“微扰量子场论研讨会”, 2021. 5.14-16, 上海

Bootstrapping a two-loop four-point form factor

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Based on the work to appear with Yuanhong Guo (郭圆宏)、Lei Wang (王磊)

Generic strategy of loop computation

Feynman integrals → **Expansion in a basis of integrals** → **Final result in functional form**

Feynman diagrams, On-shell unitarity method, …

Integral reductions: PV reduction, IBP reduction, …

Solving integrals, functional identities to simplify the result, …

$\sum \text{(integrand)}$

$\sum c_i M_i$

$\sum \text{functions}$

Generic strategy of loop computation

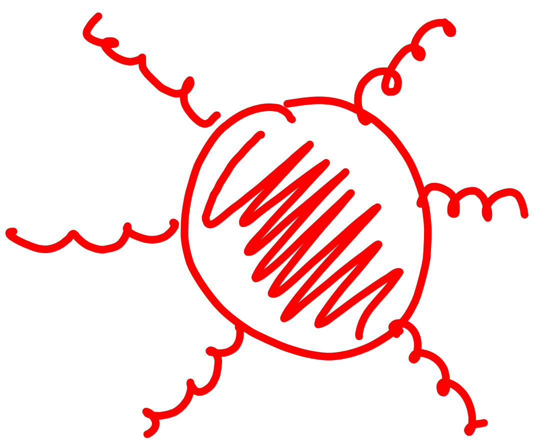
Feynman integrals $\longrightarrow$ **Expansion in a basis of integrals** $\longrightarrow$ **Final result in functional form**

Feynman integrals	Expansion in a basis of integrals	Final result in functional form
Feynman diagrams, On-shell unitarity method, …	Integral reductions: PV reduction, IBP reduction, …	Solving integrals, functional identities to simplify the result, …
$\sum \text{(integrand)}$	$\sum c_i M_i$	$\sum \textbf{functions}$

Complicated intermediate expressions

Compact analytic form

Two-loop six-gluon amplitudes in N=4



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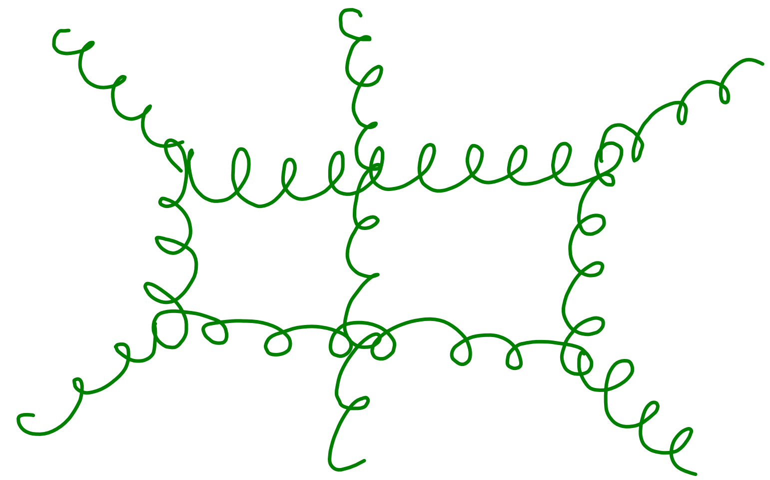
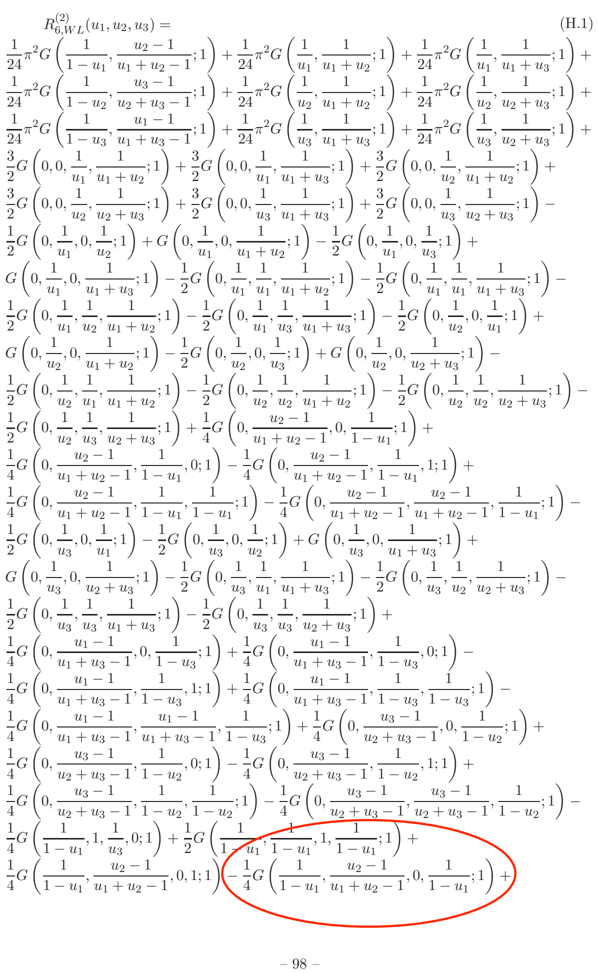
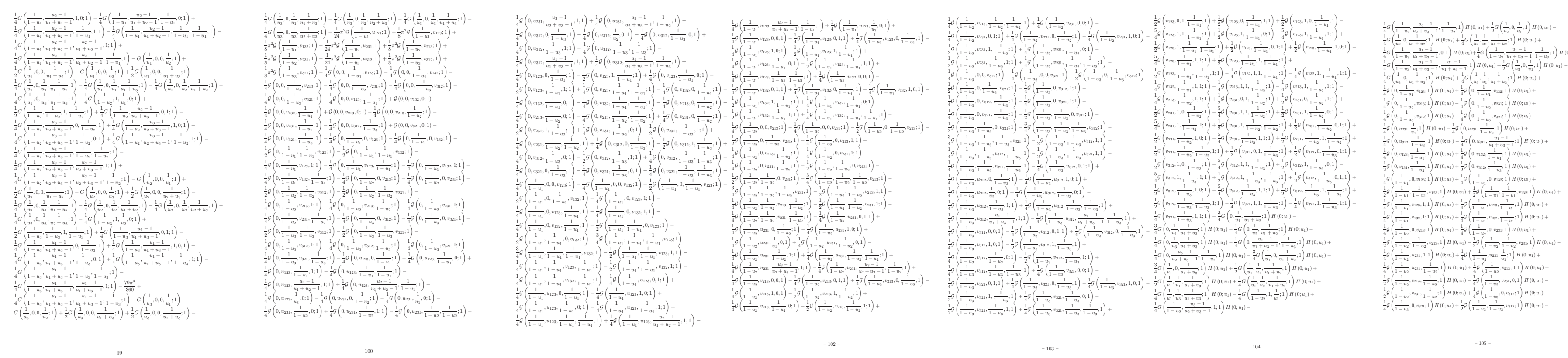


+ many more

[Del Duca, Duhr, Smirnov 2010]

17 page complicated functions

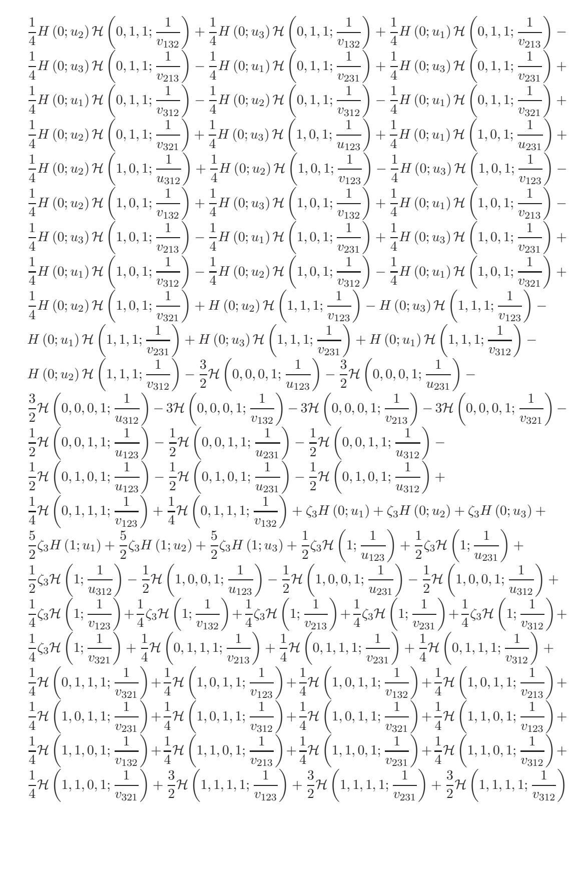




[Del Duca, Duhr, Smirnov 2010]

"multiple(Goncharov)-polylogrithm function"

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)$$



Result can be remarkably simple

17 pages =

[Goncharov, Spradlin, Vergu, Volovich 2010]

$$\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right)-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2+\frac{1}{24}J^4+\frac{\pi^2}{12}J^2+\frac{\pi^4}{72}$$

$$L_4(x^+,x^-)=\frac{1}{8!!}\log(x^+x^-)^4+\sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-)) \qquad \ell_n(x)=\frac{1}{2}\left(\mathrm{Li}_n(x)-(-1)^n\,\mathrm{Li}_n(1/x)\right) \qquad J=\sum_{i=1}^{3}(\ell_1(x_i^+)-\ell_1(x_i^-)).$$

a line result in terms of classical polylogarithms!

Such simplicity is totally unexpected using traditional Feynman diagrams!

Mathematical tool: “symbol”

From function to “Symbol”

Recursion definition of “Symbol”:

$$\mathrm{d}f_k = \sum_i f^i_{k-1}\,\mathrm{dLog}(R_i), \qquad \mathrm{Symbol}(f_k) = \sum_i \mathrm{Symbol}(f^i_{k-1}) \otimes R_i$$

Function	Differential	**symbol**
R	d R	0
log(R)	d log(R)	R
log(R1)log(R2)	logR1 dlogR2+logR2 dlogR1	R1 $\otimes$ R2 + R2 $\otimes$ R1
Li_2(R)	Li_1(R) dlogR	-(1-R) $\otimes$ R

Symbol

Algebraic relations:

$$R_1 \otimes \ldots \otimes (c\, R_i) \otimes \ldots \otimes R_n = R_1 \otimes \ldots \otimes R_i \otimes \ldots \otimes R_n$$ *c* is const

$$R_1 \otimes \ldots \otimes (R_i\, R_j) \otimes \ldots \otimes R_n = R_1 \otimes \ldots \otimes R_i \otimes \ldots \otimes R_n + R_1 \otimes \ldots \otimes R_j \otimes \ldots \otimes R_n$$

Make it easy to prove non-trivial identities, e.g.:

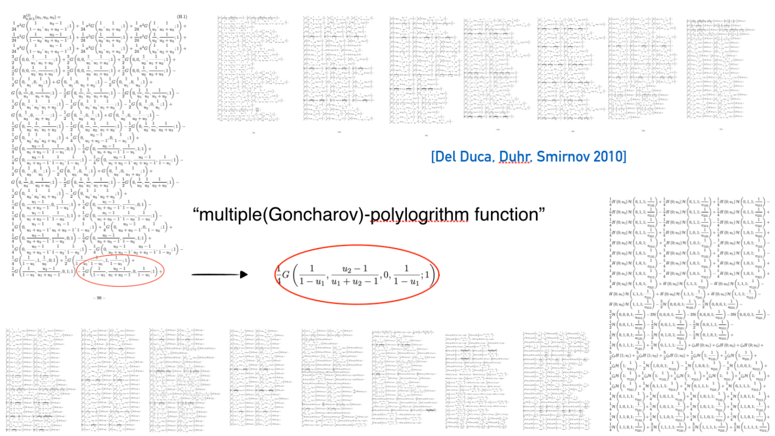
$$\mathrm{Li}_2(z) = -\mathrm{Li}_2(1-z) - \log(1-z)\log(z) + \frac{\pi^2}{6}$$

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{1}{z}\right) - \frac{1}{2}\log^2(-z) - \frac{\pi^2}{6} \;/;\; z \notin (0,\, 1)$$

$$\mathrm{Li}_2\left(\frac{x}{1-y}\right) + \mathrm{Li}_2\left(\frac{y}{1-x}\right) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \mathrm{Li}_2\left(\frac{xy}{(1-x)(1-y)}\right) = \mathrm{Log}(1-x)\mathrm{Log}(1-y)$$

Applications

Complicated expression → symbol → Simple expression



$$\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right)-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2+\frac{1}{24}J^4+\frac{\pi^2}{12}J^2+\frac{\pi^4}{72}$$

Applications

Complicated expression $\longrightarrow$ symbol $\longrightarrow$ Simple expression

A better strategy:

Derive symbol directly without knowing function in advance.

Bootstrap strategy   Dixon, Drummond, Henn 2011, ….

We will apply a different strategy based on master integrand expansion.

Outline

Background and Motivation

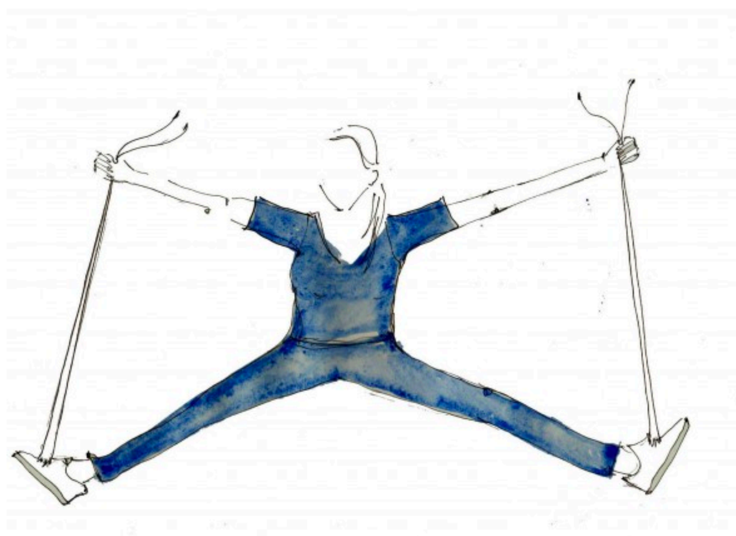
New bootstrap strategy

Two-loop four-point form factor

Summary and outlook

Bootstrap

Bootstrap



Bootstrap

Top-down

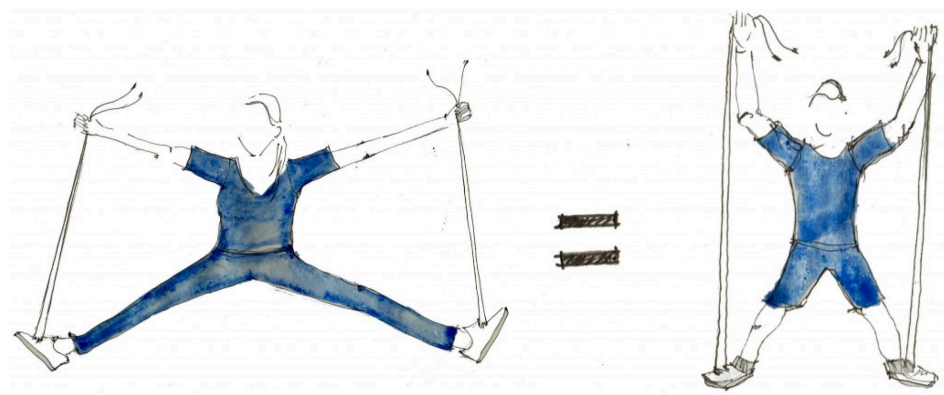
Bottom-up

S-matrix program

"One should try to calculate S-matrix elements directly, without the use of field quantities, by requiring them to have some general properties that ought to be valid, …."

— Eden et.al, "The Analytic S-matrix", 1966

Conformal bootstrap



Compute anomalous dimensions and correlation functions

Alexander M. Polyakov

Vyacheslav S. Rychkov

2-dim $\longrightarrow$ D-dim

Bootstrap of amplitudes

Symbol bootstrap

Computing the finite remainder functions using symbol techniques.

Ansatz
in symbols → **Physical constraints** → Solution

$$S_{\text{ansatz}}(R) = \sum_i c_i \left[\otimes_a W_{i,a} \right]$$

$$S(R) = \sum_i c_i \left(\otimes_a W_{i,a} \right)$$

Bootstrap of amplitudes

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Computing the finite remainder functions using symbol techniques.

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The new strategy we will use

Ansatz in master integrals → **Physical constraints** → Solution of coefficients

$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

$$\mathcal{F}^{(l)} = \sum_i C_i I_i^{(l)}$$

“master bootstrap”

Ansatz in master integral expansion → **Physical constraints** → Solution of coefficients

$$\mathscr{F}^{(l),\mathrm{ansatz}} = \sum_i C_i I_i^{(l)}$$

- Symmetry property
- IR divergences
- Collinear factorization
- Unitarity cut

$$\mathscr{F}^{(l),\mathrm{ansatz}} = \sum_i {\color{red}C_i} I_i^{(l)}$$

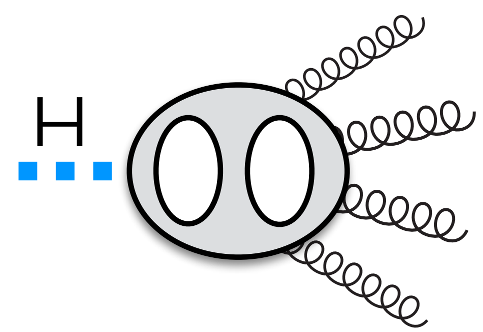
Application:
two-loop four-point form factor

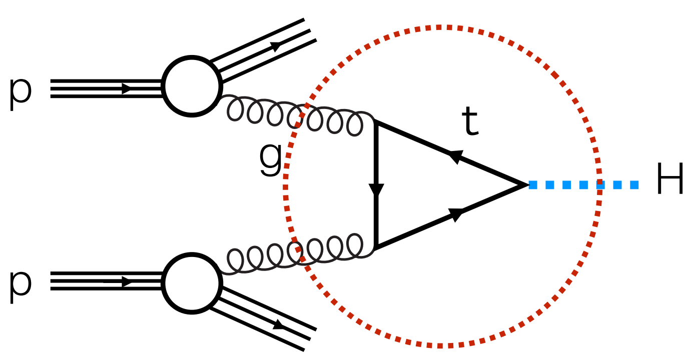
Form factors

We consider two-loop four-point form factor in N=4 SYM:

$$\mathcal{F}_{\mathcal{O},4} = \int d^4x\, e^{-iq\cdot x} \langle 1_\phi, 2_\phi, 3_\phi 4^+ \,|\, \mathrm{tr}(\phi^3)(x) \,|\, 0 \rangle$$

It is a N=4 version of Higgs+4-parton amplitudes in QCD:





$$\xrightarrow{m_t \to \infty}$$

$$\mathcal{L}_{\mathrm{eff}} = \hat{C}_0 H \mathrm{tr}(F^2) + \mathcal{O}\left(\frac{1}{m_{\mathrm{t}}^2}\right)$$

Form factors

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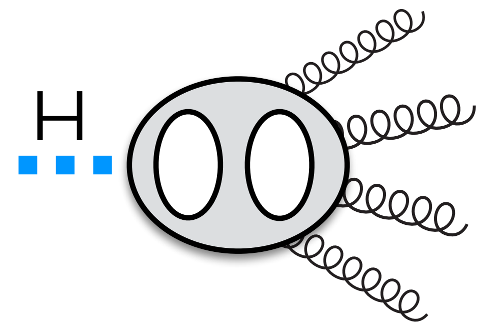


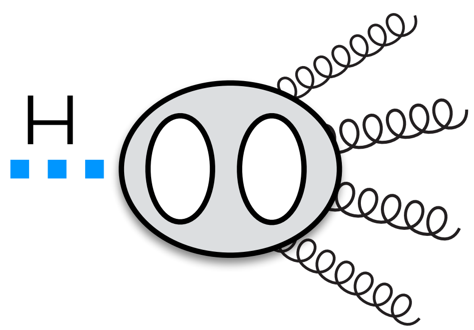
Five-point two-loop amplitudes are at frontier and under intense study:

There have been many massless five-point two-loop amplitudes obtained in analytic form. See e.g. Abreu, Dormans, Cordero, Ita, Page 2019 and many others….

For five-point two-loop amplitudes with one massive leg, so far only one result is available: $u\bar{d} \to W^+ b\bar{b}$ Badger, Hartanto, Zoia 2021

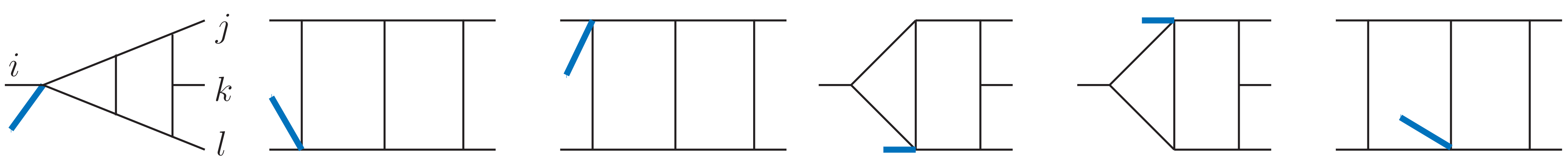
Form factors

Our result provides a first two-loop five-point example with a color-singlet off-shell leg.



$$\mathcal{F}_{\mathcal{O},4} = \int d^4x\, e^{-iq\cdot x} \langle 1_\phi, 2_\phi, 3_\phi 4^+ \,|\, \mathrm{tr}(\phi^3)(x) \,|\, 0 \rangle$$

$$\{s_{12}, s_{23}, s_{34}, s_{14}, s_{13}, s_{24}, \mathrm{tr}_5\}; \qquad \mathrm{tr}_5 = 4i\varepsilon_{p_1p_2p_3p_4}$$



Planar master integrals have been evaluated recently.

Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020

Canko, Papadopoulos, Syrrakos 2020

Ansatz

$$\mathscr{F}_{\mathscr{O},4} = \int d^4x\, e^{-iq\cdot x} \langle 1_\phi, 2_\phi, 3_\phi 4^+ \,|\, \mathrm{tr}(\phi^3)(x) \,|\, 0 \rangle$$

Tree-level: $$\mathcal{F}_4^{(0)} = \mathcal{F}^{(0)}_{\mathrm{tr}(\phi_{12}^3)}(1^\phi, 2^\phi, 3^\phi, 4^+) = \frac{\langle 31 \rangle}{\langle 34 \rangle \langle 41 \rangle}\,.$$

One-loop: $$\mathcal{F}_4^{(1)} = \mathcal{F}_4^{(0)} \mathcal{I}^{(1)} = \mathcal{F}_4^{(0)} \Big(B_1\, \mathcal{G}_1^{(1)} + B_2\, \mathcal{G}_2^{(1)} \Big)$$

$$B_1 = \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle}\,, \quad B_2 = \frac{\langle 14 \rangle \langle 23 \rangle}{\langle 13 \rangle \langle 24 \rangle}\,, \quad B_1 + B_2 = 1\,,$$

$$\Big(\sum_a B_a\, \mathcal{G}_a^{(1)} \Big)^2 - \Big[\sum_a B_a \big(\mathcal{G}_a^{(1)} \big)^2 \Big] \propto B_1 B_2 \big(\mathcal{G}_1^{(1)} - \mathcal{G}_2^{(1)} \big)$$

Ansatz

$$\mathscr{F}_{\mathscr{O},4} = \int d^4x\, e^{-iq\cdot x}\langle 1_\phi, 2_\phi, 3_\phi 4^+ \,|\, \mathrm{tr}(\phi^3)(x) \,|\, 0\rangle$$

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Two-loop ansatz:

$$\boxed{\mathcal{F}_4^{(2)} = \mathcal{F}_4^{(0)}\Big(B_1\,\mathcal{G}_1^{(2)} + B_2\,\mathcal{G}_2^{(2)}\Big)}$$

$$\mathcal{G}_a^{(2)} = \sum_{i=1}^{221} c_{a,i} I_i^{(2),\mathrm{UT}}\,, \qquad \mathcal{G}_2^{(2)} = \mathcal{G}_1^{(2)}|_{(p_1\leftrightarrow p_3)}$$

Ansatz

$$\mathscr{F}_{\mathscr{O},4} = \int d^4x\, e^{-iq\cdot x} \langle 1_\phi, 2_\phi, 3_\phi 4^+ \,|\, \mathrm{tr}(\phi^3)(x) \,|\, 0 \rangle$$

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Constraints

IR divergences

Collinear factorization

Spurious pole

Unitarity cut

BDS ansatz

$$\mathcal{I}^{(2),\mathrm{BDS}} = \frac{1}{2}\big(\mathcal{I}^{(1)}(\epsilon)\big)^2 + f^{(2)}(\epsilon)\mathcal{I}^{(1)}(2\epsilon)$$

$$\mathcal{R}_n^{(2)} = \big[\mathcal{I}^{(2)} - \mathcal{I}^{(2),\mathrm{BDS}}\big]_{\mathrm{fin}} \xrightarrow{p_i \,\|\, p_{i+1}} \mathcal{R}_{n-1}^{(2)}$$

$$\mathcal{I}^{(2),\mathrm{BDS}} = \sum_{a=1}^{2} B_a \left[\frac{1}{2}\big(\mathcal{G}_a^{(1)}(\epsilon)\big)^2 + f^{(2)}(\epsilon)\mathcal{G}_a^{(1)}(2\epsilon)\right]$$

$$\Big(\sum_a B_a\, \mathcal{G}_a^{(1)}\Big)^2 - \Big[\sum_a B_a \big(\mathcal{G}_a^{(1)}\big)^2\Big] \propto B_1 B_2 \big(\mathcal{G}_1^{(1)} - \mathcal{G}_2^{(1)}\big)$$

Constraints

IR divergences

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Unitarity cut

Constraints	Parameters left
Symmetry of $(p_1 \leftrightarrow p_3)$	221
IR (Symbol)	82
Collinear limit (Symbol)	38
Spurious pole (Symbol)	31
IR (Function)	26
Spurious pole (Funcion)	25
Collinear limit (Funcion)	18

Constraints

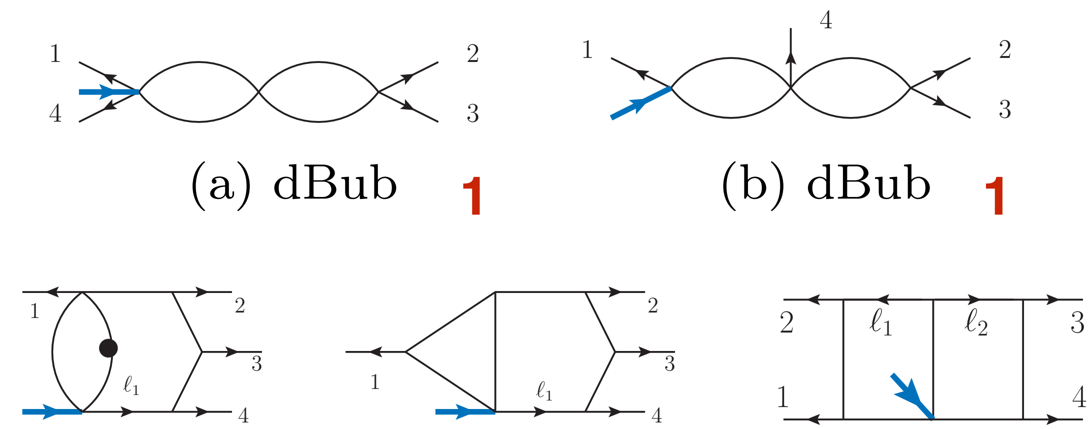
IR divergences

Collinear factorization

Spurious pole

Unitarity cut

Remaining 18 parameters can be fixed by knowing master integrals:



(a) dBub **1**

(b) dBub **1**

(c) BPb **4**

(d) TP **8**

(e) dBox2c **4**

Constraints

IR divergences

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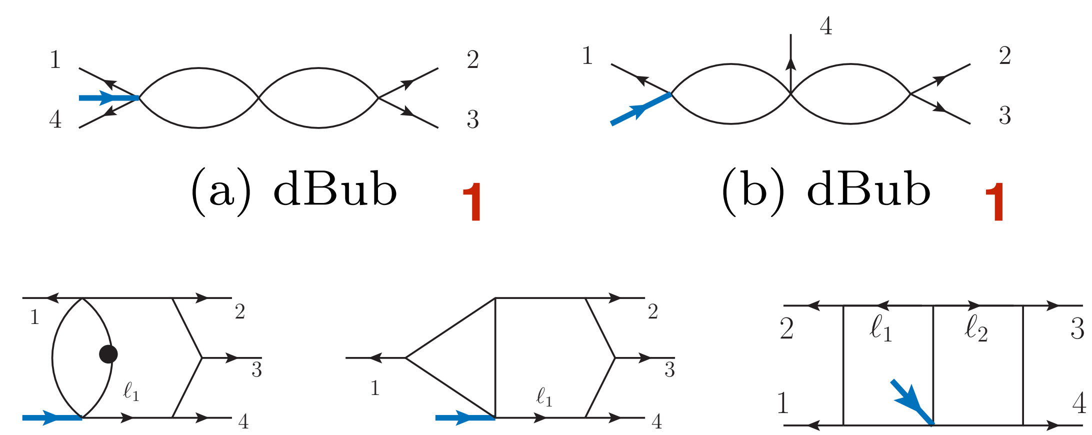


(a) dBub **1** (b) dBub **1**

(c) BPb **4** (d) TP **8** (e) dBox2c **4**

$\mathrm{tr}_5 \times \mu^2$

$\mathcal{O}(\epsilon^1)$

Constraints

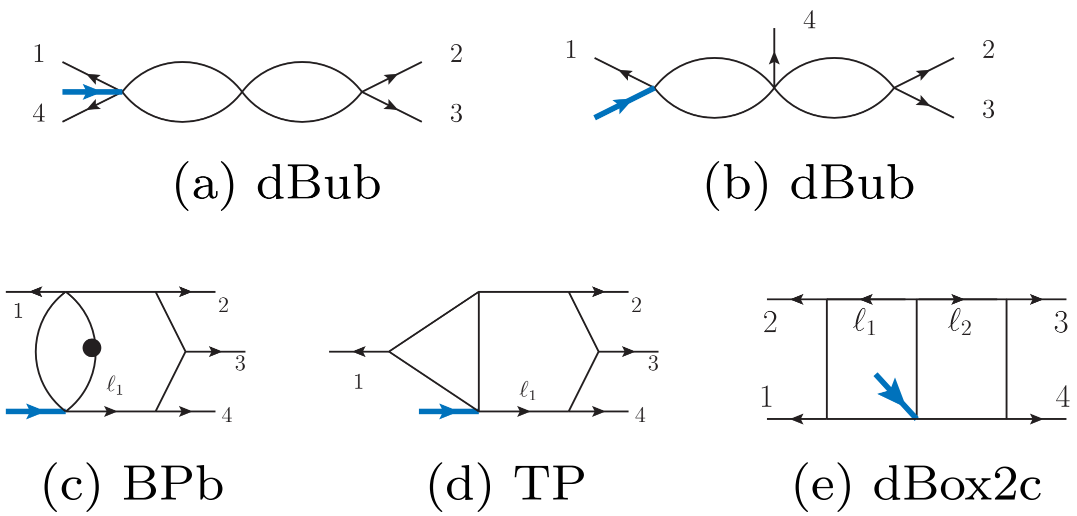
IR divergences

Collinear factorization

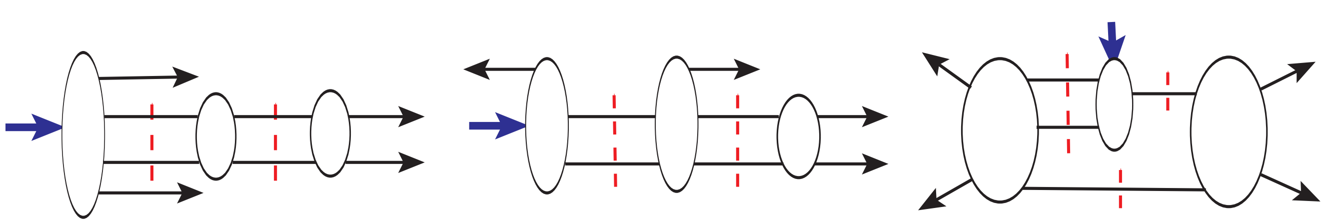
Spurious pole

Unitarity cut

Remaining 18 parameters are related to master integrals:



Can be fixed via **simple two-double cuts**:



A summary

Constraints	Parameters left
Symmetry of $(p_1 \leftrightarrow p_3)$	221
IR (Symbol)	82
Collinear limit (Symbol)	38
Spurious pole (Symbol)	31
IR (Function)	26
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If keeping only to ϵ^0 order	14
Simple unitarity cuts	0

A summary

Substituting in the master integral results, we have the full analytic form in GPLs, and they can be evaluated with GiNaC to ‘arbitrary’ high precision:

	$\mathcal{F}^{(2)}/\mathcal{F}^{(0)}$
ϵ^{-4}	8
ϵ^{-3}	$-10.88862656444854378 + 25.13274122871834590 8i$
ϵ^{-2}	$-31.87267267237051725 8 - 16.55801771198102864 4i$
ϵ^{-1}	$-24.70288908248107067 3 - 2.99232292947494907 51i$
ϵ^{0}	$-82.90201473067634238 3 - 129.78151092480602830i$

up to finite order with the kinematics: $\{s_{12} = 241/25$, $s_{23} = -377/100$, $s_{34} = 13/50$, $s_{14} = -161/100$, $s_{13} = s_{24} = -89/100$, $\mathrm{tr}_5 = \sqrt{1635802}/2500i\}$.

Technical points: symbol letters

$$\mathrm{Sym}(\mathcal{R}_4^{(2)}) = \sum_i c_i W_{i_1} \otimes W_{i_2} \otimes W_{i_3} \otimes W_{i_4}$$

Building blocks

Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020

$$x_{ijkl}^{\pm} = \frac{1 + s_{ij} - s_{kl} \pm \sqrt{\Delta_{3,ijkl}}}{2 s_{ij}},$$

$$y_{ijkl}^{\pm} = \frac{\mathrm{tr}_{\pm}(ijkl)}{2 s_{ij} s_{il}},$$

$$z_{ijkl}^{\pm\pm} = 1 + y_{ijkl}^{\pm} - x_{lijk}^{\pm},$$

$$\Delta_{3,ijkl} = \mathrm{Gram}(p_i + p_j, p_k + p_l),$$

$$\mathrm{tr}_{\pm}(ijkl) = s_{ij} s_{kl} - s_{ik} s_{jl} + s_{il} s_{jk} \pm \mathrm{tr}_5,$$

$$\mathrm{tr}_5 = 4 i \epsilon_{\mu\nu\rho\sigma} p_1^{\mu} p_2^{\nu} p_3^{\rho} p_4^{\sigma}$$

Most complicated letters:

$$X_1(p_i + p_j, p_k, p_l) = \frac{u_{ij} x_{ijkl}^{+} - u_{ijl}}{u_{ij} x_{ijkl}^{-} - u_{ijl}},$$

$$X_2(p_i + p_j, p_k + p_l) = \frac{x_{ijkl}^{+}}{x_{ijkl}^{-}},$$

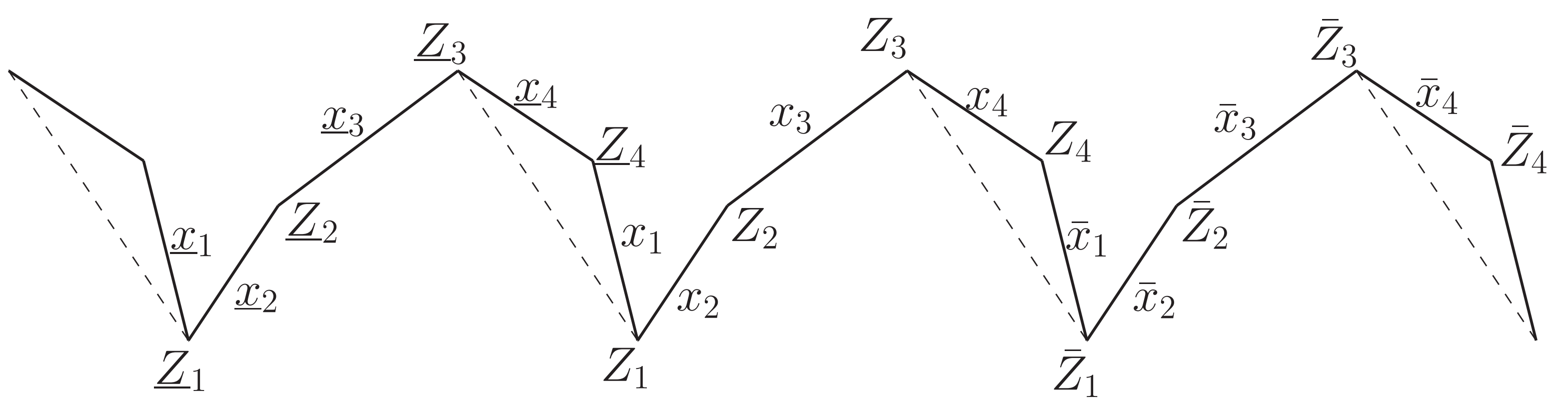
$$Y_1(p_i, p_j, p_k, p_l) = \frac{\mathrm{tr}_{+}(ijkl)}{\mathrm{tr}_{-}(ijkl)} = \frac{y_{ijkl}^{+}}{y_{ijkl}^{-}},$$

$$Y_2(p_i, p_j, p_k, p_l) = \frac{y_{ijkl}^{+} + 1}{y_{ijkl}^{-} + 1},$$

$$Z(p_i, p_j, p_k, p_l) = \frac{z_{ijkl}^{++} z_{ijkl}^{--}}{z_{ijkl}^{+-} z_{ijkl}^{-+}}.$$

Technical points: collinear limit of form factors

Dual momentum space



$$Z_4 = Z_3 + \delta \frac{\langle \bar{1}\bar{2}13 \rangle}{\langle \bar{1}\bar{2}12 \rangle} Z_2 + \tau\delta \frac{\langle \bar{2}123 \rangle}{\langle \bar{1}\bar{2}12 \rangle} \bar{Z}_1 + \eta \frac{\langle \bar{1}123 \rangle}{\langle \bar{1}\bar{2}12 \rangle} \bar{Z}_2$$

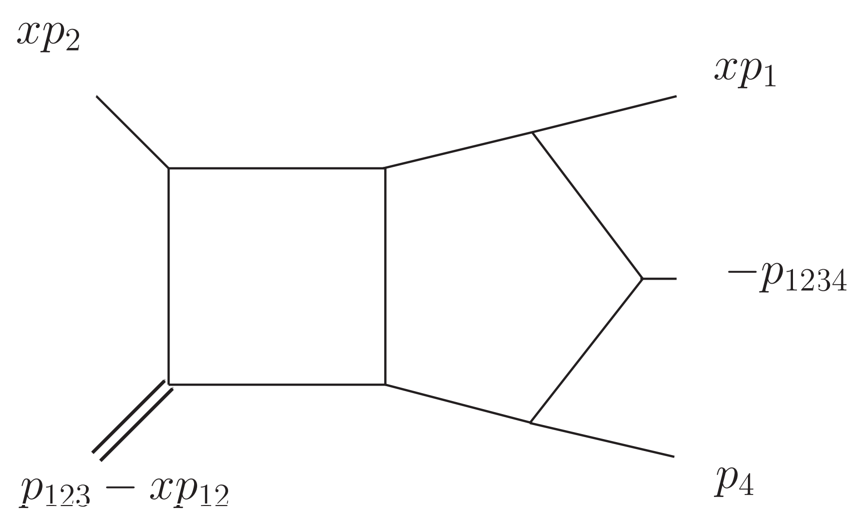
Technical points: numerical computation

Master integrals are evaluated in multiple polylogarithm.

Canko, Papadopoulos, Syrrakos 2020

A different set of kinematics are chosen.

$\{q_1, q_2, q_3, q_4, q_5\}$ with q_1 massive $\quad \{x, S_{12}, S_{23}, S_{34}, S_{45}, S_{51}\}$.



$$q_1 \to p_{123} - xp_{12},\ q_2 \to p_4,\ q_3 \to -p_{1234},\ q_4 \to xp_1$$

Summary and outlook

Summary and outlook

We present a first analytic computation of a two-loop five-point scattering with one color-singlet off-shell leg.

We develop a new bootstrap strategy based on master integral expansion, which applies efficiently for this case.

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We present a first analytic computation of a two-loop five-point scattering with one color-singlet off-shell leg.

We develop a new bootstrap strategy based on master integral expansion, which applies efficiently for this case.

Outlook:

Consider more general observables.

Study the new constraints beyond collinear limit, such as OPE limit, Regge limit.

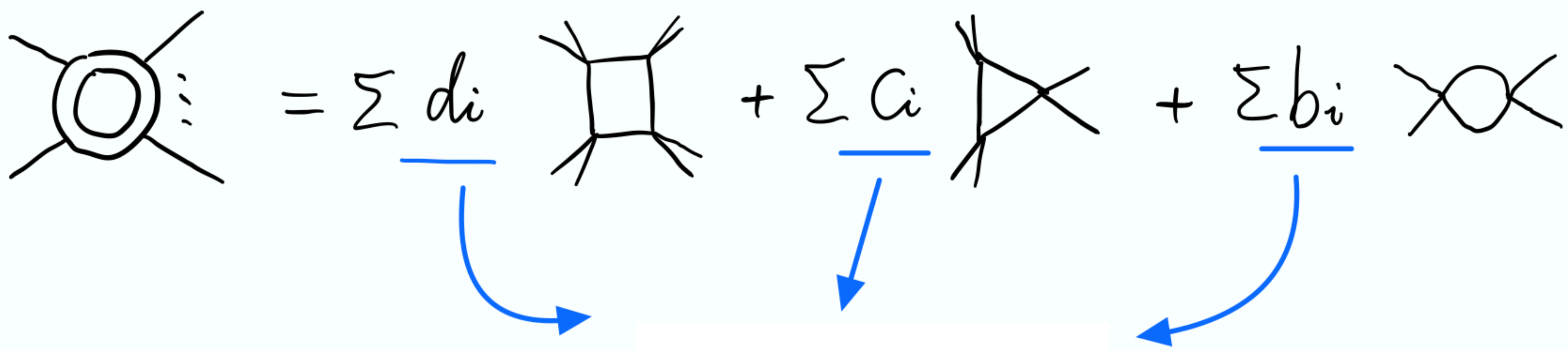
Hidden analytic structure, such as Qbar-like eqn.

Thank you!

Extra slides

Unitarity cuts

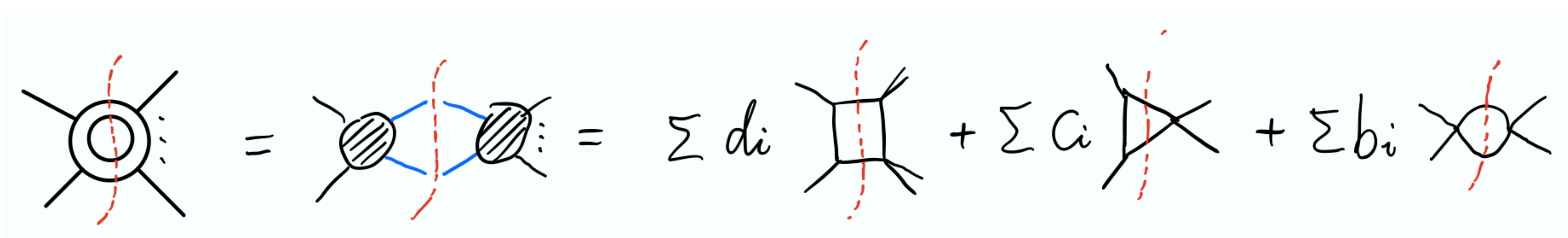
Consider one-loop amplitudes:



What we really want

Unitarity cuts

We can perform unitarity cuts:



and from tree products, we derive the coefficients more directly.

Cutkosky cutting rule: