# Lattice QCD and electroweak interactions

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# QCD: different strategies

**QCD is the fundamental theory**

$\Rightarrow$ describing strong interaction between quarks and gluons

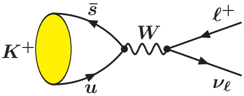
- High-Q ($>$ few GeV) $\leftrightarrow$ short distance ($<$ 0.1 fm)
  $\Rightarrow$ Theory of weakly interacting quarks and gluons
  $\Rightarrow$ (Perturbative QCD: Gross, Politzer, Wilczek for asymptotic freedom)

- Low-Q ($\ll$ 1 GeV) $\leftrightarrow$ long distance ($>$ 1 fm)
  $\Rightarrow$ Spontaneous chiral symmetry breaking
  $\Rightarrow$ EFT of weakly interacting Nambu-Goldstone bosons
  $\Rightarrow$ EFT treats hadrons as dynamical degree of freedom (no quarks, gluons)

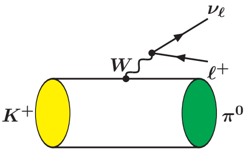
- Lattice QCD
  $\Rightarrow$ Large-scale supercomputer simulation on Euclidean spacetime lattice
  - Provide most accurate $\alpha_s$ for pQCD
  - Provide LECs for EFT

## Evaluate the hadronic matrix elements for electroweak processes

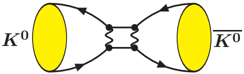
- Lattice QCD is powerful for "standard" hadronic matrix elements with




$K_{\ell 2} \to f_K$




$K_{\ell 3} \to f_+(0)$



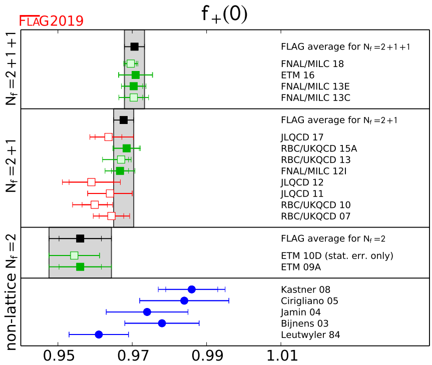

$K^0$-$\overline{K^0}$ mixing $\to B_K$

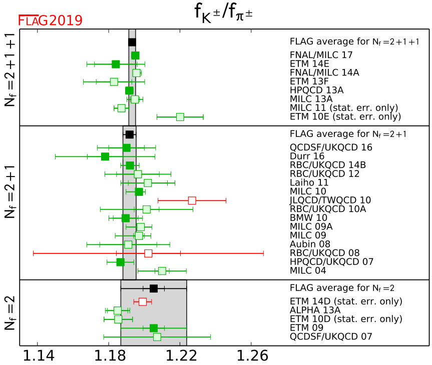
  - single local operator insertion
  - only single stable hadron or vacuum in the initial/final state
  - Requires only two- or three-point correlation functions

## Flavor Lattice Averaging Group (FLAG) average 2019

$$f_+^{K\pi}(0) = 0.9706(27) \quad \Rightarrow \quad 0.28\% \text{ error}$$

$$f_{K^\pm}/f_{\pi^\pm} = 1.1932(19) \quad \Rightarrow \quad 0.16\% \text{ error}$$




## Experimental information [arXiv:1411.5252, 1509.02220]

$$K_{\ell 3} \quad \Rightarrow \quad |V_{us}| f_+(0) = 0.2165(4) \quad \Rightarrow \quad |V_{us}| = 0.2231(7)$$

$$K_{\mu 2}/\pi_{\mu 2} \quad \Rightarrow \quad \left|\frac{V_{us}}{V_{ud}}\right| \frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.2760(4) \quad \Rightarrow \quad \left|\frac{V_{us}}{V_{ud}}\right| = 0.2313(5)$$

# Flag average 2019

## Error $< 1\%$

| | $N_f$ | FLAG average | Frac. Err. |
|---|---|---|---|
| $f_K/f_\pi$ | $2+1+1$ | 1.1932(19) | 0.16% |
| $f_+(0)$ | $2+1+1$ | 0.9706(27) | 0.28% |
| $f_D$ | $2+1+1$ | 212.0(7) MeV | 0.33% |
| $f_{D_s}$ | $2+1+1$ | 249.9(5) MeV | 0.20% |
| $f_{D_s}/f_D$ | $2+1+1$ | 1.1783(16) | 0.13% |
| $f_B$ | $2+1+1$ | 190.0(1.3) MeV | 0.68% |
| $f_{B_s}$ | $2+1+1$ | 230.3(1.3) MeV | 0.56% |
| $f_{B_s}/f_B$ | $2+1+1$ | 1.209(5) | 0.41% |

## Error $< 5\%$

| | $N_f$ | FLAG average | Frac. Err. |
|---|---|---|---|
| $\hat{B}_K$ | $2+1$ | 0.7625(97) | 1.3% |
| $f_+^{D\pi}(0)$ | $2+1$ | 0.666(29) | 4.4% |
| $f_+^{DK}(0)$ | $2+1$ | 0.747(19) | 2.5% |
| $\hat{B}_{B_s}$ | $2+1$ | 1.35(6) | 4.4% |
| $B_{B_s}/B_{B_d}$ | $2+1$ | 1.032(28) | 3.7% |
| ... | | | |

Time to go beyond local matrix elements and three-point functions

# Go beyond local hadronic matrix elements – challenges

- Computational demanding
  - Three-point function

$$\langle H_f(x_f) O(0) H_i^\dagger(x_i)\rangle \quad \Rightarrow \quad \int d^3\vec{x}_i \int d^3\vec{x}_f \quad \Rightarrow \quad \sum_{\vec{x}_i}\sum_{\vec{x}_f} \sim L^6$$

  - Four-point function

$$\langle H_f(x_f) O_1(x) O_2(0) H_i^\dagger(x_i)\rangle\rangle \quad \Rightarrow \quad \int d^3\vec{x}_i \int d^3\vec{x}_f \int d^3\vec{x} \quad \Rightarrow \quad \sum_{\vec{x}_i}\sum_{\vec{x}_f}\sum_{\vec{x}} \sim L^9$$

  with $L = 24, 32, 48, 64, 96, \cdots$

- Complicated intermediate states

$$\langle H_f|O_1(x)O_2(0)|H_i\rangle = \sum_n \langle H_f|O_1(x)|n\rangle\langle n|O_2(0)|H_i\rangle$$

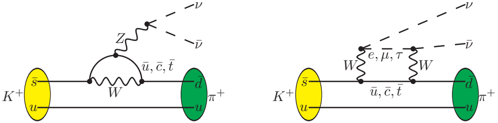
  - In Euclidean space, exponentially-growing unphysical effects from $|n\rangle$
  - Power-law FV effects if $|n\rangle$ given by low-lying multi-hadron states

- Short-distance divergence in $O_1(x)O_2(0)$ when $x \to 0$
  - Additional renormalization is required

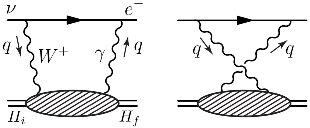
# Go beyond local hadronic matrix elements – opportunities

## Opportunities in flavor physics

- Rare decays, e.g. $\text{Br}[K^+ \to \pi^+ \nu\bar{\nu}] = 1.73^{+1.15}_{-1.05} \times 10^{-10}$



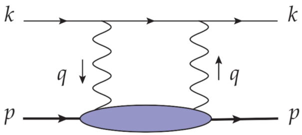

- Electroweak radative corrections to hadronic decays
  $\Rightarrow$ superallowed nuclear $\beta$ decay half-life time with precision $10^{-6}$



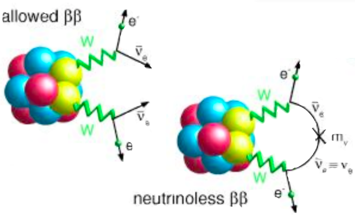

- Proton's weak charge $Q_W^p = 1 - 4\sin^2\theta_W$
  $\Rightarrow$ 0.3% measurement of $\sin^2\theta_W$ by Q-weak at JLab
  - Parity-violating e-p scattering, $\Box^V_{\gamma Z}$ contribution

## Opportunities in nuclear physics

- Muonic hydrogen spectrum $\rightarrow$ proton charge radius $r_p = 0.84087(39)$ fm
  $\Rightarrow$ 10 times more accurate than e-p scattering




- Neutrioless double beta decays




- Hadron electromagentic polarizability

# Our recent work on four-point correlation functions

- QCD+QED $\Rightarrow$ Infinite-volume reconstruction method
  [XF, L. Jin, PRD100 (2019) 094509]
- Rare kaon decays $K^+ \to \pi^+ \nu \bar{\nu}$
  [Z. Bai, XF, N. Christ, et.al. PRL118 (2017) 252001]
- Electroweak box contribution to $\pi_{\ell 3}$ and $K_{\ell 3}$ decay
  [XF, M. Gorchtein, L. Jin, P. Ma, C. Seng, PRL124 (2020) 192002]
  [P. Ma, XF, M. Gorchtein, L. Jin, C. Seng, arXiv:2102.12048]
- Neutrinoless double beta decays
  [XF, L. Jin, X. Tuo, S. Xia, PRL122 (2019) 022001]
  [X. Tuo, XF, L. Jin, PRD100 (2019) 094511]
- Work in preparation
  - Pion mass splitting
  - Pion electromagentic polarizability

Move from the pion to nucleon sector

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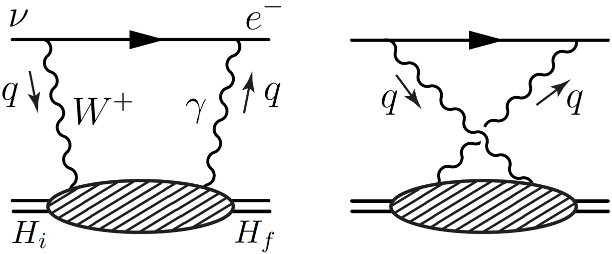
# Electroweak box diagram



# CKM unitarity – a constraint from Standard Model

## First-row CKM unitarity

$$\Delta_{\mathrm{CKM}} = |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 = 0$$

## PDG 2019 ⇒ PDG 2020

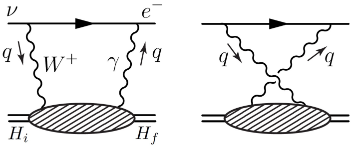
| | PDG 2019 | PDG 2020 |
|---|---|---|
| $\lvert V_{ud}\rvert$ | 0.97420(21) | 0.97370(14) |
| $\lvert V_{us}\rvert$ | 0.2243(5) | 0.2245(8) |
| $\lvert V_{ub}\rvert$ | 0.00394(36) | 0.00382(24) |
| $\Delta_{\mathrm{CKM}}$ | −0.00061(47) | −0.00149(45) |

- Main update from $|V_{ud}|$ ⇒ 3.3 $\sigma$ deviation from CKM unitarity
- $|V_{ud}|$ is from superallowed $0^+ \to 0^+$ nuclear beta decay
  - Pure vector transitions at leading order
  - Uncertainty is dominated by electroweak radiative correction

[J. Hardy, I. Towner (2015)]

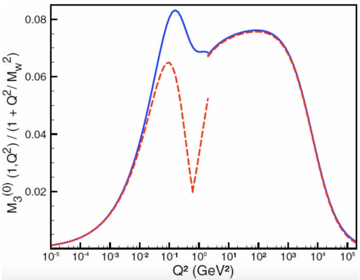
Based on current algebra, only axial $\gamma W$-box diagram sensitive to hadronic scale

[A. Sirlin, Rev. Mod. Phys. 07 (1978) 573]




$$T^{VA}_{\mu\nu} = \frac{1}{2}\int d^4x\, e^{iqx}\langle H_f(p)|T\left[J^{em}_{\mu}(x)J^{W,A}_{\nu}(0)\right]|H_i(p)\rangle$$

Re-evaluation of the $\gamma W$-box diagram




$|V_{ud}| = 0.97420(18)_{\mathrm{RC}}(10)_{\mathcal{F}t}$

Using VMD model

[Marciano & Sirlin, PRL 2006]

$\Downarrow$

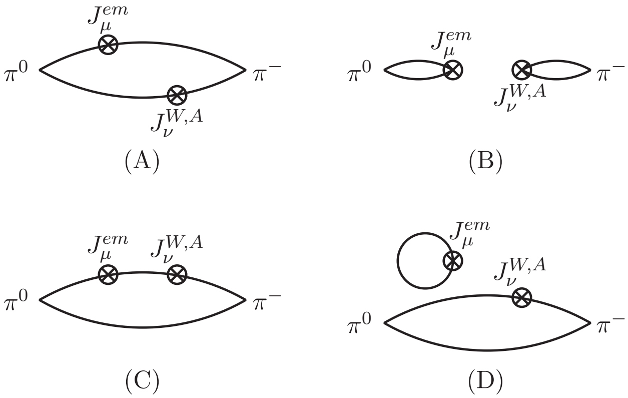
$|V_{ud}| = 0.97370(10)_{\mathrm{RC}}(10)_{\mathcal{F}t}$

Using dispsive & data-driven analysis

[Seng et.al. PRL 2018]

$> 3\sigma$ violation of CKM unitarity $\Rightarrow$ first-principle calculation

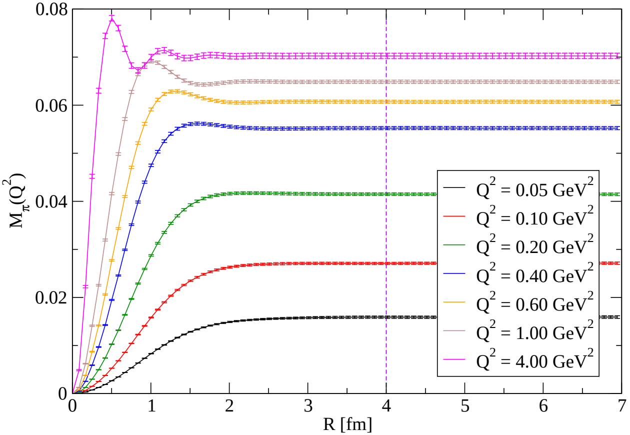
$$\mathcal{H}^{VA}_{\mu\nu}(x) = \langle \pi^0(p) | T \left[ J^{em}_\mu(x) J^{W,A}_\nu(0) \right] | \pi^-(p) \rangle$$




- Coulomb gauge fixed wall source is used for the pion interpolating field
- $J^{W,A}_\nu(0)$ is treated as a source and $J^{em}_\mu(x)$ is a sink
- Calculate $\mathcal{H}^{VA}_{\mu\nu}(x)$ as a function of $x$

## Construct the Lorentz scalar function $M_\pi(Q^2)$ from $\mathcal{H}^{VA}_{\mu\nu}(x)$

$$M_\pi(Q^2) = -\frac{1}{6\sqrt{2}}\frac{\sqrt{Q^2}}{m_\pi}\int d^4x\, \omega(Q,x)\epsilon_{\mu\nu\alpha 0}x_\alpha \mathcal{H}^{VA}_{\mu\nu}(x)$$

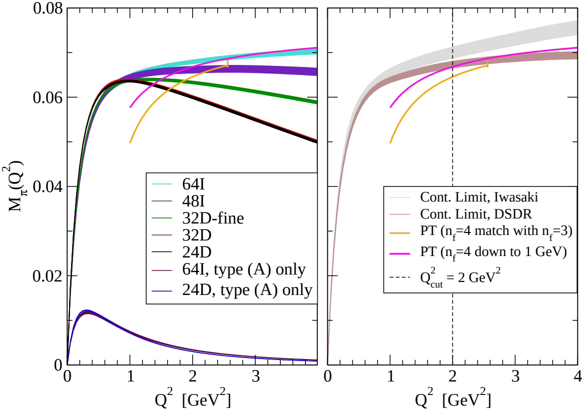


# Combine lattice results with pQCD

**Radiative correction requires the momentum integral from $0 < Q^2 < \infty$**

$$\Box^{VA}_{\gamma W} = \frac{3\alpha_e}{2\pi} \int \frac{dQ^2}{Q^2} \frac{m_W^2}{m_W^2 + Q^2} M_\pi(Q^2)$$

- Lattice data used for low-$Q^2$ region
- OPE and perturbative Wilson coefficients used for high-$Q^2$ region



# Error analysis

Use the momentum scale $Q^2_{\rm cut}$ to separate the LD and SD contributions

$$\Box^{VA}_{\gamma W} = \begin{cases} 2.816(9)_{\rm stat}(24)_{\rm PT}(18)_{\rm a}(3)_{\rm FV} \times 10^{-3} & \text{using } Q^2_{\rm cut} = 1 \text{ GeV}^2 \\ 2.830(11)_{\rm stat}(9)_{\rm PT}(24)_{\rm a}(3)_{\rm FV} \times 10^{-3} & \text{using } Q^2_{\rm cut} = 2 \text{ GeV}^2 \\ 2.835(12)_{\rm stat}(5)_{\rm PT}(30)_{\rm a}(3)_{\rm FV} \times 10^{-3} & \text{using } Q^2_{\rm cut} = 3 \text{ GeV}^2 \end{cases}$$

- When $Q^2_{\rm cut}$ increase, the lattice artifacts become larger
- When $Q^2_{\rm cut}$ decrease, systematic effects in pQCD become larger
- For 1 GeV$^2 \le Q^2_{\rm cut} \le 3$ GeV$^2$, all results are consistent within uncertainties

# Pion semileptonic $\beta$ decay

## Decay width measured by PIBETA experiment

$$\Gamma_{\pi\ell 3} = \frac{G_F^2 |V_{ud}|^2 m_\pi^5 |f_+^\pi(0)|^2}{64\pi^3}(1+\delta)I_\pi$$

- ChPT [Cirigliano et.al. (2002), Czarnecki, Marciano, Sirlin (2019)]

$$\delta = 0.0334(10)_{\mathrm{LEC}}(3)_{\mathrm{HO}}$$

- Sirlin's parametrization [A. Sirlin, Rev. Mod. Phys. 07 (1978) 573]

$$\begin{aligned}\delta &= \frac{\alpha_e}{2\pi}\left[\bar{g} + 3\ln\frac{m_Z}{m_p} + \ln\frac{M_Z}{M_W} + \tilde{a}_g\right] + \delta_{\mathrm{HO}}^{\mathrm{QED}} + 2\Box_{\gamma W}^{VA} \\ &= 0.0332(1)_{\gamma W}(3)_{\mathrm{HO}}\end{aligned}$$

where $\frac{\alpha_e}{2\pi}\bar{g} = 1.051\times 10^{-2}$, $\frac{\alpha_e}{2\pi}\tilde{a}_g = -9.6\times 10^{-5}$, $\delta_{\mathrm{HO}}^{\mathrm{QED}} = 0.0010(3)$

- Hadronic uncertainty reduced by a factor of 10, which results in

$$|V_{ud}| = 0.9739(28)_{\mathrm{exp}}(5)_{\mathrm{th}} \quad \Rightarrow \quad |V_{ud}| = 0.9739(28)_{\mathrm{exp}}(1)_{\mathrm{th}}$$

[XF, Gorchtein, Jin, Ma, Seng, PRL124 (2020) 192002]

First time to calculate $\gamma W$ box diagram $\Rightarrow$ method set up for nucleon decay

# Status of $V_{us}$

- Semileptonic decays
  - Traditionally from $K_L^0 \to \pi e \nu$ decays to avoid isospin breaking effects
  - Latest experiments justify comparison between different decay modes
  - Global average from [FlaviaNet Working Group, EPJC, 2010]

    $$K_L^0 \to \pi e \nu, \quad K_L^0 \to \pi \mu \nu, \quad K^\pm \to \pi^0 e^\pm \nu, \quad K^\pm \to \pi^0 \mu^\pm \nu, \quad K_S^0 \to \pi e \nu$$

    leads to $|V_{us}| f_+(0) = 0.2165(4)$
  - Using FLAG $N_f = 2+1+1$ lattice input of $f_+(0) = 0.9706(27) \Rightarrow$

    $$|V_{us}| = 0.2231(4)_{\mathrm{exp+RCs}}(6)_{\mathrm{lat}}$$

- Leptonic decays
  - Exp. measurements of $K \to \mu\nu(\gamma)$ and $\pi \to \mu\nu(\gamma)$ e.g. from KLOE
  - Using FLAG $N_f = 2+1+1$ lattice input of $f_K/f_\pi = 1.1932(19) \Rightarrow$

    $$|V_{us}| = 0.2252(5)_{\mathrm{lat}}$$

- $|V_{us}|$ from semileptonic and leptonic decays differs by $2.4\sigma$

Important to include QED and QCD isospin violations in the lattice calculations

# Basic idea

C.-Y. Seng, XF, M. Gorchtein, L.-C. Jin, U.-G. Meißner, JHEP 10 (2020) 179

**Combine the lattice calculation with ChPT**

- Use ChPT to determine EM correction

$$\delta_{\rm em}^{K^\pm} = 2e^2\left[-\frac{8}{3}X_1-\frac{1}{2}\tilde{X}_6^{\rm phys}(M_\rho)-2K_3^r(M_\rho)+K_4^r(M_\rho)+\frac{2}{3}K_5^r(M_\rho)+\frac{2}{3}\right.$$

$$\delta_{\rm em}^{K^0} = 2e^2\left[\frac{4}{3}X_1-\frac{1}{2}\tilde{X}_6^{\rm phys}(M_\rho)\right]+\cdots$$

$K^0$ decays are much simpler, but still require LECs $X_1$ and $\tilde{X}_6^{\rm phys}(M_\rho)$

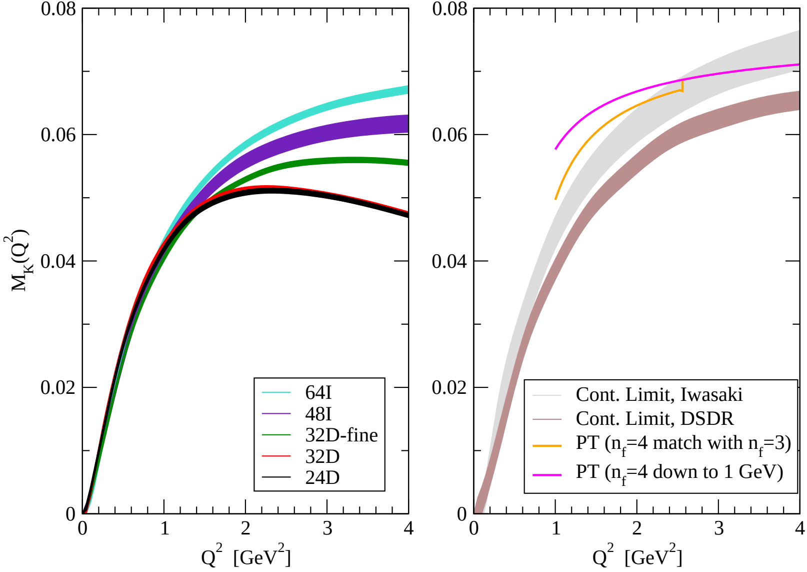
- Lattice QCD can provides LECs in principle
  - but needs to calculate all the diagrams, not only just $\gamma W$ diagram!
- Fortunately, these LECs are independent of quark mass

$K\to\pi\ell\nu$ at physical kinematics vs in flavor $SU(3)$ limit

Solution: Lattice calculation of EM correction to $K_{\ell 3}$ decay in flavor $SU(3)$ limit

# Axial $\gamma W$-box diagram contribution to $K^0 \to \pi^+$ decays

Calculation is performed in the flavor $SU(3)$ limit with $m_K = m_\pi$



# Lattice results

[P. Ma, XF, M. Gorchtein, L. Jin, C. Seng, arXiv:2102.12048]

- After combining the lattice data and PT results, we have

$$\Box^{VA}_{\gamma W}\big|_{K^0} = \begin{cases} 2.460(18)_{\mathrm{stat}}(42)_{\mathrm{PT}}(22)_a(1)_{\mathrm{FV}} \times 10^{-3} & Q^2_{\mathrm{cut}} = 1\ \mathrm{GeV}^2 \\ 2.443(20)_{\mathrm{stat}}(15)_{\mathrm{PT}}(36)_a(1)_{\mathrm{FV}} \times 10^{-3} & Q^2_{\mathrm{cut}} = 2\ \mathrm{GeV}^2 \\ 2.433(22)_{\mathrm{stat}}(7)_{\mathrm{PT}}(45)_a(1)_{\mathrm{FV}} \times 10^{-3} & Q^2_{\mathrm{cut}} = 3\ \mathrm{GeV}^2 \end{cases}$$

- The relation between box contribution and the LECs is given by

$$-\frac{8}{3}X_1 + \bar{X}_6^{\mathrm{phys}}(M_\rho) = -\frac{1}{2\pi\alpha}\left(\Box^{VA}_{\gamma W}\big|_{K^0} - \frac{\alpha}{8\pi}\ln\frac{M_W^2}{M_\rho^2}\right) + \frac{1}{8\pi^2}\left(\frac{5}{4} - \tilde{a}_g\right)$$

- This results in

$$-\frac{8}{3}X_1 + \tilde{X}_6^{\mathrm{phys}} = 0.0197(10)$$

- ChPT quoted the minimal resonance model as input

$$X_1 = -3.7(3.7) \times 10^{-3} \quad \text{and} \quad \tilde{X}_6^{\mathrm{phys}} = 10.4(10.4) \times 10^{-3}$$

$$-\frac{8}{3}X_1 + \tilde{X}_6^{\mathrm{phys}} = 0.0203(143)$$

Consistent between lattice and ChPT, but error from lattice is much smaller

# Determination of LECs

- Combine the $SU(3)$ $K^0$ decay

$$-\frac{8}{3}X_1 + \tilde{X}_6^{\mathrm{phys}} = 0.0197(10) \quad \text{for } K^0 \to \pi^+$$

with semileptonic pion decay

$$\frac{4}{3}X_1 + \tilde{X}_6^{\mathrm{phys}} = 0.0110(6) \quad \text{for } \pi^- \to \pi^0$$

- We have

$$X_1 = -2.2(4) \times 10^{-3}, \quad \tilde{X}_6^{\mathrm{phys}} = 13.9(7) \times 10^{-3}$$

- This is comparable with the minimal resonance model

| $10^3\, X_1$ | $10^3\, X_2^r$ | $10^3\, X_3^r$ | $10^3\, \tilde{X}_6^{eff}$ | $10^3 (X_6^{eff})_{\alpha_s}$ | $10^3\, X_6^{eff}$ |
|---|---|---|---|---|---|
| $-3.7$ | 3.6 | 5.00 | 10.4 | 3.0 | $-231.5$ |

# Conclusion

- Axial $\gamma W$-box contribution to $\pi_{\ell 3}$ decay

$$\Box_{\gamma W}^{VA} = 2.830(11)_{\rm stat}(26)_{\rm sys} \times 10^{-3}$$

- $K_{\ell 3}$ decay: lattice QCD interplay with EFT

$$X_1 = -2.2(4) \times 10^{-3}, \quad \tilde{X}_6^{\rm phys} = 13.9(7) \times 10^{-3}$$

- Move towards nucleon beta decay

**Four-point function: an exciting, new area for lattice QCD**