

Some results of one-loop reduction

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- The perturbative calculation of scattering amplitude is crucial for higher energy physics. using Feynman diagrams.
- The tradition way to do the calculation is to use the Feynman diagrams, but it is well known now, this method is not efficient in many situations.
- In last thirty years, various techniques have been developed to speed the computation. Now one-loop computation is considered as solved problem and the frontier is the two loop and higher, as we will hear a lot in this workshop.
- However, in this talk, I will discuss some problems left in the one-loop calculation.

Some efficient one-loop computation algorithms:

- **OPP method:** [Ossola, Papadopoulos, Pittau, 2006]
- **Unitarity cut method:** [Bern, Dixon, Dunbar, Kosower, 1994][Britto, Buchbinder, Cachazo, B.F, 2005] [C. Anastasiou, R. Britto, B.F, Z. Kunszt, P. Mastrolia, 2006]
- **Forde's method:** [D. Forde, 2007]
- **Generalized OPP method:** [R.K. Ellis, W.T. Giele, Z. Kunszt, 2007]
- **ACK method:** [N. Arkani-Hammed, F. Cachazo, J. Kaplan, 2008]

- For one-loop computation, the well established method is the reduction method.
- Now we are all known that the reduction can be divided into two categories: the reduction at the integrand level and the reduction at the integral level.
- The reduction at the integrand level is nothing, but division and separation of polynomial, for which the powerful mathematical tool is the "computational algebraic geometry".

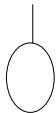
- One well known algorithm for reduction at the integrand level is the OPP method.
- OPP method has the advantage that it is easy to be implemented into program, both numerically and analytically.
- The disadvantage of OPP method is that we need to compute coefficients of spurious terms, although they do not contribute at the integral level. For practical applications, it is not a big problem since for the renormalizable theories, the spurious terms are not so much.

- However, from theoretical point of view, it is not satisfied, since the number of spurious terms increasing with the increasing of power of ℓ in numerator. Thus for arbitrary higher and higher power in numerator, there are more and more terms to be calculated, and the efficiency will be lost.

- For the reduction at the integral level, the typical algorithm is the celebrated PV-reduction method.
- For this method, we need to calculate the coefficients of masters only and the spurious terms will never show up.
- Although the algorithm of the original PV-reduction method is clear, its implement is not so easy.
- A better realization of reduction at the integral level is the **Unitarity cut method**.

PV reduction

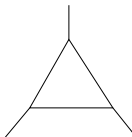
- The most basis are given by



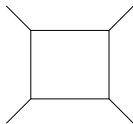
tadpole



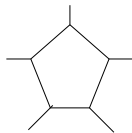
bubble



triangle



box



pentagon

- For massless inner line, there is no tadpole and massless bubble.

Unitary cut

Some facts regarding the one-loop amplitudes:

- The singular behavior of one-loop amplitudes is much more complicated than the tree-level: **we have branch cuts as well as higher dimension singular surface.**
- Under the expansion into basis, all branch cuts are given by scalar basis while **coefficients are rational functions.**
- Applying above observation we have unitarity cut method: taking imaginary part at **both sides** $\text{Im}(I) = \sum_i c_i \text{Im}(I_i)$ and comparing both sides we can get c_i if each $\text{Im}(I_i)$ is unique.

Unitary cut

- The good point for this method is that the input is the multiplication of **on-shell tree-level amplitudes** of both sides. Especially when we combine the BCFW recursion relation.
- The difficulty is how to evaluate $\text{Im}(I)$? This is solved by holomorphic anomaly: **reducing integration into reading out residues of poles**
[Cachazo, Svrcek, Witten, 2004] [Britto, Buchbinder, Cachazo, Feng, 2005]
- Current status: Now we have **well defined algebraic steps to extract coefficients from tree-level input.**

- Example: Triangle

$$\begin{aligned}
 & \text{Tri}[K_s, K] \\
 = & \frac{1}{2} \frac{(K^2)^{N+1}}{(-\beta\sqrt{1-u})^{N+1} (\sqrt{-4q_s^2 K^2})^{N+1}} \frac{1}{(N+1)! \langle P_{s,1} P_{s,2} \rangle^{N+1}} \\
 & \frac{d^{N+1}}{d\tau^{N+1}} \left(\frac{\langle \ell | K | \ell \rangle^{N+1}}{(K^2)^{N+1}} \mathcal{T}^{(N)}(\tilde{\ell}) \cdot D_s(\tilde{\ell}) \left\{ \begin{array}{l} | \ell \rangle \rightarrow | Q_s(u) | \ell \rangle \\ | \ell \rangle \rightarrow | P_{s,1} - \tau P_{s,2} \rangle \end{array} \right. \right. \\
 & \left. \left. + \{ P_{s,1} \leftrightarrow P_{s,2} \} \right) \Big|_{\tau \rightarrow 0}
 \end{aligned}$$

- Advantage: (1) we can get the wanted coefficients without calculating the spurious terms; (2) we can deal with arbitrary higher power in numerator.

However, there are some unsatisfied parts of unitarity cut method. In this talk we will discuss following two aspects:

- (A) The unitarity cut for higher poles
- (B) The tadpole coefficients

We consider the reduction of

$$\mathcal{M}[\ell] \equiv \int \frac{d^D \ell}{(2\pi)^{D/2}} \frac{\mathcal{N}[\ell]}{\prod_{j=1}^n ((\ell - K_j)^2 - m_j^2 + i\epsilon)^{a_j}}, \quad a_j \geq 1$$

- By general theory, we know that

$$\text{Im}(\mathcal{M}[\ell]) = \sum_t c_t \text{Im}(\mathcal{I}_t[\ell])$$

- The $\text{Im}(\mathcal{I}_t[\ell])$ is known, so we need to find $\text{Im}(\mathcal{M}[\ell])$

To use the unitarity cut method, we use a trick by noticing that

$$\int \frac{d^D \ell}{(2\pi)^{D/2}} \frac{\mathcal{N}[\ell]}{\prod_{j=1}^n ((\ell - K_j)^2 - m_j^2 + i\epsilon)^{a_j}}$$

$$= \left\{ \prod_{j=1}^n \frac{1}{(a_j - 1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} \int \frac{d^D \ell}{(2\pi)^{D/2}} \frac{\mathcal{N}[\ell]}{\prod_{j=1}^n ((\ell - K_j)^2 - m_j^2 - \eta_j + i\epsilon)} \right\} \Big|_{\eta_j \rightarrow 0}$$

thus

$$Re[L] + ilm[L] = \left\{ \prod_{j=1}^n \frac{1}{(a_j - 1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} (Re[R] + ilm[R]) \right\} \Big|_{\eta_j \rightarrow 0}$$

Since the η_j 's are real numbers, we have

$$\begin{aligned} \operatorname{Re}[L] + i\operatorname{Im}[L] &= \left\{ \prod_{j=1}^n \frac{1}{(a_j - 1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} \operatorname{Re}[R] \right\} \Big|_{\eta_j \rightarrow 0} \\ &+ i \left\{ \prod_{j=1}^n \frac{1}{(a_j - 1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} \operatorname{Im}[R] \right\} \Big|_{\eta_j \rightarrow 0} \end{aligned}$$

so finally

$$\operatorname{Im}[L] = \left\{ \prod_{j=1}^n \frac{1}{(a_j - 1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} \operatorname{Im}[R] \right\} \Big|_{\eta_j \rightarrow 0}$$

- For general $\mathcal{N}[\ell]$, we know the expansion

$$\text{Im}[R] = \sum_t c_t \text{Im}(\mathcal{I}_t[\ell])$$

- The action of $\frac{d}{d\eta}$ will act on both c_t and $\text{Im}(\mathcal{I}_t[\ell])$.
- Since the analytic function c_t 's are known, the unknown piece is the action of $\frac{d}{d\eta}$ on $\text{Im}(\mathcal{I}_t[\ell])$ and its expansion. In another words, we just need to consider the reduction of general power with $\mathcal{N}[\ell] = 1$ for $n \leq 5$.

Example I: bubble

$$\int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^a ((p - K)^2 - M_2^2)^b}$$

- The imaginary part is given by

$$C[\mathcal{I}_2] = (K^2)^{-1+\epsilon} \Delta^{\frac{1}{2}-\epsilon} \int_0^1 du u^{-1-\epsilon} \sqrt{1-u}$$

where

$$\begin{aligned} \Delta[K; M_1, M_2] &= (K^2)^2 + (M_1^2)^2 + (M_2^2)^2 \\ &\quad - 2M_1^2 M_2^2 - 2K^2 M_1^2 - 2K^2 M_2^2 \end{aligned}$$

- By our trick

$$C[l_2(n+1, m+1)] = \frac{1}{m!n!} \left(\frac{\partial}{\partial M_2^2} \right)^m \left(\frac{\partial}{\partial M_1^2} \right)^n C[l_2(1, 1)]$$

thus

$$c_{2 \rightarrow 2}(n+1, m+1) = \frac{1}{m!n! \Delta^{\frac{1}{2}-\epsilon}} \left(\frac{\partial}{\partial M_2^2} \right)^m \left(\frac{\partial}{\partial M_1^2} \right)^n \Delta^{\frac{1}{2}-\epsilon}$$

Recurrence relation:

$$\begin{aligned}
 I_3(1, 1, n_3) &= \frac{1}{(n_3 - 1)!} \frac{d^{n_3-1}}{d(m_1^2)^{n_3-1}} I_3(1, 1, 1) \\
 &= \frac{1}{(n_3 - 1)} \frac{d}{d(m_1^2)} \frac{1}{(n_3 - 2)!} \frac{d^{n_3-2}}{d(m_1^2)^{n_3-2}} I_3(1, 1, 1) \\
 &= \frac{1}{(n_3 - 1)} \frac{d}{d(m_1^2)} I_3(1, 1, n_3 - 1) \\
 &= \frac{1}{(n_3 - 1)} \frac{d}{d(m_1^2)} \left\{ c_{3 \rightarrow 3}(1, 1, n_3 - 1) \mathcal{I}_3 \right. \\
 &\quad \left. + \sum_{i=1}^3 c_{3 \rightarrow 2; \bar{i}}(1, 1, n_3 - 1) \mathcal{I}_{2; \bar{i}} + \dots \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(n_3 - 1)} \frac{dc_{3 \rightarrow 3}(1, 1, n_3 - 1)}{d(m_1^2)} \mathcal{I}_3 + \frac{c_{3 \rightarrow 3}(1, 1, n_3 - 1)}{(n_3 - 1)} l_3(1, 1, 2) \\
 &+ \sum_{i=1}^3 \frac{dc_{3 \rightarrow 2; \bar{i}}(1, 1, n_3 - 1)}{(n_3 - 1) d(m_1^2)} \mathcal{I}_{2; \bar{i}} \\
 &+ \frac{c_{3 \rightarrow 2; \bar{1}}(1, 1, n_3 - 1)}{(n_3 - 1)} l_{2; \bar{1}}(1, 2) + \frac{c_{3 \rightarrow 2; \bar{2}}(1, 1, n_3 - 1)}{(n_3 - 1)} l_{2; \bar{2}}(2, 1) + \dots
 \end{aligned}$$

Thus we derive

$$\begin{aligned}
 c_3(1, 1, n_3) &= \frac{1}{(n_3 - 1)} \frac{dc_{3 \rightarrow 3}(1, 1, n_3 - 1)}{d(m_1^2)} + \frac{c_{3 \rightarrow 3}(1, 1, n_3 - 1)}{(n_3 - 1)} c_{3 \rightarrow 3}(1, 1, 2) \\
 c_{3 \rightarrow 2; \bar{1}}(1, 1, n_3) &= \frac{c_{3 \rightarrow 3}(1, 1, n_3 - 1)}{(n_3 - 1)} c_{3 \rightarrow 2; \bar{1}}(1, 1, 2) + \frac{1}{(n_3 - 1)} \frac{dc_{3 \rightarrow 2; \bar{1}}(1, 1, n_3 - 1)}{d(m_1^2)} \\
 &\quad + \frac{c_{3 \rightarrow 2; \bar{1}}(1, 1, n_3 - 1)}{(n_3 - 1)} c_{2 \rightarrow 2; \bar{1}}(1, 2) \\
 c_{3 \rightarrow 2; \bar{2}}(1, 1, n_3) &= \frac{c_{3 \rightarrow 3}(1, 1, n_3 - 1)}{(n_3 - 1)} c_{3 \rightarrow 2; \bar{2}}(1, 1, 2) + \frac{1}{(n_3 - 1)} \frac{dc_{3 \rightarrow 2; \bar{2}}(1, 1, n_3 - 1)}{d(m_1^2)} \\
 &\quad + \frac{c_{3 \rightarrow 2; \bar{2}}(1, 1, n_3 - 1)}{(n - 1)} c_{2 \rightarrow 2; \bar{2}}(2, 1) \\
 c_{3 \rightarrow 2; \bar{3}}(1, 1, n_3) &= \frac{c_{3 \rightarrow 3}(1, 1, n_3 - 1)}{(n_3 - 1)} c_{3 \rightarrow 2; \bar{3}}(1, 1, 2) + \frac{1}{(n_3 - 1)} \frac{dc_{3 \rightarrow 2; \bar{3}}(1, 1, n_3 - 1)}{d(m_1^2)}
 \end{aligned}$$

Thus the key calculation is for scalar integral with one and only one propagator having power 2.

Further simplification— The dihedral symmetry D_n :

- By momentum shifting $p \rightarrow p + K_1$ we get

$$\begin{aligned}
 & I_3(n_1, n_2, n_3)[K_1, K_2, K_3; M_1, M_2, m_1] \\
 = & \int \frac{d^{4-2\epsilon} p^4}{(2\pi)^{4-2\epsilon}} \frac{1}{((p + K_1)^2 - M_1^2)^{n_1} (p^2 - M_2^2)^{n_2} ((p - K_2)^2 - m_1^2)^{n_3}} \\
 = & I_3(n_2, n_3, n_1)[K_2, K_3, K_1; M_2, m_1, M_1]
 \end{aligned}$$

- We can also consider the variable changing $p \rightarrow -p$ to get

$$\begin{aligned}
 & I_3(n_1, n_2, n_3)[K_1, K_2, K_3; M_1, M_2, m_1] \\
 = & \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^{n_1} ((p + K_1)^2 - M_2^2)^{n_2} ((p - K_3)^2 - m_1^2)^{n_3}} \\
 = & I_3(n_1, n_3, n_2)[K_3, K_2, K_1; M_1, m_1, M_2]
 \end{aligned}$$

- Thus only $I_n(1, \dots, 1, 2)$ needed to be calculated.

For triangle, we need to compute only $I_3(1, 1, 2)$. Let us show the calculation for the cut K_1 :



$$C_{K_1}(I_3(1, 1, 2)) = -\left(\frac{4K_1^2}{\Delta[K_1, M_1, M_2]}\right)^\epsilon \frac{1}{\sqrt{\Delta_{3;m=0}}} \frac{\partial}{\partial m_1^2} Tri^{(0)}(Z)$$

- With a little algebra we have

$$\begin{aligned} \frac{\partial}{\partial m_1^2} Tri^{(0)}(Z) &= \frac{2K_1^2}{\sqrt{\Delta_{3;m=0}}\Delta[K_1, M_1, M_2]} \left(\frac{2(1-2\epsilon)}{1-Z^2} Bub^{(0)} \right) \\ &+ \frac{2Z\epsilon}{1-Z^2} Tri^{(0)}(Z) \end{aligned}$$

Thus

$$c_{3 \rightarrow 3; K_1}(1, 1, 2) = \frac{4K_1^2}{\sqrt{\Delta_{3;m=0}} \Delta[K_1, M_1, M_2]} \frac{Z\epsilon}{1 - Z^2}$$

and

$$c_{3 \rightarrow 2; \bar{3}; K_1}(1, 1, 2) = -\frac{4K_1^2}{\Delta[K_1, M_1, M_2] \Delta_{3;m=0}} \frac{1 - 2\epsilon}{1 - Z^2}$$

- One of the big problem of unitarity cut method is that tadpole coefficients can not be found by this way.
- There are proposal using the single cut, but the calculation is still complicated.
- In this talk, I will present a method to give the analytic expression of tadpole coefficients

We want to find the tadpole coefficient of integral

$$I_{n+1}^{(m)}[R; \{K_i\}; M_0, \{M_i\}] \equiv \int \frac{d^D \ell}{(2\pi)^D} \frac{(2\ell \cdot R)^m}{(\ell^2 - M_0^2) \prod_{j=1}^n ((\ell - K_j)^2 - M_j^2)}$$

This expression is general

- If the numerator is $(2\ell \cdot R_1)(2\ell \cdot R_2)$, we can consider the reduction of $(2\ell \cdot R)^2$, then put $R = \alpha_1 R_1 + \alpha_2 R_2$ and expand it, thus the coefficient of $2\alpha_1 \alpha_2$ is the wanted reduction result for the original numerator.
- Similarly, if the numerator is $4\ell_\mu F^{\mu\nu} \ell_\nu$, we can consider the reduction of $(2\ell \cdot R_1)(2\ell \cdot R_2)$ first. Then for each pair of R_1, R_2 , we replace $(K_1 \cdot R)(K_2 \cdot R)$ by $(K_1)_\mu F^{\mu\nu} (K_2)_\nu$ and $R_1 \cdot R_2$ by $\eta_{\mu\nu} F^{\mu\nu}$ (please notice that $F^{\mu\nu}$ is symmetric tensor).

We will focus on

$$I_{n+1}^{(m)} = C_0(m, n+1) \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 - M_0^2)} + \dots$$

and others can be obtained by momentum shifting.

- To find the C_0 , we will use a trick, i.e., to establish some differential equations by using following differential operators:

$$\hat{D}_i \equiv K_i \cdot \frac{\partial}{\partial R}, \quad i = 1, \dots, n; \quad \hat{T} \equiv \eta^{\mu\nu} \frac{\partial}{\partial R^\mu} \frac{\partial}{\partial R^\nu}$$

$$\begin{aligned}
 K_1^\mu \frac{\partial}{\partial R^\mu} I_{n+1}^{(m)} &= \int \frac{d^D \ell}{(2\pi)^D} \frac{m(2\ell \cdot R)^{m-1} (2K_1 \cdot \ell)}{(\ell^2 - M_0^2) \prod_{j=1}^n ((\ell - K_j)^2 - M_j^2)} \\
 &= \int \frac{d^D \ell}{(2\pi)^D} \frac{m(2\ell \cdot R)^{m-1}}{\prod_{j=1}^n ((\ell - K_j)^2 - M_j^2)} \\
 &\quad - \int \frac{d^D \ell}{(2\pi)^D} \frac{m(2\ell \cdot R)^{m-1}}{(\ell^2 - M_0^2) \prod_{j=2}^n ((\ell - K_j)^2 - M_j^2)} \\
 &+ (M_0^2 + K_1^2 - M_1^2) \int \frac{d^D \ell}{(2\pi)^D} \frac{m(2\ell \cdot R)^{m-1}}{(\ell^2 - M_0^2) \prod_{j=1}^n ((\ell - K_j)^2 - M_j^2)} \\
 &= m I_{n+1; \bar{0}}^{(m-1)} - m I_{n+1; \bar{1}}^{(m-1)} + m f_1 I_{n+1}^{(m-1)}
 \end{aligned}$$

- Using

$$\widehat{D}_j I_{n+1}^{(m)} = \left\{ \widehat{D}_j C_0(m, n+1) \right\} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 - M_0^2)} + \dots$$

and comparing the tadpole coefficients, we have the equation

$$\begin{aligned} \widehat{D}_j C_0(m, n+1) &= -m C_0(m-1, n+1; \bar{j}) \\ &+ m f_j C_0(m-1, n+1) \end{aligned}$$

• Similarly

$$\begin{aligned}
 \eta^{\mu\nu} \frac{\partial}{\partial R^\mu} \frac{\partial}{\partial R^\nu} I_{n+1}^{(m)} &= \int \frac{d^D \ell}{(2\pi)^D} \frac{m(m-1)(2\ell \cdot R)^{m-2}(4\ell^2)}{(\ell^2 - M_0^2)^2 \prod_{j=1}^n ((\ell - K_j)^2 - M_j^2)} \\
 &= 4m(m-1)M_0^2 \int \frac{d^D \ell}{(2\pi)^D} \frac{(2\ell \cdot R)^{m-2}}{(\ell^2 - M_0^2)^2 \prod_{j=1}^n ((\ell - K_j)^2 - M_j^2)} \\
 &+ \int \frac{d^D \ell}{(2\pi)^D} \frac{4m(m-1)(2\ell \cdot R)^{m-2}}{\prod_{j=1}^n ((\ell - K_j)^2 - M_j^2)} \\
 &= 4m(m-1)M_0^2 I_{n+1}^{(m-2)} + 4m(m-1)I_{n+1;0}^{(m-2)}
 \end{aligned}$$

thus

$$\widehat{T} C_0(m, n+1) = 4m(m-1)M_0^2 C_0(m-2, n+1)$$

- To continue the study, we are not solve the differential equations directly, but noticing that it can be expand as following

$$C_0(m, n+1) = (M_0^2)^{-n} \sum_{\substack{2i_0 + \sum_{k=1}^n i_k = m \\ \{i_k\}, k=0, \dots, n}} c_{i_0, i_1, i_2, i_3, \dots, i_n}^{(m)} (M_0^2)^{i_0} s_{00}^{i_0} \prod_{k=1}^n s_{0k}^{i_k}$$

Using this expansion, we transfer the differential equation to the algebraic recurrence relation

Example I: tadpole



$$\begin{aligned}
 \widehat{T}C_0(m, 1)[R; M_0] &= \widehat{T} \left(c^{(m)}(M_0^2)^{\frac{m}{2}} s_{00}^{\frac{m}{2}} \right) \\
 &= c^{(m)}(M_0^2)^{\frac{m}{2}} (Dm + m(m-2)) s_{00}^{\frac{m-2}{2}} \\
 &= 4m(m-1)M_0^2 C_0(m-2, 1) = 4m(m-1)M_0^2 c^{(m-2)}(M_0^2)^{\frac{m-2}{2}} s_{00}^{\frac{m-2}{2}}
 \end{aligned}$$

which leads to the recurrence relation

$$c^{(m)} = \frac{4(m-1)}{(D+m-2)} c^{(m-2)}$$

Using the initial condition $c^{(0)} = 1$, we get immediately for

$$c^{(m=\text{even})} = 2^m \frac{(m-1)!!}{\prod_{i=1}^{\frac{m}{2}} (D+2(i-1))}, \quad c^{(m=\text{odd})} = 0$$

Example II: bubble With the expansion

$$C_0(m, 2) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} c_i^{(m)} s_{00}^i (M_0^2)^{i-1} s_{01}^{m-2i}$$

we have

- By D_1 , we get immediately than when $m = 2r$

$$2(i+1)\beta_{11}c_{i+1}^{(m)} + (m-2i)c_i^{(m)} = m\alpha_1\beta_{11}c_i^{(m-1)}, \quad i = 0, \dots, r-1$$

and when $m = 2r + 1$

$$2(i+1)\beta_{11}c_{i+1}^{(m)} + (m-2i)c_i^{(m)} = m\alpha_1\beta_{11}c_i^{(m-1)} \quad i = 0, \dots, r-1$$

$$c_r^{(m)} = -(2r+1)\beta_{11}c^{(2r)} + m\alpha_1\beta_{11}c_r^{(m-1)}$$

where $c^{(m)}$ is the tadpole expansion coefficients.

- By T , we have

$$\begin{aligned}
 & 2(i+1)(D+2m-4-2i)\beta_{11}c_{i+1}^{(m)} + (m-2i)(m-2i-1)c_i^{(m)} \\
 & = 4m(m-1)\beta_{11}c_i^{(m-2)}, \quad i = 0, \dots, \lfloor \frac{m}{2} \rfloor - 1
 \end{aligned}$$

- For $m = 2r + 1$, using the second line, we solve immediately

$$c_r^{(2r+1)} = -(2r+1)\beta_{11}c^{(2r)} + (2r+1)\alpha_1\beta_{11}c_r^{(2r)}$$

Then using the first line, we have

$$c_i^{(2r+1)} = \frac{-2(i+1)\beta_{11}}{(2r+1-2i)}c_{i+1}^{(2r+1)} + \frac{(2r+1)}{(2r+1-2i)}\alpha_1\beta_{11}c_i^{(2r)}$$

recursively from $i = r - 1$ to $i = 0$.

- For $m = 2r$, there are $(r + 1)$ unknown coefficients. Using T produce r equations. One more can be found using D with $i = r - 1$.
- Using $i = r - 1$ from T and $i = r - 1$ from D , we can solve immediately

$$c_r^{(2r)} = \frac{(2r - 1)}{(D + 2r - 3)} \left(\alpha_1 \beta_{11} c^{(2r-2)} + (4 - \alpha_1^2 \beta_{11}) c_{r-1}^{(2r-2)} \right)$$

- Having solved $c_r^{(2r)}$ we can use T relation to finally get

$$c_i^{(2r)} = \frac{8r(2r - 1)\beta_{11}c_i^{(2r-2)} - 2(i + 1)(D + 4r - 4 - 2i)\beta_{11}c_{i+1}^{(2r)}}{(2r - 2i)(2r - 2i - 1)}$$

recursively from $i = r - 1$ to $i = 0$.

Final remarks:

- Our method for tadpole is nothing, but the traditional PV-reduction method with a little deformation
- It can also be applied to find coefficients of other basis, such as bubble, triangle, box and pentagon.
- The generalization to higher loops is possible, but there are some technical difficulties.

Thanks a lot of your
attention !