

Parton Distribution Functions Matching in LaMET

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微扰量子场论研讨会 上海
2021年5月14日-16日

基于如下的工作

- Next-to-next-to-leading order corrections to non-singlet quark Quasi distribution functions
L.-B. Chen(陈龙斌), *W. Wang*(王伟), *R. Zhu*(朱瑞林),
*Phys.Rev.Lett.*126,072002(2021).
- Master Integrals for two-loop QCD corrections to Quasi PDFs
*L.-B. Chen, W. Wang, R. Zhu, JHEP*10,079(2020).
- Quasi parton distribution functions at NNLO: flavor non-diagonal quark contributions
*L.-B. Chen, W. Wang, R. Zhu, Phys.Rev.D*102,011503(2020).

部分子分布函数(PDFs)

● 最基本的物理量之一

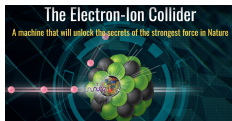
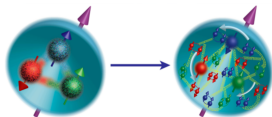
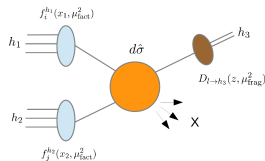
- 预测强子散射截面的一个基本输入量
- 长程关联
- 非微扰相互作用

● 一个重要问题

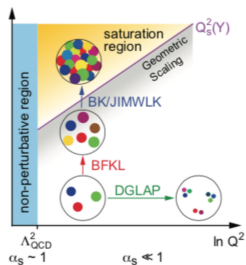
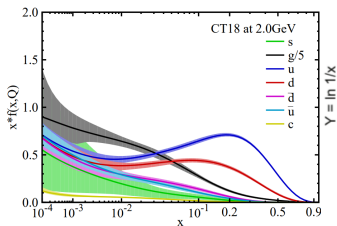
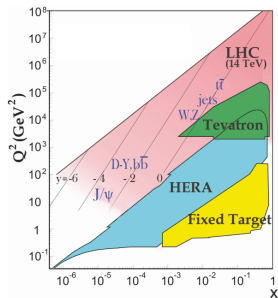
- 部分子在强子内部是如何分布的? (1-D/3-D/5-D 图像)

● 一个重要物理目标

EIC, [1602.03922](#), EICC, [2102.09222](#)



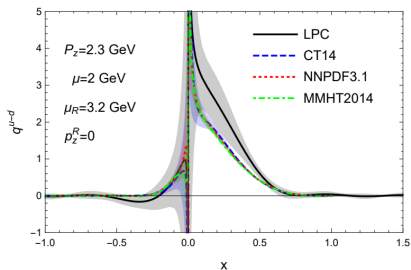
唯象模型PDFs



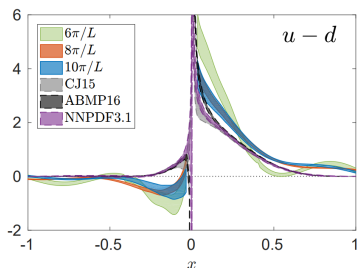
PDG2018 and 1912.10053, also see Tie-Jium Hou's talk

在各种高能强子对撞机上，经过50多年的实验测量和拟合，我们对部分子分布函数(1D)有了较为扎实的唯一认识。另一方面我们对部分子分布函数的微扰性质也有了比较深的认识。部分子分布函数是光锥关联的物理量，无法通过格点QCD第一性原理直接计算。

格点QCD+大动量有效理论计算PDFs



LPC, PRD101, 034020 (2020)



ETMC, PRL121, 112001 (2018)

目前是QCD第一性原理计算部分子分布函数的一个极具吸引力和活跃的方向 *Recent progresses see in Jian-Hui Zhang's talk*

主要内容

- 1 Large Momentum Effective Theory Introduction
- 2 Two Loop calculation
 - Two Loop Feynman Diagrams and Amplitudes
 - Calculation of Master Integrals by Differential Equations
- 3 NNLO matching for quark case
 - $q \rightarrow q'$ case
 - $q \rightarrow q$ case
 - Extraction of PDF with NNLO Matching Coefficients

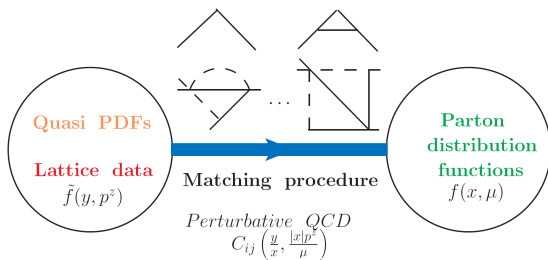
大动量有效理论(LaMET)

- LaMET 因子化公式

$$\tilde{f}_{i/H}(y, p^z) = \int_{-1}^1 \frac{dx}{|x|} \left[C_{ij} \left(\frac{y}{x}, \frac{|x|p^z}{\mu} \right) f_{j/H}(x, \mu) \right] + \mathcal{O} \left(\frac{m_h^2}{p^{z2}}, \frac{\Lambda_{\text{QCD}}^2}{p^{z2}} \right)$$

$$x \in [-1, 1], y \in [-\infty, \infty]$$

X. Ji, *PRL110,262002 (2013)*, ...



微扰论计算 $C_{ij}^{(0)}$, $C_{ij}^{(1)}$, $C_{ij}^{(2)}$, ...

- $C_{ij} = C_{ij}^{(0)} + \frac{\alpha_s}{2\pi} C_{ij}^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 C_{ij}^{(2)} + \dots$

- 领头阶(LO): $C_{ij}^{(0)}(y) = \delta(1-y)$

- 高阶匹配, 重整化方案依赖

- 次领头阶(NLO) $C_{ij}^{(1)}(y, \frac{p^z}{\mu})$

MS: Izubuchi, Ji, Jin, Stewart, Zhao, 1801.03917;

MMS: Alexandrou, Cichy, Constantinou, Jansen, Scapellato, Steffens, 1803.02685;

RI/MOM: Stewart, Zhao, 1709.04933; Wang, Zhang, Zhao, Zhu, 1904.00978;

Others: Ji, Xiong, Zhang, Zhao, 1310.7471; Ma, Qiu, 1404.6860, Wang, Zhao, Zhu, 1708.02458, ...

y 有三个区间: $[-\infty, 0]$, $[0, 1]$, $[1, +\infty]$, 色因子: C_F

- 次次领头阶(NNLO) $C_{ij}^{(2)}$ (done only for quark case)

- QCD中的高阶修正很重要

- $\mu = 2\text{GeV}$, $\alpha_s(\mu = 2\text{GeV}) \sim 0.3$, 对于精确预测, α_s^2 阶修正不能忽略

- NNLO因子化证明 **nontrivial**

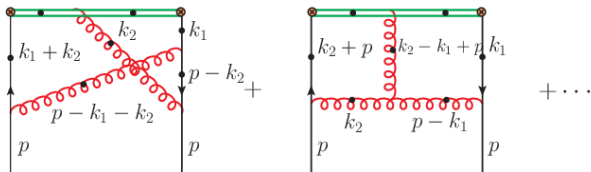
- Li, Ma, Qiu 2020; **Chen, Wang, Zhu 2020**

Outline

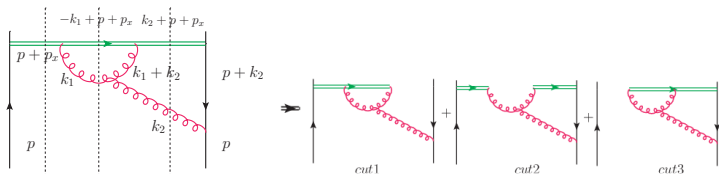
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两圈费曼图和费曼振幅

- 高阶修正产生大量费曼图(79⁺ at NNLO)
- 我们把Wilson线处理为辅助夸克场(with linear propagator)
- 使用FeynRules 和FeynArts自动生成两圈费曼图和费曼振幅
Christensen et al, 1310.1921, T. Hahn, 0012260



Feynman规范下的一个例子(I)



- 从辅助场回到Wilson线相互作用需要做切割(cut). 对于cut1, 我们有 $p_x = -p - k_2$

$$\mathcal{M}|_{cut1} = \mu^{4\epsilon} \int \int \frac{d^{4-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \text{amp}_{cut1} \times \delta(k_2^z + p^z - yp^z),$$

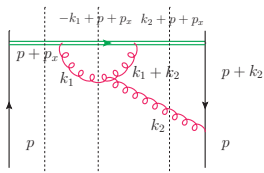
- 对于cut2, 我们有 $p_x = -p + k_1$; 两者给出实贡献

$$\mathcal{M}|_{cut2} = \mu^{4\epsilon} \int \int \frac{d^{4-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \text{amp}_{cut2} \times \delta(-k_1^z + p^z - yp^z),$$

- 对于cut3, 我们有 $p_x = -p$, 给出虚贡献

$$\mathcal{M}|_{cut3} = \mu^{4\epsilon} \int \int \frac{d^{4-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \text{amp}_{cut3} \times \delta(1 - y),$$

Feynman规范下的一个例子(II)



- 恒等式

$$\frac{1}{k_1 \cdot n k_2 \cdot n} = \frac{1}{(k_1 \cdot n + k_2 \cdot n) k_2 \cdot n} + \frac{1}{k_1 \cdot n (k_1 \cdot n + k_2 \cdot n)},$$

- 通过动量平移, 得到

$$\begin{aligned} \mathcal{M}|_{\text{cut1+cut2+cut3}} &= \left[\mu^{4\epsilon} \int \int \frac{d^{3-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \text{ampcut1}' \Big|_{k_1^z = y p_z} \right]_+ \\ &+ \left[\mu^{4\epsilon} \int \int \frac{d^{3-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \text{ampcut2}' \Big|_{k_1^z = y p_z} \right]_+ \end{aligned}$$

- 既包含实图贡献, 也包括了虚图贡献

Cutkosky 等式

- Cutkosky 等式, *J.Math.Phys.* 1,429(1960)

$$\delta(k_z - xp_z) = \frac{1}{2\pi i} \left(\frac{1}{k_z - xp_z - i0} - \frac{1}{k_z - xp_z + i0} \right).$$

- 所有的实图积分变成了协变积分
- 实图积分的实部 \Rightarrow 协变积分的虚部

$$\begin{aligned} & \mu^{4\epsilon} \int \int \frac{d^{3-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \text{amputcut1} |_{k_1^z = yp_z} \\ \Rightarrow & \text{Im} \left[\mu^{4\epsilon} \int \int \frac{d^{4-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \frac{\text{amputcut1}}{4\pi(k_1^z - yp_z)} \right] \end{aligned}$$

然后开始进行分部积分(IBP) 约化

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微分方程计算主积分

- 对主积分 f_i 按照某一个洛伦兹不变量 z 求导, 建立微分方程, 比如可以设置 $z = \frac{p^0}{p^z}$
- 当独立的主积分数量大于1, 系数 A 是一个 $n \times n$ 矩阵, 依赖于洛伦兹不变量 z 和正规化参数 ϵ

$$\frac{d}{dz} \begin{pmatrix} f_1(z, \epsilon) \\ \vdots \\ f_n(z, \epsilon) \end{pmatrix} = \begin{pmatrix} A_{11}(z, \epsilon) & \dots & A_{1n}(z, \epsilon) \\ \vdots & & \vdots \\ A_{n1}(z, \epsilon) & \dots & A_{nn}(z, \epsilon) \end{pmatrix} \begin{pmatrix} f_1(z, \epsilon) \\ \vdots \\ f_n(z, \epsilon) \end{pmatrix}$$

A. V. Kotikov, PLB254, 158(1991); PLB267, 123(1991)

- 然而对于 f_i 多元一次微分方程, 边界条件不太容易确定

一类特殊的主积分基：正则基

$$\frac{d}{dz} \begin{pmatrix} g_1(z; \epsilon) \\ \vdots \\ g_n(z; \epsilon) \end{pmatrix} = \epsilon \begin{pmatrix} B_{11}(z) & \dots & B_{1n}(z) \\ \vdots & & \vdots \\ B_{n1}(z) & \dots & B_{nn}(z) \end{pmatrix} \begin{pmatrix} g_1(z; \epsilon) \\ \vdots \\ g_n(z; \epsilon) \end{pmatrix}$$

其中

$$\vec{g} = T^{-1} \vec{f}$$

$$B = T^{-1} A T - T^{-1} \partial_z T$$

- 适用于维数正规化方案 $D = 4 - 2\epsilon$
- 任一主积分基可以通过线性变换到主积分基
- 系数矩阵 B 将只依赖于洛伦兹不变量 z

J. M. Henn, PRL110, 251601 (2013)

计算中采用的主积分基

- 对于夸克PDFs, 需要计算三组主积分基
- 比如第一组主积分基

$$I_{n_i}^1 = \int \int \frac{\mathcal{D}^D k_1 \mathcal{D}^D k_2}{(k_1^2)^{n_1} (k_2^2)^{n_2} ((k_2 - p)^2)^{n_3} ((k_1 + k_2)^2)^{n_4} ((k_1 + k_2 - p)^2)^{n_5}} \frac{1}{\left(\frac{1}{(P_1 + i0)^{n_6}} - \frac{1}{(P_1 - i0)^{n_6}} \right)} \frac{1}{4\pi i} \left(\frac{1}{(Q_1 + i0)^{n_7}} + \frac{1}{(Q_1 - i0)^{n_7}} \right),$$

其中线性传播子为

$$P_1 = n \cdot k_1 + y n \cdot p, \quad Q_1 = n \cdot k_2,$$

积分测度为

$$\mathcal{D}^D k_i = \frac{1}{i\pi^{D/2} e^{-\frac{4-D}{2}\gamma_E}} \left(\frac{p_z^2}{\mu^2} \right)^{\left(\frac{4-D}{2}\right)} d^D k_i,$$

确定边界条件

- 直接计算(Feynman/Alpha 参数化方法, 留数定理, ...)

如 $g_3^1 = \epsilon(y-1)p_z l_{0,2,0,0,2,1,0}^1$,

$$g_3^1 = \text{Sgn}(y-1) \left(-2 + \epsilon [4 \ln(4(y-1)^2)] \right. \\ \left. - \frac{1}{3} \epsilon^2 [12 \ln((y-1)^2) \ln(16(y-1)^2) + 5\pi^2 + 12 \ln^2(4)] + \mathcal{O}(\epsilon^3) \right).$$

- 正规条件

$$\frac{\partial g_7^1}{\partial z} = \frac{\epsilon}{4} \left[\frac{8g_7^1}{z} - \frac{6g_7^1 - g_8^1}{z-2y+1} - \frac{6g_7^1 + g_8^1}{z+2y-1} \right. \\ \left. + \frac{2g_3^1 - 6g_7^1 + g_8^1}{z-1} - \frac{2g_3^1 + 6g_7^1 + g_8^1}{z+1} \right].$$

对于 y 位于任一区间, 有正规点 $z = \frac{p_0}{p_z} = 0$ ($p^2 = -p_z^2$), 从而得到 $g_7^1|_{z=0} = 0$. 此外我们还有其它类似的正规点, 如 $z = 2y - 1$ ($0 < y < 1$), $z = 1$ ($y > 1$ or $y < 0$)

主积分的数值检验

- 运用FIESTA程序包，可以对解析计算的主积分进行数值检验

A. V. Smirnov, 1511.03614

$$\boxed{\frac{p^2}{p_z^2} = -\frac{1}{2}, \quad y = \frac{1}{3}, \quad \frac{p_z}{\mu} = 1}$$

Analytic:

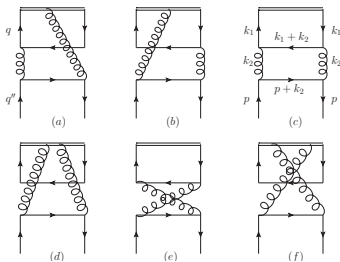
$$I_{1,1,0,0,2,1,0}^1 = \frac{-2.492900960}{\epsilon} + 0.4498613241 + \epsilon(-21.287203876),$$

FIESTA:

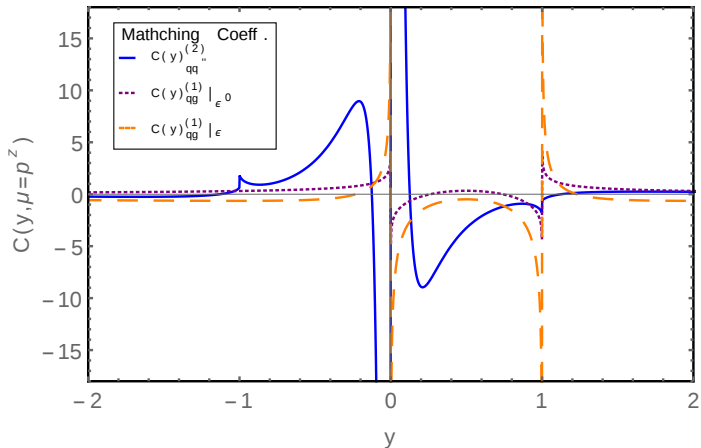
$$I_{1,1,0,0,2,1,0}^1 = \frac{-2.49290 \pm 0.0000652}{\epsilon} + 0.449836 \pm 0.000847 + \epsilon(-21.2872 \pm 0.004169).$$

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$q \rightarrow q'$ 情况 $q \rightarrow q'$ 费曼图

$$\begin{aligned}
 \tilde{f}_{q/q'}^{(2)}\left(y, \frac{p^z}{\mu}, \epsilon_{\text{IR}}\right) &= C_{qq''}^{(2)}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f_{q''/q'}^{(0)}(x, \epsilon_{\text{IR}}) \\
 &+ C_{qq''}^{(1)}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f_{q''/q'}^{(1)}(x, \epsilon_{\text{IR}}) \\
 &+ C_{qq''}^{(0)}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f_{q''/q'}^{(2)}(x, \epsilon_{\text{IR}}).
 \end{aligned}$$

$q \rightarrow q'$ 匹配系数 $q \rightarrow q'$ 双圈匹配系数

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$q \rightarrow q$ 情况, 既包含紫外又有红外发散

● 重整化方程

$$\tilde{f}(y, \frac{p^z}{\mu}, \epsilon_{\text{IR}}) = \int \frac{dy_1}{|y_1|} \left[Z_q \tilde{Z} \left(\frac{y}{y_1} \right) \right] \left[Z_q^{-1} \tilde{f} \left(y_1, \frac{p^z}{\mu}, \epsilon \right) \right].$$

这里 Z_q 是夸克重整化常数, \tilde{Z} 是准分布函数重整化常数

$$\tilde{Z}(\xi) = \delta(1-\xi) \left(1 + \frac{\alpha_s}{2\pi} \frac{\tilde{Z}^{(1)}}{\epsilon_{UV}} + \left(\frac{\alpha_s}{2\pi} \right)^2 \frac{\tilde{Z}^{(2)}}{\epsilon_{UV}^2} \right),$$

$$\tilde{Z}^{(1)} = -\frac{3C_F S_\epsilon}{2}, \quad \tilde{Z}^{(2)} = S_\epsilon^2 \left(\frac{a + 9C_F^2}{4} + \frac{b}{4}\epsilon \right)$$

X. Ji and J.H. Zhang, 1505.07699;

Braun, Chetyrkin and Kniehl, 2004.01043

准分布函数的红外行为

- 软发散严格相消
- 剩下的共线发散分两种，一种是“Reducible”

$$\tilde{f}_{q/q}^{(2)}\left(y, \frac{p^z}{\mu}, \epsilon_{\text{IR}}\right)|_{\text{div. part. 1}} = C_{qq}^{(1)}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes \left[-\frac{(1+x^2)}{(1-x)}\right]_+ \frac{1}{\epsilon_{\text{IR}}}.$$

- 另一种是“Irreducible”

与光锥分布函数严格相等，包含有 $\frac{1}{\epsilon_{\text{IR}}}$ 和 $\left(\frac{1}{\epsilon_{\text{IR}}}\right)^2$ 发散

$$\tilde{f}_{i/j}^{(2)}\left(y, \frac{p^z}{\mu}, \epsilon_{\text{IR}}\right)|_{\text{div. part. 2}} = f_{i/j}^{(2)}(x, \epsilon_{\text{IR}}).$$

$$f_{i/j}^{(2)}(x) = \frac{1}{2\epsilon_{\text{IR}}^2} \left[\sum_k P_{ik}^{(0)}(z) \otimes P_{kj}^{(0)}(x) + \beta_0 P_{ij}^{(0)}(z) \right] - \frac{P_{ij}^{(1)}(x)}{\epsilon_{\text{IR}}}$$

两圈水平下因子化公式

- 将重整化后的准分布函数，匹配到最小减除方案下的光锥分布函数：

$$\begin{aligned}\tilde{f}_{i/k}^{(0)}\left(y, \frac{p^z}{\mu}\right) &= C_{ij}^{(0)}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f_{j/k}^{(0)}(x), \\ \tilde{f}_{i/k}^{(1)}\left(y, \frac{p^z}{\mu}, \epsilon_{\text{IR}}\right) &= C_{ij}^{(1)}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f_{j/k}^{(0)}(x) \\ &\quad + C_{ij}^{(0)}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f_{j/k}^{(1)}(x, \epsilon_{\text{IR}}), \\ \tilde{f}_{i/k}^{(2)}\left(y, \frac{p^z}{\mu}, \epsilon_{\text{IR}}\right) &= C_{ij}^{(2)}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f_{j/k}^{(0)}(x) \\ &\quad + C_{ij}^{(1)}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f_{j/k}^{(1)}(x, \epsilon_{\text{IR}}) \\ &\quad + C_{ij}^{(0)}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f_{j/k}^{(2)}(x, \epsilon_{\text{IR}}).\end{aligned}$$

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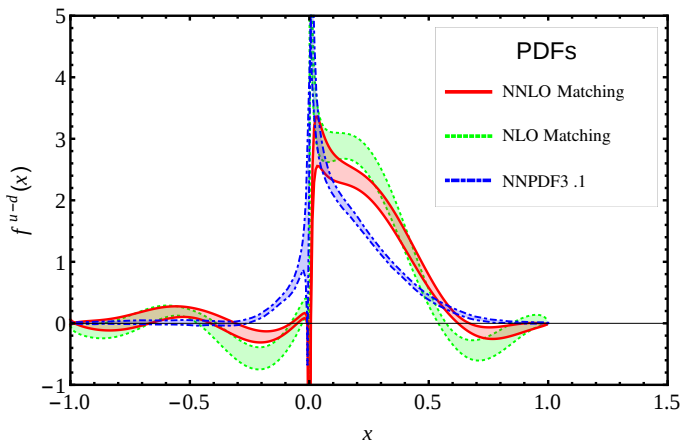
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NNLO阶匹配系数 $C_{qq}^{(2)}$

- 与Li-Ma-Qiu在 $\overline{\text{MS}}$ 方案下计算的结果一致
Phys. Rev. Lett. **126**, 072001 (2021)
- 同时我们也得到了RI/MOM和 $\overline{\text{MMS}}$ 方案下的NNLO阶匹配系数 $C_{qq}^{(2)}(y, \frac{p^z}{\mu})$
- 对于准分布动量分数 y , 我们有四个区间, 色结构有三种($C_F, C_A, nf T_F$) C_F
- 在动量分数 y 趋于无穷大的行为为 $C_{qq}^{(2), \overline{\text{MMS}}}|_{y \rightarrow \infty} \propto \frac{1}{y^2}$

$$\begin{aligned}
 & C_{qq}^{(2), \overline{\text{MMS}}}(y, \frac{p^z}{\mu}) \\
 &= [C_{qq}^{(2), \overline{\text{MMS}}}(y, \frac{p^z}{\mu})|_{y>1}]_+ + [C_{qq}^{(2), \overline{\text{MMS}}}(y, \frac{p^z}{\mu})|_{0<y<1}]_+ \\
 &+ [C_{qq}^{(2), \overline{\text{MMS}}}(y, \frac{p^z}{\mu})|_{-1<y<0}]_+ + [C_{qq}^{(2), \overline{\text{MMS}}}(y, \frac{p^z}{\mu})|_{y<-1}]_+
 \end{aligned}$$

NNLO阶匹配, 抽取PDFs



采用了ETMC格点组 \overline{MS} 重整化方案的数据, 其中 $z_{cut} = 10a$, $\mu = 2\text{GeV}$, 不确定性来自于格点QCD数据

总结

- 高阶匹配计算无论在形式理论还是实践应用上都比较重要
- 在NNLO，我们利用格点数据，模型无关地得到了非单态夸克的光锥分布函数，NNLO贡献一定程度缓解PDFs震荡行为
- 红外发散相消得到了证明，进而在两圈水平下验证了LaMET因子化公式

- 展望
 - 胶子准分布函数的NNLO计算
 - π 介子准分布振幅的NNLO计算
 - 在NNLO精度下，模型无关地抽取其它光锥关联函数

谢谢各位老师 and 同学！

Back up

Canonical bases for family-1 MIs, here only list 10 of them

$$g_1^1 = \epsilon(y+1)p_z I_{0,0,2,2,0,1,0}^1,$$

$$g_2^1 = \epsilon y p_z I_{0,2,0,2,0,1,0}^1,$$

$$g_3^1 = \epsilon(y-1)p_z I_{0,2,0,0,2,1,0}^1,$$

$$g_4^1 = \epsilon y p_z p_1^2 I_{2,2,1,0,0,1,0}^1,$$

$$g_5^1 = \epsilon^2 \sqrt{p_1^2 + p_z^2} I_{0,1,1,0,2,1,0}^1,$$

$$g_6^1 = \epsilon(p_1^2 - 4y(y-1)p_z^2) I_{0,1,1,0,2,2,0}^1 + 8\epsilon^2(2y-1)p_z I_{0,1,1,0,2,1,0}^1 \\ + \epsilon(y-1)p_z I_{0,2,0,0,2,1,0}^1 + \epsilon y p_z I_{0,2,0,2,0,1,0}^1,$$

$$g_7^1 = \epsilon^2 \sqrt{p_1^2 + p_z^2} I_{1,1,0,0,2,1,0}^1,$$

$$g_8^1 = \epsilon(p_1^2 - 4y(y-1)p_z^2) I_{1,1,0,0,2,2,0}^1 + 6\epsilon^2(2y-1)p_z I_{1,1,0,0,2,1,0}^1,$$

$$g_9^1 = \epsilon^2 \sqrt{p_1^2 + p_z^2} I_{0,1,1,2,0,1,0}^1,$$

$$g_{10}^1 = \epsilon(p_1^2 - 4y(y+1)p_z^2) I_{0,1,1,2,0,2,0}^1 + 8\epsilon^2(2y+1)p_z I_{0,1,1,2,0,1,0}^1$$

Back up

NLO matching coefficient in \overline{MS} scheme

$$\begin{aligned}
& C_{qq}^{(1),\overline{MS}}\left(y, \frac{\mu}{p^z}\right) \Big|_{\mathcal{O}(\epsilon^0)} \\
&= C_F \begin{cases} \left[\frac{1+y^2}{1-y} \ln \frac{y}{y-1} + 1 + \frac{3}{2y} \right]_{+(1)}^{[1,\infty]} - \frac{3}{2y}, & y > 1 \\ \left[\frac{1+y^2}{1-y} \left(-\ln \frac{\mu^2}{4p^{z2}} + \ln(y(1-y)) \right) - \frac{y(1+y)}{1-y} \right]_{+(1)}^{[0,1]}, & 0 < y < 1 \\ \left[-\frac{1+y^2}{1-y} \ln \frac{y}{y-1} - 1 + \frac{3}{2(1-y)} \right]_{+(1)}^{[-\infty,0]} - \frac{3}{2(1-y)}, & y < 0 \end{cases} \\
&+ C_F \left[\delta(1-y) \left(\frac{3}{2} \ln \frac{\mu^2}{4p^{z2}} + \frac{5}{2} \right) \right]
\end{aligned}$$

NLO matching coefficient in $M\overline{MS}$ scheme

$$C\left(\xi, \frac{\xi\mu}{xP_3}\right) = \delta(1-\xi) + \frac{\alpha_s}{2\pi} C_F \begin{cases} \left[\frac{1+\xi^2}{1-\xi} \ln \frac{\xi}{\xi-1} + 1 + \frac{3}{2\xi} \right]_+, & \xi > 1, \\ \left[\frac{1+\xi^2}{1-\xi} \ln \frac{x^2 P_3^2}{\xi^2 \mu^2} (4\xi(1-\xi)) - \frac{\xi(1+\xi)}{1-\xi} + 2\mu(1-\xi) \right]_+, & 0 < \xi < 1, \\ \left[-\frac{1+\xi^2}{1-\xi} \ln \frac{\xi}{\xi-1} - 1 + \frac{3}{2(1-\xi)} \right]_+, & \xi < 0, \end{cases}$$

Back up

One loop quasi pdf in RI/MOM scheme

$$\tilde{f}_{q/q,\xi}^{(1)}(y, \rho^2)/C_F = \begin{cases} -\frac{((\xi-2)\rho+2y^2-(\xi+1)\rho y+2) \log\left(\frac{\sqrt{1-\rho}+2y-1}{-\sqrt{1-\rho}+2y-1}\right)}{2(1-\rho)^{3/2}(y-1)} + \frac{\xi\rho(\rho(\rho+1)+16y^4-40y^3+4(\rho+8)y^2-2(3\rho+4)y)}{2(\rho-1)(y-1)(\rho+4y^2-4y)^2} \\ + \frac{y((4-5\rho)\rho-16y^4+8(\rho+3)y^3-24\rho y^2+4(\rho^2+3\rho-2)y)}{(\rho-1)(y-1)(\rho+4y^2-4y)^2} + \frac{3}{2-2y}, & y > 1 \\ -\frac{\xi\rho+\xi-3\rho+4y^2-2(\xi+1)y+3}{2(\rho-1)(y-1)} - \frac{\log\left(\frac{\sqrt{1-\rho}+1}{1-\sqrt{1-\rho}}\right)((\xi-2)\rho+2y^2-(\xi+1)\rho y+2)}{2(1-\rho)^{3/2}(y-1)}, & 0 < y < 1, \\ \frac{((\xi-2)\rho+2y^2-(\xi+1)\rho y+2) \log\left(\frac{\sqrt{1-\rho}+2y-1}{-\sqrt{1-\rho}+2y-1}\right)}{2(1-\rho)^{3/2}(y-1)} - \frac{\xi\rho(\rho(\rho+1)+16y^4-40y^3+4(\rho+8)y^2-2(3\rho+4)y)}{2(\rho-1)(y-1)(\rho+4y^2-4y)^2} \\ - \frac{y((4-5\rho)\rho-16y^4+8(\rho+3)y^3-24\rho y^2+4(\rho^2+3\rho-2)y)}{(\rho-1)(y-1)(\rho+4y^2-4y)^2} - \frac{3}{2-2y}, & y < 0. \end{cases}$$

$$\log\left(\frac{\sqrt{1-\rho}+1}{1-\sqrt{1-\rho}}\right) = -\log\left(\frac{1-\sqrt{z^2}}{\sqrt{2-z^2}+1}\right), \quad \log\left(\frac{\sqrt{1-\rho}+2y-1}{-\sqrt{1-\rho}+2y-1}\right) = -\log\left(\frac{2y-\sqrt{z^2}-1}{2y+\sqrt{2-z^2}-1}\right)$$

$$\int dy[h(y)]_+g(y) = \int dyh(y)[g(y) - g(1)]$$

$$iD_{\xi}^{\mu\nu}(k) = -\frac{i}{k^2} \left[g^{\mu\nu} - (1-\xi) \frac{k^{\mu}k^{\nu}}{k^2} \right].$$

Back up

NNLO matching coefficients

$$C_{qq}^{(2),\text{MMS}}(y, \frac{p^z}{\mu})|_{y>1} = C_{qq}^{(2),\text{MS}}(y, \frac{p^z}{\mu})|_{y>1} + \frac{C_F(11C_A + 9C_F - 2n_f)}{4y} \log\left(\frac{\mu^2}{p_z^2}\right) - \frac{C_F n_f(5 - 4\log(2y))}{4y} \\ - \frac{C_A C_F(132\log(2y) + 4\pi^2 - 159)}{24y} + \frac{C_F^2(-108\log(2y) + 16\pi^2 + 75)}{24y},$$

$$C_{qq}^{(2),\text{MMS}}(y, \frac{p^z}{\mu})|_{0<y<1} = C_{qq}^{(2),\text{MS}}(y, \frac{p^z}{\mu})|_{0<y<1},$$

$$C_{qq}^{(2),\text{MMS}}(y, \frac{p^z}{\mu})|_{-1<y<0} = C_{qq}^{(2),\text{MS}}(y, \frac{p^z}{\mu})|_{-1<y<0},$$

$$C_{qq}^{(2),\text{MMS}}(y, \frac{p^z}{\mu})|_{y<-1} = C_{qq}^{(2),\text{MS}}(y, \frac{p^z}{\mu})|_{y<-1} + \frac{C_F(11C_A + 9C_F - 2n_f)}{4(1-y)} \log\left(\frac{\mu^2}{p_z^2}\right) - \frac{C_F n_f(5 - 4\log(-2y))}{4(1-y)} \\ - \frac{C_A C_F(132\log(-2y) + 4\pi^2 - 159)}{24(1-y)} + \frac{C_F^2(-108\log(-2y) + 16\pi^2 + 75)}{24(1-y)}.$$

$$C_{qq}^{(2),\text{MS}}(y, \frac{p^z}{\mu})|_{y>1} = \left(C_F c_1^{C_F} + C_A c_1^{C_A} + 2T_F n_f c_1^{T_F}\right) C_F + (\Gamma_1(y)|_{y>1}) \log\left(\frac{\mu^2}{p_z^2}\right),$$

$$C_{qq}^{(2),\text{MS}}(y, \frac{p^z}{\mu})|_{0<y<1} = C_F \left(C_F c_2^{C_F} + C_A c_2^{C_A} + 2T_F n_f c_2^{T_F}\right) + (\Gamma_2(y)) \log^2\left(\frac{\mu^2}{p_z^2}\right) \\ + \left((\Gamma_1(y)|_{0<y<1}) - (P_{qq}^{(1),V}(y)|_{0<y<1})\right) \log\left(\frac{\mu^2}{p_z^2}\right),$$

$$C_{qq}^{(2),\text{MS}}(y, \frac{p^z}{\mu})|_{-1<y<0} = C_F \left(C_F c_3^{C_F} + C_A c_3^{C_A} + 2T_F n_f c_3^{T_F}\right) + \left((\Gamma_1(y)|_{-1<y<0}) - P_{qq}^{(1),V}(-y)\right) \log\left(\frac{\mu^2}{p_z^2}\right),$$

$$C_{qq}^{(2),\text{MS}}(y, \frac{p^z}{\mu})|_{y<-1} = -C_{qq}^{(2),\text{MS}}(y, \frac{p^z}{\mu})|_{y>1}.$$

The complete analytic expression can be found in our paper.   