QCD Aspects of Heavy Quark Decays

Yu-Ming Wang

Nankai University

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Why precision calculations?

- Understanding the general properties of power expansion in EFTs (HQET, SCET, NRQCD).

- Interesting to understand the strong interaction dynamics of heavy quark decays.
  - Factorization properties of the subleading-power amplitudes.
  - Renormalization and asymptotic properties of the higher-twist B-meson DAs.
  - Interplay of different QCD techniques.

- Precision determinations of the CKM matrix elements $|V_{ub}|$ and $|V_{cb}|$.
  Power corrections, QED corrections, BSM physics.

- Crucial to understand the CP violation in B-meson decays.
  Strong phase of $\mathscr{A}(B \to M_1 M_2)$ @ m_b scale in the leading power.

- Indispensable for understanding the flavour puzzles (continuously updated).
  - P_5' and $R_{K^{(*)}}$ anomalies in $B \to K^{(*)} \ell^+ \ell^-$.
  - $R_{D^{(*)}}$ anomalies in $B \to D^{(*)} \ell \bar{\nu}_\ell$.
  - Color suppressed hadronic B-meson decays.
  - Polarization fractions of penguin dominated $B_{(s)} \to VV$ decays.

Theory tools for precision flavor physics

New Physics: $\mathscr{L}_{NP}$

$\downarrow$

EW scale (m_W): $\mathscr{L}_{SM} + \mathscr{L}_{D>4}$

$\downarrow$

Heavy-quark scale (m_b): $\mathscr{L}_{eff} = -\frac{G_F}{\sqrt{2}} \sum_i C_i \, Q_i + \mathscr{L}_{eff,D>6}$

$\downarrow$

QCD scale ($\Lambda_{\rm QCD}$)

- Aim: $\langle f | Q_i | \bar{B} \rangle = ?$
- QCD factorization [Diagrammatic approach].
- SCET factorization [Operator formalism].
- TMD factorization.
- (Light-cone) QCD sum rules.
- Lattice QCD.

- Key concepts: Factorization, Resummation, Evolution.

B-meson distribution amplitudes

- The light-ray HQET matrix element [Grozin, Neubert, 1997]:

$$\langle 0|\bar{q}_\beta(z)\,[z,0]\,h_{v\,\alpha}(0)|\bar{B}(v)\rangle = -\frac{i\tilde{f}_B m_B}{4}\left[\frac{1+\not{v}}{2}\left\{2\,\tilde{\phi}_B^+(t,\mu)+\frac{\tilde{\phi}_B^-(t,\mu)-\tilde{\phi}_B^+(t,\mu)}{t}\,\not{z}\right\}\gamma_5\right]_{\alpha\beta}.$$

- Evolution equation at one loop [Lange, Neubert, 2003]:

$$\frac{d\phi_B^+(\omega,\mu)}{d\ln\mu} = -\left[\Gamma_{\rm cusp}(\alpha_s)\,\ln\frac{\mu}{\omega}+\gamma_+(\alpha_s)\right]\phi_B^+(\omega,\mu)-\omega\int_0^\infty d\eta\,\Gamma_+(\omega,\eta,\alpha_s)\,\phi_B^+(\eta,\mu)\,.$$

This is an integro-differential equation!

- (Relatively) complicated solution [Lee, Neubert, 2005]:

$$\begin{aligned}\phi_B^+(\omega,\mu) &= e^{V-2\gamma_E g}\,\frac{\Gamma(2-g)}{\Gamma(g)}\int_0^\infty\frac{d\eta}{\eta}\,\phi_B^+(\eta,\mu_0)\left(\frac{\max(\omega,\eta)}{\mu_0}\right)^g \\ &\quad\times\frac{\min(\omega,\eta)}{\max(\omega,\eta)}\,{}_2F_1\left(1-g,2-g,2,\frac{\min(\omega,\eta)}{\max(\omega,\eta)}\right), \\ g(\mu,\mu_0) &= \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha\,\frac{\Gamma_{\rm cusp}(\alpha)}{\beta(\alpha)}\approx\frac{2C_F}{\beta_0}\,\ln\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\,.\end{aligned}$$

Making the QCD resummation for enhanced logarithms complicated!

B-meson distribution amplitudes

- Better understanding from the RGE in coordinate space [Braun, Ivanov, Korchemsky, 2004]:

$$\frac{d\tilde{\phi}_B^+(t,\mu)}{d\ln\mu} = -\Gamma_{\rm cusp}(\alpha_s)\left\{\left[\ln(i\tilde{\mu}\,t) - \frac{1}{4}\right]\tilde{\phi}_B^+(t,\mu) + \int_0^1 du\,\frac{\bar{u}}{u}\,\left[\tilde{\phi}_B^+(t,\mu) - \tilde{\phi}_B^+(\bar{u}\,t,\mu)\right]\right\}.$$

  - Absence of the local operator product expansion due to the non-analytical term!
  - $\tilde{\phi}_B^+(t,\mu)$ only mixes into $\tilde{\phi}_B^+(\bar{u}\,t,\mu)$ with $0 \leq \bar{u} \leq 1$ under renormalization.

- Renormalization of $[\bar{q}_s(t\bar{n})\Gamma b_v(0)]$ does not commute with the short-distance expansion.

$$[(\bar{q}_s\,Y_s)(t\bar{n})\not{\bar{n}}\,\Gamma\,(Y_s^\dagger\,b_v)(0)]_R \neq \sum_{p=0} \frac{t^p}{p!}\left[\bar{q}_s(0)\,(n\cdot\overleftarrow{D})^p\,\not{\bar{n}}\,\Gamma\,b_v(0)\right]_R.$$

  - Many other examples in QCD physics (e.g., Light-cone projection and renormalization)!
  - Non-negative moments of the B-meson distribution amplitude ill defined.
  - Non-trivial generalization of the QCD equations of motion beyond the tree level.

B-meson distribution amplitudes

- Fourier/Mellin transformation:

$$\varphi_B^+(\theta,\mu) = \int_0^\infty \frac{d\omega}{\omega} \left(\frac{\omega}{\mu}\right)^{-i\theta} \phi_B^+(\omega,\mu) \quad \Leftrightarrow \quad \phi_B^+(\omega,\mu) = \int_{-\infty}^\infty \frac{d\theta}{2\pi} \left(\frac{\omega}{\mu}\right)^{i\theta} \varphi_B^+(\theta,\mu) .$$

- Solution to the RGE in Mellin space:

$$\varphi_B^+(\theta,\mu) = e^{V-2\gamma_E g} \left(\frac{\mu}{\mu_0}\right)^{i\theta} \frac{\Gamma(1-i\theta)}{\Gamma(1+i\theta)} \cdot \frac{\Gamma(1+i(\theta+ig))}{\Gamma(1-i(\theta+ig))} \varphi_B^+(\theta+ig,\mu_0) .$$

Already very symmetric solution in Mellin space.

- Yet simper solution to the integral-differential equation exists?

$\Leftrightarrow$ Eigenfunctions of the Lange-Neubert kernel [Bell, Feldmann, YMW and Yip, 2013].

$$\begin{aligned} \varphi_B^+(\theta,\mu) &:= \frac{\Gamma(1-i\theta)}{\Gamma(1+i\theta)} \bar{\rho}_B^+(\theta,\mu) \\ &= \frac{\Gamma(1-i\theta)}{\Gamma(1+i\theta)} \int_0^\infty \frac{d\omega'}{\omega'} \rho_B^+(\omega',\mu) \left(\frac{\omega'}{\mu}\right)^{-i\theta} . \end{aligned}$$

B-meson distribution amplitudes

- Linear differential equation [Bell, Feldmann, YMW and Yip, 2013]:

$$\frac{d\rho_B^+(\omega',\mu)}{d\ln\mu} = -\left[\Gamma_{\rm cusp}(\alpha_s)\,\ln\frac{\mu}{\hat{\omega}'} + \gamma_+(\alpha_s)\right]\rho_B^+(\omega',\mu)\,.$$

Local evolution in the dual space!

- Integral transformation:

$$\begin{aligned}
\phi_B^+(\omega,\mu) &= \int_0^\infty \frac{d\omega'}{\omega'}\sqrt{\frac{\omega}{\omega'}}\,J_1\left(2\sqrt{\frac{\omega}{\omega'}}\right)\rho_B^+(\omega',\mu)\,,\\
\rho_B^+(\omega',\mu) &= \int_0^\infty \frac{d\omega}{\omega}\sqrt{\frac{\omega}{\omega'}}\,J_1\left(2\sqrt{\frac{\omega}{\omega'}}\right)\phi_B^+(\omega,\mu)\,.
\end{aligned}$$

Eigenfunction of the Lange-Neubert kernel at one-loop is the Bessel function!

- Interesting transformation from the coordinate space to the dual space:

$$\rho_B^+(\omega',\mu) = \int\frac{dt}{2\pi}\left(1-\exp\left[-\frac{i}{\omega'\,t}\right]\right)\tilde{\phi}_B^+(t,\mu)\,.$$

$\rho_B^+(\omega',\mu)$ cannot be constructed from the local OPE of $\tilde{\phi}_B^+(t,\mu)$.

B-meson distribution amplitudes

- Solution to the RGE in dual space [Bell, Feldmann, Wang, Yip, 2013]:

$$\rho_B^+(\omega',\mu) \quad = e^V \left(\frac{\mu_0}{\hat{\omega}'}\right)^{-g} \rho_B^+(\omega',\mu_0)\,.$$

Very compact expression in a full analytical form!

- Solution to the RGE in momentum space:

$$\phi_B^+(\omega,\mu) = e^V \int_0^\infty \frac{d\omega'}{\omega'} \sqrt{\frac{\omega}{\omega'}}\, J_1\left(2\sqrt{\frac{\omega}{\omega'}}\right) \left(\frac{\mu_0}{\hat{\omega}'}\right)^{-g} \rho_B^+(\omega',\mu_0)\,.$$

Still a beautiful expression!

- The logarithmic inverse moments of LCDA and spectral function:

$$\int_0^\infty \frac{d\omega}{\omega} \ln^n\left(\frac{\omega}{\mu}\right) \phi_B^+(\omega,\mu) \overset{n=0,1,2}{=} \int_0^\infty \frac{d\omega'}{\omega'} \ln^n\left(\frac{\hat{\omega}'}{\mu}\right) \rho_B^+(\omega',\mu) \equiv L_n(\mu)\,.$$

  - Non-trivial mixing of $L_n(\mu)$ in dual space under renormalization.

$$\frac{dL_n(\mu)}{d\ln\mu} \quad = \Gamma_{\rm cusp}(\alpha_s)\, L_{n+1}(\mu) - \gamma_+(\alpha_s) L_n(\mu) - n\, L_{n-1}(\mu)\,.$$

  - More complicated RGE for the logarithmic inverse moments in momentum space.

B-meson distribution amplitudes

- Collinear conformal symmetry for the Lange-Neubert kernel [Braun, Manashov, 2014]:

$$\left(\frac{d}{d\ln\mu} + \mathscr{H}_{\rm LN}\right) O_+(z,\mu) = 0\,.$$

$\mathscr{H}_{\rm LN}$ is almost determined by the commutation relations completely [Knoedlseder, Offen, 2011]

$$[S_+,\, \mathscr{H}_{\rm LN}] = 0, \qquad [S_0,\, \mathscr{H}_{\rm LN}] = 1\,.$$

The beautiful solution in terms of S_+:

$$\mathscr{H}_{\rm LN} = \ln(i\,\mu\, S^+) - \psi(1) - \frac{5}{4}\,.$$

- Generators of the collinear conformal group:

$$S_+ = z^2\,\partial_z + 2jz\,, \qquad S_0 = z\,\partial_z + j\,, \qquad S_- = -\partial_z\,.$$

Eigenfunctions of S_+ [Braun, Manashov, 2014]:

$$i S_+\, Q_s(z) = s\, Q_s(z)\,, \qquad Q_s(z) = -\frac{1}{z^2}\, e^{is/z}\,.$$

$$\langle e^{-i\omega z} | Q_s(z)\rangle = \frac{1}{\sqrt{\omega s}}\, J_1(2\sqrt{\omega s})\,.$$

Wide applications of the conformal symmetry in high energy physics!

B-meson distribution amplitudes

- RG evolution of $\phi_B^+(\omega,\mu)$ at two loops [Braun, Ji, Manashov, 2019; Liu, Neubert, 2020]:

$$\frac{d\phi_B^+(\omega,\mu)}{d\ln\mu} = \left[\Gamma_{\rm cusp}(\alpha_s)\ln\frac{\omega}{\mu} - \gamma_\eta(\alpha_s)\right]\phi_B^+(\omega,\mu) + \Gamma_{\rm cusp}(\alpha_s)\int_0^\infty dx\,\Gamma(1,x)\,\phi_B^+(\omega/x,\mu) + \left(\frac{\alpha_s}{2\pi}\right)^2 C_F\int_0^1\frac{dx}{1-x}\,h(x)\,\phi_B^+(\omega/x,\mu)\,.$$

The last missing element for the NLL predictions of exclusive B-meson decay observables!

- The two-loop eigenfunctions depend on the strong coupling α_s [Braun, Ji, Manashov, 2019].

- Applying the Laplace transformation of the LCDA [Galda, Neubert, 2020]

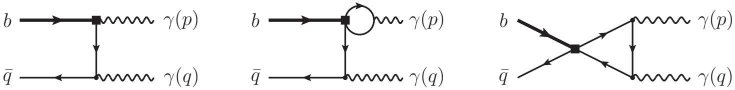
$$\tilde{\phi}_B^+(\eta,\mu) = \int_0^\infty\frac{d\omega}{\omega}\,\phi_B^+(\omega,\mu)\left(\frac{\omega}{\bar{\omega}}\right)^{-\eta},$$

$\Rightarrow$ the general solution to the two-loop RGE of $\phi_B^+(\omega,\mu)$

$$\begin{aligned}\tilde{\phi}_B^+(\eta,\mu) &= \exp\left[S(\mu_0,\mu)+a_\gamma(\mu_0,\mu)+2\,\gamma_E\,a_\Gamma(\mu_0,\mu)\right]\left(\frac{\bar{\omega}}{\mu_0}\right)^{-a_\Gamma(\mu_0,\mu)} \\ &\times\frac{\Gamma(1+\eta+a_\Gamma(\mu_0,\mu))\,\Gamma(1-\eta)}{\Gamma(1-\eta-a_\Gamma(\mu_0,\mu))\,\Gamma(1+\eta)}\exp\left[\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\alpha}{\beta(\alpha)}\,\mathcal{G}(\eta+a_\Gamma(\mu_\alpha,\mu),\alpha)\right] \\ &\times\tilde{\phi}_B^+(\eta+a_\Gamma(\mu_0,\mu),\mu_0)\,.\end{aligned}$$

Double Radiative B_q-Meson Decays

- Leading-order contributions at $\mathscr{O}(\alpha_s^0)$:



Kinematics:

$$p_\mu = \frac{n \cdot p}{2}\, \bar{n}_\mu \equiv \frac{m_{B_q}}{2}\, \bar{n}_\mu\,, \qquad q_\mu = \frac{\bar{n} \cdot q}{2}\, n_\mu \equiv \frac{m_{B_q}}{2}\, n_\mu\,.$$

Interplay of the soft and collinear QCD dynamics!

- Decay amplitude:

$$\bar{\mathscr{A}}(\bar{B}_q \to \gamma\gamma) = -\frac{4\,G_F}{\sqrt{2}}\, \frac{\alpha_{\rm em}}{4\pi}\, \varepsilon_1^{*\alpha}(p)\, \varepsilon_2^{*\beta}(q) \sum_{p=u,c} V_{pb} V_{pq}^* \sum_{i=1}^{8} C_i\, T_{i,\alpha\beta}^{(p)}\,.$$

Hadronic tensors:

$$\begin{aligned} T_{7,\alpha\beta} &= 2\,\overline{m}_b(\nu) \int d^4x\, e^{iq\cdot x} \langle 0|\mathrm{T}\left\{ j_\beta^{\rm em}(x), \bar{q}_L(0)\sigma_{\mu\alpha} p^\mu b_R(0) \right\} |\bar{B}_q(p+q)\rangle \\ & \quad + [p \leftrightarrow q, \alpha \leftrightarrow \beta]\,, \\ T_{i,\alpha\beta}^{(p)} &= -(4\,\pi)^2 \int d^4x \int d^4y\, e^{ip\cdot x}\, e^{iq\cdot y} \langle 0|\mathrm{T}\left\{ j_\alpha^{\rm em}(x), j_\beta^{\rm em}(y), P_i^{(p)}(0) \right\} |\bar{B}_q(p+q)\rangle\,, \\ & \quad (i = 1, \ldots 6, 8)\,. \end{aligned}$$

Main task: Construct the SCET factorization formulae beyond the leading power.

General aspects of $B_q \to \gamma\gamma$

- Parametrization:

$$T^{(p)}_{i,\alpha\beta} = i\, m^3_{B_q} \left[\left(g^{\perp}_{\alpha\beta} - i\, \varepsilon^{\perp}_{\alpha\beta} \right) F^{(p)}_{i,L} - \left(g^{\perp}_{\alpha\beta} + i\, \varepsilon^{\perp}_{\alpha\beta} \right) F^{(p)}_{i,R} \right] .$$

  - Only two helicity form factors due to the Ward identities and the transversity conditions.
  - Similar decomposition for the radiative leptonic B-meson decay amplitude.

- Hierarchy structure due to the chiral weak interactions and helicity conservation:

$$F^{(p)}_{i,L} : F^{(p)}_{i,R} = 1 : \left(\frac{\Lambda_{\rm QCD}}{m_b} \right) .$$

  In analogy to the two-body nonleptonic $B \to VV$ decays [Beneke, Rohrer, Yang, 2007].

- Transversity form factors:

$$F^{(p)}_{i,\parallel} = F^{(p)}_{i,L} - F^{(p)}_{i,R} , \qquad F^{(p)}_{i,\perp} = F^{(p)}_{i,L} + F^{(p)}_{i,R} .$$

  The two-photon final states as the CP eigenstates with the eigenvalues $+1$ and -1.

Current status of $B_q \to \gamma\gamma$

- QCD factorization at leading power in Λ/m_b and at NLO in α_s [Descotes-Genon, Sachrajda, 2003].
  - No collinear strong interaction dynamics at LP.
  - The two-loop $b \to q\gamma$ matrix elements of QCD penguin operators NOT included. $\Rightarrow$ A complete factorization-scale independence at NLO is absent!

- Subleading power contributions from the weak annihilation diagrams [Bosch, Buchalla, 2002].
  - Complex perturbative hard functions evaluated at one loop.
  - Diagrammatic factorization established at two loops.

- The new (technical)-ingredients from [Shen, YMW, Wei, 2020]:
  - A complete NLL calculation for the LP contribution $\Rightarrow$ 2-loop evolution of ϕ_B^+.
  - The NLP factorization for the energetic photon radiation off the light quark. The so-called "soft form factor" defined in [Beneke, Rohrwild, 2011] is factorizable!
  - The NLP factorization for the light-quark mass effect.
  - The NLP factorization for the SCET current $J^{(A2)} \supset (\bar{\xi}_{\rm hc} W_{\rm hc})\, \gamma_\alpha^\perp \, P_L \left(\frac{i \overrightarrow{\not{D}}_\top}{2 m_b} \right) h_v$.
  - The NLP factorization for the subleading twist B-meson LCDAs.
  - The resolved photon contribution with the dispersion technique.

- Referee Report: "The authors have endeavored to study the decay $B_{d,s} \to \gamma\gamma$ quite comprehensively, and achieved much progress far beyond the seminal paper by Descotes-Genon and Sachrajda on that subject".

SCET factorization at leading power

- QCD $\to$ SCET$_{\rm I}$ matching at LP:

$$\sum_{i=1}^{8} C_i \, T^{(p)}_{i,\alpha\beta} = \sum_{i=1}^{8} C_i \, H_i^{(p)} \left\{ \int d^4x \, e^{iq\cdot x} \langle 0 | \mathrm{T} \left\{ j^{\rm em}_{\beta, \rm SCET_I}(x), \left[(\bar{\xi}_{\overline{\rm hc}} W_{\overline{\rm hc}}) \gamma^{\perp}_{\alpha} P_L \, h_v \right](0) \right\} | \bar{B}_q \rangle + [p \leftrightarrow q, \alpha \leftrightarrow \beta] \right\}.$$

Universal SCET$_{\rm I}$ correlation function independent of the weak-operator index!

- Perturbative matching coefficients at NLO:

$$\sum_{i=1}^{8} C_i \, H_i^{(p)} = (-2\,i) \, \overline{m}_b(\nu) \, m_{B_q} \, V^{(p)}_{7,\rm eff} \,,$$

$$V^{(p)}_{7,\rm eff} = C_7^{\rm eff} \, C^{(\rm A0)}_{T_1} + \sum_{i=1,\ldots,6,8} \frac{\alpha_s(\mu)}{4\pi} \, C_i^{\rm eff} \, F^{(p)}_{i,7} \,.$$

  - The hard function $C^{(\rm A0)}_{T_1}$ from matching the heavy-light tensor current onto SCET$_{\rm I}$.
  - The hard functions $F^{(p)}_{i,7}$ $(i = 1, ..6, 8)$ from perturbative matching of the $b \to q\gamma$ matrix elements [Buras et al, 2002].
  - $V^{(p)}_{7,\rm eff}$ depends on both the electro-weak scale and the heavy-quark mass scale.

SCET factorization at leading power

- $\mathrm{SCET_I} \to \mathrm{SCET_{II}}$ matching in coordinate space:

$$\mathscr{T}_{\alpha\beta} = \int d^4x\, e^{iq\cdot x} \langle 0|\mathrm{T}\left\{ j^{\mathrm{em}}_{\beta,\mathrm{SCET_I}}(x),\ \left[(\bar{\xi}_{\overline{\mathrm{hc}}} W_{\overline{\mathrm{hc}}})\, \gamma_\alpha^\perp P_L h_v \right](0) \right\} |\bar{B}_q\rangle$$
$$= \int dt\, \mathscr{J}\left(\frac{\bar{n}\cdot q}{\mu^2\, v\cdot x} \right) \langle 0|(\bar{q}_s Y_s)(tn)\gamma_\beta^\perp \not{n}\, \gamma_\alpha^\perp P_L (Y_s^\dagger h_v)(0)|\bar{B}_q\rangle .$$

  - Only need the one-loop jet function (2-loop result by Liu and Neubert, 2020).

$$J = 1 + \frac{\alpha_s(\mu)\, C_F}{4\pi} \left[\ln^2\left(\frac{\mu^2}{m_b\, \omega} \right) - \frac{\pi^2}{6} - 1 \right] + \mathscr{O}(\alpha_s^2) .$$

  - The soft dynamics encoded in the twist-two HQET B-meson LCDA.
  - The resulting LP factorization formula:

$$\bar{\mathscr{A}}_{\mathrm{LP}}(\bar{B}_q \to \gamma\gamma) = i\, \frac{4\, G_F}{\sqrt{2}}\, \frac{\alpha_{\mathrm{em}}}{4\pi}\, \varepsilon_1^{*\alpha}(p)\, \varepsilon_2^{*\beta}(q) \left[g_{\alpha\beta}^\perp - i\varepsilon_{\alpha\beta}^\perp \right] e_q f_{B_q} m_{B_q}^2 K^{-1}(m_b,\mu)$$
$$\left[\sum_{p=u,c} V_{pb} V_{pq}^*\, \overline{m}_b(\nu)\, V_{7,\mathrm{eff}}^{(p)}(m_b,\mu,\nu) \right] \int_0^\infty \frac{d\omega}{\omega}\, \phi_B^+(\omega,\mu)\, J(m_b,\omega,\mu) .$$

Factorization of the hard, hard-collinear and soft dynamics.

SCET factorization at leading power

- No common choice of the factorization scale to avoid the parametrically large logarithms.
  $\Rightarrow$ QCD rsummation for the enhanced logarithms with the standard RG formalism.

- RG evolution functions for the hard functions [Beneke, Rohrwild, 2011]:

$$\begin{aligned} V_{7,\text{eff}}^{(p)}(m_b, \mu, \nu) &= \hat{U}_1(m_b, \mu_\text{h}, \mu)\, V_{7,\text{eff}}^{(p)}(m_b, \mu_\text{h}, \nu)\,, \\ K^{-1}(m_b, \mu) &= \hat{U}_2(m_b, \mu_\text{h}, \mu)\, K^{-1}(m_b, \mu_\text{h})\,. \end{aligned}$$

- Taking the factorization scale of order $\sqrt{m_b \Lambda_{\text{QCD}}}$ $\Rightarrow$ No resummation for the jet function.

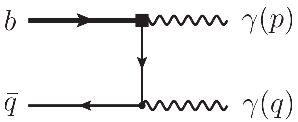
- Implementing the 2-loop evolution of $\phi_B^+(\omega, \mu)$ as discussed before.
  $\hookrightarrow$ The NLL resummation improved expression for the LP decay amplitude:

$$\sum_{i=1}^{8} C_i\, F_{i,L}^{(p),\text{LP}} = -\frac{e_q f_{B_q}}{m_{B_q}}\, \hat{U}_1(m_b, \mu_\text{h}, \mu)\, \hat{U}_2(m_b, \mu_\text{h}, \mu)\, K^{-1}(m_b, \mu_\text{h})\, \overline{m}_b(\nu)\, V_{7,\text{eff}}^{(p)}(m_b, \mu_\text{h}, \nu)\, \mathscr{R}(m_b, \mu_0, \mu)\,.$$

$\hookrightarrow$ Can be achieved alternatively in dual space [Shen, Wei, Zhao, Zhou, 2020].

SCET factorization beyond leading power

- The NLP factorization from the hard-collinear propagator:



Heavy quark expansion:

$$\frac{i\,(\not{q}-\not{k})}{(q-k)^2} = \frac{i\,\not{q}}{(q-k)^2} - \underbrace{\frac{i\,\not{k}}{(q-k)^2}}_{\text{power suppressed}} .$$

- The NLP correlation function in coordinate space:

$$T^{\mathrm{hc,NLP}}_{7,\alpha\beta} = \left[-\frac{e_q\,\overline{m}_b(\nu)\,m_{B_q}}{4\,\pi^2}\right] \int d^4x\, \frac{e^{i\,q\cdot x}}{x^2}\,(2\,v_\mu - n_\mu) \times \frac{\partial}{\partial x_\mu}\,\langle 0|\bar{q}(x)\,\gamma^\perp_\beta\,\not{n}\,\not{\bar{n}}\,\gamma^\perp_\alpha\,P_R\,h_v(0)|\bar{B}_q\rangle\,.$$

- The classical equation of motion for the non-local operator [Kawamura et al, 2001]:

$$v_\mu\,\frac{\partial}{\partial x_\mu}\,[\bar{q}(x)\,\Gamma\,h_v(0)] = i\int_0^1 du\,\bar{u}\,\bar{q}(x)\,g_s\,G_{\alpha\beta}(ux)\,x^\alpha\,v^\beta\,\Gamma\,h_v(0) + (v\cdot\partial)\,[\bar{q}(x)\,\Gamma\,h_v(0)]\,.$$

$\Rightarrow$ The yielding NLP factorization formula

$$T^{\mathrm{hc,NLP}}_{7,\alpha\beta} = \left[-2\,i\,e_q\,\overline{m}_b(\nu) f_{B_q}\,m_{B_q}\right]\left[g^\perp_{\alpha\beta} - i\,\varepsilon^\perp_{\alpha\beta}\right]\left\{\frac{1}{2} - \left(\frac{\bar{\Lambda}}{\lambda_{B_q}}\right) + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2\,\frac{1}{\omega_2}\left[\frac{1}{\omega_2}\,\ln\frac{\omega_1}{\omega_1+\omega_2} + \frac{1}{\omega_1}\right]\Psi_4(\omega_1,\omega_2,\mu)\right\}.$$

SCET factorization beyond leading power

- The NLP correction form the non-vanishing quark mass:

$$T^{m_q,\mathrm{NLP}}_{7,\alpha\beta} = \left[-i\,e_q\,\overline{m}_b(\nu)\,m_q f_{B_q}\,m_{B_q}\right]\left[g^{\perp}_{\alpha\beta} - i\,\varepsilon^{\perp}_{\alpha\beta}\right]\int_0^\infty d\omega\,\frac{\phi_B^-(\omega,\mu)}{\omega}\,.$$

- Rapidity divergence implies the breakdown of the naive soft-collinear factorization.

- Nonperturbative parametrization of the convolution integral [Beneke, Neubert, 2003]:

$$\int_0^\infty d\omega\,\frac{\phi_B^-(\omega,\mu)}{\omega} = \left[\int_0^{\Lambda_{\mathrm{UV}}} + \int_{\Lambda_{\mathrm{UV}}}^\infty\right] d\omega\,\frac{\phi_B^-(\omega,\mu)}{\omega}$$

$$= \left[\phi_B^-(0,\mu)\,X_{\mathrm{NLP}} + \int_0^{\Lambda_{\mathrm{UV}}} d\omega\,\frac{\phi_B^-(\omega,\mu)-\phi_B^-(0,\mu)}{\omega}\right] + \int_{\Lambda_{\mathrm{UV}}}^\infty d\omega\,\frac{\phi_B^-(\omega,\mu)}{\omega}\,.$$

$\Downarrow$ UV and IR Finite!

$$X_{\mathrm{NLP}} = \left[1+\rho_S\,\exp(i\,\varphi_S)\right]\ln\left(\frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{h}}}\right).$$

  - Alternative estimate from the LCSR method also possible.
  - The complete SCET factorization in demand but extremely challenging.

SCET factorization beyond leading power

- The NLP contribution from the subleading SCET current [Beneke, Feldmann, 2002]

$$J^{(A2)} \supset (\bar{\xi}_{\overline{\rm hc}} W_{\overline{\rm hc}}) \gamma_\alpha^\perp P_L \left(\frac{i \overrightarrow{\not{D}}_\top}{2 m_b} \right) h_v + \ldots , \qquad D_\top^\mu \equiv D^\mu - (v \cdot D) v^\mu .$$

Arise from the HQET representation of the QCD b-quark field.

- The resulting non-local hadronic matrix element

$$T_{7,\alpha\beta}^{A2,\rm NLP} = \left[- \frac{i e_q m_{B_q}^2}{2} \right] \int d^4x \int \frac{d^4 \ell}{(2\pi)^4} \exp[i (q-\ell) \cdot x] \frac{1}{\ell^2 + i0} \times \langle 0 | \bar{q}(x) \gamma_\beta^\perp \not{\ell} \not{\bar{n}} \gamma_\alpha^\perp \overrightarrow{\not{D}} P_L h_v(0) | \bar{B}_q \rangle .$$

$\Rightarrow$ The soft-collinear factorization formula with the aid of the HQET equations of motion

$$T_{7,\alpha\beta}^{A2,\rm NLP} = \left[i e_q f_{B_q} m_{B_q}^2 \right] \left[g_{\alpha\beta}^\perp - i \varepsilon_{\alpha\beta}^\perp \right] \left\{ \frac{1}{2} \left(\frac{\bar{\Lambda}}{\lambda_{B_q}} \right) - 1 + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \frac{1}{\omega_1 (\omega_1 + \omega_2)} \Phi_3(\omega_1, \omega_2, \mu) \right\} .$$

  - The convolution integral free of the end-point divergence.
  - In agreement with the counterpart contribution to $B \to \gamma \ell \nu$ [Beneke, Braun, Ji, Wei, 2018].

SCET factorization beyond leading power

- The two-particle higher-twist B-meson LCDAs up to the $\mathcal{O}(x^2)$ accuracy:

$$\langle 0|(\bar{q}_s Y_s)_\beta(x)\ (Y_s^\dagger h_v)_\alpha(0)|\bar{B}_q\rangle$$
$$= -\frac{i\tilde{f}_{B_q}(\mu)\, m_{B_q}}{4}\int_0^\infty d\omega\, e^{-i\omega v\cdot x}\left[\frac{1+\not{v}}{2}\left\{2\left[\phi_B^+(\omega,\mu)+x^2\, g_B^+(\omega,\mu)\right]\right.\right.$$
$$\left.\left.-\frac{1}{v\cdot x}\left[(\phi_B^+(\omega,\mu)-\phi_B^-(\omega,\mu))+x^2\,(g_B^+(\omega,\mu)-g_B^-(\omega,\mu))\right]\not{x}\right\}\gamma_5\right]_{\alpha\beta}.$$

- The three-particle higher-twist B-meson LCDAs up to twist-6 [Braun, Ji, Manashov, 2017].
  - The EOM relation between the two-particle and three-particle LCDAs.

$$-2\,\frac{d^2}{d\omega^2}\, g_B^-(\omega,\mu) = \left[\frac{3}{2}+(\omega-\bar{\Lambda})\,\frac{d}{d\omega}\right]\phi_B^-(\omega,\mu)-\frac{1}{2}\,\phi_B^+(\omega,\mu)$$
$$+\int_0^\infty \frac{d\omega_2}{\omega_2}\left[\frac{d}{d\omega}-\frac{1}{\omega_2}\right]\Psi_5(\omega,\omega_2,\mu)+\int_0^\omega \frac{d\omega_2}{\omega_2^2}\,\Psi_5(\omega-\omega_2,\omega_2,\mu).$$

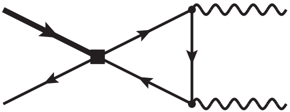
  - The systematic parametrization requires 8 independent LCDAs.

- The higher-twist factorization formula at tree level:

$$T_{7,\alpha\beta}^{\rm HT,NLP} \simeq \left[-i\,e_q\,\overline{m}_b(\nu) f_{B_q}\, m_{B_q}\right]\left[g_{\alpha\beta}^\perp - i\,\varepsilon_{\alpha\beta}^\perp\right]\left\{-1+2\int_0^\infty d\omega\,\ln\omega\,\Delta\phi_B^-(\omega,\mu)\right.$$
$$\left.-2\int_0^\infty d\omega_2\,\frac{1}{\omega_2}\,\Phi_4(\omega_1=0,\omega_2,\mu)\right\}.$$

SCET factorization beyond leading power

- The NLP contribution from the weak annihilation diagram:



The resulting helicity form factors:

$$\sum_{i=1}^{6} C_i\, F_{i,L}^{(p),\,\mathrm{WA,NLP}} = \frac{f_{B_q}}{m_{B_q}} \left[\mathscr{F}_V^{(p),\,\mathrm{WA}} - \mathscr{F}_A^{(p),\,\mathrm{WA}} \right],$$

$$\sum_{i=1}^{6} C_i\, F_{i,R}^{(p),\,\mathrm{WA,NLP}} = \frac{f_{B_q}}{m_{B_q}} \left[\mathscr{F}_V^{(p),\,\mathrm{WA}} + \mathscr{F}_A^{(p),\,\mathrm{WA}} \right].$$

  - The weak-annihilation effect will spoil the large-recoil symmetry.
  - Massive quark loops generate the non-trivial strong phase.
    $\Leftrightarrow$ Dual to the final-state rescattering $\bar{B}_q \to H_c H'_{\bar{c}} \to \gamma\gamma$ at hadronic level.
  - Tree-operator contributions consistent with [Bosch, Buchalla, 2002].

- The NLP contribution from the (anti)-collinear photon radiation off the bottom-quark:

$$T_{7,\alpha\beta}^{e_b,\,\mathrm{NLP}} = \left[-i\, e_q f_{B_q} m_{B_q}^2 \right] \left[g_{\alpha\beta}^{\perp} - i\, \varepsilon_{\alpha\beta}^{\perp} \right].$$

  - Local correction preserves the large-recoil symmetry!
  - In analogy to the P_7 contribution to $B_q \to \gamma \ell \bar{\ell}$ with the B-type insertion [Beneke, Bobeth, YMW, 2020].

The resolved photon contribution

- The NLP contribution from the "hadronic" component of the on-shell photon [Ball, Braun, 2002].
- The dispersion technique [Khodjamirian, 1999; Braun, Khodjamirian, 2013; YMW, 2016]:

$$\begin{aligned}\tilde{T}_{7,\alpha\beta} &= 2\,\overline{m}_b(\nu)\int d^4x\, e^{iq\cdot x}\,\langle 0|\mathrm{T}\left\{j_\beta^{\mathrm{em}}(x), \bar{q}_L(0)\sigma_{\mu\alpha}p^\mu b_R(0)\right\}|\bar{B}_q(p+q)\rangle\Big|_{q^2<0} \\ &\quad +[p\leftrightarrow q, \alpha\leftrightarrow\beta] \\ &= -\frac{i\,e_q\,\overline{m}_b(\nu)\,m_{B_q}^2}{2}\left\{\left(g_{\alpha\beta}^{\perp}-i\,\varepsilon_{\alpha\beta}^{\perp}\right)\tilde{F}_{7,L}(\bar{n}\cdot q, n\cdot q)+[p\leftrightarrow q, \alpha\leftrightarrow\beta]\right\}.\end{aligned}$$

Power countering scheme:

$$\bar{n}\cdot q\sim\mathcal{O}(m_b)\,,\qquad n\cdot q\sim\mathcal{O}(\Lambda)\,.$$

- The hadronic dispersion relation:

$$\begin{aligned}\tilde{T}_{7,\alpha\beta} &= -i\,\overline{m}_b(\nu)\,m_{B_q}^2\left(g_{\alpha\beta}^{\perp}-i\,\varepsilon_{\alpha\beta}^{\perp}\right)\left\{\sum_V \frac{c_V\,f_V\,m_V\,T_1^{B_q\to V}(0)}{\bar{n}\cdot q\left(m_V^2/\bar{n}\cdot q-n\cdot q-i0\right)}\right. \\ &\quad \left.+\int_{\omega_s}^{\infty} d\omega'\,\frac{\rho^{\mathrm{had}}(\bar{n}\cdot q,\omega')}{\omega'-n\cdot q-i0}\right\}.\end{aligned}$$

The constant c_V determined by the flavour factor and the electric charge of the QED quark-current.

- LCSR for the tensor $B\to V$ form factors:

$$\sum_V \frac{c_V\,f_V\,m_V}{\bar{n}\cdot q}\exp\left[-\frac{m_V^2}{\bar{n}\cdot q\,\omega_M}\right]T_1^{B_q\to V}(0)=\frac{e_q}{2}\,\frac{1}{\pi}\int_0^{\omega_s} d\omega'\,e^{-\omega'/\omega_M}\,\mathrm{Im}_{\omega'}\,\tilde{F}_{7,L}(\bar{n}\cdot q,\omega')\,.$$

The resolved photon contribution

- Improved dispersion relations (setting $n \cdot q = 0$) [Master formula]:

$$T_{7,\alpha\beta} = -\frac{i\, e_q\, \overline{m}_b(\nu)\, m_{B_q}^2}{2} \left(g_{\alpha\beta}^{\perp} - i\,\varepsilon_{\alpha\beta}^{\perp}\right) \Bigg\{ \underbrace{\frac{1}{\pi} \int_0^{\infty} \frac{d\omega'}{\omega'} \, \mathrm{Im}_{\omega'} \tilde{F}_{7,L}(\bar{n}\cdot q, \omega')}_{\text{LP factorization formulae}}$$

$$+ \underbrace{\frac{1}{\pi} \int_0^{\omega_s} d\omega' \left[\frac{\bar{n}\cdot q}{m_V^2} \exp\left(\frac{m_V^2 - \bar{n}\cdot q\, \omega'}{\bar{n}\cdot q\, \omega_M} \right) - \frac{1}{\omega'} \right] \mathrm{Im}_{\omega'} \tilde{F}_{7,L}(\bar{n}\cdot q, \omega')}_{\text{nonperturbative modification}} \Bigg\} + [p \leftrightarrow q, \alpha \leftrightarrow \beta] \, .$$

- Power counting scheme for the sum-rule parameters:

$$\omega_s = \frac{s_0}{\bar{n}\cdot q} \sim \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{m_{B_q}}\right), \qquad \omega_M = \frac{M^2}{\bar{n}\cdot q} \sim \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{m_{B_q}}\right).$$

$\Rightarrow$ Nonperturbative modification yields the soft non-factorizable contribution.

- Spectral density at tree level:

$$\frac{1}{\pi} \mathrm{Im}_{\omega'} \tilde{F}_{7,L}(\bar{n}\cdot q, \omega') = f_{B_q} \underbrace{\phi_B^+(\omega', \mu)} + \mathcal{O}(\alpha_s, \Lambda/m_b) \, .$$

of $\mathcal{O}(1/\Lambda)[\mathcal{O}(1/m_b)]$ for $\omega' \sim \mathcal{O}(\Lambda)[\omega' \sim \mathcal{O}(\Lambda^2/m_b)]$

Power suppressed soft contribution!

- Alternative LCSR calculation with the subleading-twist photon LCDAs [Shen, YMW, 2018].

Summary for the helicity amplitudes of $B_q \to \gamma\gamma$

- Final expressions for the factorized NLP corrections:

$$\sum_{i=1}^{8} C_i \, F_{i,L}^{(p),\,\mathrm{fac,NLP}} = C_7^{\mathrm{eff}} \left[F_{7,L}^{\mathrm{hc,NLP}} + F_{7,L}^{m_q,\mathrm{NLP}} + F_{7,L}^{A2,\mathrm{NLP}} + F_{7,L}^{\mathrm{HT,NLP}} + F_{7,L}^{e_b,\mathrm{NLP}} \right] + \frac{f_{B_q}}{m_{B_q}} \left[\mathscr{F}_V^{(p),\,\mathrm{WA}} - \mathscr{F}_A^{(p),\,\mathrm{WA}} \right],$$

$$\sum_{i=1}^{8} C_i \, F_{i,R}^{(p),\,\mathrm{fac,NLP}} = \frac{f_{B_q}}{m_{B_q}} \left[\mathscr{F}_V^{(p),\,\mathrm{WA}} + \mathscr{F}_A^{(p),\,\mathrm{WA}} \right].$$

Large-recoil symmetry violation from the weak annihilation correction completely.

- Final expressions for the two helicity amplitudes:

$$\bar{\mathscr{A}}_L = \sum_{p=u,c} V_{pb} \, V_{pq}^* \sum_{i=1}^{8} C_i \left[F_{i,L}^{(p),\,\mathrm{LP}} + F_{i,L}^{(p),\,\mathrm{fac,NLP}} + F_{i,L}^{(p),\,\mathrm{soft,NLP}} \right],$$

$$\bar{\mathscr{A}}_R = \sum_{p=u,c} V_{pb} \, V_{pq}^* \sum_{i=1}^{8} C_i \left[F_{i,R}^{(p),\,\mathrm{LP}} + F_{i,R}^{(p),\,\mathrm{fac,NLP}} + F_{i,R}^{(p),\,\mathrm{soft,NLP}} \right].$$

- The fundamental nonperturbative functions: HQET B-meson LCDAs.
  $\Rightarrow$ Key hadronic inputs for exclusive B-meson decay phenomenologies.

B-meson distribution amplitudes in HQET

- The LP contribution depends on λ_{B_q} and the inverse-logarithmic moments.

- Applying the general ansatz of the 2- and 3- particle B-meson LCDAs

$$
\begin{aligned}
&\phi_B^+(\omega,\mu) = \omega\, f(\omega)\,, \; \Phi_3(\omega_1,\omega_2,\mu_0) = -\frac{1}{2}\,\kappa(\mu_0)\,\left[\lambda_E^2(\mu_0) - \lambda_H^2(\mu_0)\right]\,\omega_1\,\omega_2^2 f'(\omega_1+\omega_2)\,, \\
&\Phi_4(\omega_1,\omega_2,\mu_0) = \frac{1}{2}\,\kappa(\mu_0)\,\left[\lambda_E^2(\mu_0) + \lambda_H^2(\mu_0)\right]\,\omega_2^2 f(\omega_1+\omega_2)\,, \\
&\Psi_4(\omega_1,\omega_2,\mu_0) = \kappa(\mu_0)\,\lambda_E^2(\mu_0)\,\omega_1\,\omega_2 f(\omega_1+\omega_2)\,.
\end{aligned}
$$

The factorized NLP corrections can then be parameterized by the local HQET parameters.

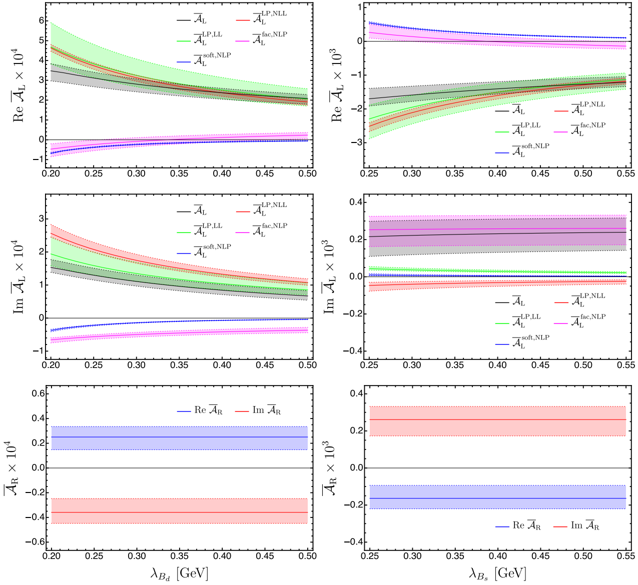
- The NLP soft contribution sensitive to the precise shape of $\phi_B^+(\omega,\mu)$.

$$
\phi_B^+(\omega,\mu_0) = \int_0^\infty ds\,\sqrt{w\,s}\,J_1(2\sqrt{w\,s})\,\eta_+(s,\mu_0)\,, \qquad \eta_+(s,\mu_0) = {}_1F_1(\alpha;\beta;-s\,\omega_0)\,.
$$

Such three-parameter ansatz [Beneke, Braun, Ji, Wei, 2018] is advantageous, since the resulting RG evolution can be done analytically in terms of ${}_2F_2$ functions.

Theory predictions for the helicity amplitudes

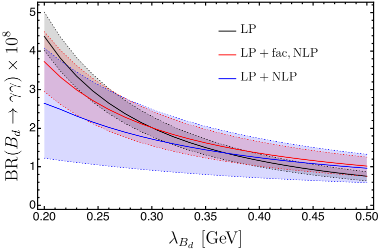
- Breakdown of the various QCD mechanisms:

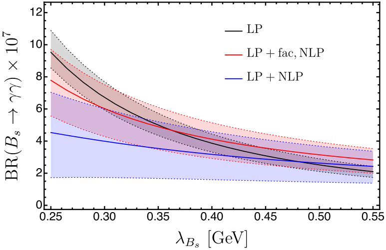


- NLL effects stabilize the factorization-scale dependence.
- Factorizable NLP effects around $\mathcal{O}(30\%)$.
- Destructive effects from the NLP soft corrections.
- Strong phase from the 2-loop matrix element of P_2^c and the weak annihilation.

Phenomenological observables for $B_q \to \gamma\gamma$

- Time-integrated branching fraction (for the flavour-tagged measurement):





- The yielding theory predictions

$$\mathcal{BR}(B_d \to \gamma\gamma) = \left(1.44^{+1.35}_{-0.80}\right) \times 10^{-8}, \qquad \mathcal{BR}(B_s \to \gamma\gamma) = \left(3.17^{+1.96}_{-1.74}\right) \times 10^{-7}.$$

  with the dominant uncertainties from λ_{B_q}, $\widehat{\sigma}^{(1)}_{B_q}$, $\widehat{\sigma}^{(2)}_{B_q}$ and the QCD renormalization scale ν.
- Both the factorizable and soft NLP corrections are numerically important.
- The ratio of the two branching ratios for $B_{d,s} \to \gamma\gamma$

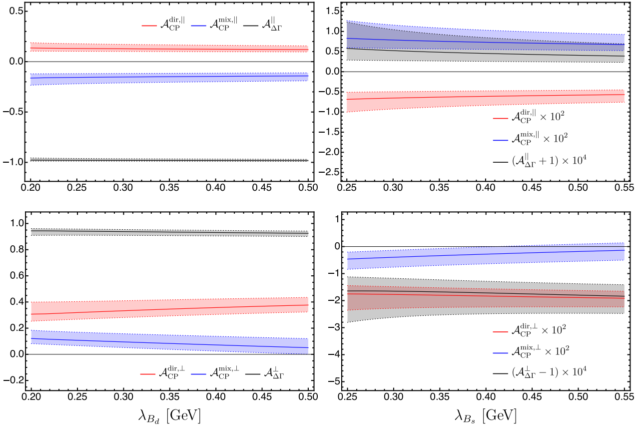
$$\frac{\mathcal{BR}(B_s \to \gamma\gamma)}{\mathcal{BR}(B_d \to \gamma\gamma)} = 33.80 \left(\frac{\lambda_{B_d}}{\lambda_{B_s}}\right)^2 + \mathcal{O}\left(\frac{\Lambda}{m_b}, \alpha_s\right).$$

  The λ_{B_q}-scaling violation effect due to the NLP contributions approximately $(10-20)\,\%$.

Phenomenological observables for $B_q \to \gamma\gamma$

- Time-dependent CP asymmetries:

$$A_{\rm CP}^{\chi}(t) = \frac{\bar{\Gamma}^{\chi}(\bar{B}_q(t) \to \gamma\gamma) - \Gamma^{\chi}(B_q(t) \to \gamma\gamma)}{\bar{\Gamma}^{\chi}(\bar{B}_q(t) \to \gamma\gamma) + \Gamma^{\chi}(B_q(t) \to \gamma\gamma)} = -\frac{\mathscr{A}_{\rm CP}^{\rm dir,\,\chi}\cos(\Delta m_q\, t) + \mathscr{A}_{\rm CP}^{\rm mix,\,\chi}\sin(\Delta m_q\, t)}{\cosh(\Delta\Gamma_q\, t/2) + \mathscr{A}_{\Delta\Gamma}^{\chi}\sinh(\Delta\Gamma_q\, t/2)}\,.$$



- $|\mathscr{A}_{\rm CP}^{\rm dir,\,\chi}|$ and $|\mathscr{A}_{\rm CP}^{\rm mix,\,\chi}|$ around $(10-40)\%$ for $B_d \to \gamma\gamma$.
- Tiny CP asymmetries for $B_s \to \gamma\gamma$ as expected.
- The difference between $\mathscr{A}_{\rm CP}^{\rm dir,\,\parallel}$ and $\mathscr{A}_{\rm CP}^{\rm dir,\,\perp}$ due to the NLP corrections.

Current status of $B \to \gamma \ell \bar{\nu}_\ell$ (for Belle II)

- Factorization properties at leading power [Korchemsky, Pirjol, Yan, 2000; Descotes-Genon, Sachrajda, 2002; Lunghi, Pirjol, Wyler, 2003; Bosch, Hill, Lange, Neubert, 2003].

- Leading power contributions at NLL and (partial)-subleading power corrections at tree level [Beneke, Rohrwild, 2011].

- Subleading power corrections from the dispersion technique:
  - Soft two-particle correction at tree level [Braun, Khodjamirian, 2013].
  - Soft two-particle correction at one loop [Wang, 2016].
  - Three-particle B-meson DA's contribution at tree level [Wang, 2016; Beneke et al, 2018].
  - Subleading effective current and twist-5 and 6 corrections at tree level. [Beneke et al, 2018].

- Subleading power corrections from the direct QCD approach:
  - Hadronic photon corrections at tree level up to the twist-4 accuracy [Khodjamirian, Stoll, Wyler, 1995; Ali, Braun, 1995; Eilam, Halperin, Mendel, 1995].
  - Hadronic photon corrections of twist-two at one loop and of higher-twist at tree level [Ball, Kou, 2003; Wang, Shen, 2018].

- Leading power contributions at NNLL and the updated NLP corrections:
  - Two-loop RG evolution of $\phi_B^+(\omega, \mu)$ derived in [Braun, Ji, Manashov, 2019].
  - Two-loop jet function obtained in [Liu, Neubert, 2020].
  - Further improvement on [Cui, Shen, Wang, Wang, Wei, 2021] will appear soon.

Theoretical wishlist

- Systematic understanding of the (high-twist) B-meson distribution amplitudes.
  - Renormalization properties beyond the one-loop approximation [conformal symmetry].
  - Perturbative constraints at large ω_i [OPE technique].
  - Renormalon analysis and the renormalization-scheme dependence.
  - Precision determinations of the inverse moment λ_B.

- QCD factorization for the subleading power corrections.
  - SCET analysis for the pion-photon form factor as the first step [operator structures, symmetry constraints, etc].
  - General treatment of the rapidity divergences in the (naïve)-factorization formulae.
  - Rigorous factorization proof taking into account the Glauber gluons.
  - Novel resummation techniques for enhanced logarithms.

- Technical issues for future improvements.
  - Factorization techniques for the electromagnetic corrections.
  - NNLO QCD computations for $B \to V\gamma$ and $B \to V\ell\ell$.
  - QCD factorization for the radiative and electroweak penguin decays of the Λ_b-baryon.
  - Improved understanding of the parton-hadron duality violation.

- Very promising future for QCD aspects of heavy-quark physics!