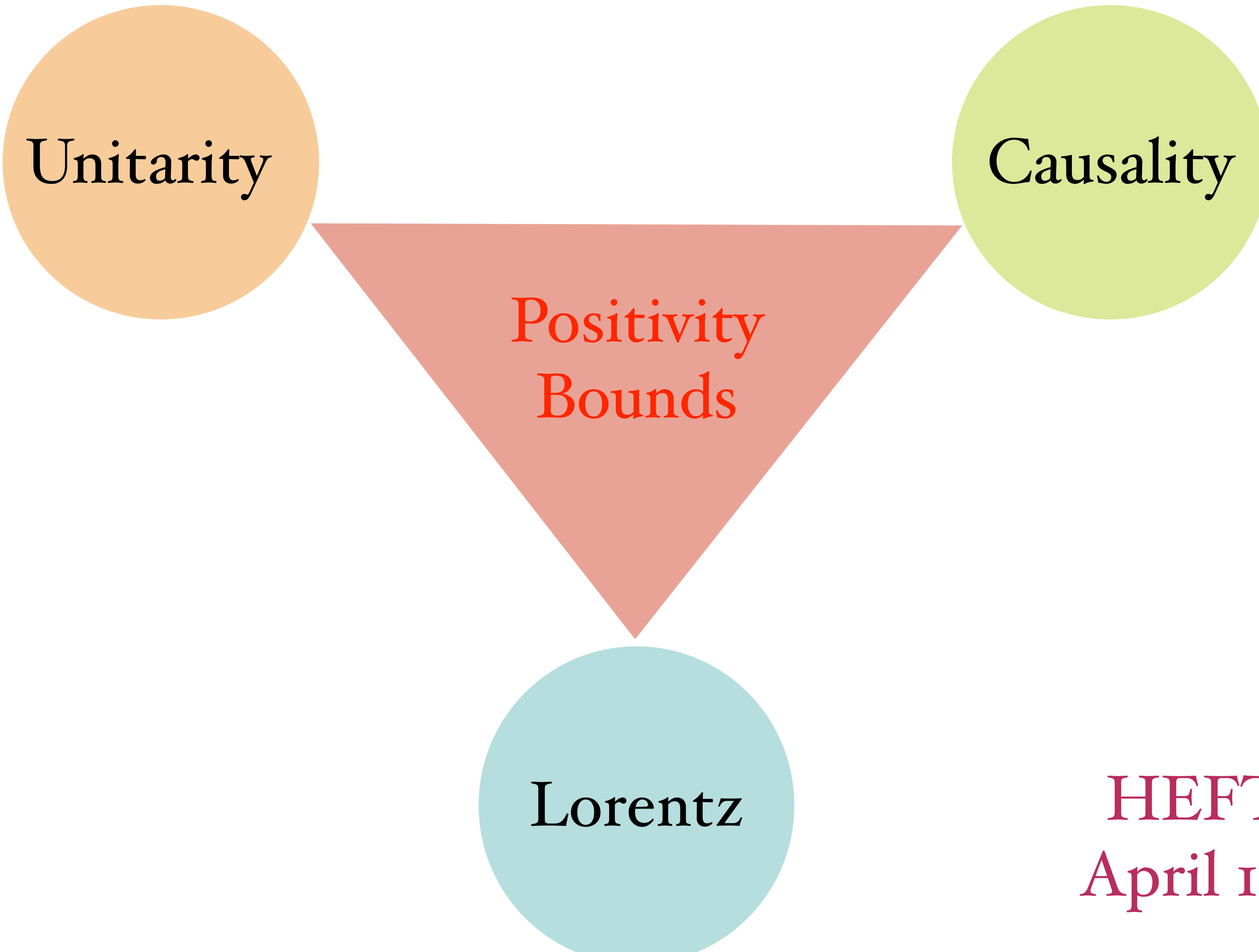
Effective Field Theories and Positivity Bounds

Andrew J. Tolley
Imperial College London



HEFT 2021
April 16, 2021

Are all EFTs allowed? - SWAMPLAND

With typical assumption that:
UV completion is Local, Causal, Poincare Invariant and Unitary

Answer: NO! Certain low energy effective theories do not admit well defined UV completions

Kallen-Lehmann Spectral Representation

Together with Poincare invariance these imply:

$$i\langle 0|\hat{T}\hat{O}(x)\hat{O}(y)|0\rangle = \int \frac{d^d k}{(2\pi)^d} e^{ik.(x-y)} G_O(k)$$

$$G_O(k) = \frac{Z}{k^2+m^2-i\epsilon} + S(-k^2) + (-k^2)^N \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N(k^2+\mu-i\epsilon)}$$

$$S(-k^2) = \sum_{k=0}^{N-1} c_k (-k^2)^k \qquad \lim_{\mu\to\infty} \rho(\mu) \sim \mu^{\Delta-d/2} \qquad N = [\Delta - d/2 + 1]$$

Δ UV Conformal weight

Positive Spectral Density
as a result of Unitarity

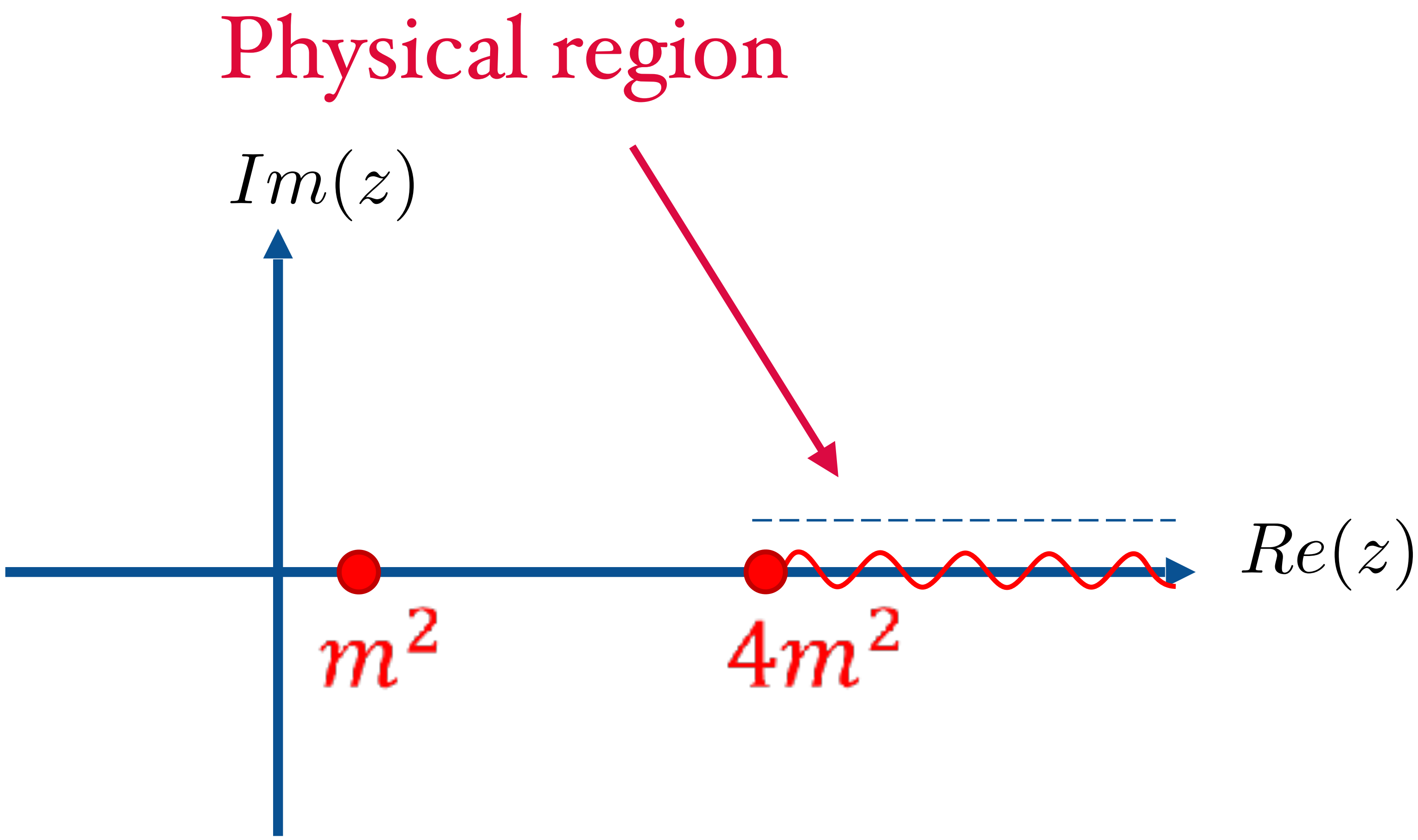
$$\rho(\mu) \geq 0$$

Analytic Structure

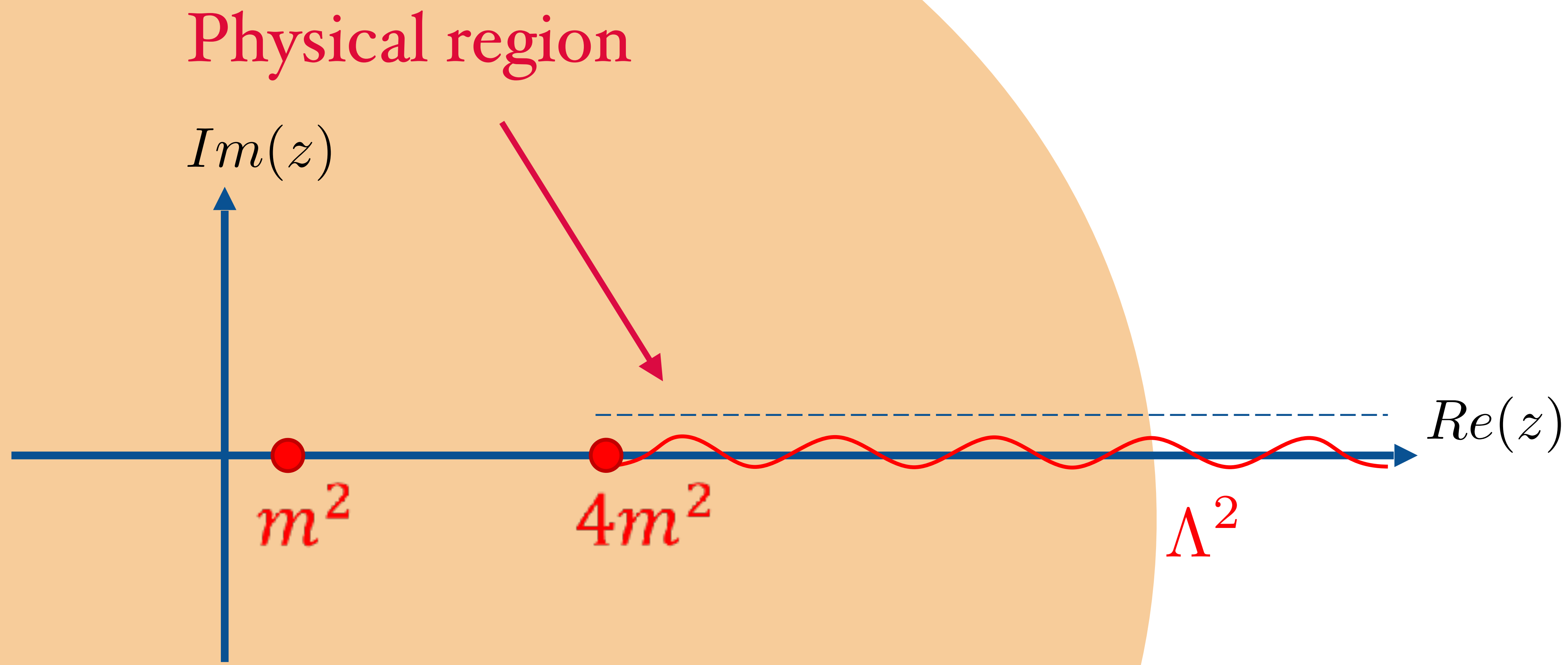
Define complex momenta squared $z = -k^2 + i\epsilon$

Pole — Branch cut

$$G_O(z) = \frac{Z}{m^2 - z} + S(z) + z^N \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N(\mu - z)}$$



Region of Validity of EFT



EFT valid here, can calculate pole and 'low energy' part of cut

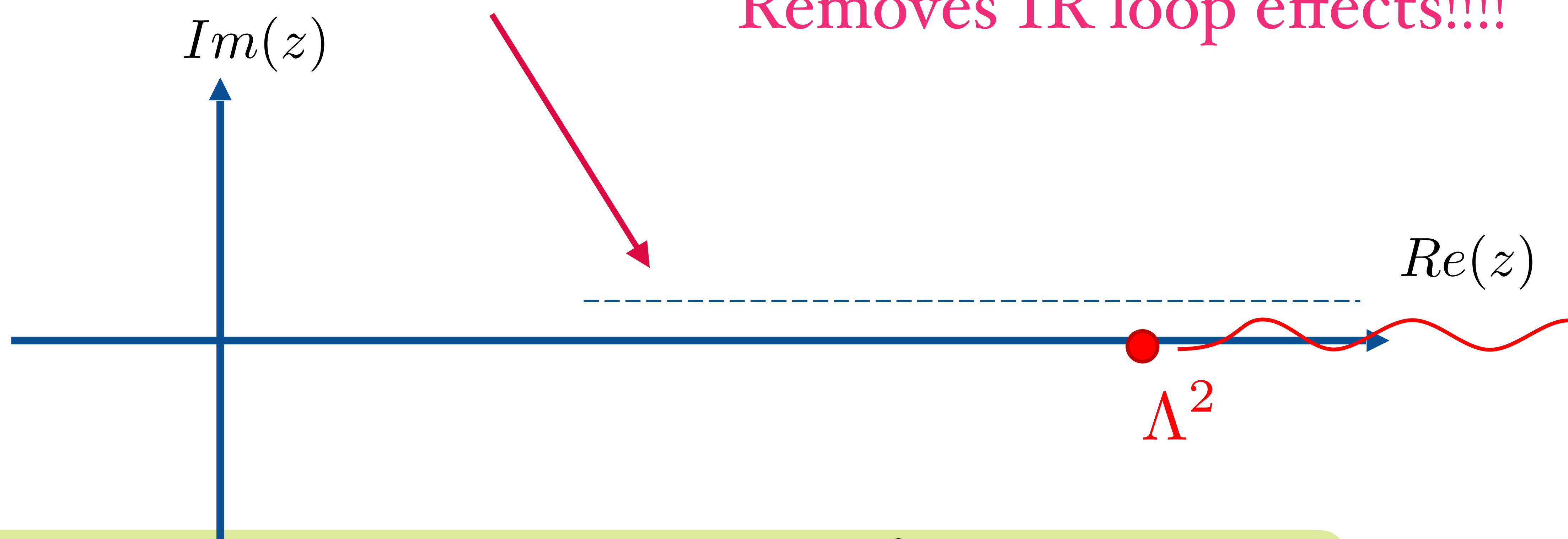
UV completion - unknown?

Analytic Structure 2: Move the branch cut!

de Rham, Melville, AJT, Zhou 1702.08577
Bellazzini et al 1710.02539
de Rham, Melville, AJT 1710.09611

Physical region

Removes IR loop effects!!!!



$$G'_O(z) = G_O(z) - \frac{Z}{m^2 - z} - z^N \int_{4m^2}^{\Lambda^2} d\mu \frac{\rho(\mu)}{\mu^N(\mu - z)}$$

$$G'_O(z) = S(z) + z^N \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N(\mu - z)}$$

Calculable in EFT

Linear (Improved) Positivity Bounds

$$G'_O(z) = S(z) + z^N \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N(\mu - z)}$$

$M \geq N$

$$D_M(z) = \frac{1}{M!}\frac{d^M}{dz^M}G'_O(z) = \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{(\mu - z)^{M+1}}$$

$$D_M(0) = \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^{M+1}}$$

$$D_M(0) > 0 \qquad D_M(0) \geq \Lambda^2 D_{M+1}(0)$$

Positivity of these integrals enforces positivity of combinations of Wilson coefficients for Irrelevant operators

Nonlinear (Improved) Positivity Bounds

Maths by Stieltjes in 1890s, applied to amplitudes positivity in 1970s!! Recently rediscovered ..

$$D_M(0) = \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^{M+1}} = \langle \frac{1}{\mu^M} \rangle$$

Arkani-Hamed, Huang, Huang EFT-Hedron 2020

see also Bellazzini et al, Positive Moments .., 2020

$$y^T D_M y = \sum_{p,q=0}^{N} D_{M+p+q} y^p y^q = \langle \mu^{-M} (\sum_{p=0}^{N} y^p \mu^{-p})^2 \rangle > 0$$

$$\det(D_M) > 0$$

'positivity of N x N Hankel matrix'

$$(D_M)_{pq} = D_{M+p+q}$$

Simply example **Cauchy-Schwarz:**

$$\langle (\mu^{-M} + \lambda \mu^{-N})^2 \rangle \geq 0$$

$$D_{2M} D_{2N} \geq (D_{N+M})^2$$

$$\begin{pmatrix} D_{2N} & D_{N+M} \\ D_{N+M} & D_{2M} \end{pmatrix}$$

'positivity of 2 x 2 Hankel matrix'

What does this tell us about EFT?

e.g. Suppose scalar field in EFT with tree level action ....

$$S = \int d^4x \hat{O}(x)[\Box + a_1 \frac{\Box^2}{\Lambda^2} + a_2 \frac{\Box^3}{\Lambda^4} + \ldots]\hat{O}(x)$$

Tree level Feynman propagator is

$$G_O(z) = -\frac{1}{z + a_1 \frac{z^2}{\Lambda^2} + a_2 \frac{z^3}{\Lambda^4} + a_3 \frac{z^4}{\Lambda^6} + a_4 \frac{z^5}{\Lambda^8} \ldots}$$

$$\downarrow$$

Assume no subtractions needed .....

$$G'_O(z) = \frac{a_1}{\Lambda^2} + \frac{(a_2 - a_1^2)}{\Lambda^4} z + \frac{a_1^3 - 2a_1 a_2 + a_3}{\Lambda^6} z^2 + \frac{a_4 - 2a_1 a_3 - a_2^2 + 3a_1^2 a_2 - a_1^4}{\Lambda^8} z^3 + \mathcal{O}(z^4)$$

What does this tell us about EFT?

$$G'_O(z) = \frac{a_1}{\Lambda^2} + \frac{(a_2 - a_1^2)}{\Lambda^4} z + \frac{a_1^3 - 2a_1 a_2 + a_3}{\Lambda^6} z^2 + \frac{a_4 - 2a_1 a_3 - a_2^2 + 3a_1^2 a_2 - a_1^4}{\Lambda^8} z^3 + \mathcal{O}(z^4)$$

assuming no subtractions

$$N = 0$$

$$D_2 D_0 > D_1^2$$

NonLinear (Improved) Positivity Bounds:

$$(a_1^3 - 2a_1 a_2 + a_3) a_1 - (a_2 - a_1^2)^2 > 0$$

↓

$$a_1 a_3 - a_2^2 > 0$$

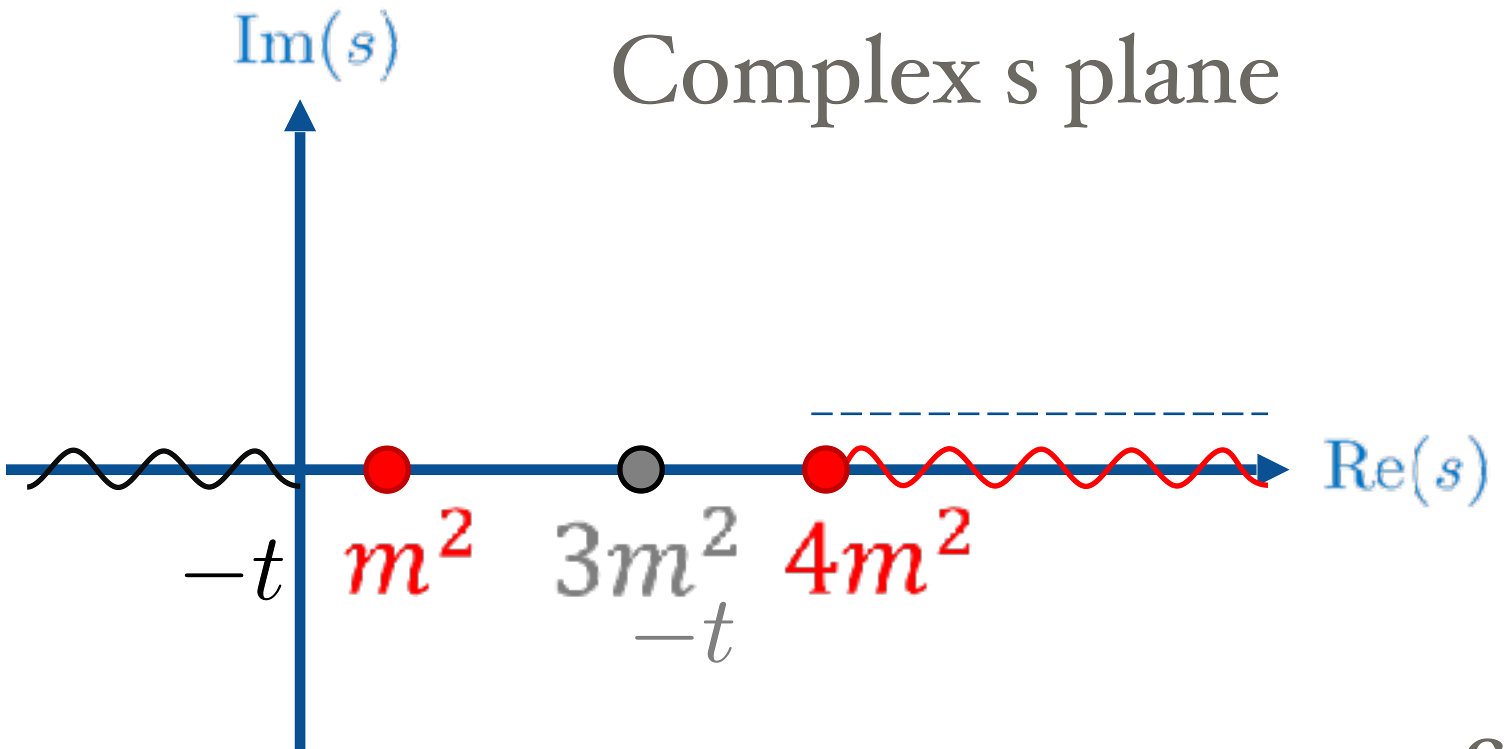
$$D_3 D_0^2 - D_1^3 + 2D_0^2 (D_2 D_0 - D_1^2) > 0$$

Linear (Improved) Positivity Bound

↓

$$a_4 a_1^2 - a_2^3 > 0$$

Scattering Amplitude Analyticity



Complex s plane

Physical scattering region is $s \geq 4m^2$

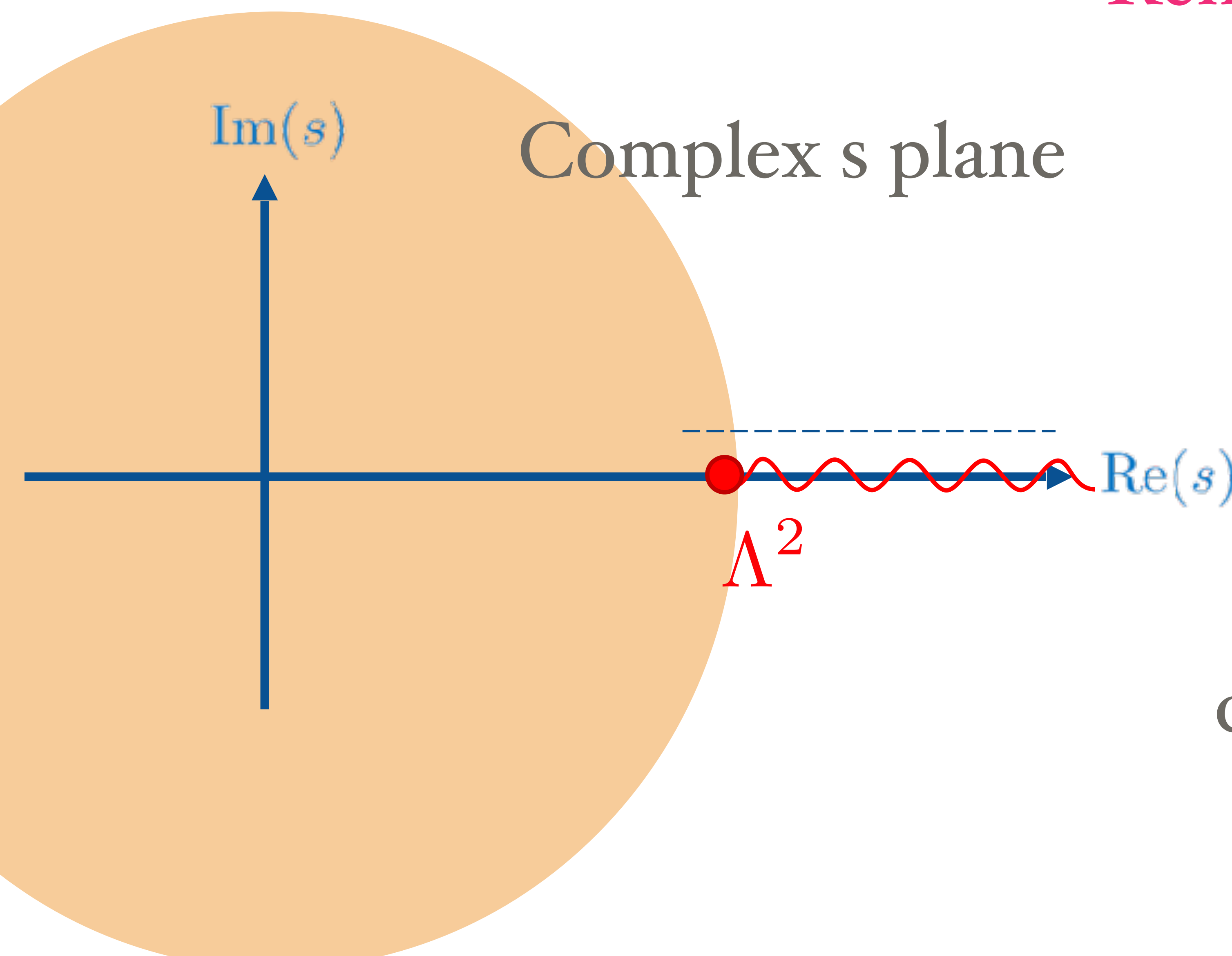
crossing: $u = 4m^2 - s - t$

$$\mathcal{A}_s(s,t) = \frac{\lambda_s(t)}{m^2 - s} + \frac{\lambda_u(t)}{m^2 - u} + (c_0(t) + c_1(t)s) + \frac{s^2}{\pi}\int_{4m^2}^{\infty} d\mu \frac{Im(A_s(\mu,t))}{\mu^2(\mu - s)} + \frac{u^2}{\pi}\int_{4m^2}^{\infty} d\mu \frac{Im(A_u(\mu,t))}{\mu^2(\mu - u)}$$

Poles Subtractions Branch cuts

‘Improved’ Scattering Amplitude Analyticity

Removes IR loop effects!!!!



Physical scattering region is $s \geq 4m^2$

crossing: $u = 4m^2 - s - t$

$$\mathcal{A}'_s(s,t) = c_0(t) + c_1(t)s + \frac{s^2}{\pi}\int_{\Lambda^2}^{\infty} d\mu \frac{Im(\mathcal{A}_s(\mu,t))}{\mu^2(\mu - s)} + \frac{u^2}{\pi}\int_{\Lambda^2}^{\infty} d\mu \frac{Im(\mathcal{A}_u(\mu,t))}{\mu^2(\mu - u)}$$

Fixed t (improved) linear Positivity Bounds

$$\mathcal{A}'_s(s,t) = c_0(t) + c_1(t)s + \frac{s^2}{\pi}\int_{\Lambda^2}^{\infty} d\mu \frac{Im(\mathcal{A}_s(\mu,t))}{\mu^2(\mu-s)} + \frac{u^2}{\pi}\int_{\Lambda^2}^{\infty} d\mu \frac{Im(\mathcal{A}_u(\mu,t))}{\mu^2(\mu-u)}$$

$\downarrow$

$$\frac{1}{M!}\frac{d^M}{ds^M}\mathcal{A}'_s(2m^2 - t/2, t) = \frac{1}{\pi}\int_{\Lambda^2}^{\infty} d\mu \frac{Im\mathcal{A}_s(\mu,t) + Im\mathcal{A}_s(\mu,t)}{(\mu - 2m^2 + t/2)^{M+1}} > 0$$

$M \geq 2$ $\quad$ $0 \leq t < 4m^2$ $\quad$ RH Cut $\quad$ LH Cut

Even M

Fixed t (improved) 'Stieltjes' Positivity Bounds

$$\frac{1}{M!}\frac{d^M}{ds^M}\mathcal{A}'_s(2m^2 - t/2, t) = \frac{1}{\pi}\int_{\Lambda^2}^{\infty} d\mu \frac{Im\mathcal{A}_s(\mu,t) + Im\mathcal{A}_s(\mu,t)}{(\mu - 2m^2 + t/2)^{M+1}} > 0$$

$$0 \leq t < 4m^2$$

$$\det_{pq}\left(\frac{1}{(M+p+q)!}\frac{d^{M+p+q}}{ds^{M+p+q}}\mathcal{A}'_s(2m^2 - t/2, t)\right) > 0$$

$$0 \leq t < 4m^2$$

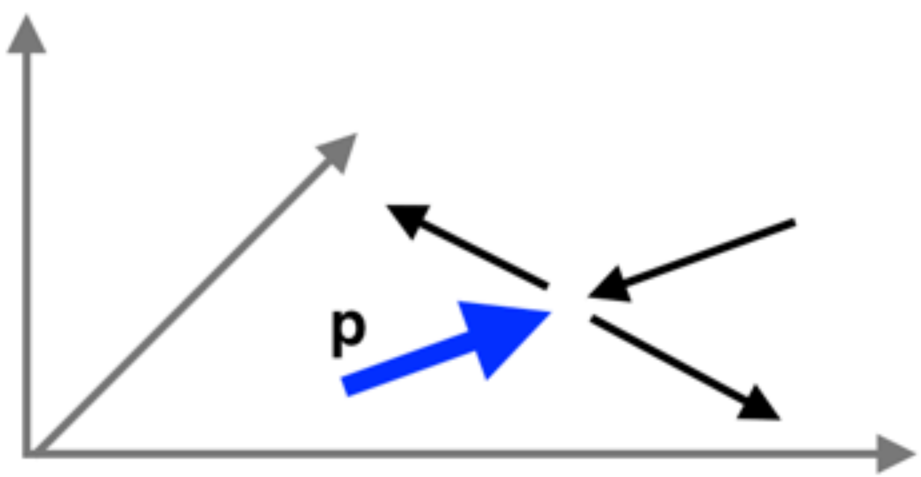
Even M+p+q

de Rham, Melville, AJT, Zhou 1706.02712

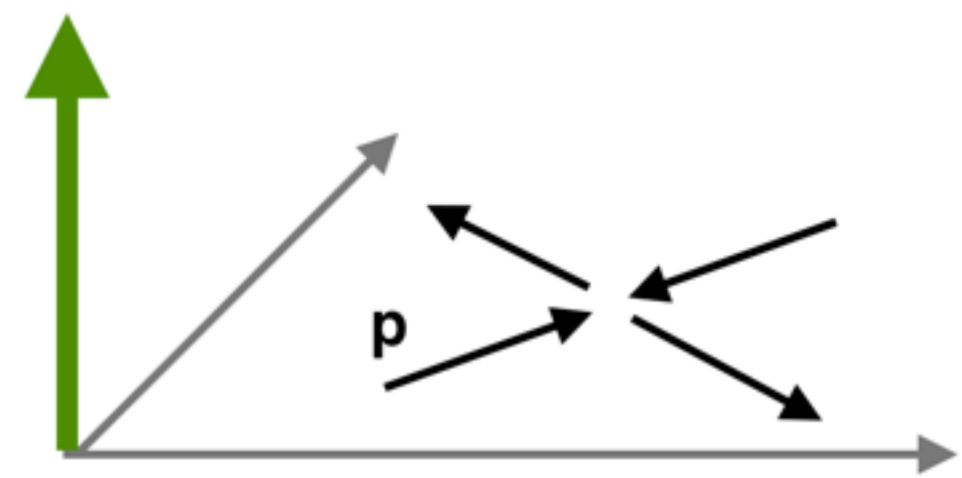
Scattering of all spins

Kotanski, 1965

Helicity



Transversity



$$\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4} = \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} u^{S_1}_{\lambda_1\tau_1} u^{S_2}_{\lambda_2\tau_2} u^{S_1 *}_{\tau_3\lambda_3} u^{S_2 *}_{\tau_4\lambda_4} \mathcal{H}_{\lambda_1\lambda_2\lambda_3\lambda_4}$$

Change of Basis $u^S_{\lambda\tau} = \langle S, \lambda | e^{-i\frac{\pi}{2}\hat{J}_z} e^{-i\frac{\pi}{2}\hat{J}_y} e^{i\frac{\pi}{2}\hat{J}_z} | S, \tau \rangle$

$$T^s_{\tau_1\tau_2\tau_3\tau_4}(s,t,u) = e^{-i\sum_i \tau_i \chi} T^u_{-\tau_1-\tau_4-\tau_3-\tau_2}(u,t,s)$$

Crossing is Simple!!

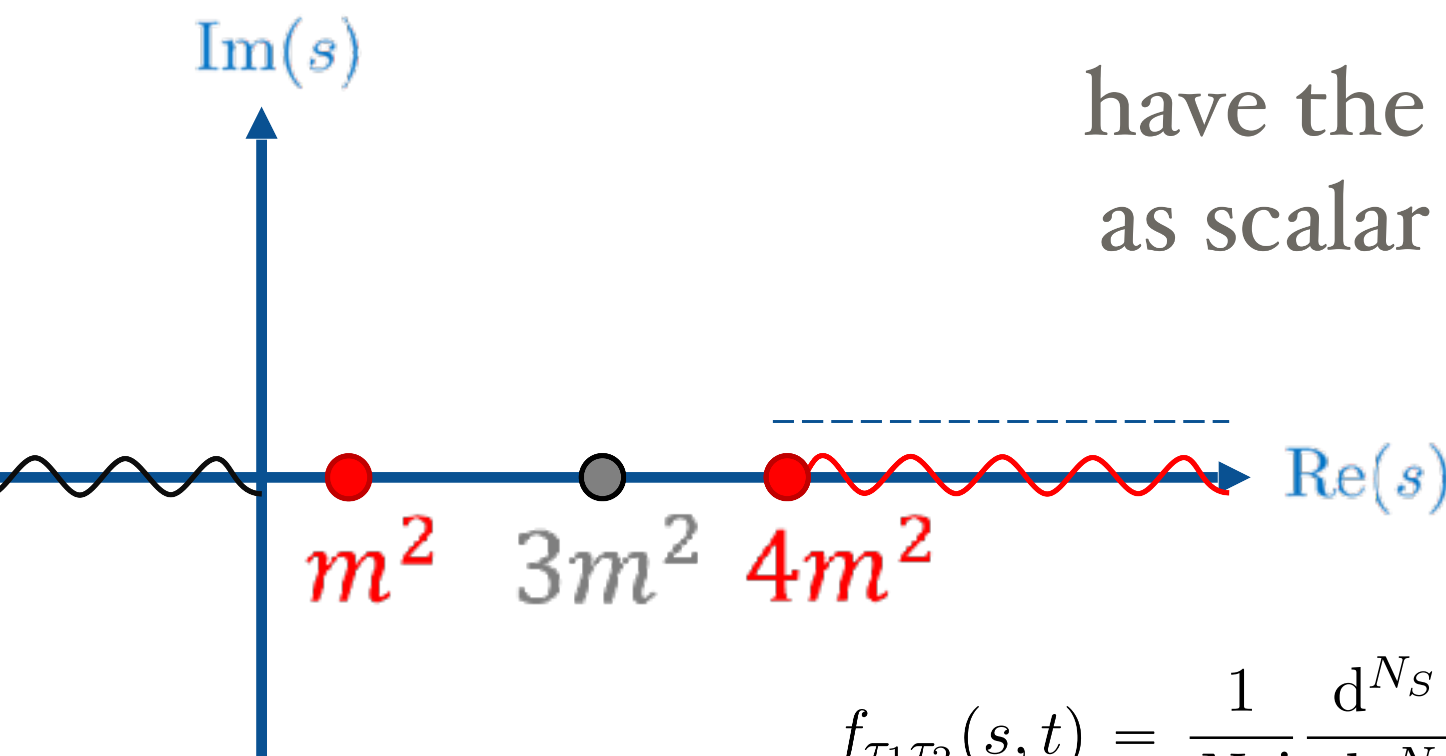
Dispersion Relation with Positivity along BOTH cuts

de Rham, Melville, AJT, Zhou 1706.02712

Punch line: The specific combinations:

$$\mathcal{T}^{+}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) = \left(\sqrt{-su}\right)^{\xi}\mathcal{S}^{S_1+S_2}\left(\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) + \mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,-\theta)\right)$$

have the same analyticity structure as scalar scattering amplitudes!!!!!!!



Implies Dispersion Relation

$$f_{\tau_1\tau_2}(s,t) = \frac{1}{N_S!}\frac{\mathrm{d}^{N_S}}{\mathrm{d}s^{N_S}}\tilde{\mathcal{T}}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t)$$

$$f_{\tau_1\tau_2}(v,t) = \frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}\mu\frac{\mathrm{Abs}_s\mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(\mu,t)}{(\mu-2m^2+t/2-v)^{N_S+1}} + \frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}\mu\frac{\mathrm{Abs}_u\mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(4m^2-t-\mu,t)}{(\mu-2m^2+t/2+v)^{N_S+1}}$$

Positive partial wave Moments

Partial wave expansion:

$$A(s,t) = F(\alpha)\frac{s^{1/2}}{(s-4m^2)^\alpha}\sum_{\ell=0}^{\infty}(2\ell+2\alpha)C_\ell^{(\alpha)}(\cos\theta)a_\ell(s), \quad \alpha = \frac{D-3}{2}$$

Define

$$\rho_{\ell,\alpha}(\mu) = \frac{F(\alpha)}{(\mu-\mu_p)^3}\frac{\mu^{1/2}}{(\mu-4m^2)^\alpha}(2\ell+2\alpha)\mathrm{Im}a_\ell(\mu)C_\ell^{(\alpha)}(1)$$

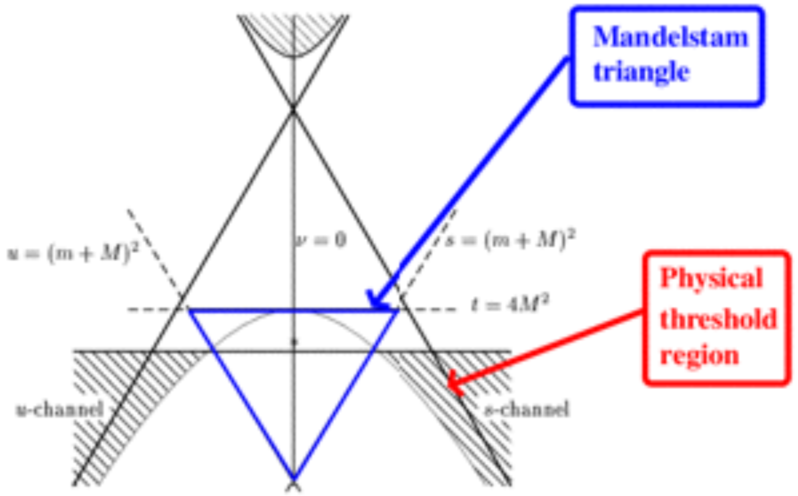
$$\frac{1}{2}\partial_s^2\mathcal{A}'(s,t) = \sum_\ell\int_0^\infty d\mu\left[\frac{1}{(\mu-s)^3}+\frac{1}{(\mu-s-t)^3}\right]\frac{\mu^3\rho_{\ell,\alpha}(\mu)}{C_\ell^{(\alpha)}(1)}C_\ell^{(\alpha)}\left(1+\frac{2t}{\mu}\right)$$

$$f^{(2N,M)} \equiv \frac{1}{2(2N+2)!}\partial_t^M\partial_s^{2N+2}\mathcal{A}'(s,t)$$

$$f^{(2N,0)} = \sum_\ell\int \mathrm{d}\mu\rho_{\ell,\alpha}(\mu)\frac{1}{\mu^{2N}} > 0, \quad N = 0,1,2,...,$$

$$\langle\!\langle X(\mu,l)\rangle\!\rangle = \frac{\sum_\ell\int \mathrm{d}\mu\rho_{\ell,\alpha}(\mu)X(\mu,l)}{\sum_\ell\int \mathrm{d}\mu\rho_{\ell,\alpha}(\mu)}$$

$$f^{(2N,0)} = \langle\langle\frac{1}{\mu^{2N}}\rangle\rangle$$



Crossing Symmetry

AJT, Zi-Yue Wang, Shuang-Yong Zhou
arXiv:2011.02400

Simon Caron-Huot, Vincent Van Duong
arXiv:2011.02957

Aninda Sinha, Ahmadullah Zahed
arXiv:2012.04877

$$A_s(s,t,u) = A_t(t,s,u)$$

Null-constraints

$$0 = \mathcal{A}(s,t) - \mathcal{A}(t,s) = \sum_\ell \int d\mu \rho_{\ell,\alpha}(\mu) \left[\frac{2H_{D,\ell}\, st(s^2-t^2)}{(D-2)D\mu^2} + \dots \right]$$

$$\sum_\ell \int \mathrm{d}\mu\ \rho_{\ell,\alpha}(\mu) \frac{H_{D,\ell}}{\mu^2} = 0$$

$$H_{D,\ell} = \ell(\ell+D-3)[4-5D-2(3-D)\ell+2\ell^2]$$

$$\langle\langle \frac{H_{D,\ell}}{\mu^2} \rangle\rangle = 0$$

Key Idea

Make Maximal use of null constraints to strengthen positivity bounds

$$\langle\langle \frac{H_{D,\ell}}{\mu^2} \rangle\rangle = 0$$

$$H_{D,\ell} = \ell(\ell + D - 3)[4 - 5D - 2(3 - D)\ell + 2\ell^2]$$

$$\langle\langle X(\mu, l) \rangle\rangle = \frac{\sum_\ell \int \mathrm{d}\mu \rho_{\ell,\alpha}(\mu) X(\mu, l)}{\sum_\ell \int \mathrm{d}\mu \rho_{\ell,\alpha}(\mu)}$$

Example

$$f^{(2N,M)} \equiv \frac{1}{2(2N+2)!} \partial_t^M \partial_s^{2N+2} \mathcal{A}'(s,t)\Big|_{s,t\to 0}$$

$$\frac{f^{(0,1)}}{f^{(0,0)}} + \left\langle\!\!\left\langle \frac{3}{2\mu} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \frac{2(-3+D)\ell + 2\ell^2}{(D-2)\mu} \right\rangle\!\!\right\rangle$$

Cauchy-Schwarz

$$\langle\!\langle X(\mu,l)\rangle\!\rangle^2 \leq \langle\!\langle X(\mu,l)^2\rangle\!\rangle$$

$$\left(\frac{f^{(0,1)}}{f^{(0,0)}} + \left\langle\!\!\left\langle \frac{3}{2\mu} \right\rangle\!\!\right\rangle\right)^2 = \left\langle\!\!\left\langle \frac{2(D-3)\ell + 2\ell^2}{(D-2)\mu} \right\rangle\!\!\right\rangle^2 \leq \left\langle\!\!\left\langle \left(\frac{2(D-3)\ell + 2\ell^2}{(D-2)\mu}\right)^2 \right\rangle\!\!\right\rangle$$

ZERO!!!

BUT!!!

$$(2(D-3)\ell + 2\ell^2)^2 = (5D-4)\left[2(D-3)\ell + 2\ell^2\right] + 2H_{D,\ell}$$

hence:

$$\left(\frac{f^{(0,1)}}{f^{(0,0)}} + \left\langle\!\!\left\langle \frac{3}{2\mu} \right\rangle\!\!\right\rangle\right)^2 \leq \frac{5D-4}{D-2}\left\langle\!\!\left\langle \frac{2(D-3)\ell + 2\ell^2}{(D-2)\mu^2} \right\rangle\!\!\right\rangle$$

AJT, Zi-Yue Wang, Shuang-Yong Zhou
arXiv:2011.02400

Upper and Lower Bound

given:

$$\left\langle\!\!\left\langle \frac{2(D-3)\ell + 2\ell^2}{(D-2)\mu^2} \right\rangle\!\!\right\rangle < \frac{1}{\Lambda^2} \left\langle\!\!\left\langle \frac{2(D-3)\ell + 2\ell^2}{(D-2)\mu} \right\rangle\!\!\right\rangle$$

then:

$$\left(\frac{f^{(0,1)}}{f^{(0,0)}} + \left\langle\!\!\left\langle \frac{3}{2\mu} \right\rangle\!\!\right\rangle \right)^2 < \frac{5D-4}{(D-2)\Lambda^2} \left(\frac{f^{(0,1)}}{f^{(0,0)}} + \left\langle\!\!\left\langle \frac{3}{2\mu} \right\rangle\!\!\right\rangle \right)$$

$$-\frac{3}{2\Lambda^2} f^{(0,0)} < f^{(0,1)} < \frac{5D-4}{(D-2)\Lambda^2} f^{(0,0)}$$

Weakly Broken Galileon

$$\Lambda_3^{4-D}\mathcal{L}_{\rm mg} = -\frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{1}{2}m^2\pi^2 + \sum_{n=3}^{D+1}\frac{g_n}{\Lambda_3^{3n-3}}\pi\partial^{\mu_1}\partial_{[\mu_1}\pi\partial^{\mu_2}\partial_{\mu_2}\pi\cdots\partial^{\mu_n}\partial_{\mu_n]}\pi + \sum_i \mathcal{O}_i\left(\frac{\partial^2\pi}{\Lambda_3^3},\frac{\partial^3\pi}{\Lambda_3^4},\frac{\partial^4\pi}{\Lambda_3^5},\ldots\right),$$

$$\Lambda_3^{4-D}\mathcal{L}_{\rm wbg} = -\frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{\alpha}{\Lambda_3^4}(\partial\pi)^4 + \sum_{n=3}^{D+1}\frac{g_n}{\Lambda_3^{3n-3}}\pi\partial^{\mu_1}\partial_{[\mu_1}\pi\partial^{\mu_2}\partial_{\mu_2}\pi\cdots\partial^{\mu_n}\partial_{\mu_n]}\pi + \sum_i \mathcal{O}_i\left(\frac{\partial^2\pi}{\Lambda_3^3},\frac{\partial^3\pi}{\Lambda_3^4},\frac{\partial^4\pi}{\Lambda_3^5},\ldots\right),$$

AJT, Zi-Yue Wang, Shuang-Yong Zhou
arXiv:2011.02400

Weakly Broken Galileon

$$\mathcal{A}'(s,t) \sim \frac{1}{\Lambda_3^{D-4}} \left(\frac{m^2}{\Lambda_3^6} x + \frac{1}{\Lambda_3^6} y + \frac{1}{\Lambda_3^8} x^2 + \dots \right)$$

$$y = stu$$
$$x = s^2 + t^2 + u^2$$

not suppressed

suppressed

$$-\frac{3}{2\Lambda^2} f^{(0,0)} < f^{(0,1)} < \frac{5D-4}{(D-2)\Lambda^2} f^{(0,0)}$$

Contradiction!!!

No local UV completion for weakly broken Galileons

see also Bellazzini et al, Positive Moments .., 2020

Extended bounds

AJT, Zi-Yue Wang, Shuang-Yong Zhou
arXiv:2011.02400

See also

Simon Caron-Huot, Vincent Van Duong
arXiv:2011.02957

$$\mathcal{A}'(s,t) = \sum_{p.q=0}^{\infty} c_{p,q} w^p t^q \qquad w = -(s-2m^2)(u-2m^2)$$

General Idea:

I: Given a polynomial $Poly(l)$ whose highest power is positive

$$\langle \mu^{-M} Poly(l) \rangle \geq \langle \mu^{-M} Min(Poly(l)) \rangle$$

Low orders in l ⟶ Lower t derivatives

II: Use null constraints to define new polynomials

$$\pm\langle \mu^{-M} Poly(l) \rangle + \langle \mu^{-M} NullPoly(l) \rangle = \langle \mu^{-M} Poly'(l) \rangle \geq \langle \mu^{-M} Min(Poly'(l)) \rangle$$

0!

Extended bounds

AJT, Zi-Yue Wang, Shuang-Yong Zhou
arXiv:2011.02400

See also

Simon Caron-Huot, Vincent Van Duong
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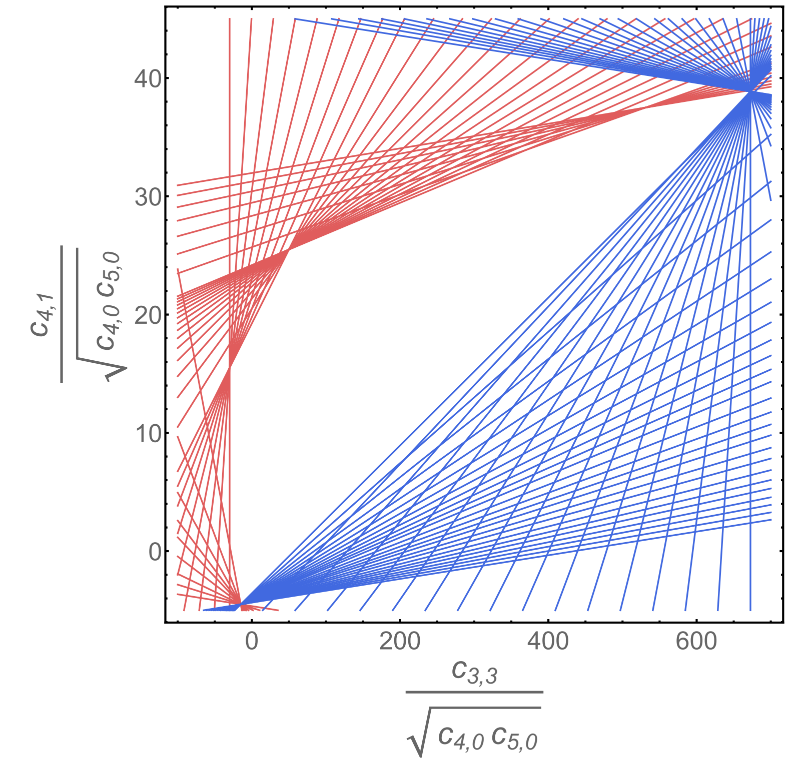
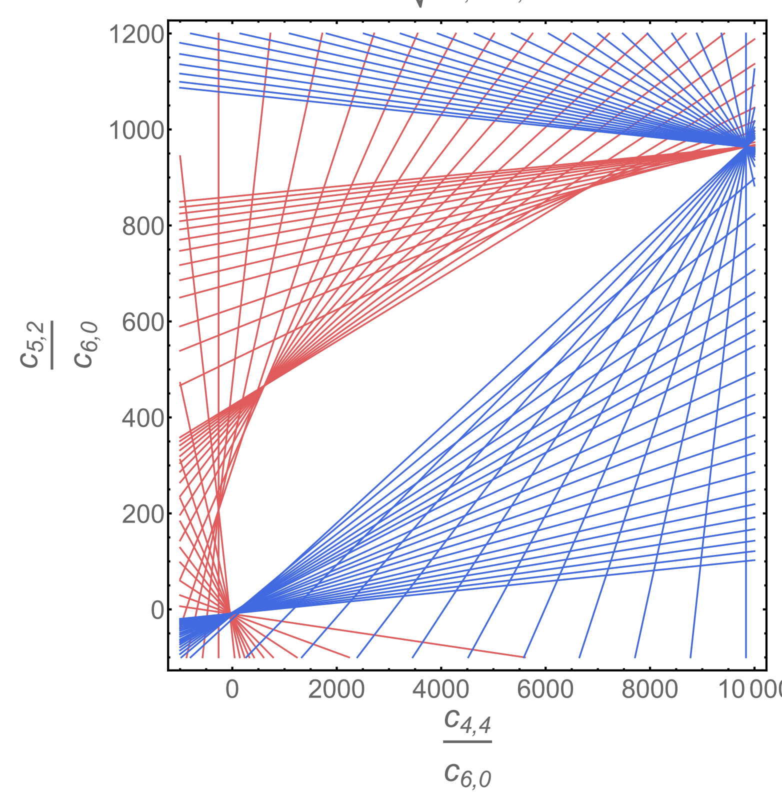
Bellazzini et al, Positive Moments .., 2020

$$\mathcal{A}'(s,t) = \sum_{p.q=0}^{\infty} c_{p,q} w^p t^q$$

$$w = -(s-2m^2)(u-2m^2)$$

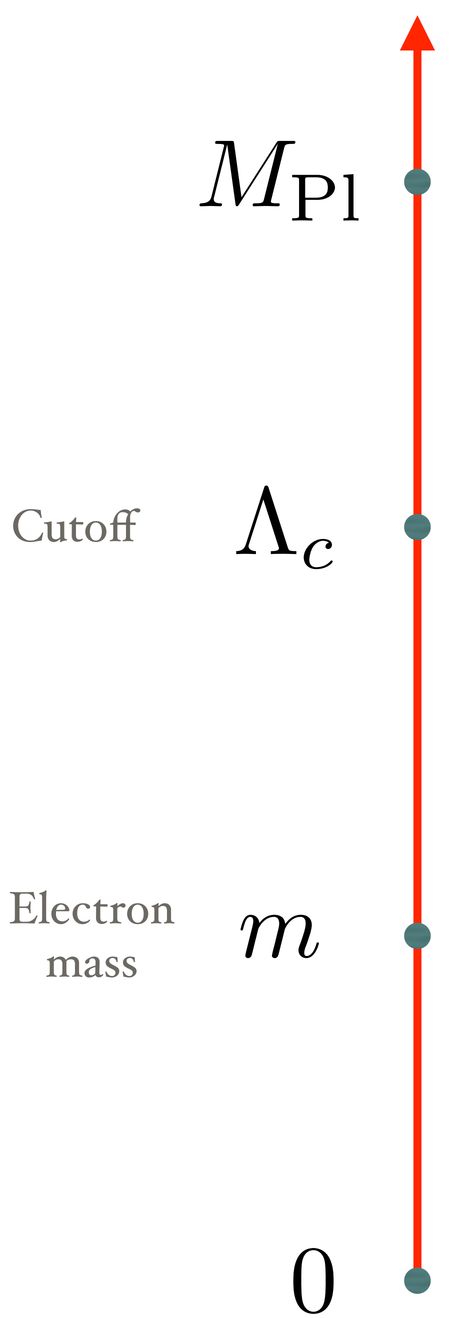
(m,n)	$D^{\text{stu}}_{m,n}$ bound	$\bar{D}^{\text{stu}}_{m,n}$ bound
$(1,1)$	$c_{1,1} > -\frac{3}{2}\sqrt{c_{1,0}c_{2,0}}$	$c_{1,1} < 8\sqrt{c_{1,0}c_{2,0}}$
$(2,1)$	$c_{2,1} > -\frac{5}{2}\sqrt{c_{2,0}c_{3,0}}$	$c_{2,1} < \frac{465}{38}\sqrt{c_{2,0}c_{3,0}}$
$(2,2)$	$c_{2,2} > -\frac{9}{2}c_{3,0}$	$c_{2,2} < \frac{2961}{58}c_{3,0}$
$(3,1)$	$c_{3,1} > -\frac{7}{2}\sqrt{c_{3,0}c_{4,0}}$	$c_{3,1} < \frac{1097}{58}\sqrt{c_{3,0}c_{4,0}}$
$(3,2)$	$c_{3,2} > -7c_{4,0}$	$c_{3,2} < \frac{10027}{59}c_{4,0}$
$(3,3)$	$c_{3,3} + \frac{3}{4}c_{4,1} > -\frac{147}{8}\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - 8c_{4,1} > -154\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - \frac{481}{12}c_{4,1} > -\frac{7777}{8}\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - 104c_{4,1} > -3369\sqrt{c_{4,0}c_{5,0}}$	$c_{3,3} - \frac{650}{41}c_{4,1} < \frac{2310}{41}\sqrt{c_{4,0}c_{5,0}}$
$(4,2)$	$c_{4,2} > -\frac{17}{2}c_{5,0}$	$c_{4,2} < \frac{3923}{12}c_{5,0}$
$(4,3)$	$c_{4,3} + \frac{3}{4}c_{5,1} > -\frac{253}{8}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{180}{41}c_{5,1} > -\frac{8705}{82}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{325}{12}c_{5,1} > -\frac{16825}{24}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{169}{2}c_{5,1} > -\frac{11187}{4}\sqrt{c_{5,0}c_{6,0}}$ $c_{4,3} - \frac{743}{4}c_{5,1} > -\frac{63279}{8}\sqrt{c_{5,0}c_{6,0}}$	$c_{4,3} - \frac{73153}{1748}c_{5,1} < \frac{708543}{3496}\sqrt{c_{5,0}c_{6,0}}$
$(4,4)$	$c_{4,4} + \frac{25}{24}c_{5,2} > -\frac{147}{8}c_{6,0}$, $c_{4,4} - \frac{125}{37}c_{5,2} > -\frac{71175}{74}c_{6,0}$, $c_{4,4} - \frac{785}{52}c_{5,2} > -\frac{83490}{13}c_{6,0}$, $c_{4,4} - \frac{2485}{69}c_{5,2} > -\frac{1144125}{46}c_{6,0}$	$c_{4,4} - 15c_{5,2} < \frac{195}{2}c_{6,0}$, $c_{4,4} + \frac{368085}{36544}c_{5,2} < \frac{2365845}{18272}c_{6,0}$





What about coupling to gravity?

gravitational QED scales



Unknown UV completion

gravitational QED EFT

gravitational Euler-Heisenberg EFT

gravitational Euler-Heisenberg

$$\mathcal{L}_{\rm QED} = \sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}(i\nabla\!\!\!/ + m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi\right]$$

$$S_{\rm Eul-Heis,1} = \int {\rm d}^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a_1}{m^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{a_2}{m^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right.$$
$$\frac{b_1}{m^2}RF_{\mu\nu}F^{\mu\nu} + \frac{b_2}{m^2}R_{\mu\nu}F^{\mu\lambda}F^{\nu}{}_{\lambda} + \frac{b_3}{m^2}R_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}$$
$$\left.+c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + c_3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \cdots\right],$$

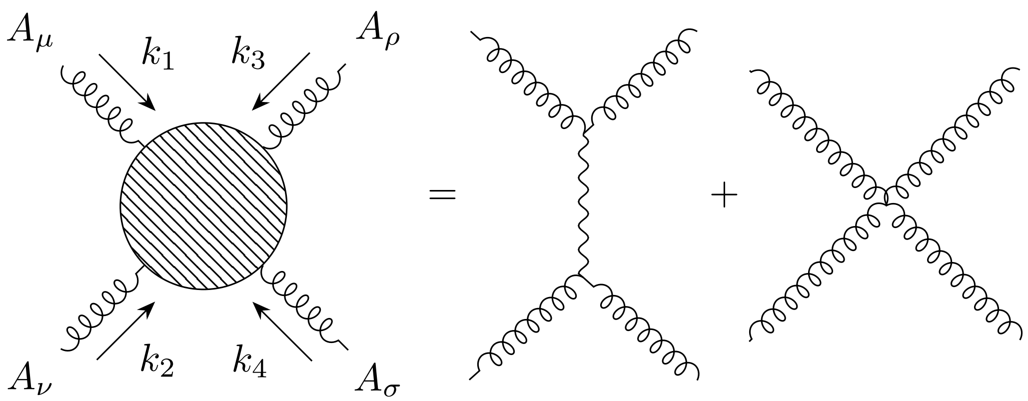
$$\mathcal{L}_{\rm Eul-Heis,2} = \sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a_1'}{m^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{a_2'}{m^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \frac{b_3}{m^2}F_{\mu\nu}F_{\rho\sigma}C^{\mu\nu\rho\sigma}\right]$$

$$a_1' = a_1 + \frac{1}{4}\frac{m^2}{M_{\rm Pl}^2}b_2 + \frac{1}{2}\frac{m^2}{M_{\rm Pl}^2}b_3\,, \qquad a_2' = a_2 + \frac{1}{4}\frac{m^2}{M_{\rm Pl}^2}b_2 + \frac{1}{2}\frac{m^2}{M_{\rm Pl}^2}b_3$$

Relevant for discussions of Weak Gravity Conjecture

Hamada, Noumi, Shui 1909.01352
Bellazzini et al 1902.03250

Positivity (t-channel pole removed)



Cheung, Remmen 1407.7865

$$\mathcal{A}_{\text{Eul-Heis}}(++--) = \mathcal{A}_{\text{Eul-Heis}}(--++) = \frac{s^4}{M_{\text{Pl}}^2 stu} + \frac{8(a_1' + a_2')}{m^4} s^2$$

$$a_1' + a_2' > 0$$

For spinor QED

$$\frac{e^4}{5760 M_{\text{Pl}}^2 \pi^2}\left(-24\frac{m^2}{e^2} + 11 M_{\text{Pl}}^2\right) > 0$$

For scalar QED

$$\frac{e^4}{2880 M_{\text{Pl}}^2 \pi^2}\left(-2\frac{m^2}{e^2} + M_{\text{Pl}}^2\right) > 0$$

$$e/m \gtrsim \sqrt{2}/M_{\text{Pl}}$$

Weak Gravity Conjecture!!!

Problem!

Non-Gravitational part positive

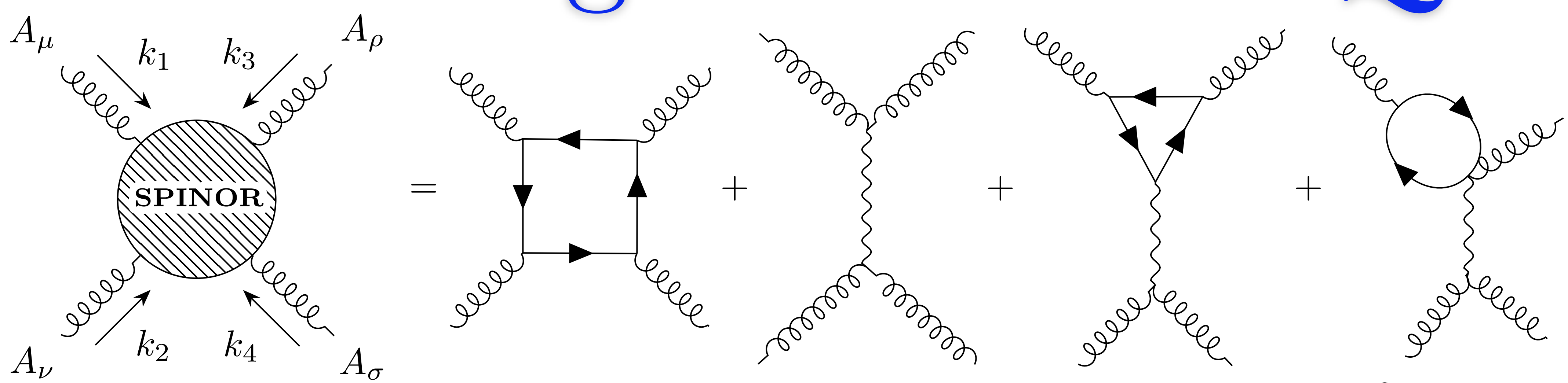
Gravitational part negative

$$\frac{e^4}{5760 M_{\rm Pl}^2 \pi^2}\left(-24\frac{m^2}{e^2} + 11 M_{\rm Pl}^2\right) > 0$$

'Known' contribution from electron loops

This contribution can be **removed**, by applying (improved) positivity bounds directly to gravitational QED EFT!!

Cutoff of gravitational QED



To order $1/M^2_{\rm Planck}$

$$0 < \partial_s^2 \tilde{\mathcal{A}}^{\rm I}(0,0,0) - \frac{2}{\pi}\int_0^{\epsilon^2\Lambda_c^2} {\rm d}s' \, \frac{{\rm Disc}_s \mathcal{A}^{\rm I}(s',0,u')}{s'^3} - \frac{2}{\pi}\int_0^{\epsilon^2\Lambda_c^2} {\rm d}u' \, \frac{{\rm Disc}_u \mathcal{A}^{\rm I}(s',0,u')}{u'^3}$$

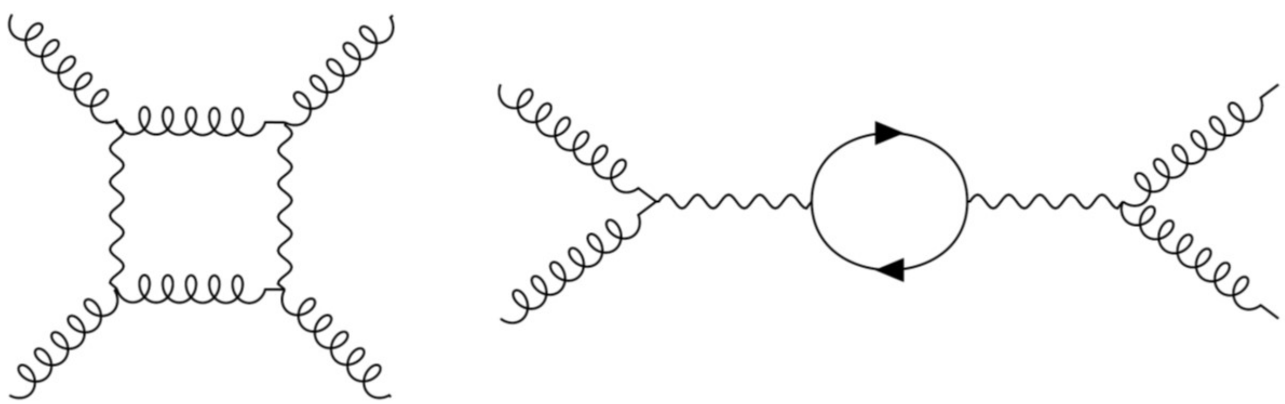
$$0 < \frac{11e^4}{360\pi^2 m^4} - \frac{11e^2}{180\pi^2 m^2 M_{\rm Pl}^2} - \frac{2}{\pi}\int_0^{\epsilon^2\Lambda_c^2} {\rm d}s' \, \frac{{\rm Disc}_s \mathcal{A}^{\rm I}(s',0,u')}{s'^3} - \frac{2}{\pi}\int_0^{\epsilon^2\Lambda_c^2} {\rm d}u' \, \frac{{\rm Disc}_u \mathcal{A}^{\rm I}(s',0,u')}{u'^3}$$

$$0 < -\frac{11e^2}{360\pi^2 m^2 M_{\rm Pl}^2} - \frac{e^2}{3\pi^2\Lambda^2 M_{\rm Pl}^2} - \frac{e^4}{4\pi^2\Lambda^4} - \frac{e^2 m^2}{4\pi^2\Lambda^4 M_{\rm Pl}^2} + \frac{e^4}{\pi^2\Lambda^4}\ln\frac{\Lambda}{m} + \frac{e^2 m^2}{\pi^2\Lambda^4 M_{\rm Pl}^2}\ln\frac{\Lambda}{m},$$

$$\frac{e^4}{\pi^2\Lambda^4}\left(\ln\frac{\Lambda}{m} - \frac{1}{4}\right) - \frac{11e^2}{360\pi^2 m^2 M_{\rm Pl}^2} > 0$$

$$\epsilon\Lambda_c \lesssim (emM_{\rm Pl})^{1/2}$$

Higher order gravitational contributions



$$\gamma(\mu) = \gamma_m - B\ln(\mu/m)$$

Coefficient of R squared terms

$$\frac{e^4}{\pi^2\Lambda^4}\left(\ln\frac{\Lambda}{m} - \frac{1}{4}\right) - \frac{11e^2}{360\pi^2 m^2 M_{\rm Pl}^2} - \frac{B}{M_{\rm Pl}^4}\ln\left(\frac{\Lambda}{m}\right) + \frac{\gamma_m}{M_{\rm Pl}^4} > 0$$

Noted in 3D in
Chen et al. 1901.11480

$$-\frac{11e^2}{360\pi^2 m^2 M_{\rm Pl}^2} + \frac{\gamma_\Lambda}{M_{\rm Pl}^4} > 0$$

$$m \gtrsim \frac{eM_{\rm Pl}}{\sqrt{N_*}}$$

Alternative Explanation - mild negativity allowed

Decoupling limits consistent with $c > -\frac{\mathcal{O}(1)}{M^2 M_{\rm Pl}^2}$

Alberte et al. 2007.12667
Tokuda et al. 2007.15009
Herrero-Valea et al. 2011.11652

Conjecture

Positivity Bounds and the Massless Spin-2 Pole
Lasma Alberte, Claudia de Rham, Sumer Jaitly, Andrew J. Tolley
arXiv:2007.12667

For a weakly coupled (tree level) UV completion, given

$$A(s,t) = -\frac{s^2}{M_{\rm Pl}^2 t} + \frac{\tilde{c}}{M^4} s^2 + \dots$$

Conjecture! $\tilde{c} > -\frac{M^2}{M_{\rm Pl}^2} \times \mathcal{O}(1)$

Recently 'Proven'! $\tilde{c} > -\frac{M^2}{M_{\rm Pl}^2} 17 \log(1.7 M b_{\rm max})$

Sharp Boundaries for the Swampland
Simon Caron-Huot, Dalimil Mazac, Leonardo Rastelli, David Simmons-Duffin

arXiv:2102.08951

Conclusions

- Positivity Bounds are very powerful at constraining irrelevant operators in a low energy EFT
- Full crossing symmetry implies upper and lower bounds on Wilson coefficients
- Strong constraints on interacting massive spin theories and supersoft theories
- Full understanding of extension to massless gravity (no mass gap) unclear, although recent exciting progress
- Small amount of *'negativity'* allowed with gravity, without contradicting unitarity and causality