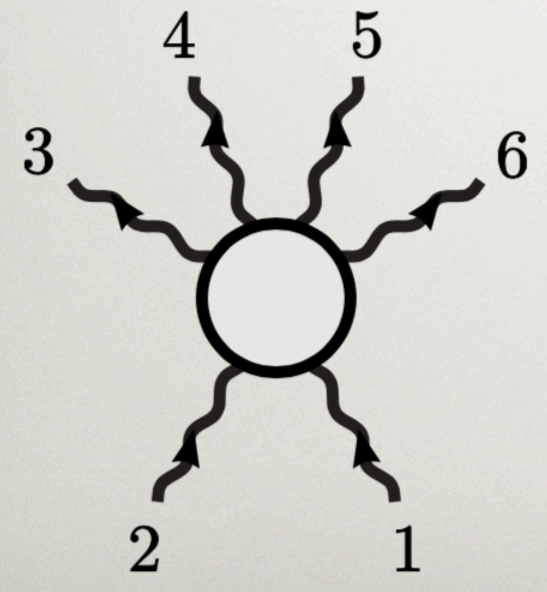
MHV: Maximally helicity violating
(all out-going) amps for all + or one - vanish!

Spinor-helicity variables

$$p^\mu = \sigma^\mu_{a\dot{a}} \lambda_a \tilde{\lambda}_{\dot{a}}$$
$$\langle 12 \rangle = \epsilon_{ab} \lambda^{(1)}_a \lambda^{(2)}_b$$
$$[12] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}^{(1)}_{\dot{a}} \tilde{\lambda}^{(2)}_{\dot{b}}$$

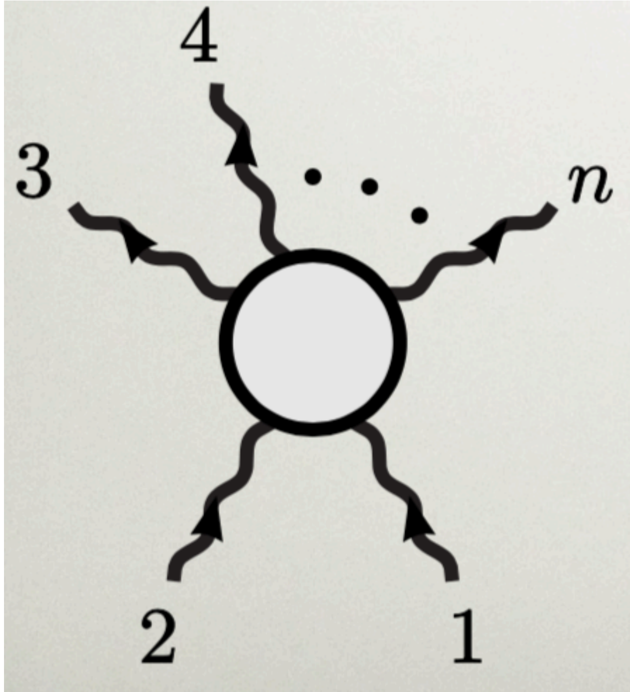
(Mangano, Parke, Xu 1987)

6 months later they realized



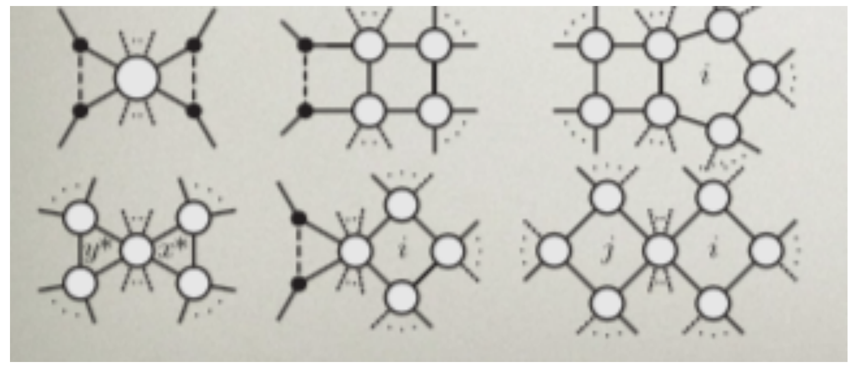
$$= \frac{\langle 1\,2 \rangle^4}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,4 \rangle \langle 4\,5 \rangle \langle 5\,6 \rangle \langle 6\,1 \rangle}$$

conjecture for n-pt MHV amp [Parke,Taylor]

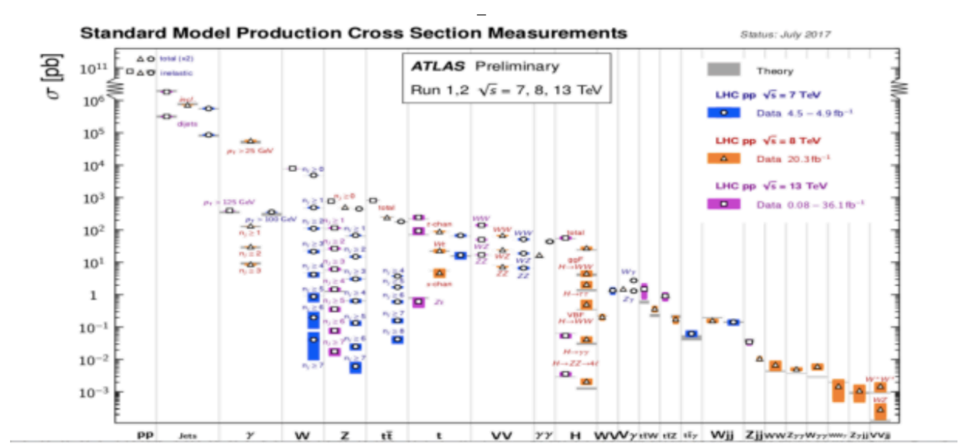


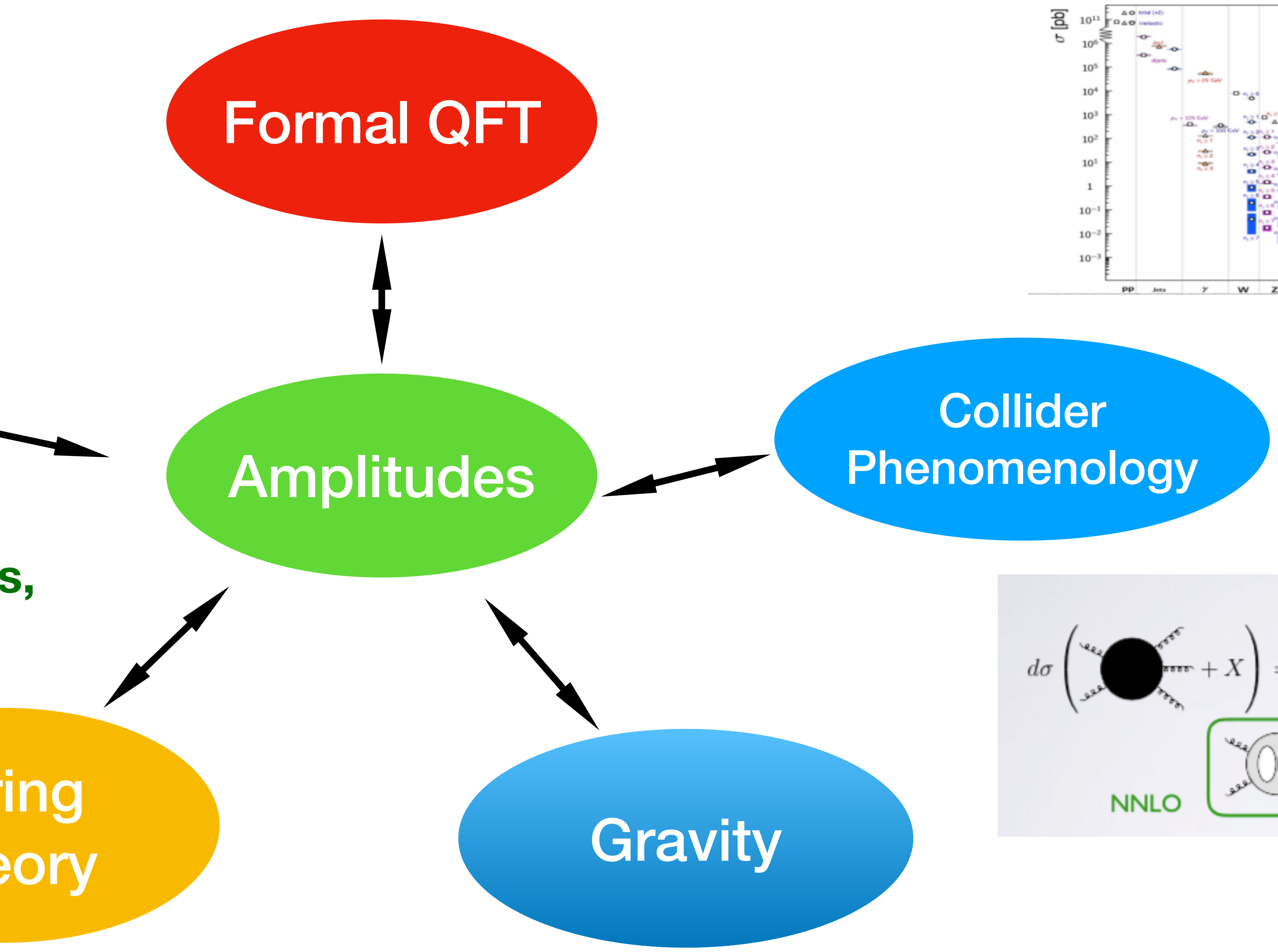
$$= \frac{\langle 1\,2 \rangle^4}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,4 \rangle \langle 4\,5 \rangle \cdots \langle n\,1 \rangle}$$

Who do we connect to?

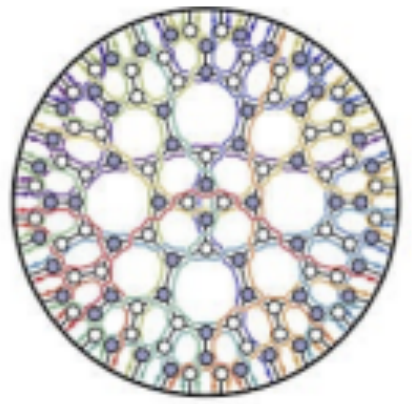


trees, integrands, integrals …

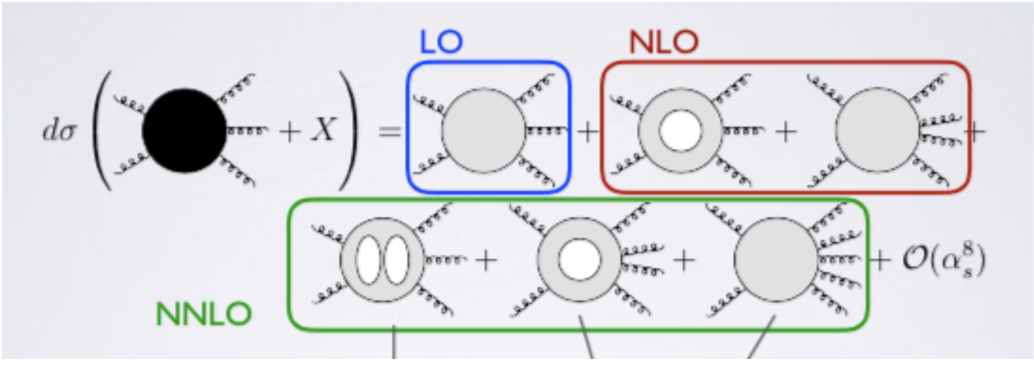


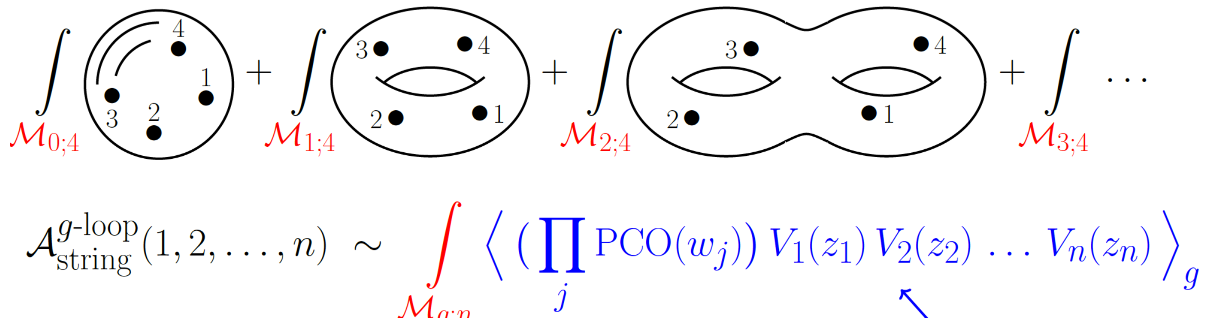


precision frontier

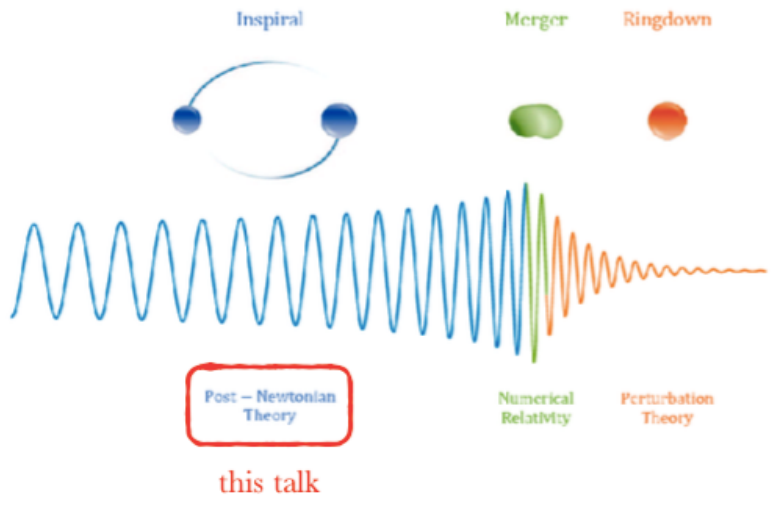


geometries, combinatorics, number theory …





string-inspired: KLT, BCJ, CHY …



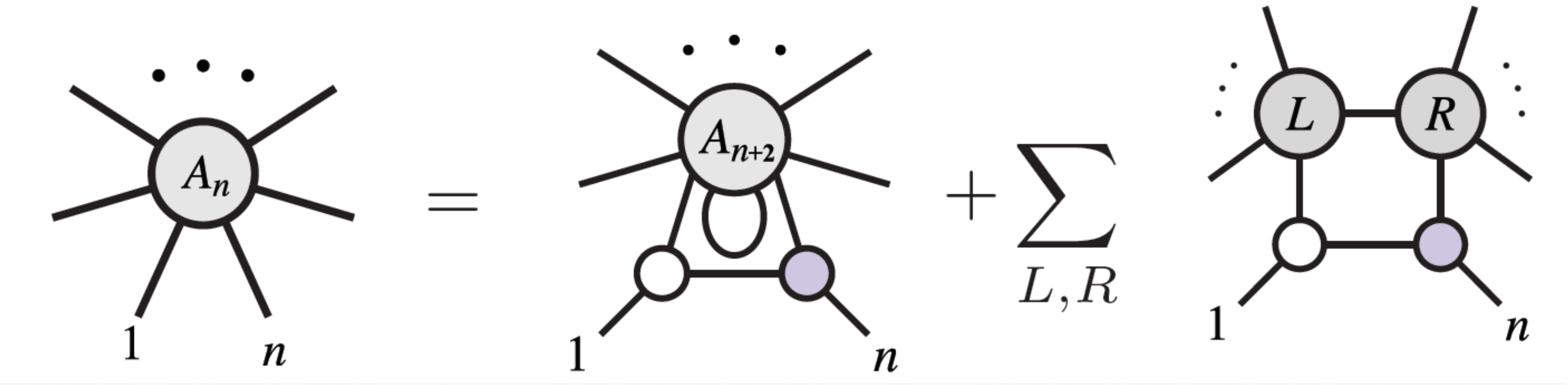
double copy, gravity amps
-> post-Newtonian

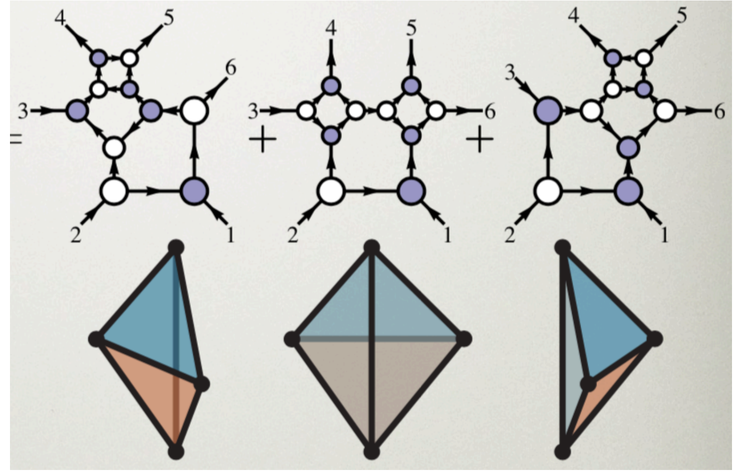
Scattering amplitudes from WL

The simplest QFT

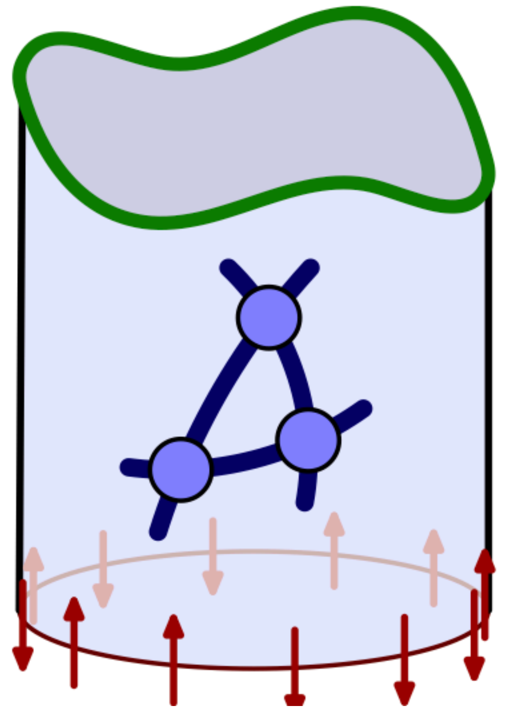
hydrogon atom in 21st centrury: hidden simplicity + new structure in $\mathcal{N} = 4$ SYM (esp. planar limit)

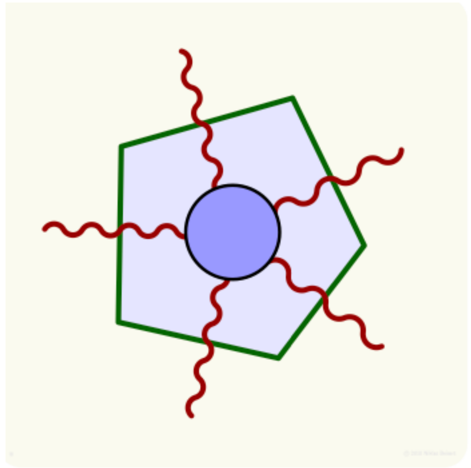
on-shell diagrams + all-loop recursion <-> pos. Grassmannian + amplituhedron [Arkani-Hamed, Trnka]





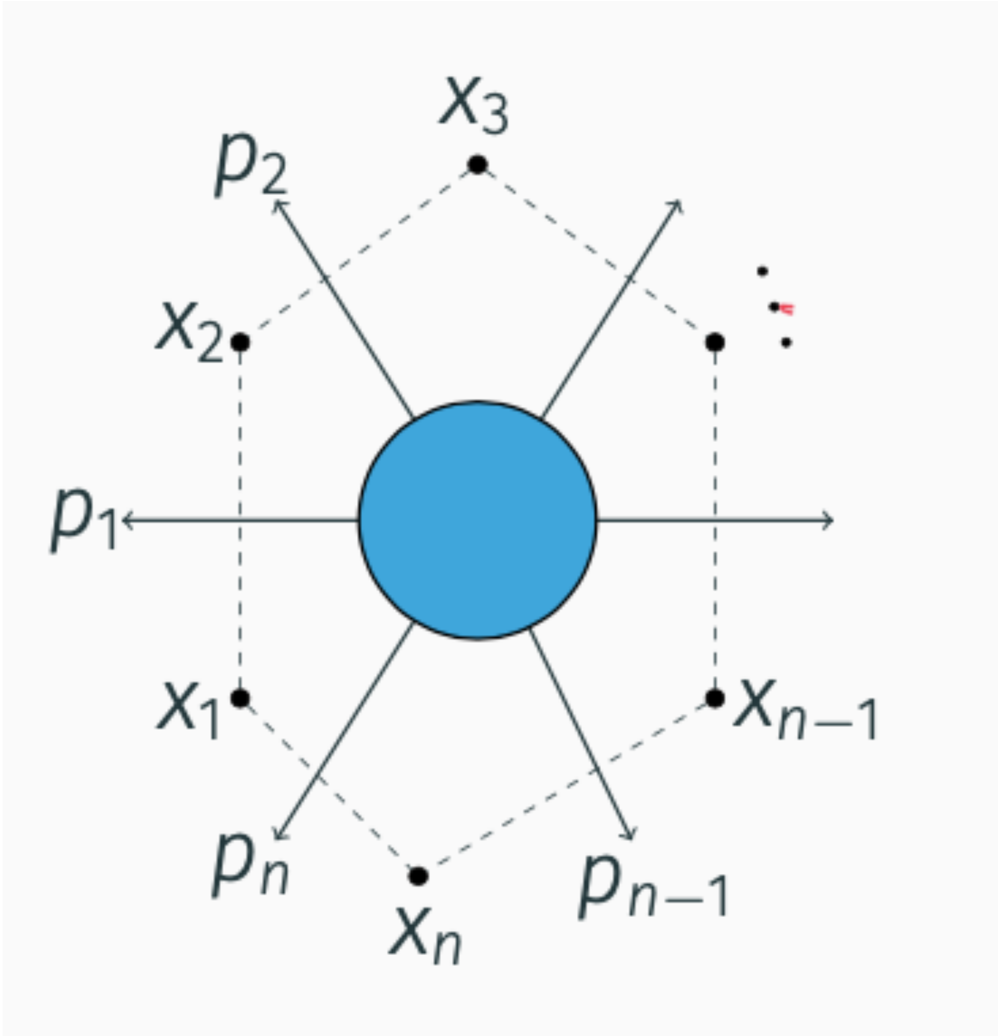
Integrability (planar limit): strong coupling via AdS/CFT, Wilson loops & OPE
Yangian symmetry … Ising model of gauge theories!





(Integrated) amplitudes + Feynman Integrals: extremely rich laboratory for perturbative QFT!
iterated integrals (polylogs + beyond), symbology, differential eqs, Qbar + bootstrap, cluster algebra,…

Wilson loop & symmetries



MHV amplitudes (tree stripped) = null polygonal Wilson loops (strong+ weak coupling)

[Alday, Maldacena][Brandhuber, Heslop, Travaglini] [Drummond, Henn, Korchemski, Sokatchev][…]

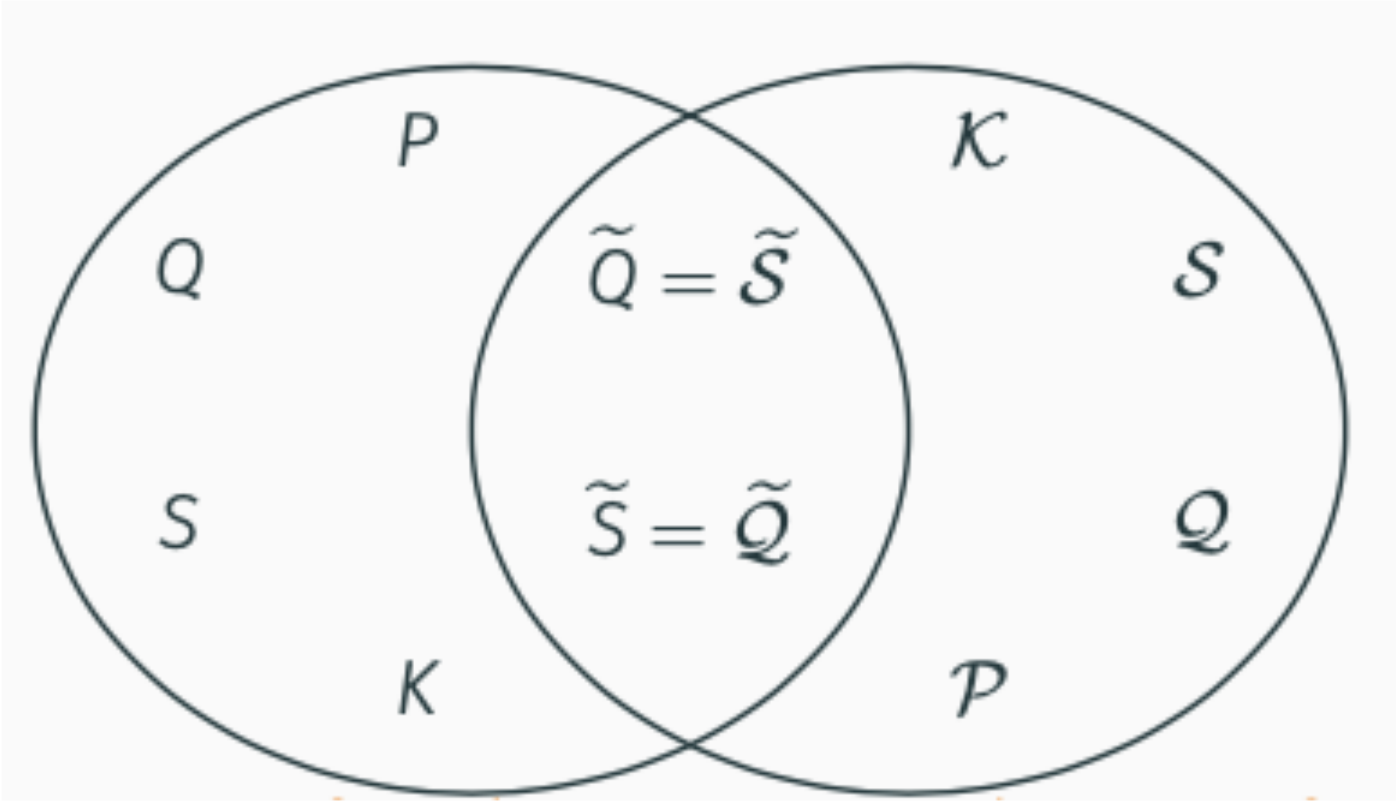
$$A_n(p_1, p_2, \cdots, p_n) \leftrightarrow W_n(x_1, x_2, \cdots, x_n) \sim \langle \mathrm{Tr}\mathscr{P} \exp\, (i \oint \mathbf{A} \cdot dx) \rangle$$

super-amplitudes $\mathscr{A}_{n,k}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$ = super-Wilson loops $\mathscr{W}[\mathbb{A} := \mathbf{A}^{\alpha,\dot{\alpha}} + \bar{\psi}^{\dot{\alpha}}\theta^{\alpha} + \cdots]$ [Mason, Skinner] [Caron-Huot]

dual space w. $\mathcal{N} = 4$ SUSY extension: $(x_{i+1} - x_i)^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}\,, \quad (\theta_{i+1} - \theta_i)^{\alpha A} = \lambda_i^{\alpha}\eta_i^A$

superconformal (amps) + dual superconformal (WL)
→Yangian symmetry (infinite dim.) → integrability
[Drummond, Korchemski, Sokatchev; + Henn, Smirnov] [Drummond, Henn, Plefka]

PSU(2,2|4)



Loop-level: symmetry broken by IR/UV divergence! Yangian inv.=leading singularity
=contour integral over $G_+(k, n)$ [Arkani-Hamed, Bourjaily, Cacahzo, Goncharov, Postnikov, Trnka] [Drummond, Ferro] […]

loop amplitudes

BDS ansatz [Bern, Dixon, Smirnov]: $A_n^{\text{BDS}} \sim \exp(\frac{1}{4}\Gamma_{\text{cusp}} F_n^{1-\text{loop}}) \implies$ BDS-normalized amps: $R_{n,k} = \mathcal{A}_{n,k}/A_n^{\text{BDS}}$

- Dual conformal invariant (DCI) function of $3n - 15$ cross-ratios ($n = 4{,}5$ trivial)

interestingly only invariant under chiral half of dual SUSY, not the other half!

- natural separation into transcendental (w. discontinuities) & algebraic part (only poles)

$$R_{n,k} \sim \underbrace{\text{(Yangian invariants)}}_{\text{helicity, "rational/algebraic"}} \times \underbrace{\text{(Transcendental functions)}}_{\text{DCI, "uniform weight"=2 L}}$$

Yangian inv. classified, e.g. MHV, $R_{n,0} \sim$ pure functions, similarly for NMHV

These two cases expected to be simplest: only generalized polylogarithms!
for $k \geq 2$ (NNMHV…): elliptic integrals & other monsters will appear

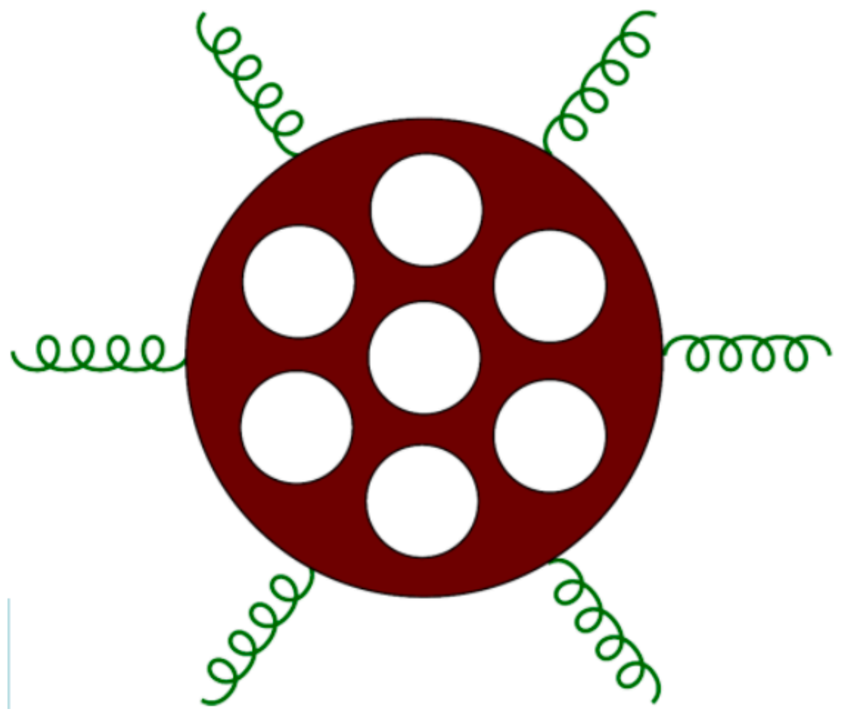
Symbology & bootstrap

Multiple polylogs: $G(\mathbf{a}, t_0) = \int_0^{t_0} \frac{dt_1}{t_1 - a_1} \int_0^{t_1} \frac{dt_2}{t_2 - a_2} \cdots \int_0^{t_{w-1}} \frac{dt_w}{t_w - a_w}$ → symbol & letters [Goncharov, Spradlin, Vergu, Volovich]

$$dG^{(w)} = \sum_i G_i^{(w-1)} d\log x_i \implies \mathcal{S}(G^{(w)}) = \sum_i \mathcal{S}(G_i^{(w-1)}) \otimes x_i$$ e.g. $\mathcal{S}(\log(x)) = x, \ \mathcal{S}(\mathrm{Li}_2(x)) = -(1-x) \otimes x$

trivialize relations between gen. polylogs; 1st entry: physical discontinuities, last entry: differential

For n=6,7: only 9 & 42 letters! conjecture: to all loops only cluster variables of $G_+(4,n)$
finite-type A_3 for n=6, E_6 for n=7 (infinite starting n=8 -> finite alphabet) [Golden et al][…]



hexagon/heptagon bootstrap: ansatz with alphabet + conditions (Qbar + collinear etc.)
-> unique answer to 7/4 loops respectively! [Dixon et al] [Caron-Huot, Dixon, Dulat, McLeod, von Hippel, Papathanasiou][…]

Momentum twistors [Hodges]

- Unconstrained variables for any massless kinematics (useful for QCD as well)
- "Light-rays" of dual space, linearly realize dual symmetry SL(4|4): $\mathcal{Z}_i = (Z_i^a \mid \chi_i^A) := (\lambda_i^\alpha, x_i^{\alpha,\dot\alpha}\lambda_{i,\alpha} \,|\, \theta_i^{\alpha,A}\lambda_i^\alpha)$
- Basic SL(4) invariant: 4-bracket $\langle i\,j\,k\,l\rangle := \varepsilon_{abcd}Z_i^a Z_j^b Z_k^c Z_l^d$ e.g. $\langle i-1\, i\, j-1\, j\rangle \propto (x_i - x_j)^2$
- Dual symmetries: $K_b^a = \sum_i Z_i^a \frac{\partial}{\partial Z_i^b}$, $\quad R_B^A = \sum_i \chi_i^A \frac{\partial}{\partial \chi_i^B}$, $\quad Q_A^a = (Q_a^\alpha, \bar{S}_A^{\dot\alpha}) = \sum_i Z_i^a \frac{\partial}{\partial \chi_i^A}$ annihilate $R_{n,k}$, but not $\bar{Q}_a^A = (\bar{Q}_\alpha^A, S_{\dot\alpha}^A) = \sum_i \chi_i^A \frac{\partial}{\partial Z_i^a}$ (chiral nature of super WL!)
- Usual symmetry generators become level-1, just need one: $s_A^\alpha = \sum_i \frac{\partial}{\partial \lambda_{i,\alpha}} \frac{\partial}{\partial \eta_i^A}$ (parity of $S_{\dot\alpha}^A$)

Yangian anomaly equations [Caron-Huot, SH]

$$\bar{Q}_a^A R_{n,k} = a\, \mathrm{Res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(d^{2|3}\mathcal{Z}_{n+1}\right)_a^A \left[R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\mathrm{tree}}\right] + \text{cyclic},$$

loop parameter $$a := \frac{1}{4}\Gamma_{\mathrm{cusp}} = g^2 - \frac{\pi^2}{3} g^4 + \frac{11\pi^4}{45} g^6 + \ldots,$$

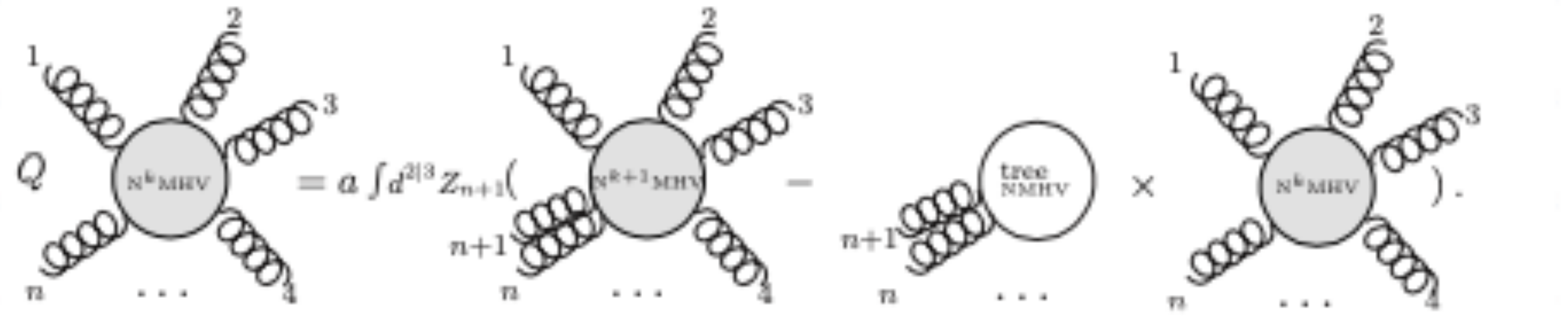


Figure 1. All-loop equation for planar $\mathcal{N} = 4$ S-matrix.

- first all-loop equations for BDS-normalized amplitudes
- Insert fermion on edges of WL -> collinear WL τ-integral

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon(\mathcal{Z}_{n-1} - \tau \mathcal{Z}_1) + \mathcal{O}(\epsilon^2)$$

- 2nd term on RHS: $\bar{Q}$ on BDS-ansatz (fixes Γ_{cusp})

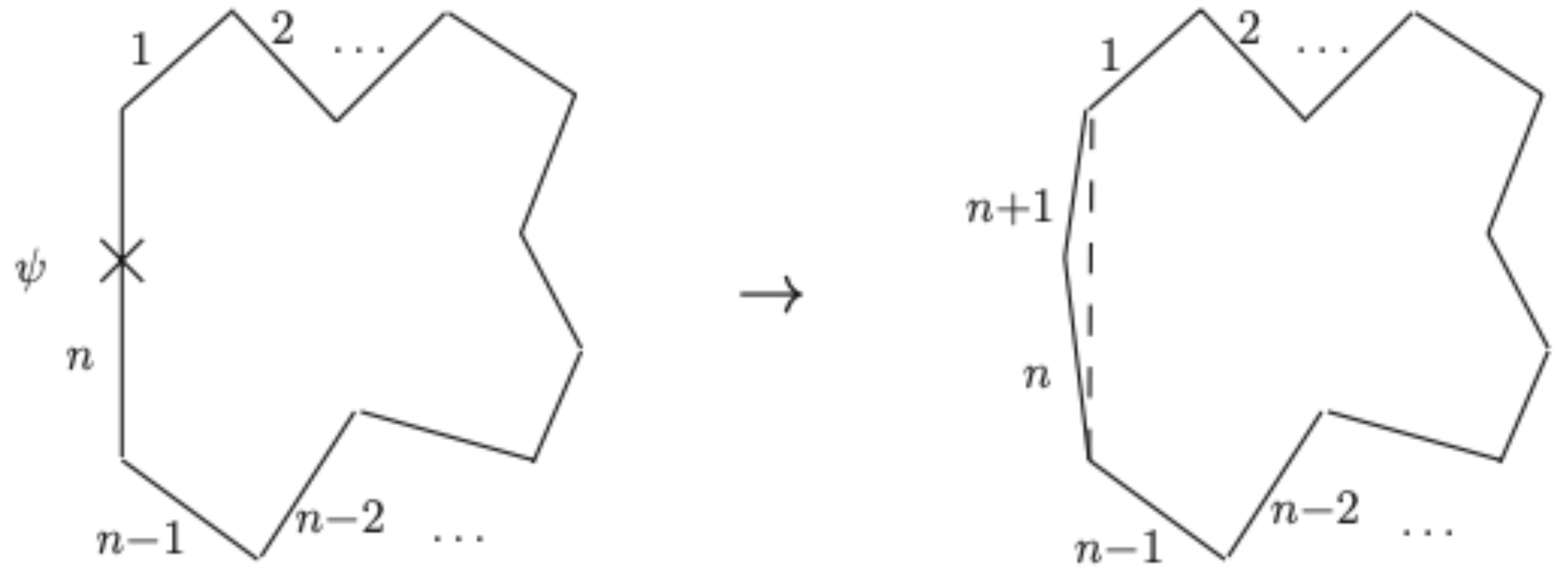


Figure 2. Fermion insertion on the Wilson loop versus kink insertion

space-time parity $\Longrightarrow$ $\bar{Q}^{(1)}$ equations; determine "anomalies" of all Yangian generators!

to LHS as quantum corrections → exact symmetries for S-matrix: $\hat{\bar{Q}}\mathcal{M} = \widehat{Q^{(1)}}\mathcal{M} = \hat{K}\mathcal{M} = Q\mathcal{M} = 0$

Jumpstarting all-loop amplitudes [Caron-Huot, SH]

differential equations (1st order) determine the all-loop S-matrix (b.c. fixed by collinear limits)

In practice, $\bar{Q}$ alone determine MHV & NMHV amps uniquely, given lower-loop amps

Last entry for all loops (important for bootstrap) MHV and NMHV (to all n) [SH, Z. Li, C. Zhang]

become straightforward to compute 2-loop n-point MHV (1-d integral of 1-loop NMHV x last entry)

e.g. 1-loop NNMHV octagon -> 2-loop NMHV heptagon -> 3-loop MHV hexagon [Caron-Huot, SH]

Starting n=8: algebraic letters (square roots) need rationalization; predict symbol alphabet

2-loop NMHV octagon & n-gon [SH, Z. Li, C. Zhang]

$$
\begin{aligned}
&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{c-1}_{a,b,c,d} \otimes x^{c-1}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,b\,c-1]\\
&- \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{c}_{a,b,c,d} \otimes x^{c}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,b\,c]\\
&+ \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{b-1}_{a,b,c,d} \otimes x^{b-1}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,c-1\,c]\\
&- \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{b}_{a,b,c,d} \otimes x^{c}_{a,b,c,d}\,[a{-}1\,a\,b\,c{-}1\,c]\\
&+ \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{a-1}_{a,b,c,d} \otimes x^{a-1}_{a,b,c,d}\,[a{-}1\,b{-}1\,b\,c-1\,c]\\
&- \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{a}_{a,b,c,d} \otimes x^{a}_{a,b,c,d}\,[a\,b{-}1\,b\,c{-}1\,c]\,,
\end{aligned}
$$

$$
\mathcal{X}^{*}_{a,b,c,d} := \frac{(x^{*}_{a,b,c,d}+1)^{-1}-\bar{z}_{d,a,b,c}}{(x^{*}_{a,b,c,d}+1)^{-1}-z_{d,a,b,c}}\,, \qquad \tilde{\mathcal{X}}^{*}_{a,b,c,d} := \frac{(x^{*}_{a,b,c,d-1}+1)^{-1}-z_{d,a,b,c}}{(x^{*}_{a,b,c,d-1}+1)^{-1}-\bar{z}_{d,a,b,c}}
$$

with 6 choices $a{-}1, a, b{-}1, b, c{-}1, c$ of the superscript, where

$$
x^{a}_{a,b,c,d} = \frac{\langle \bar{d}(c{-}1\,c)\cap(a\,b{-}1\,b)\rangle}{\langle \bar{d}\,a\rangle\langle b{-}1\,b\,c{-}1\,c\rangle}\,, \qquad x^{a-1}_{a,b,c,d} = x^{a}_{a,b,c,d}|_{a\leftrightarrow a-1}
$$

$$
x^{b}_{a,b,c,d} = \frac{\langle \bar{d}(c{-}1\,c)\cap(a{-}1\,a\,b)\rangle}{\langle \bar{d}(a{-}1\,a)\cap(b\,c{-}1\,c)\rangle}\,, \qquad x^{b-1}_{a,b,c,d} = x^{b}_{a,b,c,d}|_{b\leftrightarrow b-1}
$$

$$
x^{c}_{a,b,c,d} = \frac{\langle \bar{d}\,c\rangle\langle a{-}1\,a\,b{-}1\,b\rangle}{\langle \bar{d}(a{-}1\,a)\cap(b{-}1\,b\,c)\rangle}\,, \qquad x^{c-1}_{a,b,c,d} = x^{c}_{a,b,c,d}|_{c\leftrightarrow c-1}
$$

Remarkably constrained & compact "algebraic part": 4-mass $\otimes$ algebraic^i $\otimes$ final^i $\times R_i$ (all correlated!)

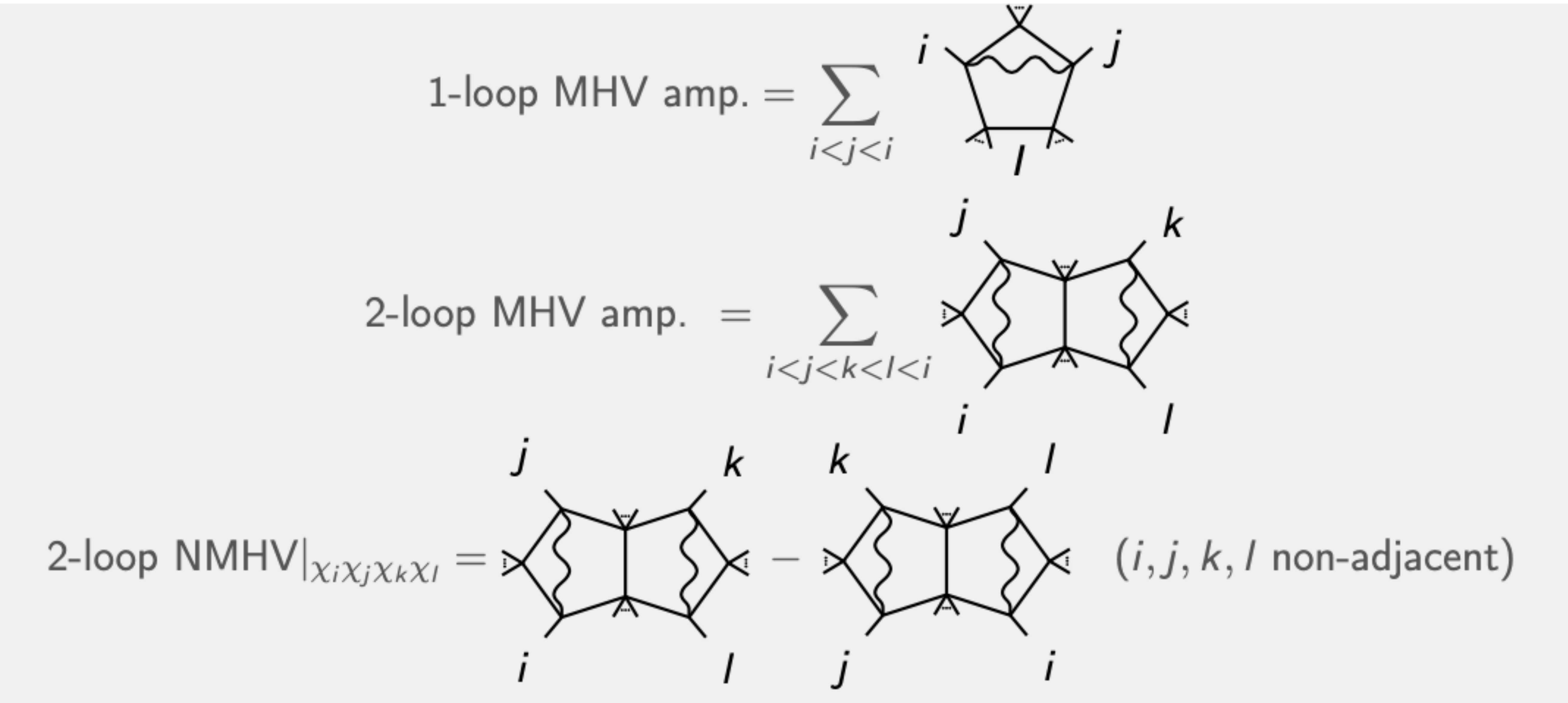
For each $\Delta_{a,b,c,d}$: generically 17 multiplicative independent algebraic letters $\dfrac{a^i - \bar{z}}{a^i - z}$

(most degenerate) $n = 8$: $\Delta_{1,3,5,7}$ & $\Delta_{2,4,6,8}$, 9+9 independent algebraic letters (+ 180 rational letters)

Origin of alphabet: Landau equations [Spradlin et al] tropical $G_+(4,8)$ etc. [Drummond, et al] [Arkani-Hamed, Lam, Spradlin]
poles/"letters" of Yangian invariants [SH, Z. Li] [Mago, Schreiber, Spradlin, Volovich][…]

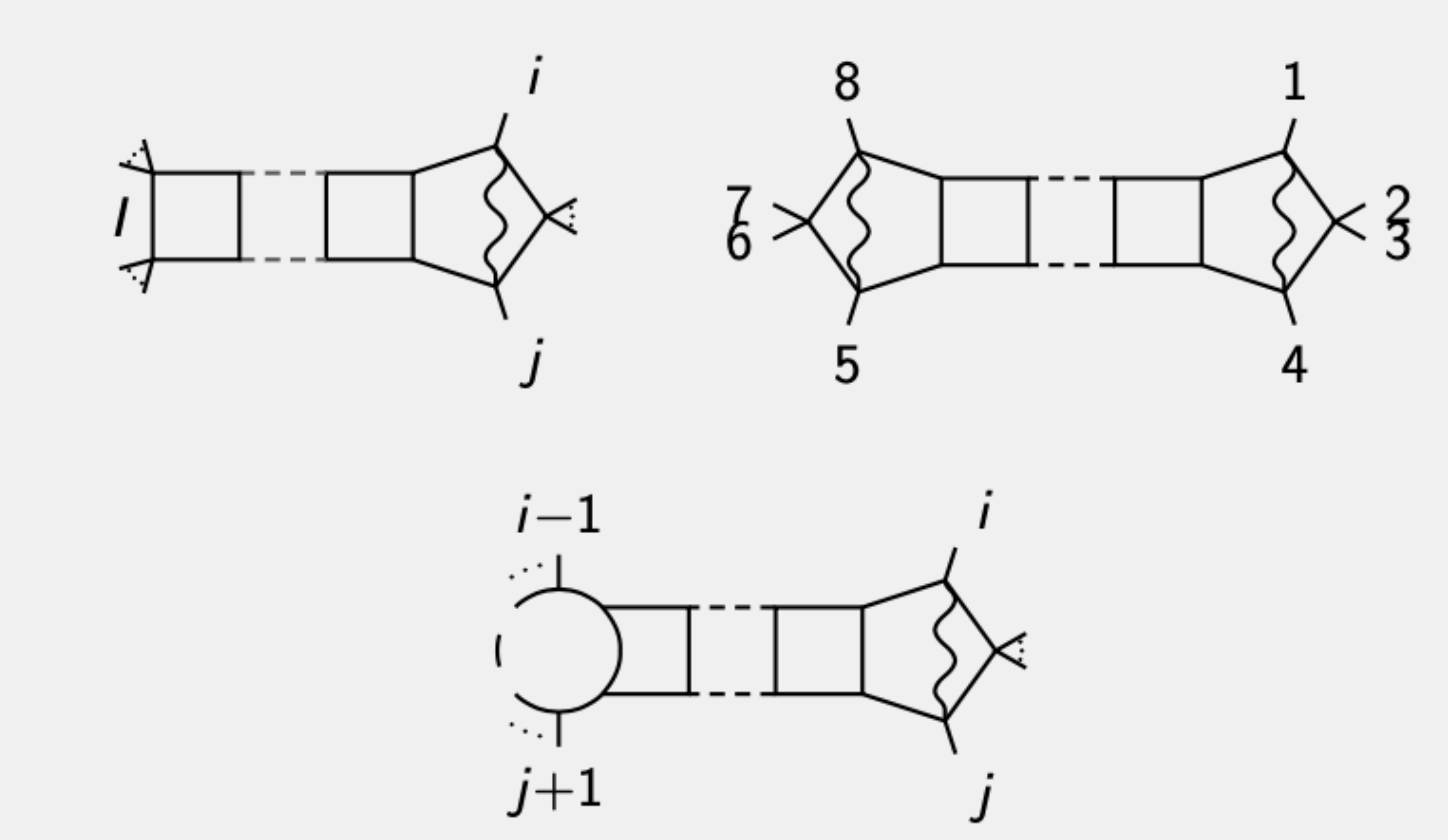
WL rep. of Feynman integrals

Uniform transcendental integrals



pentagon & double-pentagon for MHV/NMHV amps [Arkani-Hamed et al]

(e.g. numerators $\langle \ell_1 \bar{i} \cap \bar{j} \rangle, \quad \langle \ell_2 \bar{k} \cap \bar{l} \rangle$)



(double-) penta-ladder [Drummond, Henn, Trnka] & gen. penta-ladders

uniform transcendentality vs. dlog (focus on IR finite integrals e.g. $i < j-1$ & $k < l-1$ for dp)

-> similar integrals in N=4 SYM; also play an important role beyond N=4 SYM (master integrals)

Feynman integrals from WL

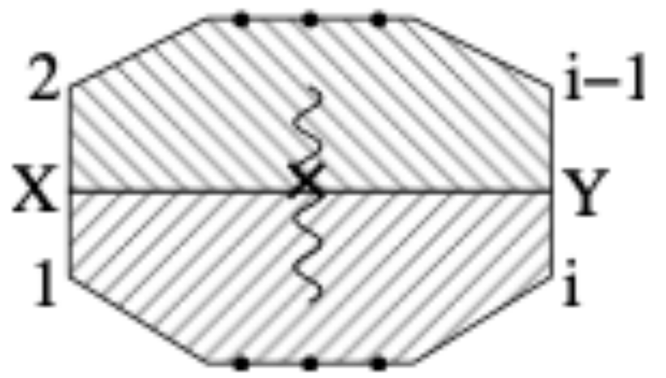
WL powerful for not only (full) amps, also a large class of Feynman integrals (=WL diagrams)

How we originally computed 2-loop MHV: $dR_{n,0} = \sum_{i<j} C_{i,j} d\log\langle \bar{i} j\rangle$ w. $C_{i,j}$ (super-) WL diagrams

L=1:

$$C_{2,i} = \int_0^\infty d\tau_X d\tau_Y \frac{\langle \bar{2} i\rangle\langle \bar{i} 2\rangle}{\langle XY\rangle^2} = \log u_{2,i-1,i,1}$$

L=2: 1-d τ-integral of box integrals



simplest NMHV: difference of two WL diagrams=double pentagon!

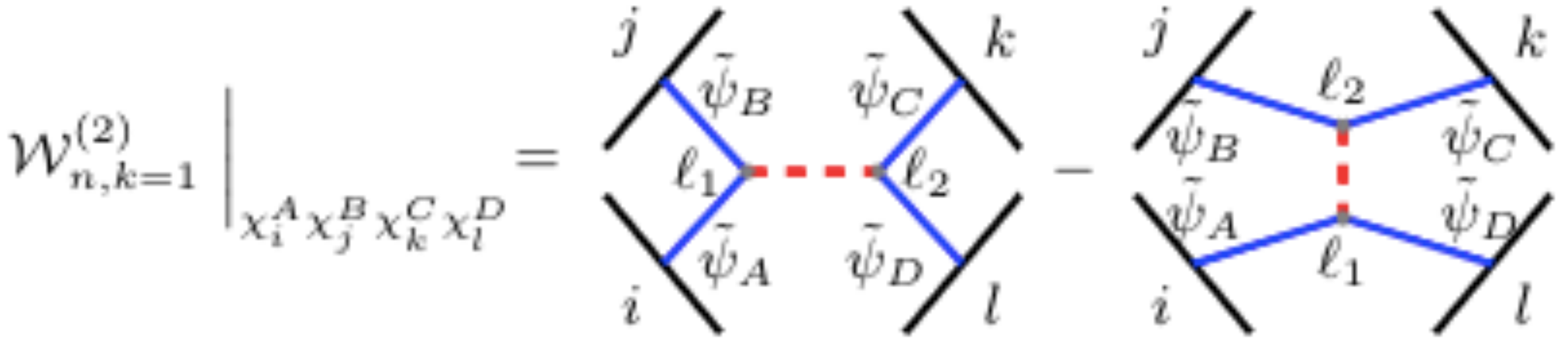


FIG. 2. NMHV component of super-WL as difference of two diagrams, each equals to a double-pentagon integral.

WL $d\log$-integral: 1-loop examples [SH, Z. Li, Y. Tang, Q. Yang]

Why useful? swap order of integrals, left with simple line integrals (“smart parametrization”)

chiral pentagon: $\frac{1}{\langle \ell i-1 i\rangle \langle \ell i i+1\rangle} = \int_0^\infty \frac{d\tau}{\langle \ell i X(\tau)\rangle^2}$, $X(\tau) := Z_{i-1} + \tau_X Z_{i+1}$ “fermions” at $x := (iX)$ & $y := (jY)$

$$\Longrightarrow I_{\text{pent.}} = \int d\tau_X d\tau_Y \int \frac{d^4\ell \langle \ell \bar{i} \cap \bar{j}\rangle \langle I i j\rangle}{\langle \ell i X\rangle^2 \langle \ell j Y\rangle^2 \langle \ell I\rangle} = \int d^2\tau \frac{\langle I \bar{i} \cap \bar{j}\rangle \langle I i j\rangle}{\langle I i X\rangle \langle I j Y\rangle \langle i X j Y\rangle}$$ (star-triangle identity)

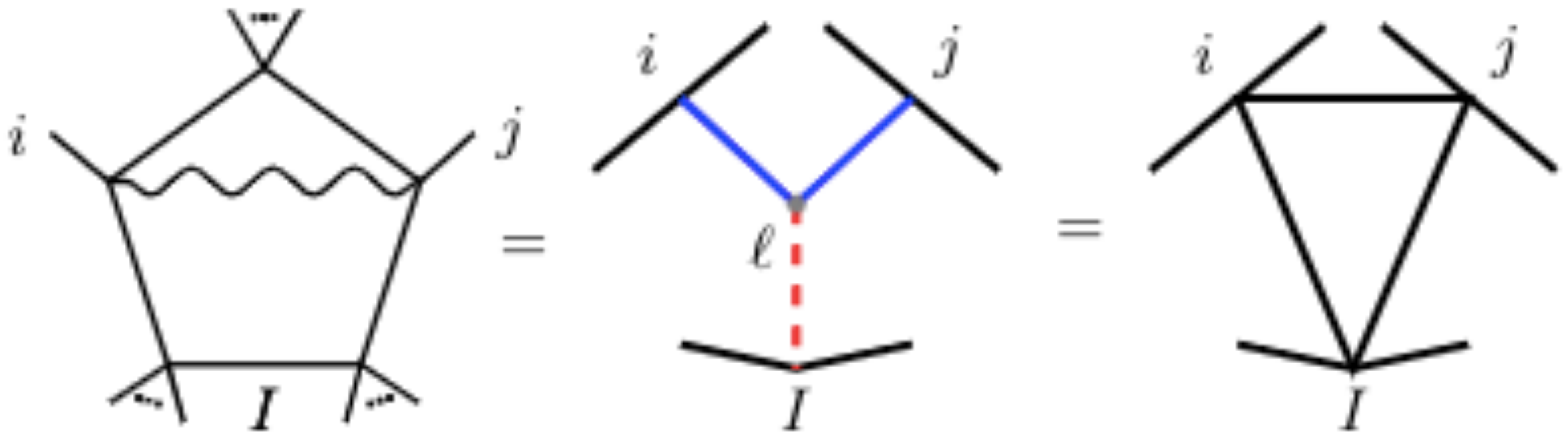


FIG. 3. The chiral pentagon written as a WL diagram, and loop integral performed using “star-triangle” identity.

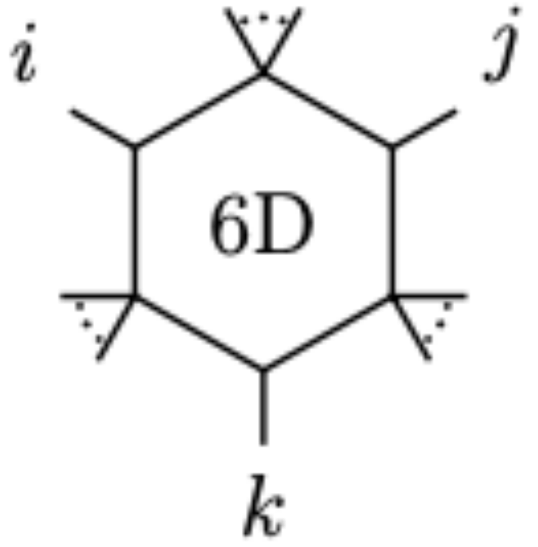
Nice $d\log$ 2-form: $\int_{(i,j)} d\log \frac{\langle IjY\rangle}{\langle \bar{i}(jY)\cap(iI)\rangle} d\log \frac{\langle iXjY\rangle}{\langle IiX\rangle}$

Trivially give well-known dilog (manifest DCI + weight-2)

Geometry: integrating $\Omega(\Delta')$ in Δ (similar to Aomoto)

6d 3-mass-easy hexagon [Del Duca et al]
(weight-3 polylog of 9 cross-ratios)

$$\Omega_1^{(6D)}(i,j,k) :=$$



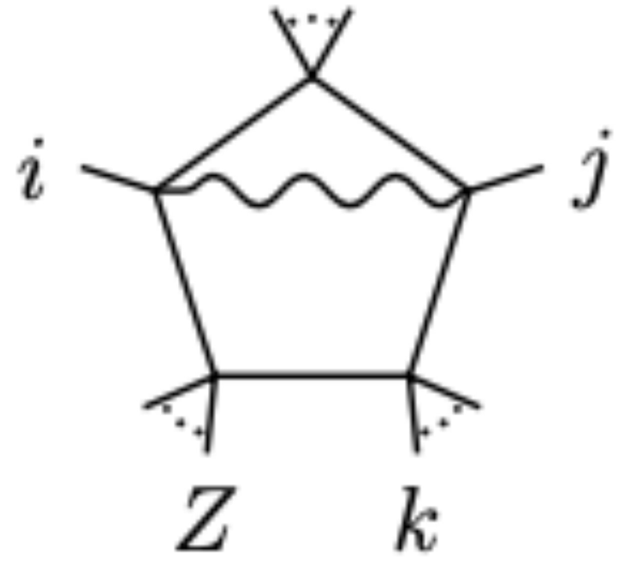
$$= \int \frac{d^6 x_0}{\pi^3} \frac{x_{i,j+1}^2 x_{j,k+1}^2 x_{k,i+1}^2 \sqrt{\Delta_9}}{x_{0,i}^2 x_{0,i+1}^2 x_{0,j}^2 x_{0,j+1}^2 x_{0,k}^2 x_{0,k+1}^2}.$$

momentum twistors (external 4d kinematics): all square roots disappear

nice 3-fold dlog integral
(1-d integral of “deformed” pentagon)

using symbol integration [Caron-Huot SH][...]
straightforward to get weight-3 symbol
without performing any integral

$$\Omega_1^{(6D)}(i,j,k) = \int_0^\infty d\tau_z \frac{\langle (ijk)\bar{i} \cap \bar{j} \cap \bar{k}\rangle}{\langle kZ\bar{i}\cap \bar{j}\rangle\langle kZij\rangle}$$



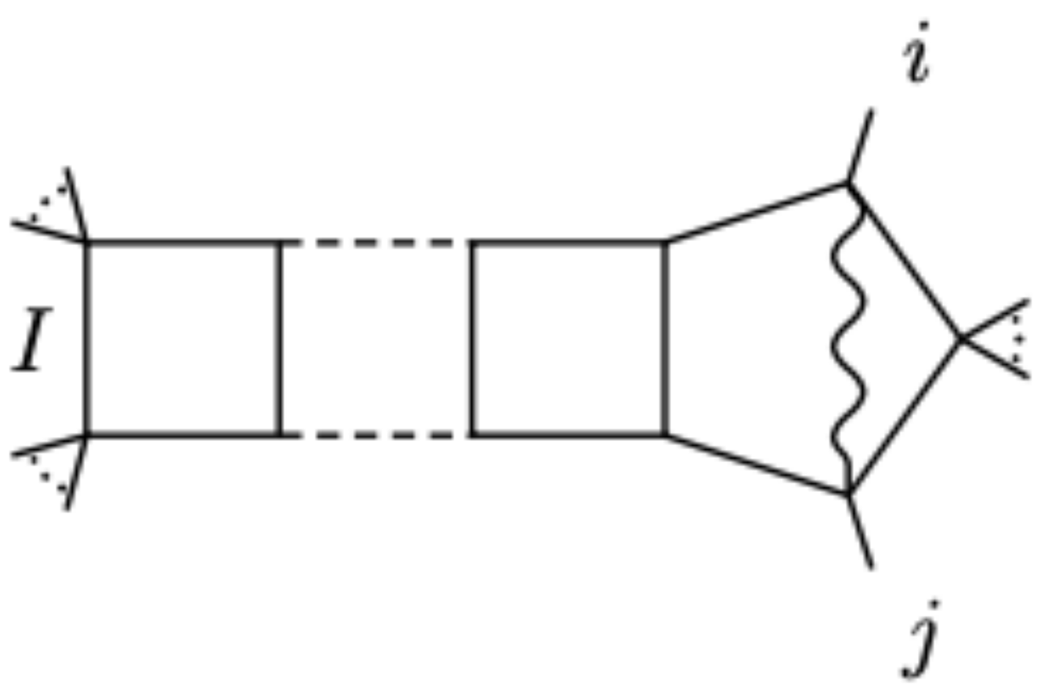
$$= \int_{\mathbb{R}^3_{\geq 0}} d\log \frac{\langle kZij\rangle}{\langle kZ\bar{i} \cap \bar{j}\rangle}\left(d\log\frac{\langle jYkZ\rangle}{\langle jYi(kZ)\cap \bar{i}\rangle}\, d\log \frac{\langle iXjY\rangle}{\langle iXkZ\rangle}\right).$$

$$\int_a^b d\log(t+c)\,(F(t)\otimes w(t)) \implies \{F(t)\otimes w(t)\otimes (t+c))|_{t=a}^{t=b},\quad \left(\int_a^b d\log(t+c)\,F(t)\right)\otimes w,\quad \left(\int_a^b d\log\frac{t+c}{t+d}\,F(t)\right)\otimes (c-d)\}$$

Recursion for all-loop ladder [SH, Z. Li, Y. Tang, Q. Yang]

Simplest multi-loop application: penta-box ladder integral

$\Psi_L(i,j;I) :=$



recursion: L-loop as 2-fold dlog integral of deformed (L-1)-loop
1-loop pentagon=2 dlog -> 2L-fold dlog integral

$$u=\frac{\langle i{-}1iI\rangle\langle jj{+}1ii{+}1\rangle}{\langle i{-}1ijj{+}1\rangle\langle Iii{+}1\rangle},\ v=\frac{\langle jj{+}1I\rangle\langle i{-}1ij{-}1j\rangle}{\langle jj{+}1i{-}1i\rangle\langle Ij{-}1j\rangle},\ w=\frac{\langle i{-}1ijj{+}1\rangle\langle j{-}1jii{+}1\rangle}{\langle i{-}1ij{-}1j\rangle\langle jj{+}1ii{+}1\rangle}.$$

$$\Psi_L(i,j,I)=\int\left[\prod_{a=1}^{L-1}\mathrm{d}\log\langle i{-}1ijY_a\rangle\mathrm{d}\log\frac{\langle iX_ajY_a\rangle}{\tau_{X_a}}\right]\mathrm{d}\log\frac{\langle jY_LI\rangle}{\langle jY_LiI\cap\bar{i}\rangle}\mathrm{d}\log\frac{\langle iX_LjY_L\rangle}{\langle iX_LI\rangle}.$$

beautiful DCI form: simple deform & "odd" weight objects in between (tree=$1-u-v+uvw$)

$$\Psi_{L+\frac{1}{2}}(u,v,w)=\int\mathrm{d}\log\frac{\tau_X+1}{\tau_X}\,\Psi_L\left(\frac{u(\tau_X+w)}{\tau_X+uw},v,\frac{w(\tau_X+1)}{\tau_X+w}\right)$$

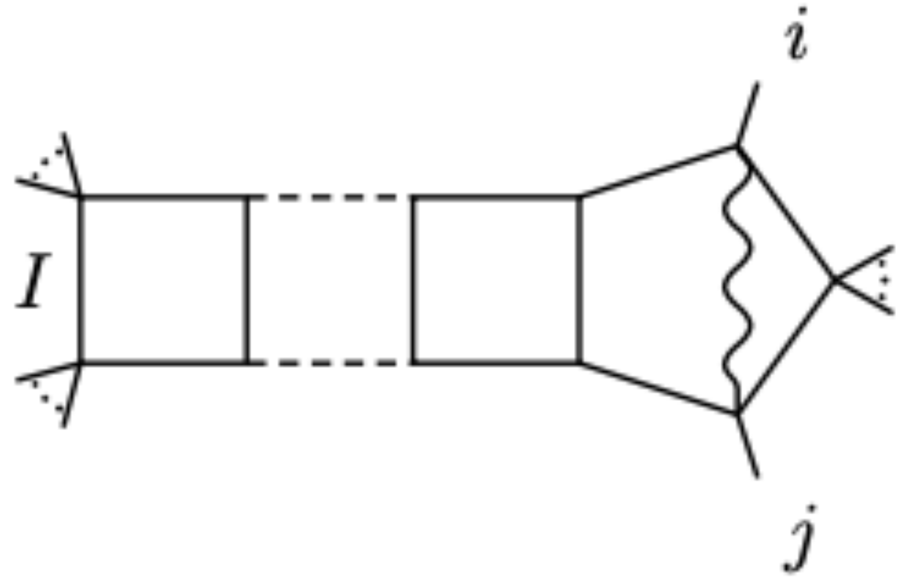
straightforward to get symbol to all loops w. 9 letters
$\{u,v,w,1-u,1-v,1-w,1-uw,1-vw,1-u-v+uvw\}$

$$\Psi_{L+1}(u,v,w)=\int\mathrm{d}\log(\tau_Y+1)\,\Psi_{L+\frac{1}{2}}\left(u,\frac{v(\tau_Y+1)}{v\tau_Y+1},\frac{\tau_Y+w}{\tau_Y+1}\right)$$

Differential eq. & resummation

easy to show the recursion satisfy beautiful diff. eq. [Drummond, Henn, Trnka]

$$(1-u-v+uvw)uv\partial_u\partial_v\Psi_{L+1}(u,v,w)=\Psi_L(u,v,w)$$

$\Psi_L(i,j;I) :=$ 

recursion helps to resum the ladders: define $\Psi_g := \sum_{L=1}^{\infty} g^{2L}\Psi_L$ (w. coupling const.), it satisfies

$$\Psi_g(u,v,w)=g^2\Psi_1(u,v,w)+g^2\int \mathrm{d}\log(\tau_Y+1)\,\mathrm{d}\log\frac{\tau_X+1}{\tau_X}\Psi_g(\tilde{u},\tilde{v},\tilde{w})$$

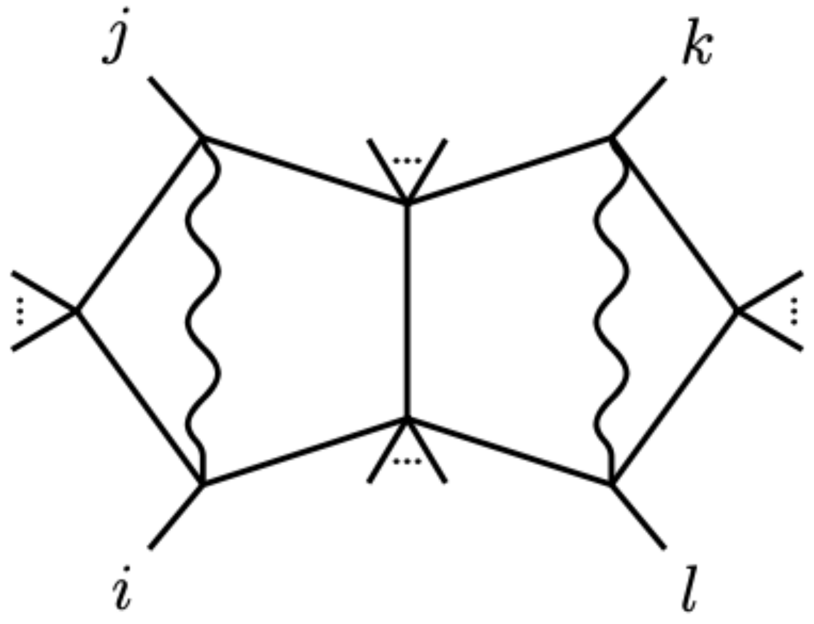
this can be solved by series expansion of kinematic var. $x=1-u^{-1}, y=1-v^{-1}, z=1-w$

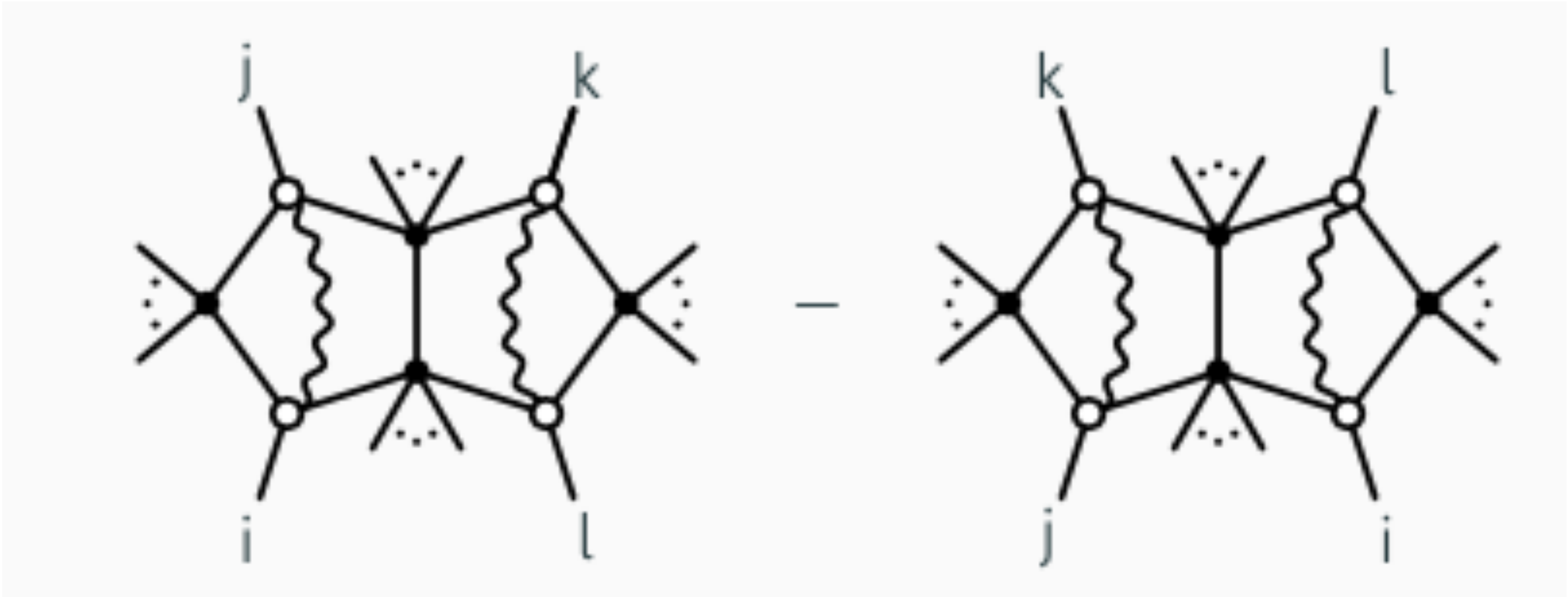
$$\Psi_g=g^2\sum_{k,l=1}^{\infty}\frac{x^k y^l}{kl+g^2}-g^2\sum_{k,l=0,m=1}^{\infty}\frac{x^k y^l z^m}{kl+g^2}\frac{g^2}{(k+m)(l+m)}\prod_{n=1}^{m}\frac{(k+n)(l+n)}{(k+n)(l+n)+g^2}\,.$$

Two-loop all-multiplicity amplitudes

Longstanding problem: compute generic double-pentagon analytically (12 legs, lots of square roots)?

All we need for MHV: $A_{n,\mathrm{MHV}}^{2-\mathrm{loop}} = \sum_{i<j<k<l} I_{\mathrm{dp}}(i,j,k,l)$ how to see cancellation of square roots?

$I_{dp}(i,j,k,l) =$ 



Component $\chi_i^1 \chi_j^2 \chi_k^3 \chi_l^4$ for non-adjacent i, j, k, l (vanishes for L=0,1) given by 2 double-pentagon integrals

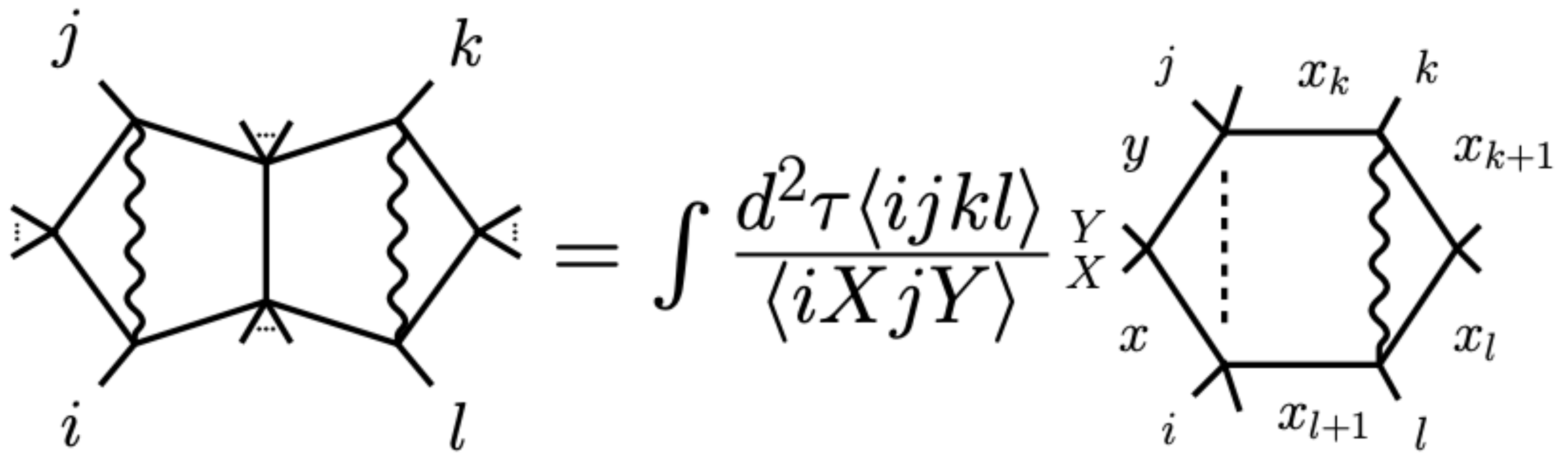
Surprise: $\bar{Q}$ result for the components free of roots (algebraic words vanish)!

For n=8: also observed in [Bourjaily et al] by evaluating $I_{\mathrm{dp}}(1,3,5,7)$ at a numeric point? Why cancel?

Generic double pentagon [SH, Z. Li, Q. Yang, C. Zhang]

Exactly the same method: 2d integral of a hexagon
($k = j+1,\ l = i-1$: chiral hexagon)

$$X := Z_{i-1} - \tau_X Z_{i+1}\,, \quad Y := Z_{j-1} - \tau_Y Z_{j+1}$$



New: hexagon not "pure", 15 boxes w. 2-form "leading singularities"

$$[x,y] = \mathrm{d}\log\frac{\langle iXkl\rangle}{\langle iXjY\rangle}\mathrm{d}\log\frac{\langle \bar{i}(jY)\cap(ikl)\rangle}{\langle jYkl\rangle}\,,$$

$$[x,x_k] = \mathrm{d}\log\frac{\langle jYil\rangle}{\langle jYkl\rangle}\mathrm{d}\log\frac{\langle iXjY\rangle}{\langle l(iX)(jY)(kk{+}1)\rangle}\,,$$

straightforward to compute 2d integral except for the issue:

4-mass box has prefactor $\gamma = \dfrac{r_1 - r_2}{r_1 + r_2}$ -> not d log ?!

$$\int([x,x_k]I_{x,x_k} - (k{-}1 \leftrightarrow k{+}1)) - (\bar{k} \leftrightarrow \bar{l}) + [x,y]I_{x,y}$$

$$\begin{aligned}
I_{x,x_k} :=\ & \tilde{F}(x,y,x_{k+1},x_l) - \tilde{F}(x,y,x_{k+1},x_{l+1}) \\
& - \mathrm{L}_2(l{+}1,x,y,l) + \mathrm{L}_2(l{+}1,x,k{+}1,l) \\
& - \mathrm{L}_2(l{+}1,y,k{+}1,l) + \log u_{l+1,x,y,l}\log u_{x,y,k+1,l+1}\,, \\
I_{x,y} :=\ & \mathrm{L}_2(x,k,k{+}1,l) - \mathrm{L}_2(x,k,k{+}1,l{+}1) \\
& - \mathrm{L}_2(l{+}1,x,k,l) + \mathrm{L}_2(l{+}1,x,k{+}1,l) \\
& - \mathrm{L}_2(l{+}1,k,k{+}1,l) + \log u_{l+1,x,k,l}\log u_{x,k,k+1,l+1}
\end{aligned}$$

$$\mathrm{L}_2(a,b,c,d) := \mathrm{Li}_2(1 - u_{a,b,c,d})$$

Rationalization

$$F(u,v) := \mathrm{Li}_2(1-z) - \mathrm{Li}_2(1-\bar{z}) + \tfrac{1}{2}\log\left(\frac{z}{\bar{z}}\right)\log(v)$$

$$u = u_{a,b,c,d} = z\bar{z}\,,\ v = u_{b,c,d,a} = (1-z)(1-\bar{z})$$

Key: need to rationalize $\int d\log\frac{\tau+a}{\tau+b}\,\gamma(\tau)\ F(z(\tau),\bar{z}(\tau))$

change of var. $\tau \to z(\tau)$ (note $\bar{z} = \dfrac{az+b}{cz+d}$ w. constants) $\Longrightarrow$ integral becomes manifestly pure

$$\int_{z^{-1}(0)}^{z^{-1}(\infty)} d\log\frac{z-w}{z-\bar{w}}\ \mathrm{dilog}(z,\bar{z}) + (z \leftrightarrow \bar{z})\ \text{ with } \bar{w} = \frac{aw+b}{cw+d}$$

beautiful weight-3 symbol: “4-mass box ⊗ algebraic letter”, $\left(u\otimes\frac{1-z}{1-\bar{z}} + v\otimes\frac{\bar{z}}{z}\right)\otimes\frac{(z-w)(\bar{z}-\bar{w})}{(\bar{z}-w)(z-\bar{w})}$

reminiscent to τ-integral of $\bar{Q}$: 2-loop NMHV from 1-loop NNMHV (γ in leading singularities)

already this level: (algebraic symbol)$\big|_{(i,j,k,l)-(j,k,l,i)}$ =0 ! nicely explain cancellation of square roots

next step (no γ): nicely no more algebraic letters; conjecture to work to all loops (e.g. for ladder)!

Final answer

Weight 4 → final-entry free of square roots (similar to 2-loop NMHV to 3-loop MHV!)

remarkably compact algebraic words : sum of 16 blocks with square roots $\Delta(a,b,c,d)$

$$\sum_{\sigma_a\in\{0,1\}} (-)^{\sigma_1+\sigma_2+\sigma_3+\sigma_4}\, S^{4-\mathrm{m}}(i+\sigma_1, j+\sigma_2, k+\sigma_3, l+\sigma_4) \otimes W^{i,j,k,l}_{\sigma_1,\cdots,\sigma_4}$$

explain cancellation of square roots!

total differential nicely written w. only two weight-3 functions: $dI_{\mathrm{dp}}(i,j,k,l) =$

$$\frac{1}{2} R^{\bar{i}}_{j-1j} \mathrm{d}\log \frac{\langle i(i-1i+1)(j-1j)(kl)\rangle}{\langle \bar{i}j\rangle\langle j-1jkl\rangle} + M^{ikl}_{j-1j} \mathrm{d}\log \frac{\langle ij-1jk\rangle}{\langle j-1jkl\rangle} - (j-1j \leftrightarrow jj+1) + (\bar{i} \leftrightarrow \bar{j}) + (\bar{k} \leftrightarrow \bar{l}) + (i\,j \leftrightarrow k\,l)$$

generic (n=12): 6*16 algebraic letters + 164 rational letters -> what is the origin?

first two-loop all-multiplicity amplitudes from integrals for MHV and NMHV components (agree w. $\bar{Q}$)

Discussions

- Wilson loops powerful for perturbative computations: amplitudes, Feynman integrals, …
- Yangian anomaly eqs ($\bar{Q}$ + parity): all-loop equations, e.g. 3-loop MHV octagon (& higher) [Z. Li, C. Zhang]
- Geometries: connections to amplituhedron? cluster algebra & non-perturbative S-matrix?
- WL rep for integrals: systematic "smart parametrization"? apply to more integrals (IR divergent, mass etc.)

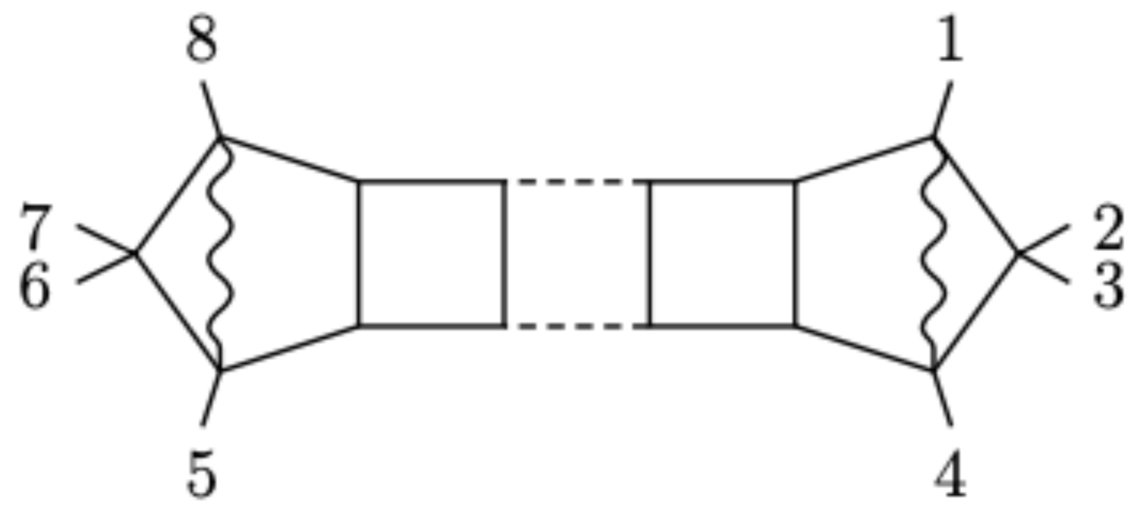
Thank You!

Generalized penta ladders [SH, Z. Li, Y. Tang, Q. Yang]

applicable to a large class of integrals w. “pentagon handles” (or similar 1-loop)—> reduce loop orders
in particular a recursion for “generalized penta ladders”: 2(L-1)-fold dlog of some 1-loop integrals

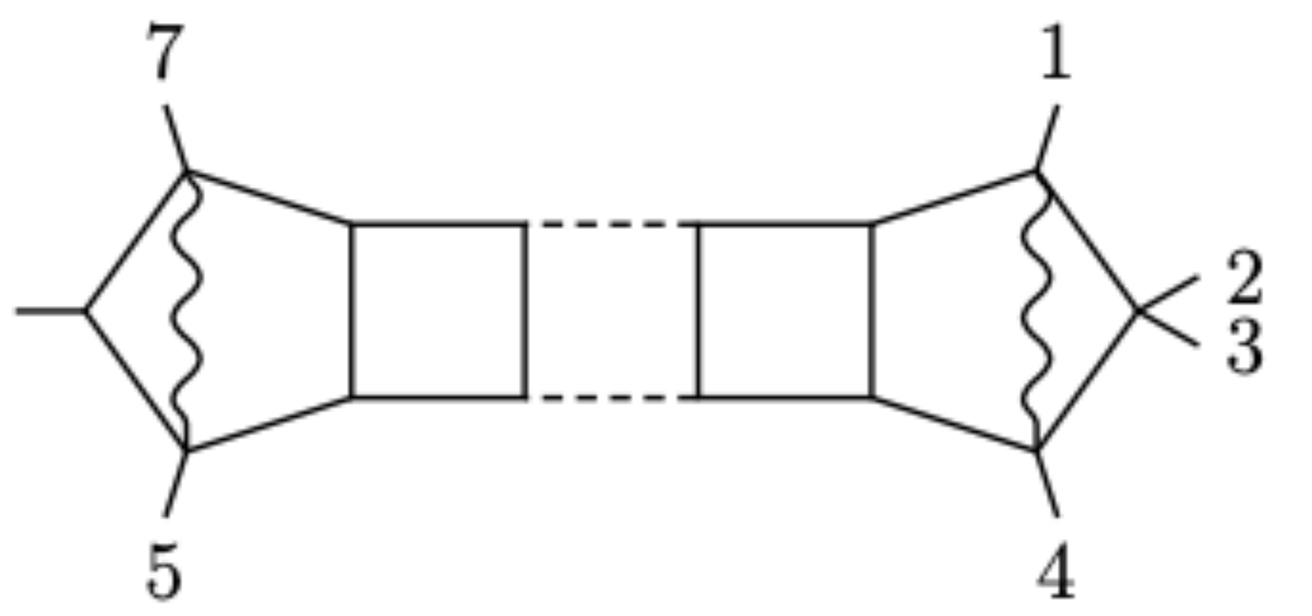
straightforward to obtain symbol if no square roots involved
but need “rationalization” otherwise

e.g. $\Omega_L(1{,}4{,}5{,}8)$ involves square root $\sqrt{(1-u-v)^2-4uv}$

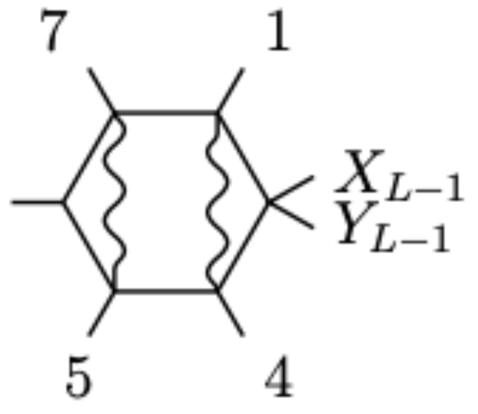


focus on $\Omega_L(1,4,5,7)$: 2(L-1) fold d log integral of 1-loop (7-pt) hexagon

$$\Omega_L(1,4,5,7) = 6$$



$$\Omega_L(1,4,5,7) = \int \prod_{a=1}^{L-1} \mathrm{d}\log\langle 147 Y_a\rangle \mathrm{d}\log\frac{\langle 1X_a 4Y_a\rangle}{\tau_{X_a}} \times 6$$



$$u_1 = \frac{\langle 1245\rangle\langle 5671\rangle}{\langle 1256\rangle\langle 4571\rangle},\quad u_2 = \frac{\langle 3471\rangle\langle 4567\rangle}{\langle 3467\rangle\langle 4571\rangle},\quad u_3 = \frac{\langle 1267\rangle\langle 3456\rangle}{\langle 1256\rangle\langle 3467\rangle},\quad u_4 = \frac{\langle 1234\rangle\langle 4571\rangle}{\langle 1245\rangle\langle 3471\rangle}.$$

beautiful DCI form: essentially same deform

$$\Omega_{L+\frac{1}{2}}(u_1,u_2,u_3,u_4) = \int \mathrm{d}\log\frac{\tau_X+1}{\tau_X}\Omega_L\left(\frac{u_1(\tau_X+u_4)}{\tau_X+u_1u_4}, u_2, \frac{\tau_X u_3}{\tau_X+u_1u_4}, \frac{u_4(\tau_X+1)}{\tau_X+u_4}\right),$$

$$\Omega_{L+1}(u_1,u_2,u_3,u_4) = \int \mathrm{d}\log(\tau_Y+1)\Omega_{L+\frac{1}{2}}\left(u_1, \frac{u_2(\tau_Y+1)}{u_2\tau_Y+1}, \frac{u_3}{1+\tau_Y u_2}, \frac{\tau_Y+u_4}{\tau_Y+1}\right),$$

straightforward to obtain symbol to all loops w. 16 letters $u_1, u_2, u_3, u_4, 1-u_1, 1-u_2, 1-u_3, 1-u_4, 1-u_1u_4, 1-u_2u_4, 1-u_3-u_1u_4, 1-u_3-u_2u_4; y_1, y_2, y_3, y_4$

nicely, alphabet of D_4 cluster algebra [WIP w. Z. Li, Q. Yang] also appear for 6d 1-mass hexagon [Chicherin, Henn, Papathanasiou]
w. $u_3 \to 0$ back to the 9 letters of Ψ_L : sub-algebra A_3

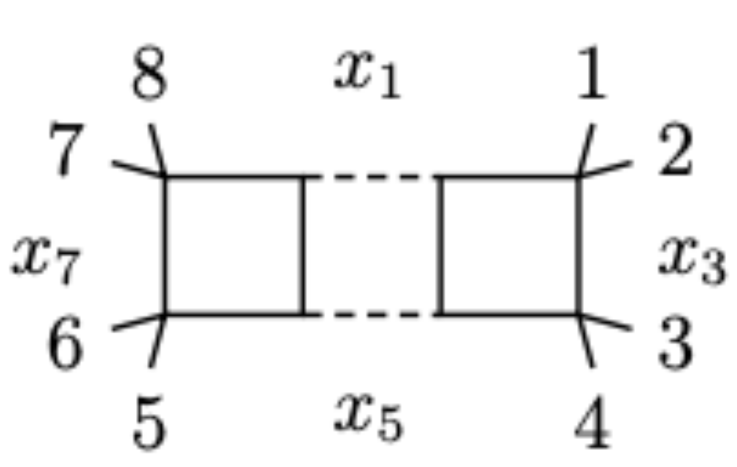
cluster algebras for integrals [SH, Z. Li, Q. Yang]

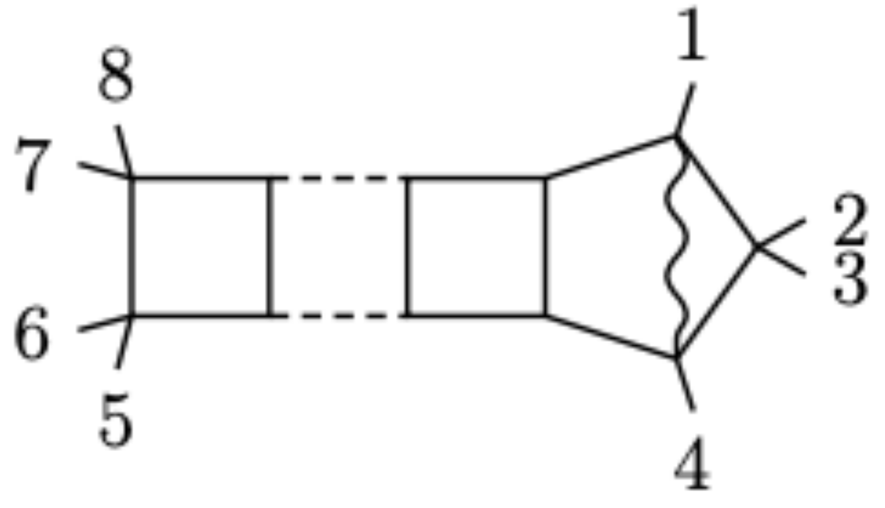
Not only (n=6,7) amplitudes, but a large class of integrals (beyond N=4 SYM) have alphabet of cluster algebra!
e.g. n=6 double-pentagon has A_3 to all loops; systematically studied (4pt, 5pt, various 1-loop) in [Chicherin, Henn, Papathanasiou

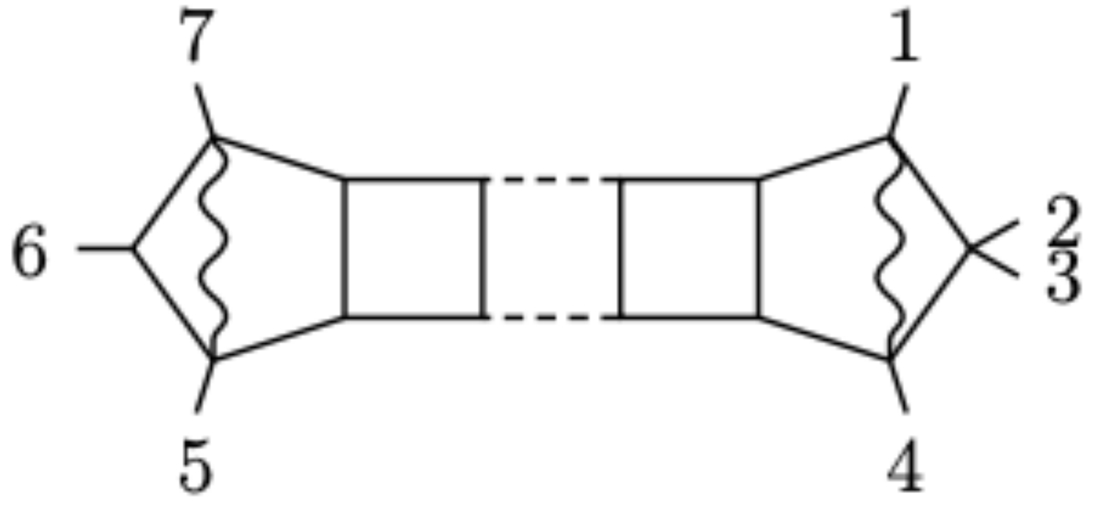
another (trivial) example: 8-pt box ladders (2 cross-ratios), $\{z, \bar{z}, 1-z, 1-\bar{z}\} \sim D_2 = A_1^2$ (not part of $G_+(4,8)$ c.a.)

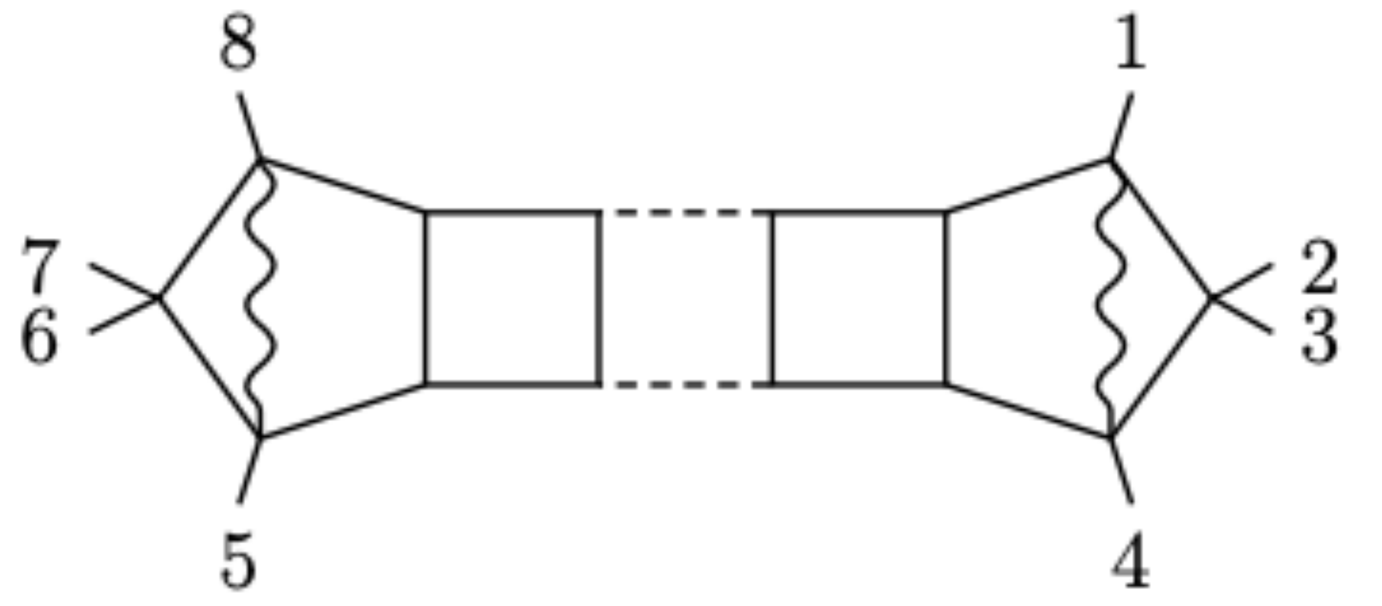
more all-loop ex.: 8-pt penta-box ladder $D_3 = A_3$ (not hexagon A_3), 7-pt double-penta ladder D_4 (emb. in heptagon E_6)

"good" variables: last entries (useful for resummation), e.g. $x = 1 - u^{-1}, y = 1 - v^{-1}, z = 1 - w$ for A3









what about more general integrals w. square roots? embed in trop. $G_+(4,8)$ etc.?