A Long Journey (Odyssey) to a Full NLO Calculation in CGC Formalism

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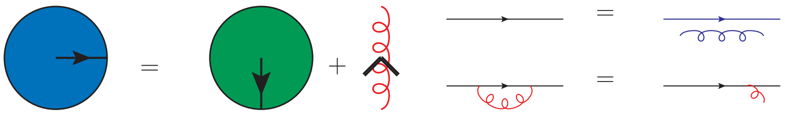
Kinoshita-Lee-Nauenberg Theorem



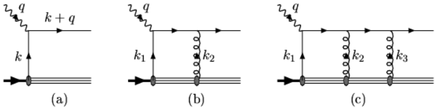
KLN theorem: In a theory with massless fields, transition rates are free of the infrared divergence (soft and collinear) if the summation over initial and final degenerate states is carried out.

- Infrared safe observables. e.g, Jet observables and e^+e^- total cross section.
- The KLN theorem: infrared divergences appear because some of states are physically "degenerate", but we treat them as different.
- A state with a quark accompanied by a collinear gluon is degenerate with a state with a single quark.
- A state with a soft gluon is degenerate with a state with no gluon (virtual).

The gauge invariant definition of parton distributions

The integrated quark distribution

$$f_q(x) = \int \frac{d\xi^-}{4\pi} e^{ixP^+\xi^-} \langle P \left| \bar{\psi}(0)\gamma^+ \mathcal{L}(\xi^-)\psi(0,\xi^-) \right| P \rangle$$



- The gauge links come from the sum over all degenerate quark states.

$$|\psi_q(k)\rangle_{GI} = |\psi_q(k)\rangle + |\psi_q(k_1)g(k-k_1)\rangle + |\psi_q(k_1)g(k_2)g(k-k_1-k_2)\rangle + \cdots .$$

- Gauge invariant definition with $\mathcal{L}(\xi^-) \equiv \mathrm{P}\exp\left[-ig\int_0^{\xi^-} d\xi^{-\prime} A^+(\xi^{-\prime})\right]$.
- Light-cone gauge together with proper B.C. ⇒ parton density interpretation.
  The unintegrated (Transverse Momentum Dependent (TMD)) quark distribution

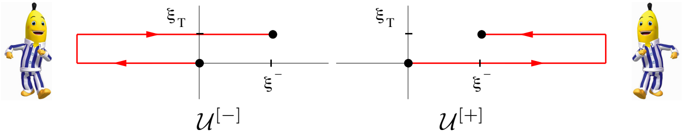
Two Different Gauge Invariant Operator Definitions

[F.Dominguez, BX and F. Yuan, 11] I. Weizsäcker Williams gluon distribution:

$$xG^{(1)} = 2\int \frac{d\xi^- d\xi_\perp}{(2\pi)^3 P^+} e^{ixP^+\xi^- - ik_\perp\cdot\xi_\perp} \mathrm{Tr}\langle P|F^{+i}(\xi^-,\xi_\perp)\mathcal{U}^{[+]\dagger}F^{+i}(0)\mathcal{U}^{[+]}|P\rangle.$$

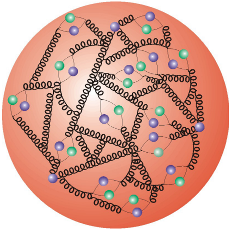
II. Color Dipole gluon distributions:

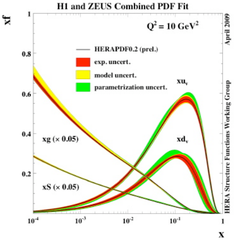
$$xG^{(2)} = 2\int \frac{d\xi^- d\xi_\perp}{(2\pi)^3 P^+} e^{ixP^+\xi^- - ik_\perp\cdot\xi_\perp} \mathrm{Tr}\langle P|F^{+i}(\xi^-,\xi_\perp)\mathcal{U}^{[-]\dagger}F^{+i}(0)\mathcal{U}^{[+]}|P\rangle.$$

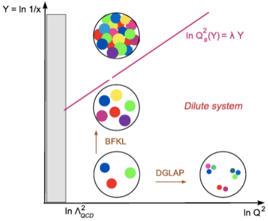


- The WW gluon distribution is the conventional gluon distributions.
- The dipole gluon distribution has no such interpretation.
- Sudakov resummation in small-x physics. *[Mueller, BX and Yuan, 13]*

Deep into small-x region





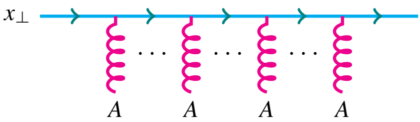


- Partons in the low-x region is dominated by gluons. See HERA data.
- BFKL equation ⇒ Resummation of the $\alpha_s \ln \frac{1}{x}$.
- When too many gluons squeezed in a confined hadron, gluons start to overlap and recombine ⇒ Non-linear dynamics ⇒ BK (JIMWLK) equation
- Use $Q_s(x)$ to separate the saturated dense regime from the dilute regime.
- Core ingredients: Multiple interactions + Small-x (high energy) evolution

Wilson Lines in Color Glass Condensate Formalism

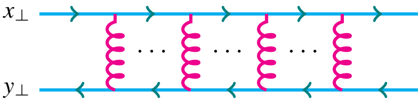
We use Wilson line to represent the multiple scattering between the fast moving quark and target background gluon fields.

$$U(x_\perp) = \mathcal{P} \exp\left(-ig \int dz^+ A^-(x_\perp, z^+)\right)$$



The Wilson loop (color dipole) in McLerran-Venugopalan (MV) model

$$\frac{1}{N_c}\left\langle \mathrm{Tr} U(x_\perp) U^\dagger(y_\perp)\right\rangle = e^{-\frac{Q_s^2 (x_\perp - y_\perp)^2}{4}}$$



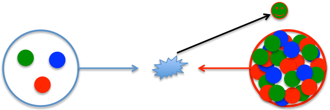
- Dipole amplitude $S^{(2)}$ then produces the quark k_T spectrum via Fourier transform

$$\mathcal{F}(k_\perp) \equiv \frac{dN}{d^2k_\perp} = \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^2} e^{-ik_\perp \cdot (x_\perp - y_\perp)} \frac{1}{N_c} \left\langle \mathrm{Tr} U(x_\perp) U^\dagger(y_\perp)\right\rangle.$$

Forward hadron production in pA collisions

[Dumitru, Jalilian-Marian, 02] Inclusive forward hadron production in pA collisions

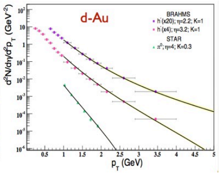
$$\frac{d\sigma_{\rm LO}^{pA\to hX}}{d^2p_\perp dy_h} = \int_\tau^1 \frac{dz}{z^2}\left[x_1 q_f(x_1,\mu)\mathcal{F}_{x_2}(k_\perp)D_{h/q}(z,\mu) + x_1 g(x_1,\mu)\tilde{\mathcal{F}}_{x_2}(k_\perp)D_{h/g}(z,\mu)\right].$$



projectile: $x_1 \sim \frac{p_\perp}{\sqrt{s}}e^{+y} \sim 1$ valence

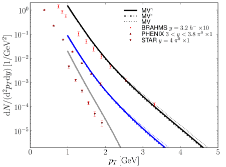
target: $x_2 \sim \frac{p_\perp}{\sqrt{s}}e^{-y} \ll 1$ gluon

Albaete-Marquet, 2010



Lappi & Mantysaari 13

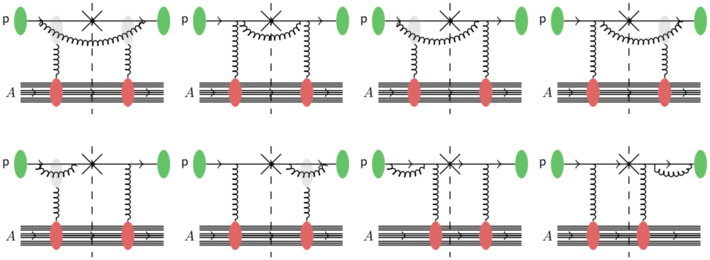
$d + Au \to \pi^0/h^- + X$, $\sqrt{s} = 200\,\mathrm{GeV}$, $K = 2.5$



- $\mathcal{F}(k_\perp)$ (dipole gluon distribution) encodes dense gluon info.
- *[Dumitru, Hayashigaki, Jalilian-Marian, 06; Altinoluk, Kovner 11] [Altinoluk, Armesto, Beuf, Kovner, Lublinsky, 14]*; Full NLO *[Chirilli, BX and Yuan, 12]*

NLO diagrams in the $q \to q$ channel

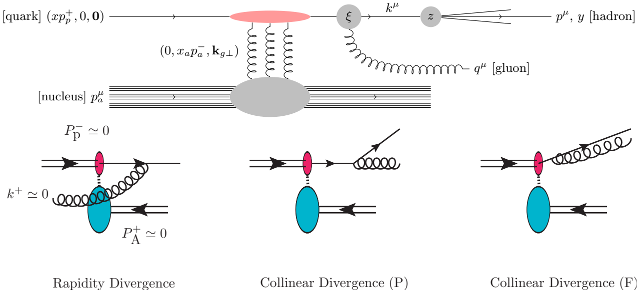
[Chirilli, BX and Yuan, 12]



- Take into account real (top) and virtual (bottom) diagrams together!
- Multiple interactions inside the grey blobs!
- Integrate over gluon phase space $\Rightarrow$Divergences!.

Factorization for single inclusive hadron productions

Factorization for the $p + A \to H + X$ process [Chirilli, BX and Yuan, 12]



- Need to include all real and virtual graphs in all four channel $q \to q$, $q \to g$, $g \to q(\bar{q})$ and $g \to g$.
- Gluons in different kinematical regions give different divergences due to degeneracy. KLN
- 1. collinear to the target nucleus; $\Rightarrow$ BK evolution for UGD $\mathcal{F}(k_\perp)$.
- 2. collinear to the initial quark; $\Rightarrow$ DGLAP evolution for PDFs
- 3. collinear to the final quark. $\Rightarrow$ DGLAP evolution for FFs.
- Divergence $\Rightarrow$ Renormalization $\Rightarrow$ Resummation!

Hard Factors

For the $q \to q$ channel, the factorization formula can be written as

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \int \frac{dz}{z^2}\frac{dx}{x}\xi xq(x,\mu)D_{h/q}(z,\mu)\int\frac{d^2x_\perp d^2y_\perp}{(2\pi)^2}\left\{S_Y^{(2)}(x_\perp,y_\perp)\left[\mathcal{H}_{2qq}^{(0)}+\frac{\alpha_s}{2\pi}\mathcal{H}_{2qq}^{(1)}\right]\right.$$
$$\left. + \int\frac{d^2b_\perp}{(2\pi)^2}S_Y^{(4)}(x_\perp,b_\perp,y_\perp)\frac{\alpha_s}{2\pi}\mathcal{H}_{4qq}^{(1)}\right\}$$

with $\mathcal{H}_{2qq}^{(0)} = e^{-ik_\perp\cdot r_\perp}\delta(1-\xi)$ and

$$\mathcal{H}_{2qq}^{(1)} = C_F\mathcal{P}_{qq}(\xi)\ln\frac{c_0^2}{r_\perp^2\mu^2}\left(e^{-ik_\perp\cdot r_\perp}+\frac{1}{\xi^2}e^{-i\frac{k_\perp}{\xi}\cdot r_\perp}\right) - 3C_F\delta(1-\xi)e^{-ik_\perp\cdot r_\perp}\ln\frac{c_0^2}{r_\perp^2k_\perp^2}$$
$$-(2C_F-N_c)\,e^{-ik_\perp\cdot r_\perp}\left[\frac{1+\xi^2}{(1-\xi)_+}\widetilde{I}_{21}-\left(\frac{\left(1+\xi^2\right)\ln(1-\xi)^2}{1-\xi}\right)_+\right]$$

$$\mathcal{H}_{4qq}^{(1)} = -4\pi N_c e^{-ik_\perp\cdot r_\perp}\left\{e^{-i\frac{1-\xi}{\xi}k_\perp\cdot(x_\perp-b_\perp)}\frac{1+\xi^2}{(1-\xi)_+}\frac{1}{\xi}\frac{x_\perp-b_\perp}{(x_\perp-b_\perp)^2}\cdot\frac{y_\perp-b_\perp}{(y_\perp-b_\perp)^2}\right.$$
$$\left.-\delta(1-\xi)\int_0^1 d\xi'\frac{1+\xi'^2}{(1-\xi')_+}\left[\frac{e^{-i(1-\xi')k_\perp\cdot(y_\perp-b_\perp)}}{(b_\perp-y_\perp)^2}-\delta^{(2)}(b_\perp-y_\perp)\int d^2r'_\perp\frac{e^{ik_\perp\cdot r'_\perp}}{r'^2_\perp}\right]\right\},$$

where $$\widetilde{I}_{21} = \int\frac{d^2b_\perp}{\pi}\left\{e^{-i(1-\xi)k_\perp\cdot b_\perp}\left[\frac{b_\perp\cdot(\xi b_\perp-r_\perp)}{b_\perp^2\,(\xi b_\perp-r_\perp)^2}-\frac{1}{b_\perp^2}\right]+e^{-ik_\perp\cdot b_\perp}\frac{1}{b_\perp^2}\right\}.$$

Clear physical interpretation in coordinate space. However, need to go to momentum space for numerical evaluation!

Factorization and NLO Calculation

- Factorization is about separation of short distant physics (perturbatively calculable hard factor) from large distant physics (Non perturbative).

$$\sigma \sim xf(x) \otimes \mathcal{H} \otimes D_h(z) \otimes \mathcal{F}(k_\perp)$$

- NLO (1-loop) calculation always contains various kinds of divergences.
  - Some divergences can be absorbed into the corresponding evolution equations.
  - The rest of divergences should be cancelled.
- Hard factor

$$\mathcal{H} = \mathcal{H}^{(0)}_{\mathrm{LO}} + \frac{\alpha_s}{2\pi}\mathcal{H}^{(1)}_{\mathrm{NLO}} + \cdots$$

  should always be finite and free of divergence of any kind.
- NLO vs NLL Naive α_s expansion sometimes is not sufficient!

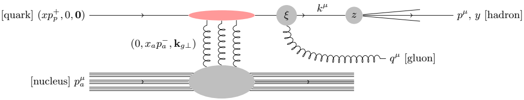
	LO	NLO	NNLO	$\cdots$
LL	1	$\alpha_s L$	$(\alpha_s L)^2$	$\cdots$
NLL		α_s	$\alpha_s\,(\alpha_s L)$	$\cdots$
$\cdots$			$\cdots$	$\cdots$

- Evolution $\rightarrow$ Resummation of large logs.
  LO evolution resums LL; NLO $\Rightarrow$ NLL.

Numerical implementation of the NLO result

Single inclusive hadron production up to NLO

$$\begin{aligned} d\sigma \; &= \int x f_a(x) \otimes D_a(z) \otimes \mathcal{F}_a^{x_g}(k_\perp) \otimes \mathcal{H}^{(0)} \\ &\quad + \frac{\alpha_s}{2\pi} \int x f_a(x) \otimes D_b(z) \otimes \mathcal{F}_{(N)ab}^{x_g} \otimes \mathcal{H}_{ab}^{(1)} . \end{aligned}$$

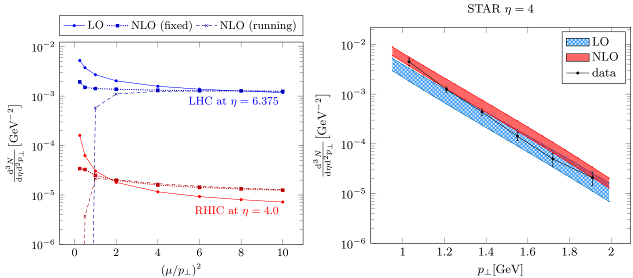


Consistent implementation should include all the NLO α_s corrections.

- NLO parton distributions. (MSTW or CTEQ)
- NLO fragmentation function. (DSS or others.)
- Use NLO hard factors. Partially by *[Albacete, Dumitru, Fujii, Nara, 12]*
- Use the one-loop approximation for the running coupling
- rcBK evolution equation for the dipole gluon distribution *[Balitsky, Chirilli, 08; Kovchegov, Weigert, 07]*. Full NLO BK evolution not available.
- Saturation physics at One Loop Order (SOLO). *[Stasto, Xiao, Zaslavsky, 13]*

Numerical implementation of the NLO result

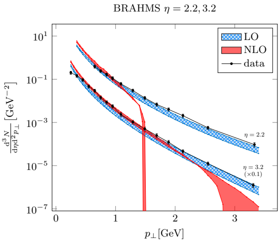
Saturation physics at One Loop Order (SOLO). *[Stasto, Xiao, Zaslavsky, 13]*

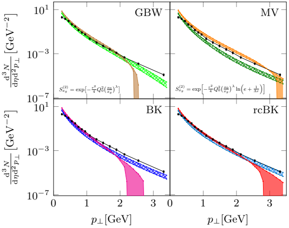


- Agree with data for $p_\perp < Q_s(y)$, and reduced scale dependence, no K factor.
- For more forward rapidity, the agreement gets better and better.

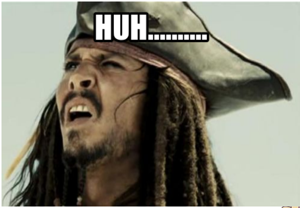
Numerical implementation of the NLO result

Saturation physics at One Loop Order (SOLO). *[Stasto, Xiao, Zaslavsky, 13]*





- The abrupt drop at NLO when $p_\perp > Q_s$ was surprising and puzzling.
- Fixed order calculation in field theories is not guaranteed to be positive.
- Failure of positivity is also seen in TMD, where Y-term is devised to match collinear factorization.
  [Collins, Foundations of perturbative QCD]



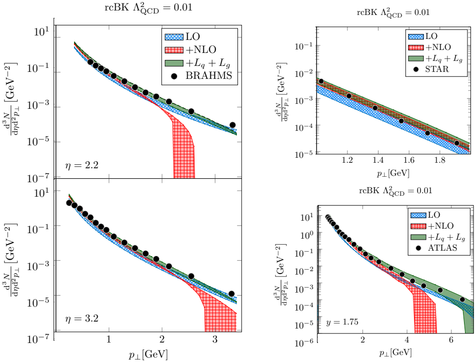
Extending the applicability of CGC calculation

Some thoughts:

- Towards a more complete framework. *[Altinoluk, Armesto, Beuf, Kovner and Lublinsky, 14; Kang, Vitev and Xing, 14; Ducloue, Lappi and Zhu, 16, 17; Iancu, Mueller and Triantafyllopoulos, 16; Liu, Ma, Chao, 19; Kang, Liu, 19; Kang, Liu, Liu, 20;]*
- To solve this problem, needs to find a solution within our current factorization to extend the applicability of CGC.
- More than just negativity problem. Need to work reliably (describe data) from RHIC to LHC, low p_T to high p_T.
- Additional consideration: solution needs to be easy to be implemented numerically due to limited computing resources.
- A lot of logs occur in pQCD loop-calculations: DGLAP, small-x, threshold, Sudakov.
- Breakdown of pQCD expansion often happens due to the appearance of logs in certain phase spaces.

NLO hadron productions in pA collisions: An Odyssey

[Watanabe, Xiao, Yuan, Zaslavsky, 15]



- Work in low $p_\perp \leq Q_s$ region!
- Including the kinematical constraints. (Originally assume the limit $s \to \infty$)

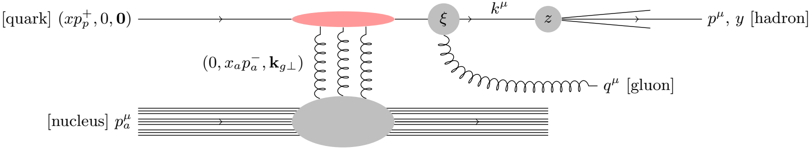
$$\ln \frac{1}{x_g} + \underbrace{\ln \frac{k_\perp^2}{q_\perp^2}}_{\text{missed earlier}} \Rightarrow$$

  New terms: $L_q + L_g$.

  Related to threshold double logs!
- SOLO (1.0 and 2.0) break down in the large $p_\perp \geq Q_s$ region.
- Approach threshold at high $k_\perp$. Threshold resummation (Sudakov)! *[Xiao, Yuan, 18; work in process]*
  Another method: X. Liu's talk

Gluon Radiation at the Threshold

Near threshold: radiated gluon has to be soft!



- Gluon momentum: $q^+ = (1-\xi)p_q^+ \to 0$ with two regions of q^- and $q_\perp$.
- If $q_\perp \sim k_\perp$, then $q^- \to \infty$, this is part of the small-x evolution.
- If $q_\perp \to 0$ as well, this gives large log like $\ln \frac{k_\perp^2}{q_\perp^2}$.
- KLN $\Rightarrow$ complete cancellation between real and virtual.
- Introduce an additional factorization scale Λ for soft gluon q_perp when incomplete cancellation occurs and logs start to appear.

Threshold resummation in the saturation formalism

- $\ln(1-x_p)$ and $\ln k_\perp^2/Q_s^2$ in the large $k_\perp$ region ($k_\perp \gg Q_s$) near threshold

$$\int_x^1 \frac{d\xi}{(1-\xi)_+} f(\xi) = \int_x^1 d\xi \frac{f(\xi)-f(1)}{1-\xi} + f(1)\ln(1-x)$$

- In fact, these two types of logs seem to always appear together in our calculation and soft-collinear effective theory (SCET) in almost identical pattern.
- Remarkable similarities between the threshold resummation in CGC formalism (fixed k_T) and that in SCET*[Becher, Neubert, 06]*.
- Threshold resummation: Sudakov soft gluon part and plus-function part.
- The forward threshold jet function $\Delta(\mu^2, \Lambda^2, z)$ satisfies an almost identical RGE equation. The solution helps to resum threshold logs.

$$\frac{d\Delta(\mu^2,\Lambda^2,z)}{d\ln\mu} = -\frac{2\alpha_s N_c}{\pi}\left[\ln z + \beta_0\right]\Delta(\mu^2,\Lambda^2,z) + \frac{2\alpha_s N_c}{\pi}\int_0^z dz' \frac{\Delta(\mu^2,\Lambda^2,z)-\Delta(\mu^2,\Lambda^2,z')}{z-z'},$$

Solution: $$\Delta(\mu^2,\Lambda^2,z=\ln\frac{x}{\tau}) = \frac{e^{(\beta_0-\gamma_E)\gamma_{\mu,\Lambda}}}{\Gamma[\gamma_{\mu,\Lambda}]} z^{\gamma_{\mu,\Lambda}-1}.$$

Numerical challenges

[Watanabe, Xiao, Yuan, Zaslavsky, 15] Several numerical tricks to help compute the NLO corrections more precisely.

- Numerical calculation (8-d in total) is notoriously hard in coordinate space. Go to momentum space.
- There are terms which have strong cancellation ($1/k_T^2 \to 1/k_T^4$), need to combine them in numerics.
- Work in finite integration range, need to identify the peaks of each term!
- A couple of identities in Fouier transformations

$$\int \frac{d^2x_\perp}{(2\pi)^2} S(x_\perp) \ln \frac{c_0^2}{x_\perp^2 \mu^2} e^{-ik_\perp \cdot x_\perp} = \int \frac{d^2 l_\perp}{\pi l_\perp^2} \left[F(k_\perp + l_\perp) - J_0(\frac{c_0}{\mu} l_\perp) F(k_\perp) \right]$$

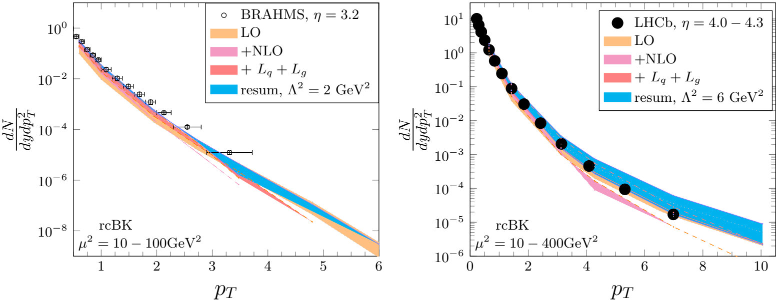
$$= \frac{1}{\pi} \int \frac{d^2 l_\perp}{(l_\perp - k_\perp)^2} \left[F(l_\perp) - \frac{\Lambda^2}{\Lambda^2 + (l_\perp - k_\perp)^2} F(k_\perp) \right] + F(k_\perp) \ln \frac{\Lambda^2}{\mu^2}.$$

- Introduce a semi-hard (additional) scale $\Lambda^2 \sim (1-\xi) k_\perp^2 \sim Q_s^2$ which is analogous to the intermediate jet scale μ_i^2 in SCET *[Becher, Neubert, 06]*. (Sudakov soft part!)
- μ^2 and Λ^2 dependences cancel order by order! At fixed order, need to choose the "natural" values for them.

Preliminary Results

[Xiao, Yuan, 18; Shi, Wang, Wei, Xiao, Yuan,numerical work in process]

$$d\sigma = \int xf_a(x) \otimes D_a(z) \otimes \mathcal{F}_a^{x_g}(k_\perp) \otimes \mathcal{H}^{(0)} \otimes \Delta(\mu,\Lambda) \otimes S_{\rm Sud}(\mu,\Lambda) + \frac{\alpha_s}{2\pi}\int xf_a(x) \otimes D_b(z) \otimes \mathcal{F}_{(N)ab}^{x_g} \otimes \mathcal{H}_{ab}^{(1)}(\mu,\Lambda).$$



- Set $\mu \sim Q \sim k_\perp$ and $\Lambda \ll \mu$. Slightly increase σ ($e^{-x} \geq 1 - x$)
- $\Delta(\mu,\Lambda)$ and $S_{\rm Sud}(\mu,\Lambda)$ satisfy RGEs.

Conclusion

- Factorization for single and dihadron productions in pA collisions in the small-x saturation formalism at one-loop order. (More interesting).
- Towards the quantitative test of saturation physics beyond LL. (More precise).
- One-loop calculation for hard processes in CGC, Sudakov factor. (More complete understanding of TMD or UGD).
- Extension to larger $k_\perp$ region and QCD threshold resummation. Low-$k_\perp$ $\Leftrightarrow$ saturation; High-$k_\perp$ $\Leftrightarrow$ pQCD + Resummation.
- Gluon saturation could be the next interesting discovery at the LHC and future EIC.