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Bootstrapping a two-loop four-point form factor

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ITP, CAS

Based on the work to appear with Yuanhong Guo (郭圆宏)、Lei Wang (王磊)

Generic strategy of loop computation



Generic strategy of loop computation



Complicated intermediate expressions

Compact analytic form

Tw**Olassipat**i**Retylogarithmps**litudes in N=4 for Amplitudes and Wilson Loops



[[]Del Duca, Duhr, Smirnov 2010]

17 page complicated functions

 $R_{6,WL}^{(2)}(u_1, u_2, u_3) =$ $(\frac{u_1}{u_1}, \frac{u_1 + u_2}{u_1 + u_2};$ $+\frac{1}{24}\pi^2 G$ $+\frac{1}{24}\pi^2 G$ $\overline{u_1}^{, \overline{u_1 + u_3}}_{1 \ 1 \ 1}$ $-u_1'u_1 + u_2$ $u_3 - 1$ $\overline{u_2}' \, \overline{u_1 + u_2}$ $1 - u_2$, $u_2 + u_3 - 1$ 24 $\begin{array}{c} \left(0, \frac{1}{u_1}, 0, \frac{1}{u_1 + u_2}; 1\right) = \frac{1}{2} \bigcirc \left(1, u_1, \frac{1}{u_1}, \frac{1}{u_1}; 1\right) = \frac{1}{2} G\left(0, \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1\right) = \frac{1}{2} G\left(0, \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1 + u_3}; 1\right) \\ \left(1, \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1 + u_1}; 1, \frac{1}{u_1 + u_3}; 1\right) = \frac{1}{2} G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_1}; 1\right) + \frac{1}{u_1 + u_2}; 1 = \frac{1}{2} G\left(0, \frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1, \frac{1}{u_1 + u_3}; 1\right) = \frac{1}{2} G\left(0, \frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1, \frac{1}{u_1 + u_2}; 1\right) = \frac{1}{2} G\left(0, \frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1, \frac{1}{u_1 + u_2}; 1\right) = \frac{1}{2} G\left(0, \frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1, \frac{1}{u_1 + u_2}; 1, \frac{1}{u_1 + u_2}; 1, \frac{1}{u_1 + u_2}; 1, \frac{1}{u_1 + u_2}; 1\right) = \frac{1}{2} G\left(0, \frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1, \frac{1}{u_1 + u_2};$ $(1)^{-1} + G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_2 + u_3}; 1\right)$ -, 1; 1) + $\begin{array}{c} \overline{(1-1)}, \overline{(1-u_1)}, \overline$ $\left(\begin{array}{c} 0, \\ \overline{u_1 + u_3 - 1}, \\ \overline{1 - u_3}, \\ \overline{1 - u_3}, \\ \overline{1 - u_3} \end{array} \right)$ $\begin{array}{c} 1 \\ \hline 1 \\ \hline -u_3;1 \\ \end{array} + \frac{1}{4}G\left(0, \frac{u_3 - 1}{u_2 + u_3 - 1}, 0, \frac{1}{1 - u_3}, \frac{1}{$ $\frac{1}{G}\left(0, \frac{u_3 - 1}{2}\right)$ $u_3 - 1$ $(\underbrace{u_2+u_3-1}_{1}, \underbrace{u_2+u_3-1}_{1}, \frac{1}{1-u_2}; 1)$ $\frac{1}{u_3}, 0; 1 + \frac{1}{2}G$ $-\frac{1}{4}G$ 0.1.1 $(\overline{1-u_1}, \overline{u_1+u_2-1})$

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n=1) 1 (1 n=1 1)		
$\frac{1}{u_1 + u_2 - 1}$, $1, 0; 1 - \frac{1}{4}G\left(\frac{1}{1 - u_1}, \frac{1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}, 0; 1\right) +$	$\frac{1}{4}G$	÷
$\frac{u_2-1}{1}$ $\frac{1}{1}$ $\frac{1}{1}$ $\frac{1}{2}$ $G\left(\frac{1}{1}$ $\frac{u_2-1}{1}$ $\frac{1}{1}$ 1	$\frac{1}{G}$	1.1
$u_1 + u_2 - 1^{-}1 - u_1^{} / 4^{-} (1 - u_1^{-}u_1 + u_2 - 1^{-}1 - u_1^{-}1 - u_1^{} / 1 - u_1^{} / 1$	11	2.1
$\frac{u_1}{u_1+u_2-1}$, $\frac{u_2}{u_1+u_2-1}$, 1; 1 +	20	7 (1-1
$u_2 - 1$, $u_2 - 1$, 1 , 1 , $-G(\frac{1}{2}, 0, 0, \frac{1}{2}, 1) +$	40	: (1
$i_1 + i_2 - 1$ $i_1 + i_2 - 1$ $1 - i_1$ $(i_1 - i_2)$	8.1	Y-1
$u_1 + u_2$; 1) - G $\left(\overline{u_1}, 0, 0, \overline{u_2}, 1 \right)$ + $\frac{2}{2}$ G $\left(\overline{u_1}, 0, 0, \overline{u_1 + u_2}, 1 \right)$ -	313	$\left(\frac{1}{1-1}\right)$
$\frac{1}{1-1} = \frac{1}{G} \left(\frac{1}{2} \otimes \frac{1}{2} + \frac{1}{2} = 1 \right) = \frac{1}{G} \left(\frac{1}{2} \otimes \frac{1}{2} + \frac{1}{2} = 1 \right) = \frac{1}{G} \left(\frac{1}{2} \otimes \frac{1}{2} + \frac{1}{2} = 1 \right) = \frac{1}{G} \left(\frac{1}{2} \otimes \frac{1}{2} + \frac{1}{2}$	100	0.0
u_1+u_2 / 4 $(u_1 \cdots u_1 \cdot u_1 + u_1 \cdot)$ 4 $(u_1 \cdots u_2 \cdot u_1 + u_2 \cdot)$	11	
$\left(\frac{1}{u_1+u_2}, 1\right) - \frac{1}{4}G\left(\frac{1}{1-u_2}, 1, \frac{1}{u_1}, 0, 1\right) +$	- 29 (0,0,
$\frac{1}{1}$, $\frac{1}{1}$, $\frac{1}{1}$, $\frac{1}{1}$, $\frac{1}{2}$,	100	0.0.112
$1-u_2$, $1-u_2$, $1-4$, $(1-u_2, u_2+u_3-1)$, $(1-u_2, u_3+u_3-1)$	11	
$\frac{a_1}{a_2+a_1-1}$, $0, \frac{a_1}{1-a_2}$, $1 + \frac{a}{4}G\left(\frac{a_1}{1-a_2}, \frac{a_2}{a_2+a_3-1}, 1, 0; 1\right) -$	- 29 (0,0,121
$u_1 - 1$ $(u_1 - 1) + \frac{1}{G}(1) - u_2 - 1$ $(u_1 - 1) + \frac{1}{G}(1) - u_3 - 1$ $(u_1 - 1) + \frac{1}{G}(1) - \frac{1}{G}(1) + \frac{1}{G}(1) - \frac{1}{G}(1) + $	100	0.0
u + u - 1 1 - u / 4 (1 - u - u + u - 1 1 - u /) u - 1 1 1)	1.7	
$u_1 + u_1 - 1$, $1 - u_2$, $1 - u_2$, 1) -	29	$0, \frac{1}{1-x}$
$u_1 - 1$ $u_2 - 1$ $1:1 +$	100	1
$u_1 + u_2 - 1$ $u_2 + u_3 - 1$ / $u_2 - 1$ $u_2 - 1$ 1) . (1 1)	1.7	1
$u_2 + u_3 - 1$, $u_2 + u_3 - 1$, $1 - u_2$; $1 - G \left(u_2, 0, 0, -c; 1 \right) + G \left(u_3, 0, 0, -c; 1 \right)$	491	0.1-1
$\frac{1}{1}$ = 1 - G $\left(\frac{1}{2}, 0, 0, \frac{1}{2}, 1\right)$ + $\frac{1}{2}G \left(\frac{1}{2}, 0, 0, \frac{1}{2}, 1\right)$ -	1c(0. 1
i_1+i_2 / $(i_2 - i_1 / 2 - (i_2 - i_2+i_1 / 2 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 -$	1.1	1
$(\overline{u_1 + u_2}; 1) - \overline{4}G(\overline{u_2}; 0; \overline{u_2}; \overline{u_1 + u_2}; 1) - \overline{4}G(\overline{u_2}; 0; \overline{u_2}; \overline{u_2 + u_3}; 1) -$	±2	1-1
$-\frac{1}{-1}(1) - \frac{1}{-G}(\frac{1}{-1}, 1, \frac{1}{-0}, 1) +$	÷9(0.
u_2+u_3 / 4 (1- u_3 u_2 / 1 1) 1.(1 u_3-1)	1.1	1
$(1-u_1, 1, \frac{1}{1-u_1}, 1) + \frac{1}{4}G\left((1-u_1, \frac{1}{u_1+u_2-1}, 0, 1, 1)\right) -$	321	1-4
$\frac{u_1 - 1}{1 + u_1 - 1}$, $0, \frac{1}{1 + u_2}$, 1 + $\frac{1}{2}G\left(\frac{1}{1 + u_2}, \frac{u_1 - 1}{1 + u_2}, 1, 0; 1\right)$ -	÷9(0.
$u_1 + u_2 - 1 = 1 = u_1 / u_1 + u_2 - 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1$	1.01	11
$u_1 + u_2 - 1$, $\overline{1 - u_3}$, $\overline{v_1, u_1}$, $\overline{1 - u_2}$, $\overline{u_1 + u_2 - 1}$, $\overline{1 - u_3}$, $\overline{u_1, u_2}$, $\overline{u_1 + u_2 - 1}$, $\overline{1 - u_3}$, $\overline{u_1, u_2}$, $\overline{u_2, u_2}$, $\overline{u_1, u_2}$, $\overline{u_2, u_2}$	1	71-4
$\frac{u_1-1}{1-u_1-1}$, $\frac{1}{1-u_1}$,	÷9(0, 4120.
$u_1 - 1$ $u_1 - 1$, 79 π^4	10	
$u_1 + u_2 - 1$ $u_1 + u_3 - 1$ $u_1 + u_3 - 1$ 300 $+$	11	
$\frac{u_1 - 1}{u_1 + u_2 - 1}$, $\frac{u_1 - 1}{u_2 + u_2 - 2}$, $\frac{1}{1 - u_2}$; 1) - G($\frac{1}{u_2}$, 0, 0, $\frac{1}{u_2}$; 1) -	₩(0, 4120-
$(1) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\left(\frac{1}{2} + 0 + \frac{1}{2} + 1 \right) + \frac{1}{2} \left(\frac{1}{2} + 1 \right) + 1$	10	0.0222.
$(1)^{+} = \frac{1}{2} \left(\frac{u_1}{u_2}, 0, 0, \frac{u_1 + u_2}{u_1 + u_2}, 1 \right)^{+} = \frac{1}{2} \left(\frac{u_1}{u_2}, 0, 0, \frac{u_2 + u_2}{u_2 + u_2}, 1 \right)^{-}$	4-1	

 $\begin{array}{c} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1$

$$\begin{split} & \left| c \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) - \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) - \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} + 1 \right) + \frac{1}{2^2} \left(\frac{1}{1+2} + \frac{1}{1+2} +$$

 $\begin{array}{c} \frac{m-1}{m+1} (1) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right) - \frac{1}{2^{2}} \left(k = \frac{m-1}{2}, \frac{1}{m}, 1 \right$

$1_{c}(1, 1, 1, 1, 1)$	1 _c (1 - aa1)	$\frac{1}{2}g\left(p_{12}, 0, 1, \frac{1}{2}, 1\right) + \frac{1}{2}g\left(p_{12}, 0, \frac{1}{2}, \frac{1}$
$\frac{4}{1-u_2} \left(\frac{1-u_2}{1-u_2}, \frac{v_{213}}{1-u_2}, \frac{1-u_2}{1-u_2}, \frac{1}{1-u_2} \right)$	$\left(\frac{1}{4}\right)\left(\frac{1}{1-u_2}, \frac{v_{221}, v, v, 1}{1}\right)^{-1}$	2° (
$\frac{1}{4}G\left(\frac{1}{1-u_1}, v_{211}, 0, 1; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_2}, v_{211}, 0, 1; 1\right)$	$u_2, v_{221}, 0, \frac{1}{1-u_2}, 1 = \frac{1}{4} - \frac{G}{4} \left(\frac{1}{1-u_2}, v_{221}, 1, 0, 1 \right) - \frac{1}{4} - \frac{1}$	$\left(\frac{1}{4}\right) \left(\frac{1}{122}, \frac{1}{2}, \frac{1}{2}, \frac{1}{1}, \frac{1}{2}, \frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
$-\frac{1}{2}G\left(\frac{1}{1-u_2}, v_{211}, 1, \frac{1}{1-u_2}; 1\right) + \frac{1}{4}G$	$\left(\frac{1}{1-u_2}, v_{210}, \frac{1}{1-u_2}, 0; 1\right)$ -	$\frac{1}{2}\mathcal{G}\left(v_{120}, 1, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{120}, \frac{1}{1-u_1}; 1\right)$
$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-r}, v_{211}, \frac{1}{1-r}, 1; 1\right) + \frac{1}{2}\mathcal{G}$	$\left(\frac{1}{1-r_{12}}, r_{22}, \frac{1}{1-r_{12}}, \frac{1}{1-r_{12}}, 1\right) -$	$-\frac{5}{4}\mathcal{G}\left(v_{123}, \frac{1}{1-w}, 1, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{123}, \frac{1}{1-w}\right)$
$\frac{1}{2}\left(\frac{1}{1}, 0, 0, v_{v+1}\right) - \frac{1}{2}\left(\frac{1}{1}\right)$	$= 0.0, rm; 1 - \frac{1}{2} c \left(\frac{1}{1}, 0, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, 0, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, 0, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, 0, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, 0, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, 0, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, 0, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, 0, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, rm; 1 \right) - \frac{1}{2} c \left(\frac{1}{1}, \frac{1}{1$	$\frac{1}{2}\mathcal{G}\left(v_{122}, \frac{1}{1-v_{1}}, \frac{1}{1-v_{1}}, 1; 1\right) - \frac{1}{2}\mathcal{G}\left(v_{122}, \frac{1}{1-v_{1}}, \frac{1}{1-v_{1$
$\frac{1}{1}\left(1-u_{1}-1\right)$	$(1 - u_1 - 1 - u_2)$	1 (
$\frac{1}{2} \left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}, v_{221}; 1 \right) - \frac{1}{4} \right)$	$\left(\frac{1-u_0}{1-u_0}, 0, v_{212}, 1, 1\right)$ -	4^{μ} $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 $
$\frac{1}{4}G\left(\frac{1}{1-u_1}, 0, v_{312}, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}G$	$\left(\frac{1}{1-u_{0}}, 0, v_{221}, 1; 1\right) -$	$\frac{1}{4}$ $\left(v_{210}, \frac{1}{1-u_2}, 1, 1, 1\right) + \frac{1}{2}$ $\left(v_{210}, 0, 1, \frac{1}{1}\right)$
$-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, v_{321}, \frac{1}{1-u_1}; 1\right) - \frac{1}{2}\mathcal{G}$	$\left(\frac{1}{1-u_0}, \frac{1}{1-u_1}, 0, v_{312}; 1\right)$ -	$\frac{1}{2}\mathcal{G}\left(v_{233}, 1, 0, \frac{1}{1-u_2}, 1\right) - \frac{2}{4}\mathcal{G}\left(v_{233}, 1, 1, \frac{1}{1-u_2}, 1\right)$
$\frac{1}{3}\mathcal{G}\left(\frac{1}{1}, \frac{1}{1}, 0, v_{331}; 1\right) - \frac{3}{2}\mathcal{G}$	$\left(\frac{1}{1-1}, \frac{1}{1-1}, \frac{1}{1-1}, v_{111}; 1\right) -$	$-\frac{5}{4}\mathcal{G}\left(van, 1, \frac{1}{1-w}, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(van, 1, \frac{1}{1-w}\right)$
$\frac{1}{2}\left(\frac{1-u_1}{1},\frac{1-u_3}{1},\frac{1}{1},\frac{1}{1}\right)$	$-\frac{1}{6}\left(\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}\right) -$	$\frac{1}{2}\mathcal{G}\left(\operatorname{vun}, \frac{1}{1-\varepsilon}, 1, 0; 1\right) - \frac{5}{2}\mathcal{G}\left(\operatorname{vun}, \frac{1}{1-\varepsilon}\right)$
$\frac{1}{1}$ $\begin{pmatrix} 1 - u_1 \cdot 1 - u_3 \cdot 1 - u_1 \cdot 1 - u_1 \\ 1_{ab} \begin{pmatrix} 1 & 1 & 1 \\ & 1 & & 1 \end{pmatrix}$	$\frac{1}{1}$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ \end{pmatrix}$	$\frac{1}{2}(r_{m}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) + \frac{1}{2}(r_{m}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$\frac{1}{4} \left(\frac{1-u_1}{1-u_2}, \frac{1-u_3}{1-u_3}, \frac{v_{312}}{1-u_3}, \frac{1-u_3}{1-u_3}, \frac{1}{1} \right)$	$-\frac{2}{2} \frac{\nu}{(1-u_1}, \frac{1-u_2}{1-u_3}, \frac{v_{223}}{1-u_1}) -$	2° $\begin{pmatrix} m & 1 - m & 1 - m \\ 1 & 0 & 1 $
$\frac{1}{4}$ $\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{321}, \frac{1}{1-u_3}, 1\right)$	$-\frac{1}{4}G\left(\frac{1}{1-u_2}, u_{312}, 0, 1; 1\right) +$	$\frac{1}{2} \left(\left(1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 $
$-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{312}, 0, \frac{1}{1-u_1}, 1\right) - \frac{1}{4}\mathcal{G}$	$\left(\frac{1}{1-u_2}, u_{312}, 1, 0; 1\right) +$	$-\frac{1}{4}\mathcal{G}\left(v_{312}, 1, \frac{1}{1-u_3}, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{312}, 1, \frac{1}{1-u_3}, 1; 1\right)$
$\frac{1}{4}\hat{\varphi}\left(\frac{1}{1-u_{312}}, u_{312}, \frac{1}{u_{312}}, 0; 1\right) + \frac{1}{4}\hat{\varphi}\left(\frac{1}{u_{312}}, \frac{1}{u_{312}}, \frac{1}{u_{312}}, 0; 1\right)$	1	$-\frac{1}{2}\mathcal{G}\left(v_{312}, \frac{1}{1-u_2}, 1, 0; 1\right) - \frac{3}{4}\mathcal{G}\left(v_{312}, \frac{1}{1-u_1}\right)$
$\frac{1}{2}\left(\frac{1}{1}, v_{32}, \frac{1}{1}, 1; 1\right) + \frac{1}{2}$	$\left(\frac{1}{1}, u_{22}, \frac{1}{1}, \frac{1}{1}, 1\right) +$	$\frac{1}{2}\mathcal{G}\left(v_{112}, \frac{1}{1-v_{1}}, \frac{1}{1-v_{2}}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{211}, \frac{1}{1-v_{2}}, 1; 1\right)$
$\frac{4}{1}\begin{pmatrix} 1-u_1 & 1-u_1 \\ 1 & u_1-1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1-u_2 & 1-u_1 & 1-u_2 \\ 1_{c} \begin{pmatrix} 1 & & & u_1-1 & 1 \\ & & & u_1-1 & 1 \\ \end{pmatrix}$	$\frac{1}{2}g\left(v_{m}, \frac{1}{1}, 1, 1\right) - \frac{3}{2}g\left(0, \frac{1}{1}, \frac{1}{1}\right)$
$\frac{4}{1-u_1} \left(\frac{1-u_2}{u_1+u_2-1}, \frac{u_1+u_2-1}{u_1+u_2-1} \right)$	$\frac{1}{4} \sqrt{\left(1 - u_1^{-1} - u_1^{-1}\right)^{-1}} = \frac{1}{u_1 + u_1 - 1} \frac{1}{1 - u_1^{-1}} + \frac{1}{1 - u_1^{-1}} + \frac{1}{1 - u_1^{-1}}$	$\frac{1}{2} \begin{pmatrix} -1 & -u_1 \\ -1 & -u_2 \end{pmatrix} = \begin{pmatrix} -1 & -u_1 \\ -1 & -u_1 \end{pmatrix} = \begin{pmatrix} -u_1 & -u_1 \\ -u_1 & -u_1 \end{pmatrix} = \begin{pmatrix} -u_1 & -u_1 \\ -u_1 & -u_2 $
$\frac{1}{4}G\left(\frac{1-u_3}{1-u_3}, v_{312}, 0, 0; 1\right) - \frac{1}{4}G\left(\frac{1}{1-u_3}, v_{312}, 0, 0; 1\right)$	$\overline{u_1}$, v_{212} , $0, 1; 1$ + $\overline{4}G\left(\overline{1-u_1}, v_{212}, 0, \overline{1-u_2}, 1\right)$ -	$-\frac{4}{4}$ $\left(0, \frac{u_1}{u_1}, \frac{u_1 + u_2}{u_1 + u_2}, \frac{1}{2} \right) $ $M(0, u_1) = \frac{4}{4}$ $\left(0, \frac{1}{4} \right)$
$-\frac{1}{4}G\left(\frac{1}{1-u_2}, v_{312}, 1, 0; 1\right) - \frac{1}{2}G\left(\frac{1}{1-u_2}, v_{312}, 1, 0; 1\right)$	$\frac{1}{u_1}$, v_{212} , 1 , $\frac{1}{1-u_1}$, 1) +	$\frac{1}{4}G\left(0, \frac{1}{u_1}, \frac{1}{u_1 + u_1}; 1\right)H(0; u_1) - \frac{1}{4}G\left(0, \frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1\right)H(0; u_1) - \frac{1}{4}G\left(0, \frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1\right)H(0; u_1) - \frac{1}{4}G\left(0, \frac{1}{u_2}; 1, \frac{1}{u_1 + u_2}; 1, \frac{1}{u_1 $
$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-y_{1}}, v_{312}, \frac{1}{1-y_{2}}, 0; 1\right) - \frac{1}{2}\mathcal{G}$	$\left(\frac{1}{1-u_{1}}, v_{312}, \frac{1}{1-u_{2}}, 2; 1\right) +$	$-\frac{1}{4}G\left(0, \frac{u_2-1}{u_2+u_3-1}, \frac{1}{1-u_2}, 1\right)H(0; u_1) - \frac{1}{4}$
$\frac{1}{2}\left(\frac{1}{1}, r_{112}, \frac{1}{1}, \frac{1}{1}, 1\right)$	$\frac{1}{2} \mathcal{G} \left(\frac{1}{1}, v_{221}, 0, 0; 1 \right) -$	$\frac{3}{4}G\left(\frac{1}{u_1}, 0, \frac{1}{u_1 + u_2}; 1\right)H(0; u_1) + \frac{1}{7}G\left(\frac{1}{u_2}; 1\right)$
$\frac{1}{1}$ $\begin{pmatrix} 1 \\ -u_1 \\ -u_2 \\ -u_1 \\ -u_2 \\ -u_1 \\ -u_1 \\ -u_2 \\ -u_1 \\ -u_2 \\ -u_1 \\ -u_2 \\ -u_2 \\ -u_1 \\ -u_2 $	$\frac{4}{1-u_1}$ $\frac{1}{1-u_1}$	$\frac{1}{2}G\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)H(0; u_1) + \frac{1}{2}G\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{$
$\frac{4^{\mu}(1-u_1^{-1},u_2^{-1},u_1^{-1},u_1^{-1})}{1_{\mu}(1-u_1^{-1},u_1^{-1},u_1^{-1})} + \frac{4^{\mu}(1-u_1^{-1},u_1^{-1})}{1_{\mu}(1-u_1^{-1},u_1^{-1},u_1^{-1})}$	$u_1^{(1)} = u_1^{(1)} + u_2^{(1)} + u_3^{(1)} + u_3^$	$\frac{1}{2} \left(\frac{1}{2} + 1$
$\frac{1}{2} \left\{ \frac{1-u_1}{1-u_2}, v_{B1}, 1, \frac{1-u_1}{1-u_1}, 1 \right\} + \frac{1}{4} G$	$\left(\frac{1-u_1}{1-u_1}, v_{121}, \frac{1-u_1}{1-u_1}, w, 1\right) =$	4^{-1} $(u_1, u_3, u_1 + u_3, 1)$ $m(0, u_1) = \frac{1}{4} O(\frac{1}{1})$ $1 + (1 + u_1, u_3 + u_3, 1)$
$\frac{1}{2}G\left(\frac{1}{1-m}, v_{BB}, \frac{1}{1-m}, 1; 1\right) + \frac{1}{4}G$	$\left[\frac{1}{1-m_1}, \pi_{322}, \frac{1}{1-m_2}, \frac{1}{1-m_2}; 1\right] +$	$\frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{1}{u_2+u_3-1}, 1; 1\right)H(0; u_1) -$

 $\begin{array}{c} & = \left\{ \left(\sum_{i=1}^{n} \sum_{m \in m_{i} \to m_{i}}^{n} \right) \left(m + m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i} \to m_{i}}^{n} \right) \left(m + m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i} \to m_{i}}^{n} \right) \left(m + m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i} \to m_{i}}^{n} \right) \left(m + m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i} \to m_{i}}^{n} \right) \left(m + m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i} \to m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i} \to m_{i}}^{n} \right) \left(m + m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i} \to m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i}}^{n} \right) \left(m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \right) \right) \left(m + \frac{1}{2} \left(\sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \sum_{m \in m_{i}}^{n} \sum_$

[Del Duca, Duhr, Smirnov 2010]

"multiple(Goncharov)-polylogrithm function"



$$\begin{split} & \frac{1}{2} \left(\frac{1}{1+\alpha} - \frac{1}{1+\alpha} - \frac{1}{1+\alpha} - \frac{1}{1+\alpha} \right) (B(n_1) - \frac{1}{2} \left(\frac{1}{1+\alpha} - \frac{1}{1+\alpha} - \frac{1}{1+\alpha} + \frac{1}{1+\alpha} \right) (B(n_2) - \frac{1}{2} \left(\frac{1}{1+\alpha} - \frac{1}{1+\alpha} - \frac{1}{1+\alpha} + \frac{1}$$

$$\begin{split} & \frac{1}{12} (\max) = 1 \left\{ \begin{array}{l} \left\{ \max_{i=1}^{n} \left\{ \max$$

$$\begin{split} & \left[\left(\max_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^$$

$$\begin{split} & \frac{1}{4} \mathcal{G} \left(z_{22,1} + \frac{1}{1-z_{2}} + 1 \right) H \left(0, z_{1} \right) + \frac{1}{4} \mathcal{G} \left(z_{22,1} + \frac{1}{1-z_{2}} + 1 \right) \\ & \frac{1}{4} \mathcal{G} \left(z_{22,1} + \frac{1}{1-z_{2}} + 1 \right) H \left(0, z_{1} \right) + \frac{1}{4} \mathcal{G} \left(z_{22,1} + \frac{1}{1-z_{2}} + 1 \right) \\ & \frac{1}{4} \mathcal{G} \left(\frac{1}{z_{1}-1} - \frac{1}{z_{2}-1} + 1 \right) H \left(0, z_{1} \right) H \left(0, z_{2} \right) + \\ & \frac{1}{4} \mathcal{G} \left(\frac{1}{z_{1}-1} - \frac{1}{z_{2}-1} + 1 \right) H \left(0, z_{1} \right) H \left(0, z_{2} \right) + \\ & \frac{1}{4} \mathcal{G} \left(\frac{1}{z_{1}-1} - \frac{1}{z_{2}-1} + 1 \right) H \left(0, z_{1} \right) H \left(0, z_{2} \right) + \\ & \frac{1}{4} \mathcal{G} \left(\frac{1}{z_{1}-1} - \frac{1}{z_{2}-1} + 1 \right) H \left(0, z_{1} \right) H \left(0, z_{2} \right) + \\ & \frac{1}{4} \mathcal{G} \left(\frac{1}{z_{1}-1} - \frac{1}{z_{2}-1} + 1 \right) H \left(0, z_{1} \right) H \left(0, z_{2} \right) + \\ & \frac{1}{2} \mathcal{G} \left(\frac{1}{z_{1}-1} - \frac{1}{z_{2}-1} + 1 \right) H \left(z_{2} - \frac{1}{z_{2}-1}$$

$$\begin{split} & \frac{1}{2} \left(\frac{1}{2} \frac{1}{\sqrt{1-1}} - \frac{1}{\sqrt{1-1}} \right) \left[\frac{1}{2} \left(\frac{1}{\sqrt{1-1}} - \frac{1}{\sqrt{1-1}} \right) \left[\frac{1}{\sqrt{1-1}} - \frac{1}{\sqrt{1-1}} - \frac{1}{\sqrt{1-1}} - \frac{1}{\sqrt{1-1}} - \frac{1}{\sqrt{1-1}} \right] \right] \\ & + \frac{1}{\sqrt{1-1}} \left[\frac{1}{\sqrt{1-1}} - \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} - \frac{1}{\sqrt{1-1}} - \frac{1}{\sqrt{1-1}} - \frac{1}{\sqrt{1-1}} \right] \right] \\ & + \frac{1}{\sqrt{1-1}} \left[\frac{1}{\sqrt{1-1}} - \frac{1}{\sqrt{1-1}} + \frac{1}{\sqrt{1-1}} +$$

$$\begin{split} & \frac{n + n - 1}{n} + \frac{1}{n} \left(\frac{1}{n} e^{-2} B(\lambda, | \lambda_{1, n} + n_{1} \rangle - \frac{1}{n} e^{-2} B(\lambda, | \lambda_{1, n} + n_{1} \rangle + B(\lambda, | \lambda_{1, n} + n_{$$

$$\begin{split} & I(n,u) \in R(n,k) = -n_0^{-1} - \frac{1}{2} H(n,u) \in \left\{ (n,k) - \frac{n_0(n+1)}{2} - \frac{1}{2} H(n,u) \in I(n,k) - \frac{1}{2} H(n,u) = H(n,u) \in I(n,k) - \frac{1}{2} H(n,u) \in I(n,k) - \frac{1}{2$$

$$\begin{split} & \frac{1}{2} \operatorname{He}(\operatorname{He}(\operatorname{He}(\frac{1}{2}))) = \frac{1}{2} \operatorname{He}(\operatorname{He}(\operatorname{He}(\frac{1}{2}))) = \frac{1}{2} \operatorname{He}(\operatorname{He}(\operatorname{He}(\frac{1}{2}))) = \frac{1}{2} \operatorname{He}(\operatorname{He}(\frac{1}{2})) = \frac{1}{2} \operatorname{He}(\operatorname{He}((1+\frac{1}{2}))) = \frac{1}{2} \operatorname{He}(\operatorname{He}$$

 $\frac{1}{4}H(0; u_2) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{122}}\right) + \frac{1}{4}H(0; u_3) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{122}}\right) + \frac{1}{4}H(0; u_1) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{213}}\right)$ $\frac{1}{4}H(0; u_3) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{213}}\right) - \frac{1}{4}H(0; u_1) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{231}}\right) + \frac{1}{4}H(0; u_3) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{231}}\right)$ $\frac{1}{4}H(0; u_1) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{312}}\right) - \frac{1}{4}H(0; u_2) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{312}}\right) - \frac{1}{4}H(0; u_1) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{312}$ $\frac{1}{4}H(0; u_2) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{321}}\right) + \frac{1}{4}H(0; u_3) \mathcal{H}\left(1, 0, 1; \frac{1}{u_{123}}\right) + \frac{1}{4}H(0; u_1) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{123}}\right) + \frac{1}{4}H(0; u_1) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{123}$ $\frac{1}{4}H(0; u_2) \mathcal{H}\left(1, 0, 1; \frac{1}{u_{312}}\right) + \frac{1}{4}H(0; u_2) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{123}}\right) - \frac{1}{4}H(0; u_3) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{123}$ $\frac{1}{4}H(0; u_2) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{122}}\right) + \frac{1}{4}H(0; u_3) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{122}}\right) + \frac{1}{4}H(0; u_1) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{122}}\right)$ $\frac{1}{4}H(0; u_3) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{213}}\right) - \frac{1}{4}H(0; u_1) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{231}}\right) + \frac{1}{4}H(0; u_3) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{231}$ $\frac{4}{1}H(0;u_1)\mathcal{H}\left(1,0,1;\frac{v_{213}}{v_{312}}\right) - \frac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right) - \frac{1}{4}H(0;u_1)\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right) - \frac{1}{4}H(0;u_$ $\frac{1}{4}H(0;u_2)\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right) + H(0;u_2)\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right) - H(0;u_3)\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)$ $H(0; u_1) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{231}}\right) + H(0; u_3) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{231}}\right) + H(0; u_1) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{312}}\right)$ $H(0; u_2) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{312}}\right) - \frac{3}{2} \mathcal{H}\left(0, 0, 0, 1; \frac{1}{u_{123}}\right) - \frac{3}{2} \mathcal{H}\left(0, 0, 0, 1; \frac{1}{u_{233}}\right)$ $\begin{array}{c} \frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{312}}\right) - 3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{132}}\right) - 3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{132}}\right) - 3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{213}}\right) - 3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{213}}\right) - 3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{322}}\right) - \frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{233}}\right) - \frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{312}}\right) - \frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{323}}\right) - \frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{323}}\right) + \frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{333}}\right) + \frac$ $\frac{1}{4}\mathcal{H}\left(0, 1, 1, 1; \frac{1}{v_{123}}\right) + \frac{1}{4}\mathcal{H}\left(0, 1, 1, 1; \frac{1}{v_{132}}\right) + \zeta_3 H\left(0; u_1\right) + \zeta_3 H\left(0; u_2\right) + \zeta_3 H\left(0; u_3\right) + \zeta_3 H\left(0; u_2\right) + \zeta_3 H\left(0; u_3\right) + \zeta_3 H\left(0; u_3\right$ $\frac{5}{2}\zeta_{3}H(1;u_{1}) + \frac{5}{2}\zeta_{3}H(1;u_{2}) + \frac{5}{2}\zeta_{3}H(1;u_{3}) + \frac{1}{2}\zeta_{3}\mathcal{H}\left(1;\frac{1}{u_{123}}\right) + \frac{1}{2}\zeta_{3}\mathcal{H}\left(1;\frac{1}{u_{231}}\right) + \frac{1}{2}\zeta_{3}\mathcal{H}\left(1;\frac{1}{u_{231$ $\frac{1}{2}\zeta_{3}\mathcal{H}\left(1;\frac{1}{u_{312}}\right) - \frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{123}}\right) - \frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{231}}\right) - \frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{312}}\right) - \frac{1}{2}\mathcal{H}\left(1,0,$ $\frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{123}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{132}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{213}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{213}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{231}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{312}}\right) + \frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{31$ $\frac{1}{4}\zeta_{3}\mathcal{H}\left(1;\frac{1}{v_{321}}\right) + \frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{213}}\right) + \frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{231}}\right) + \frac{1}{4}\mathcal{H}\left(0,1,$ $\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{321}}\right) + \frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{123}}\right) + \frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{132}}\right) + \frac{1}{4}\mathcal{H}\left(1,0$ $\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{231}}\right) + \frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{332}}\right) + \frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{232}}\right) + \frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{123}}\right) - \frac{1}{4}\mathcal{H}\left(1,1$ $\begin{array}{c} \overset{\mathbf{a}}{} \\ \overset{\mathbf{a}}{} & \overset{\mathbf{a}}{} \\ \overset{\mathbf{a}}{} \\ \overset{\mathbf{a}}{} \\ \overset{\mathbf{a}}{} \\ \overset{\mathbf{a}}{} &$

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Result can be remarkably simple

17 pages =

[Goncharov, Spradlin, Vergu, Volovich 2010]

$$\begin{split} &\sum_{i=1}^{3} \left(L_4 \left(x_i^+, x_i^- \right) - \frac{1}{2} \operatorname{Li}_4 \left(1 - 1/u_i \right) \right) - \frac{1}{8} \left(\sum_{i=1}^{3} \operatorname{Li}_2 \left(1 - 1/u_i \right) \right)^2 + \frac{1}{24} J^4 + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72} J^4 + \frac{\pi^2}{12} J^4 + \frac{\pi^2}{12} J^4 + \frac{\pi^2}{12} J^4 + \frac{\pi^4}{12} J^4 + \frac{\pi^4}{1$$

a line result in terms of classical polylogarithms!

Such simplicity is totally unexpected using traditional Feynman diagrams!

Mathematical tool: "symbol"

From function to "Symbol"

Recursion definition of "Symbol":

$$df_k = \sum_i f_{k-1}^i dLog(R_i),$$
 Symbol $(f_k) = \sum_i Symbol(f_{k-1}^i) \otimes R_i$

Function	Differential	symbol
R	d R	0
log(R)	d log(R)	R
log(R1)log(R2)	logR1 dlogR2+logR2 dlogR1	$R1 \otimes R2 + R2 \otimes R1$
Li2(R)	Li1(R) dlogR	-(1-R)⊗ R

Symbol

Algebraic relations:

 $R_1 \otimes \ldots \otimes (c R_i) \otimes \ldots \otimes R_n = R_1 \otimes \ldots \otimes R_i \otimes \ldots \otimes R_n \qquad \textbf{C is const}$ $R_1 \otimes \ldots \otimes (R_i R_j) \otimes \ldots \otimes R_n = R_1 \otimes \ldots \otimes R_i \otimes \ldots \otimes R_n + R_1 \otimes \ldots \otimes R_j \otimes \ldots \otimes R_n$

Make it easy to prove non-trivial identities, e.g.:

$$\begin{aligned} \operatorname{Li}_{2}(z) &= -\operatorname{Li}_{2}(1-z) - \log(1-z)\log(z) + \frac{\pi^{2}}{6} \\ \operatorname{Li}_{2}(z) &= -\operatorname{Li}_{2}\left(\frac{1}{z}\right) - \frac{1}{2}\log^{2}(-z) - \frac{\pi^{2}}{6} / ; z \notin (0, 1) \\ \operatorname{Li}_{2}\left(\frac{x}{1-y}\right) + \operatorname{Li}_{2}\left(\frac{y}{1-x}\right) - \operatorname{Li}_{2}(x) - \operatorname{Li}_{2}(y) - \operatorname{Li}_{2}\left(\frac{xy}{(1-x)(1-y)}\right) = \operatorname{Log}(1-x)\operatorname{Log}(1-y) \end{aligned}$$

Applications





A better strategy:

Derive symbol directly without knowing function in advance.

Bootstrap strategy Dixon, Drummond, Henn 2011,

We will apply a different strategy based on master integrand expansion.

Outline

Background and Motivation

New bootstrap strategy

Two-loop four-point form factor

Summary and outlook

Bootstrap

Bootstrap







S-matrix program

The Analytic S-Matrix

R.J. EDEN P.V. LANDSHOFF D.I.OLIVE J.C.POLKINGHORNE

Cambridge University Press

"One should try to calculate S-matrix elements directly, without the use of field quantities, by requiring them to have some general properties that ought to be valid,"

- Eden et.al, "The Analytic S-matrix", 1966

Conformal bootstrap



Compute anomalous dimensions and correlation functions



Vyacheslav S. Rychkov

Alexander M. Polyakov

2-dim \longrightarrow D-dim

Bootstrap of amplitudes

Symbol bootstrap

Computing the finite remainder functions using symbol techniques.



Bootstrap of amplitudes

Symbol bootstrap

Computing the finite remainder functions using symbol techniques.



"master bootstrap"



Application: two-loop four-point form factor

Form factors

We consider two-loop four-point form factor in N=4 SYM:

$$\mathscr{F}_{\mathcal{O},4} = \left[d^4 x \, e^{-iq \cdot x} \langle 1_{\phi}, 2_{\phi}, 3_{\phi} 4^+ | \operatorname{tr}(\phi^3)(x) | 0 \rangle \right]$$

It is a N=4 version of Higgs+4-parton amplitudes in QCD:





Form factors

We consider two-loop four-point form factor in N=4 SYM:

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It is a N=4 version of Higgs+4-parton amplitudes in QCD:



Five-point two-loop amplitudes are at frontier and under intense study:

There have been many massless five-point two-loop amplitudes obtained in analytic form. See e.g. Abreu, Dormans, Cordero, Ita, Page 2019 and many others....

For five-point two-loop amplitudes with one massive leg, so far only one result is available: $u\bar{d} \rightarrow W^+ b\bar{b}$ Badger, Hartanto, Zoia 2021

Form factors

Our result provides a first two-loop five-point example with a **color-singlet** off-shell leg.



Planar master integrals have been evaluated recently.

Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020 Canko, Papadopoulos, Syrrakos 2020



$$\mathcal{F}_{\mathcal{O},4} = \int d^4x \, e^{-iq \cdot x} \langle 1_{\phi}, 2_{\phi}, 3_{\phi} 4^+ | \operatorname{tr}(\phi^3)(x) | 0 \rangle$$

Tree-level:
$$\mathcal{F}_4^{(0)} = \mathcal{F}_{\mathrm{tr}(\phi_{12}^3)}^{(0)}(1^\phi, 2^\phi, 3^\phi, 4^+) = \frac{\langle 31 \rangle}{\langle 34 \rangle \langle 41 \rangle} \,.$$

One-loop:

$$\mathcal{F}_{4}^{(1)} = \mathcal{F}_{4}^{(0)} \mathcal{I}^{(1)} = \mathcal{F}_{4}^{(0)} \left(B_{1} \mathcal{G}_{1}^{(1)} + B_{2} \mathcal{G}_{2}^{(1)} \right)$$

$$B_{1} = \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle}, \quad B_{2} = \frac{\langle 14 \rangle \langle 23 \rangle}{\langle 13 \rangle \langle 24 \rangle}, \quad B_{1} + B_{2} = 1,$$

$$\left(\sum_{a} B_a \mathcal{G}_a^{(1)}\right)^2 - \left[\sum_{a} B_a \left(\mathcal{G}_a^{(1)}\right)^2\right] \propto B_1 B_2 \left(\mathcal{G}_1^{(1)} - \mathcal{G}_2^{(1)}\right)$$



 $\mathscr{F}_{\mathcal{O},4} = \left[d^4 x \, e^{-iq \cdot x} \langle 1_{\phi}, 2_{\phi}, 3_{\phi} 4^+ | \operatorname{tr}(\phi^3)(x) | 0 \rangle \right]$

$$\mathcal{F}_{4}^{(0)} = \mathcal{F}_{\mathrm{tr}(\phi_{12}^{3})}^{(0)}(1^{\phi}, 2^{\phi}, 3^{\phi}, 4^{+}) = \frac{\langle 31 \rangle}{\langle 34 \rangle \langle 41 \rangle}$$



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Two-loop ansatz:

$$\mathcal{F}_4^{(2)} = \mathcal{F}_4^{(0)} \left(B_1 \, \mathcal{G}_1^{(2)} + B_2 \, \mathcal{G}_2^{(2)} \right)$$

$$\mathcal{G}_{a}^{(2)} = \sum_{i=1}^{221} c_{a,i} I_{i}^{(2),\text{UT}}, \qquad \mathcal{G}_{2}^{(2)} = \mathcal{G}_{1}^{(2)}|_{(p_{1}\leftrightarrow p_{3})}$$



 $\mathscr{F}_{\mathcal{O},4} = \left[d^4 x \, e^{-iq \cdot x} \langle 1_{\phi}, 2_{\phi}, 3_{\phi} 4^+ | \operatorname{tr}(\phi^3)(x) | 0 \rangle \right]$

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$$\mathcal{F}_{4}^{(1)} = \mathcal{F}_{4}^{(0)} \mathcal{I}^{(1)} = \mathcal{F}_{4}^{(0)} \left(B_1 \,\mathcal{G}_1^{(1)} + B_2 \,\mathcal{G}_2^{(1)} \right)$$
$$B_1 = \frac{\langle 12 \rangle \,\langle 34 \rangle}{\langle 13 \rangle \,\langle 24 \rangle}, \quad B_2 = \frac{\langle 14 \rangle \,\langle 23 \rangle}{\langle 13 \rangle \,\langle 24 \rangle}, \quad B_1 + B_2 = 1,$$

Two-loop ansatz:

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Constraints

IR divergences

Collinear factorization

BDS ansatz

$$\mathcal{I}^{(2),\text{BDS}} = \frac{1}{2} \left(\mathcal{I}^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) \mathcal{I}^{(1)}(2\epsilon)$$

$$\mathcal{R}_n^{(2)} = \left[\mathcal{I}^{(2)} - \mathcal{I}^{(2),\text{BDS}} \right]_{\text{fin}} \xrightarrow{p_i \parallel p_{i+1}} \mathcal{R}_{n-1}^{(2)}$$

Spurious pole

Unitarity cut

$$\mathcal{I}^{(2),\text{BDS}} = \sum_{a=1}^{2} B_a \left[\frac{1}{2} \left(\mathcal{G}_a^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) \mathcal{G}_a^{(1)}(2\epsilon) \right]$$
$$\left(\sum_a B_a \mathcal{G}_a^{(1)} \right)^2 - \left[\sum_a B_a \left(\mathcal{G}_a^{(1)} \right)^2 \right] \propto B_1 B_2 \left(\mathcal{G}_1^{(1)} - \mathcal{G}_2^{(1)} \right)$$

IR divergences

Collinear factorization

Spurious pole

Constraints Parameters left Symmetry of $(p_1 \leftrightarrow p_3)$ 221IR (Symbol) 82 Collinear limit (Symbol) 38 Spurious pole (Symbol) 31IR (Function) 26Spurious pole (Funcion) 25Collinear limit (Funcion) 18

Unitarity cut

IR divergences

Collinear factorization

Spurious pole

Unitarity cut

Remaining 18 parameters can be fixed by knowing master integrals:





(a) dBub 1

(b) dBub 1



IR divergences

Collinear factorization

Spurious pole

Unitarity cut

Remaining 18 parameters can be fixed by knowing master integrals:





Unitarity cut





Can be fixed via **simple two-double cuts**:





Constraints	Parameters left
Symmetry of $(p_1 \leftrightarrow p_3)$	221
IR (Symbol)	82
Collinear limit (Symbol)	38
Spurious pole (Symbol)	31
IR (Function)	26
Spurious pole (Funcion)	25
Collinear limit (Funcion)	18
If keeping only to ϵ^0 order	14
Simple unitarity cuts	0

A summary

Substituting in the master integral results, we have the full analytic form in GPLs, and they can be evaluated with GiNaC to 'arbitrary' high precision:

	$\mathcal{F}^{(2)}/\mathcal{F}^{(0)}$
ϵ^{-4}	8
ϵ^{-3}	-10.888626564448543787 + 25.132741228718345908i
ϵ^{-2}	-31.872672672370517258 - 16.558017711981028644i
ϵ^{-1}	-24.702889082481070673 - 2.9923229294749490751 i
ϵ^0	-82.902014730676342383 - 129.78151092480602830i

up to finite order with the kinematics: $\{s_{12} = 241/25, s_{23} = -377/100, s_{34} = 13/50, s_{14} = -161/100, s_{13} = s_{24} = -89/100, tr_5 = \sqrt{1635802}/2500i\}.$

Technical points: symbol letters

 $\operatorname{Sym}(\mathcal{R}_4^{(2)}) = \sum_i c_i W_{i_1} \otimes W_{i_2} \otimes W_{i_3} \otimes W_{i_4}$

,

Building blocks

Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020

$$\begin{aligned} x_{ijkl}^{\pm} &= \frac{1 + s_{ij} - s_{kl} \pm \sqrt{\Delta_{3,ijkl}}}{2s_{ij}} \\ y_{ijkl}^{\pm} &= \frac{\mathrm{tr}_{\pm}(ijkl)}{2s_{ij}s_{il}} \,, \\ z_{ijkl}^{\pm\pm} &= 1 + y_{ijkl}^{\pm} - x_{lijk}^{\pm} \,, \end{aligned}$$

 $\Delta_{3,ijkl} = \operatorname{Gram}(p_i + p_j, p_k + p_l),$ $\operatorname{tr}_{\pm}(ijkl) = s_{ij}s_{kl} - s_{ik}s_{jl} + s_{il}s_{jk} \pm \operatorname{tr}_5,$ $\operatorname{tr}_5 = 4i\epsilon_{\mu\nu\rho\sigma}p_1^{\mu}p_2^{\nu}p_3^{\rho}p_4^{\sigma}$

Most complicated letters:

$$\begin{aligned} X_1(p_i + p_j, p_k, p_l) &= \frac{u_{ij} x_{ijkl}^+ - u_{ijl}}{u_{ij} x_{ijkl}^- - u_{ijl}}, \\ X_2(p_i + p_j, p_k + p_l) &= \frac{x_{ijkl}^+}{x_{ijkl}^-}, \\ Y_1(p_i, p_j, p_k, p_l) &= \frac{\operatorname{tr}_+(ijkl)}{\operatorname{tr}_-(ijkl)} = \frac{y_{ijkl}^+}{y_{ijkl}^-}, \\ Y_2(p_i, p_j, p_k, p_l) &= \frac{y_{ijkl}^+ + 1}{y_{ijkl}^- + 1}, \\ Z(p_i, p_j, p_k, p_l) &= \frac{z_{ijkl}^{++} z_{ijkl}^{--}}{z_{ijkl}^{+-} z_{ijkl}^{-+}}. \end{aligned}$$

Technical points: collinear limit of form factors



Technical points: numerical computation

Master integrals are evaluated in multiple polylogarithm. Canko, Papadopoulos, Syrrakos 2020

A different set of kinematics are chosen.

 $\{q_1, q_2, q_3, q_4, q_5\}$ with q_1 massive $\{x, S_{12}, S_{23}, S_{34}, S_{45}, S_{51}\}.$



 $q_1 \rightarrow p_{123} - xp_{12}, q_2 \rightarrow p_4, q_3 \rightarrow -p_{1234}, q_4 \rightarrow xp_1$

Summary and outlook

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Outlook:

Consider more general observables.

Study the new constraints beyond collinear limit, such as OPE limit, Regge limit.

Hidden analytic structure, such as Qbar-like eqn.



Extra slides

Unitarity cuts

Consider one-loop amplitudes:



Unitarity cuts

We can perform unitarity cuts:

$$\mathcal{A} = \mathcal{A} \mathcal{A} = \Sigma di \mathcal{A} + \Sigma G \mathcal{A} + \Sigma bi \mathcal{A}$$

and from tree products, we derive the coefficients more directly.

Cutkosky cutting rule:
$$\frac{1}{p^2} = \longrightarrow \Rightarrow = 2\pi i \delta^{\dagger}(p^2)$$