Causality Constraints for EFTs and Selection Rules

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Based on 2011.10058

Introduction

- Effective Field Theories provide systematic parameterisation of the impact of heavy states on local, low energy corrections.

- If UV is unknown, IR symmetries can sometimes be highly predictive…

$$M_{Pl}^2 R \qquad \frac{1}{\Lambda}(LL)(HH)$$

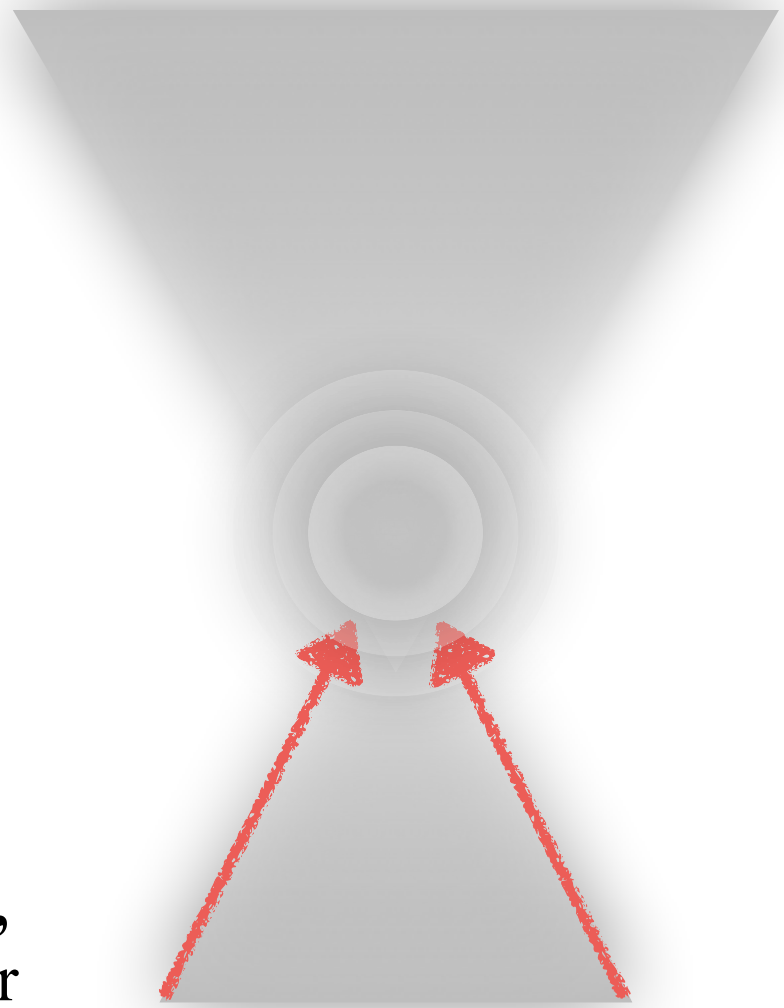
- …but sometimes not: e.g. Dim 8 SMEFT see pg 17-34 of **Murphy 2005.00059**.

#	Class	N_{type}	N_{term}	N_{op} [10]	Table(s)
1	X^4	7	43	43	2
2	H^8	1	1	1	2
3	H^6D^2	1	2	2	2
4	H^4D^4	1	3	3	2
5	X^3H^2	3	6	6	3
6	X^2H^4	5	10	10	3
7	$X^2H^2D^2$	4	18	18	3
8	XH^4D^2	2	6	6	3
9	ψ^2X^2H	16	96	$96n_g^2$	4
10	ψ^2XH^3	8	22	$22n_g^2$	5
11	$\psi^2H^2D^3$	6	16	$16n_g^2$	5
12	ψ^2H^5	3	6	$6n_g^2$	5
13	ψ^2H^4D	6	13	$13n_g^2$	5
14	ψ^2X^2D	21	57	$57n_g^2$	6, 7
15	ψ^2XH^2D	16	92	$92n_g^2$	7, 8
16	ψ^2XHD^2	8	48	$48n_g^2$	9
17	$\psi^2H^3D^2$	3	36	$36n_g^2$	9
18(B)	ψ^4H^2	19	75	$n_g^2(67n_g^2+n_g+7)$	10, 11
18($\not B$)		$4+3$	$12+8$	$\frac{1}{3}n_g^2(43n_g^2-9n_g+2)$	10
19(B)	ψ^4X	$40+5$	$156+12$	$4n_g^2(40n_g^2-1)$	12, 13, 14
19($\not B$)		4	$44+12$	$2n_g^3(21n_g+1)$	15
20(B)	ψ^4HD	16	$134+2$	$n_g^3(135n_g-1)$	16, 17
20($\not B$)		7	32	$n_g^3(29n_g+3)$	17
21(B)	ψ^4D^2	18	55	$\frac{11}{2}n_g^2(9n_g^2+1)$	10, 18
21($\not B$)		4	$10+2$	$n_g^3(11n_g-1)$	10
	B	$204+5$	$895+14$	$895(36971)$, $n_g=1(3)$	
	$\not B$	$19+3$	$98+22$	$98(7836)$, $n_g=1(3)$	
	Total	$223+8$	$993+36$	$993(44807)$, $n_g=1(3)$	

Introduction

- Local contact terms in effective action, but not sufficient for causality. Interactions that fuzz-out the light cone may still coherently mediate superluminal excitations.

- In QFT, "believed" that microcausality
$$(x-y)^2 > 0 \;\Rightarrow\; [\phi(x), \phi(y)] = 0$$
and local contact interactions implies polynomial strength high-energy scaling and analytic momentum dependence of the S-matrix.

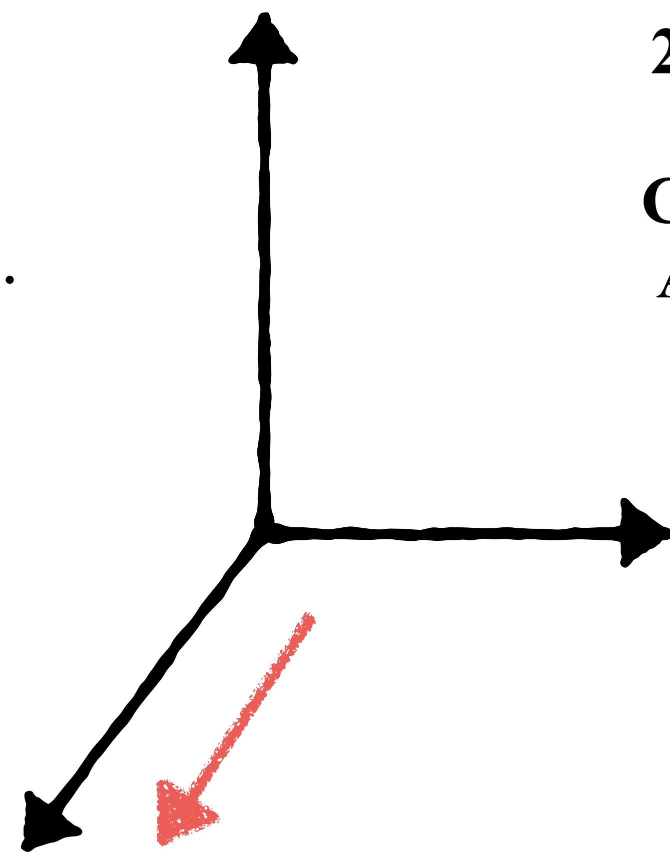


Gell-Mann, Goldberger, Thirring (1954), Martin (1962), Bros, Epstein, Glaser (1963), much work in chiral PT, later **Adams, Arkani-Hamed, et.al. hep-th/0602178,** recent bootstrap papers e.g. **Hartman, Jain, Kundu 1509.00014** etc.

Introduction

Space of interactions in EFTs.

Angular Momentum



Dimension

Species

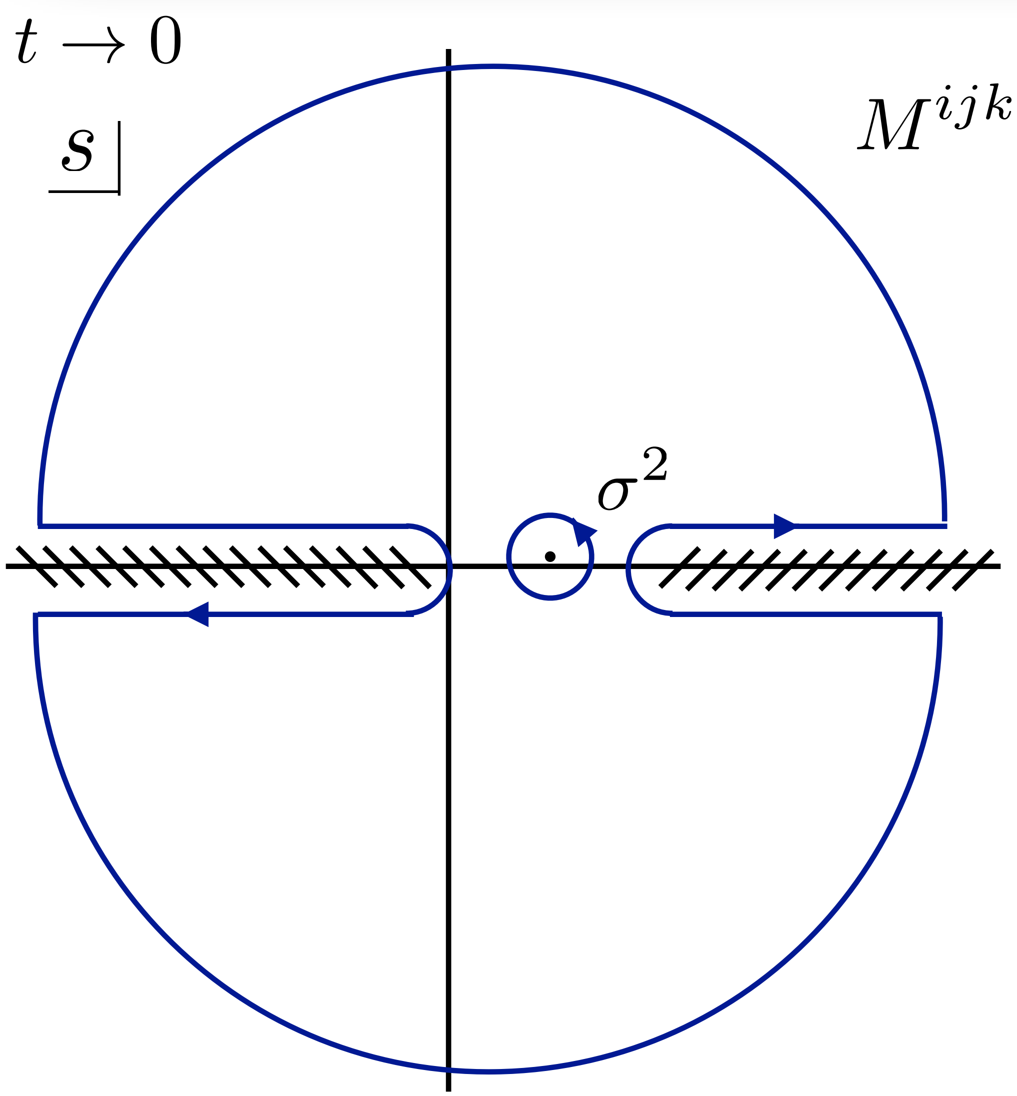
This talk (and the next).

Bellazzini, Elias Miro, et.al. 2011.00037, Tolley, Wang, Zhou 2011.02400, Caron-Huot, Duong 2011.02957, Arkani-Hamed, Huang, Huang 2012.15849 and more.

Outline

1. Sum rule and constraints from causality and unitary on multi-species theories.
2. Internal symmetries and entanglement.
3. Spin and supersymmetry.
4. Loops and renormalisation.

Dispersion Relation



$$M^{ijkl} = \frac{d^2}{ds^2} A(i,j \to k,l)(s)|_{s=\sigma^2}$$

$$= \frac{1}{\pi i} \int_{4m^2}^{\infty} \left(\frac{\mathrm{Disc} A^{ijkl}(s)}{(s-\sigma^2)^3} + \frac{\mathrm{Disc} A^{i\bar{l}k\bar{j}}(s)}{(s+\sigma^2-4m^2)^3} \right) ds$$

$$+ \text{residues at poles}$$

Assume conditions under which this all works (e.g. LSZ constructible, Wightman axioms etc.)

$$\left(A^{ijkl}(s)\right)^* = A^{klij}(s^*)$$

Hermitian analyticity (analytic requirement for S-matrix unitarity) **Olive (1962)**

$A^{ijkl}(s) = o(s^2)$ for $s \to \infty$

Froissart bound etc.

Take crossing-symmetric insertion, assume all masses small for simplicity.

$$\sigma^2 = 2m^2 \to 0$$

Following notation of
Zhang, Zhou 2005.03047

Sum Rule

Move IR segment of dispersion integral over to IR side of equation, leave implicit if necessary.

$$M^{ijkl}(0) = \frac{1}{\pi}\int_{\lambda^2}^{\infty}\frac{1}{s^3}\sum_X \left(\mathcal{M}^{ij\to X}\left(\mathcal{M}^{kl\to X}\right)^* + \mathcal{M}^{i\bar{l}\to X}\left(\mathcal{M}^{k\bar{j}\to X}\right)^*\right) ds$$

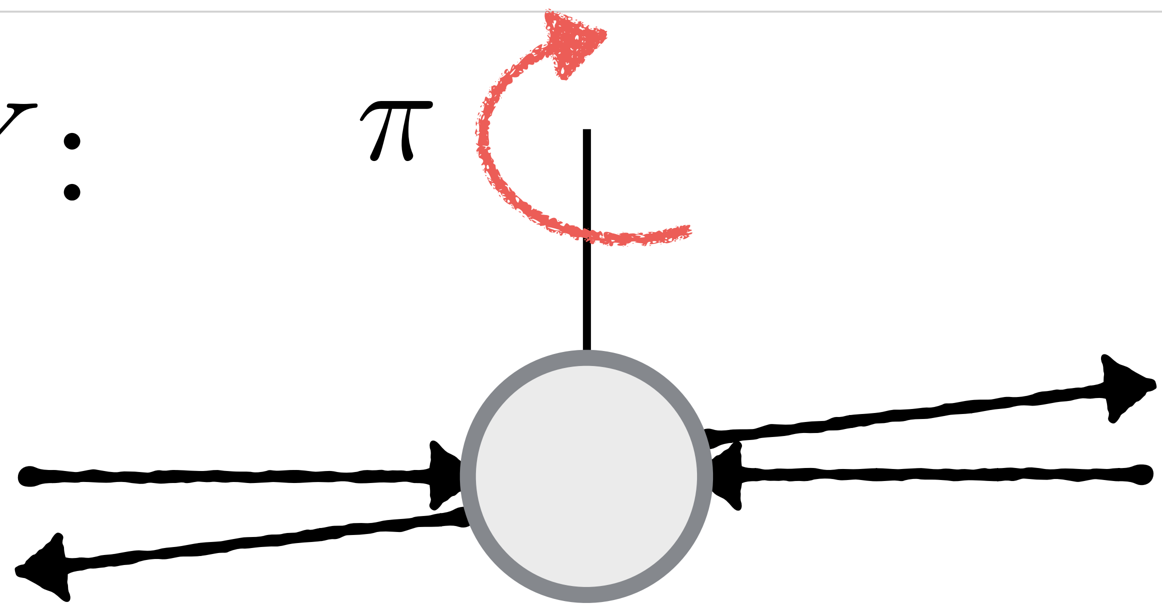
$$M^{ijkl} = \boldsymbol{m}^{kl}\cdot\boldsymbol{m}^{ij} + \boldsymbol{m}^{k\bar{j}}\cdot\boldsymbol{m}^{i\bar{l}}$$

$$\boldsymbol{m}^{ij} = [\mathcal{M}^{ij\to X}(s)]$$

Vector (continuous, infinite) of couplings of states (i, j) to UV state X at energy s.

Discrete Symmetries

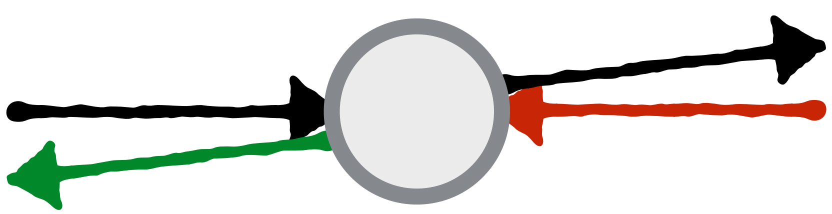
$Y:$



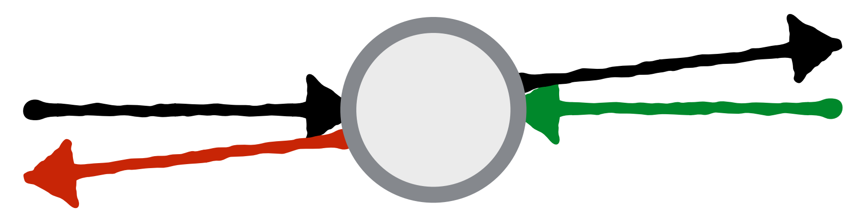
$$M^{ijkl} = M^{jilk}$$

$$\boldsymbol{m}^{kl} \cdot \boldsymbol{m}^{ij} = \boldsymbol{m}^{lk} \cdot \boldsymbol{m}^{ji}$$

Crossing:



$=$



$$M^{ijkl} = M^{i\bar{l}k\bar{j}}$$

$CPT:$ $\quad M^{ijkl} = M^{\bar{k}\bar{l}\bar{i}\bar{j}} \qquad \boldsymbol{m}^{kl} \cdot \boldsymbol{m}^{ij} = \boldsymbol{m}^{\bar{i}\bar{j}} \cdot \boldsymbol{m}^{\bar{k}\bar{l}}$

General Causality Constraints

- Problem: Reformulate sum rule as a set of inequalities among Wilson coefficients (the IR amplitudes).

$$M^{ijkl} = \boldsymbol{m}^{kl} \cdot \boldsymbol{m}^{ij} + \boldsymbol{m}^{k\bar{j}} \cdot \boldsymbol{m}^{i\bar{l}}$$

- Can draw some simple and immediate conclusions from unitarity:

$$M^{ijij} = |\boldsymbol{m}^{ij}|^2 + |\boldsymbol{m}^{i\bar{j}}|^2 > 0$$

Forward "elastic" amplitudes are positive.

$$|M^{ijkl}| + |M^{ilkj}| \leq \sqrt{M^{ijij} M^{klkl}} + \sqrt{M^{ilil} M^{kjkj}}$$

"Inelastic" amplitudes are bounded in size by elastic ones.

See incarnations in **Remmen, Rodd 1908.09845, 2004.02885 Yamashita, Zhang, Zhou 2009.04490**

- More generally, impose symmetries to relate UV couplings and search for stronger bounds in multistate theories.

See recent advances **Li, Yang, et.al. 2101.01191**

Simple Example: Complex Goldstone

- Assume charge conjugation $C : \phi \leftrightarrow \bar{\phi}$, but not charge conservation.

	$\phi\phi$	$\phi\overline{\phi}$	$\overline{\phi}\phi$	$\overline{\phi\phi}$
$\phi\phi$	$\lvert \boldsymbol{m}^{\phi\phi}\rvert^2 + \lvert \boldsymbol{m}^{\phi\bar{\phi}}\rvert^2$	$2\boldsymbol{m}^{\phi\bar{\phi}} \cdot \boldsymbol{m}^{\phi\phi}$	$2\boldsymbol{m}^{\phi\bar{\phi}} \cdot \boldsymbol{m}^{\phi\phi}$	$2\boldsymbol{m}^{\bar{\phi}\bar{\phi}} \cdot \boldsymbol{m}^{\phi\phi}$
$\phi\overline{\phi}$	.	$\lvert \boldsymbol{m}^{\phi\phi}\rvert^2 + \lvert \boldsymbol{m}^{\phi\bar{\phi}}\rvert^2$	$2\boldsymbol{m}^{\bar{\phi}\phi} \cdot \boldsymbol{m}^{\phi\bar{\phi}}$	$2\boldsymbol{m}^{\phi\bar{\phi}} \cdot \boldsymbol{m}^{\phi\phi}$
$\overline{\phi}\phi$	.	.	$\lvert \boldsymbol{m}^{\bar{\phi}\phi}\rvert^2 + \lvert \boldsymbol{m}^{\bar{\phi}\bar{\phi}}\rvert^2$	$2\boldsymbol{m}^{\phi\bar{\phi}} \cdot \boldsymbol{m}^{\phi\phi}$
$\overline{\phi\phi}$	.	.	.	$\lvert \boldsymbol{m}^{\bar{\phi}\phi}\rvert^2 + \lvert \boldsymbol{m}^{\bar{\phi}\bar{\phi}}\rvert^2$

Table is self-conjugate by Hermitian analyticity. Partly simplified with discrete symmetries.

$$CPT \Longrightarrow |\boldsymbol{m}^{\phi\phi}| = |\boldsymbol{m}^{\bar{\phi}\bar{\phi}}|, \; |\boldsymbol{m}^{\bar{\phi}\phi}| = |\boldsymbol{m}^{\phi\bar{\phi}}|$$

$$|M^{\phi\phi\phi\overline{\phi}}| \leq M^{\phi\overline{\phi}\phi\overline{\phi}}, \quad |M^{\phi\phi\overline{\phi\phi}}| + |M^{\phi\overline{\phi\phi}\phi}| \leq 2M^{\phi\overline{\phi}\phi\overline{\phi}}$$

- Sum rule fundamentally limits the extent to which the $U(1)$ symmetry can be broken.

Symmetries

Wigner-Eckart Theorem: Only transitions between the same initial and final state representations and components are permitted. Transitions between different components related by Clebsch-Gordan coefficients.

$$\begin{aligned} A^{abcd} &= {}_{\text{out}}\left(\langle r4; d|\langle r3; c|\right)\left(|r1; a\rangle|r2; b\rangle\right)_{\text{in}} \\ &= \sum_{R,\xi,\xi'} P^{abcd}_{R_{\xi\xi'}} A_{R_{\xi\xi'}}, \qquad A_{R_{\xi\xi'}} = {}_{\text{out}}\langle R_{\xi'}; \iota | R_{\xi}; \iota\rangle_{\text{in}} \end{aligned}$$

For special cases where all transitions are rigidly fixed by symmetries (no degeneracy) and these are obeyed in the UV, sum rule becomes

$$M^{abcd} = \sum_R P^{abcd}_R |\boldsymbol{m}_R|^2 + \sum_{R'} P^{a\bar{d}c\bar{b}}_{R'} |\boldsymbol{m}_{R'}|^2$$

By WE theorem, u-channel projectors can be decomposed into s-channel projectors:

$$P^{a\bar{d}c\bar{b}}_{R'} = \sum_R C_{R'R} P^{abcd}_R$$

Crossing matrix, determined entirely by symmetry group.

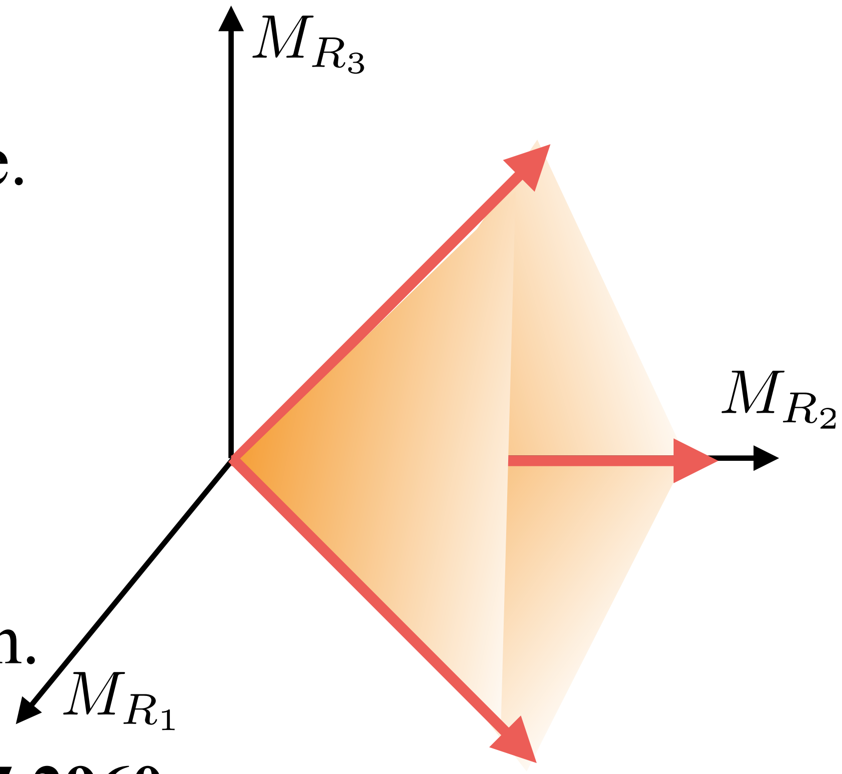
Internal Symmetries

$$M^{abcd} = \sum_R M_R P_R^{abcd} = \sum_R \left(|\boldsymbol{m}_R|^2 + \sum_{R'} C_{R'R} |\boldsymbol{m}_{R'}|^2 \right) P_R^{abcd}$$

- Could just invert equations to solve for positive linear combinations of amplitudes if there were fewer $|\boldsymbol{m}_R|^2$ terms than component amplitudes.

- Instead employ convex geometry interpretation: space of allowed amplitudes $\{M_R\}$ is a convex cone. The UV couplings $|\boldsymbol{m}_R|^2$ represent positive linear combinations of a set of a rays whose convex hull generates the cone.

- No degeneracy $\Leftrightarrow$ polyhedrality. Use vertex enumeration to convert to hyperplane representation.



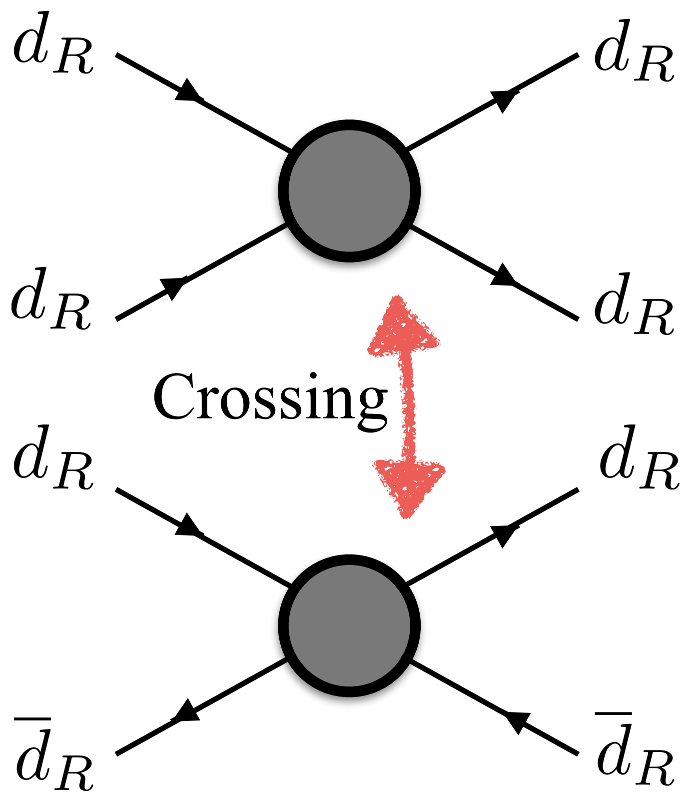
Bellazzini, Martucci, Torre 1405.2960,
Zhang, Zhou 2005.03047

Internal Symmetries

- Right-handed quarks: $\mathbf{3} \sim SU(3)$

$$\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6}$$

Antisymmetric ($\overline{\mathbf{3}}$) Symmetric ($\mathbf{6}$)



Hypercharge and angular momentum conservation ensure that cone is polyhedral for (unflavoured) SM fermion scattering.

$$M_{\overline{\mathbf{3}}} P_{\overline{\mathbf{3}}} + M_{\mathbf{6}} P_{\mathbf{6}} = \left(|\boldsymbol{m}_{\overline{\mathbf{3}}}|^2 - \frac{1}{3}|\boldsymbol{m}_{\mathbf{1}}|^2 + \frac{4}{3}|\boldsymbol{m}_{\mathbf{8}}|^2 \right) P_{\overline{\mathbf{3}}}$$
$$+ \left(|\boldsymbol{m}_{\mathbf{6}}|^2 + \frac{1}{3}|\boldsymbol{m}_{\mathbf{1}}|^2 + \frac{2}{3}|\boldsymbol{m}_{\mathbf{8}}|^2 \right) P_{\mathbf{6}}$$

2 possible transitions, 4 UV variables, but two redundant.

$$u^1 v^1 \in \mathbf{6} \implies M_{\mathbf{6}} > 0$$

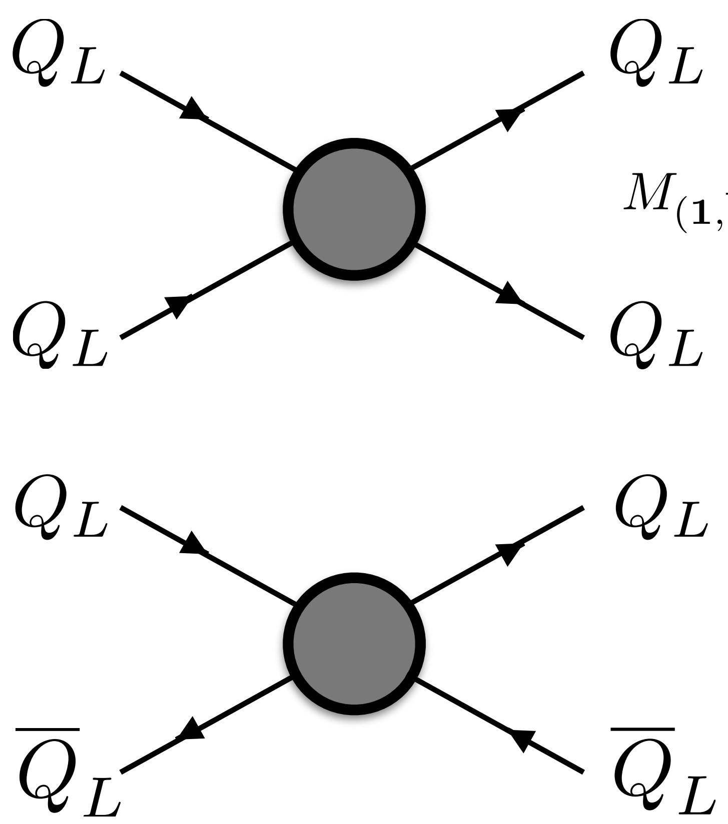
Remmen, Rodd 2004.02885

Can derive these from positivity of elastic scattering of pure states:

$$u^1 v^2 = \frac{1}{2}\left(u^1 v^2 + u^2 v^1\right) \implies M_{\overline{\mathbf{3}}} + M_{\mathbf{6}} > 0$$
$$+ \frac{1}{2}\left(u^1 v^2 - u^2 v^1\right) \in \overline{\mathbf{3}} + \mathbf{6}$$

Internal Symmetries

- Left-handed quarks: $(\mathbf{2},\mathbf{3}) \sim SU(2)\times SU(3)$



4 possible transitions, 8 UV variables:

$$
\begin{aligned}
&M_{(\mathbf{1},\overline{\mathbf{3}})}P_{\mathbf{1}} + M_{(\mathbf{1},\mathbf{6})}P_{\mathbf{6}} + M_{(\mathbf{3},\overline{\mathbf{3}})}P_{\mathbf{3}}P_{\overline{\mathbf{3}}} + M_{(\mathbf{1},\mathbf{6})}P_{\mathbf{3}}P_{\mathbf{6}} = \\
&\quad \left(|\boldsymbol{m}_{(\mathbf{1},\overline{\mathbf{3}})}|^2 + \frac{1}{6}|\boldsymbol{m}_{(\mathbf{1},\mathbf{1})}|^2 - \frac{2}{3}|\boldsymbol{m}_{(\mathbf{1},\mathbf{8})}|^2 - \frac{1}{2}|\boldsymbol{m}_{(\mathbf{3},\mathbf{1})}|^2 + 2|\boldsymbol{m}_{(\mathbf{3},\mathbf{8})}|^2\right) P_{\mathbf{1}}P_{\overline{\mathbf{3}}} \\
&\quad + \left(|\boldsymbol{m}_{(\mathbf{1},\mathbf{6})}|^2 - \frac{1}{6}|\boldsymbol{m}_{(\mathbf{1},\mathbf{1})}|^2 - \frac{1}{3}|\boldsymbol{m}_{(\mathbf{1},\mathbf{8})}|^2 + \frac{1}{2}|\boldsymbol{m}_{(\mathbf{3},\mathbf{1})}|^2 + |\boldsymbol{m}_{(\mathbf{3},\mathbf{8})}|^2\right) P_{\mathbf{1}}P_{\mathbf{6}} \\
&\quad + \left(|\boldsymbol{m}_{(\mathbf{3},\overline{\mathbf{3}})}|^2 - \frac{1}{6}|\boldsymbol{m}_{(\mathbf{1},\mathbf{1})}|^2 + \frac{2}{3}|\boldsymbol{m}_{(\mathbf{1},\mathbf{8})}|^2 - \frac{1}{6}|\boldsymbol{m}_{(\mathbf{3},\mathbf{1})}|^2 + \frac{2}{3}|\boldsymbol{m}_{(\mathbf{3},\mathbf{8})}|^2\right) P_{\mathbf{3}}P_{\overline{\mathbf{3}}} \\
&\quad + \left(|\boldsymbol{m}_{(\mathbf{3},\mathbf{6})}|^2 + \frac{1}{6}|\boldsymbol{m}_{(\mathbf{1},\mathbf{1})}|^2 + \frac{1}{3}|\boldsymbol{m}_{(\mathbf{1},\mathbf{8})}|^2 + \frac{1}{6}|\boldsymbol{m}_{(\mathbf{3},\mathbf{1})}|^2 + \frac{1}{3}|\boldsymbol{m}_{(\mathbf{3},\mathbf{8})}|^2\right) P_{\mathbf{3}}P_{\mathbf{6}}
\end{aligned}
$$

Extremal Rays:

$$
\begin{aligned}
(M_{(\mathbf{1},\overline{\mathbf{3}})}, M_{(\mathbf{1},\mathbf{6})}, M_{(\mathbf{3},\overline{\mathbf{3}})}, M_{(\mathbf{1},\mathbf{6})}) \in \langle\{&(1,0,0,0),(0,1,0,0),(0,0,1,0),\\
&(1,-1,-1,1),(-2,-1,2,1),(-3,3,-1,1)\}\rangle_+
\end{aligned}
$$

Entanglement

- Use vertex enumeration to convert extremal rays into linear inequalities describing facets:

Remmen, Rodd
2004.02885

$$\begin{cases} M_{(\mathbf{1},\mathbf{6})} + M_{(\mathbf{3},\mathbf{6})} > 0 \\ M_{(\mathbf{3},\mathbf{6})} > 0 \\ M_{(\mathbf{1},\overline{\mathbf{3}})} + M_{(\mathbf{1},\mathbf{6})} + M_{(\mathbf{3},\overline{\mathbf{3}})} + M_{(\mathbf{3},\mathbf{6})} > 0 \\ M_{(\mathbf{3},\overline{\mathbf{3}})} + M_{(\mathbf{3},\mathbf{6})} > 0 \end{cases}$$

$$4M_{(\mathbf{1},\overline{\mathbf{3}})} + M_{(\mathbf{1},\mathbf{6})} + 9M_{(\mathbf{3},\mathbf{6})} > 0$$

$$M_{(\mathbf{1},\overline{\mathbf{3}})} + 3M_{(\mathbf{3},\mathbf{6})} > 0$$

- Find six constraints, two of which cannot be found from scattering of pure states. All pure states decompose into linear combinations of irreps given by the first four inequalities. The last two are only accessible with entangled states.

Crossing matrix (determined here purely by group theory) encodes entangled states satisfying positivity. Examples of entangled states given by corresponding CG coefficients e.g.

$$\frac{1}{2}\left(|\mathbf{2};1\rangle|\mathbf{2};2\rangle - |\mathbf{2};2\rangle|\mathbf{2};1\rangle\right)\left(|\mathbf{3};1\rangle|\mathbf{3};2\rangle - |\mathbf{3};2\rangle|\mathbf{3};1\rangle\right)$$

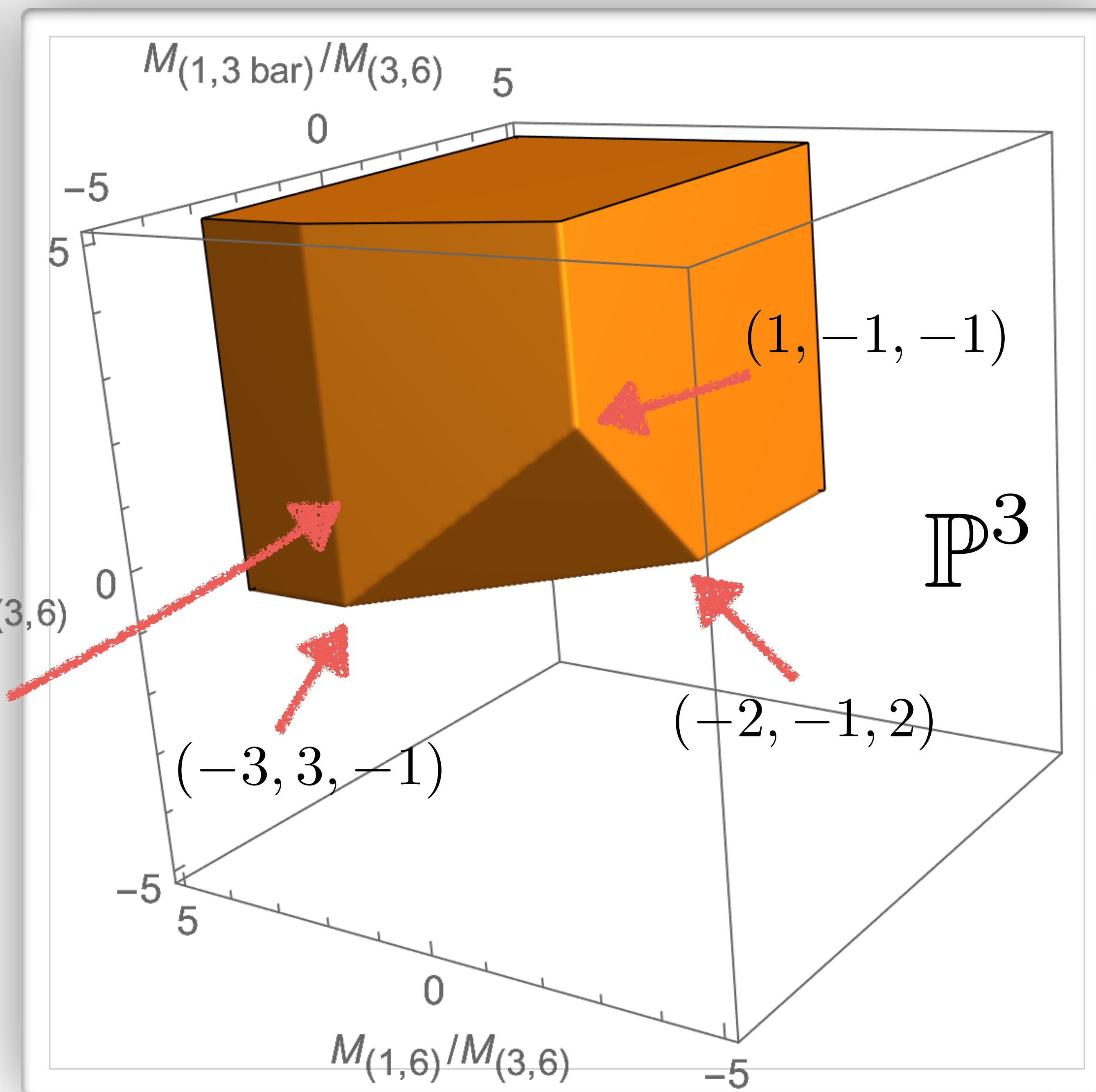
$$+\frac{3}{2}\left(|\mathbf{2};1\rangle|\mathbf{2};2\rangle + |\mathbf{2};2\rangle|\mathbf{2};1\rangle\right)\left(|\mathbf{3};1\rangle|\mathbf{3};2\rangle + |\mathbf{3};2\rangle|\mathbf{3};1\rangle\right)$$

Internal Symmetries

- Left-handed quarks: $(\mathbf{2},\mathbf{3}) \sim SU(2)\times SU(3)$

$$\begin{aligned}
M_{(\mathbf{1},\mathbf{6})} + M_{(\mathbf{3},\mathbf{6})} &> 0\\
M_{(\mathbf{3},\mathbf{6})} &> 0\\
M_{(\mathbf{1},\bar{\mathbf{3}})} + M_{(\mathbf{1},\mathbf{6})} + M_{(\mathbf{3},\bar{\mathbf{3}})} + M_{(\mathbf{3},\mathbf{6})} &> 0\\
M_{(\mathbf{3},\bar{\mathbf{3}})} + M_{(\mathbf{3},\mathbf{6})} &> 0\\
4M_{(\mathbf{1},\bar{\mathbf{3}})} + M_{(\mathbf{1},\mathbf{6})} + 9M_{(\mathbf{3},\mathbf{6})} &> 0\\
M_{(\mathbf{1},\bar{\mathbf{3}})} + 3M_{(\mathbf{3},\mathbf{6})} &> 0
\end{aligned}$$

Entangled lower bounds on $M_{(\mathbf{1},\bar{\mathbf{3}})}/M_{(\mathbf{3},\mathbf{6})}$



Extremal Rays become vertices in projective space:

$\{(1,0,0,0),(0,1,0,0),(0,0,1,0),$
$(1,-1,-1,1),(-2,-1,2,1),(-3,3,-1,1)\}$

Internal Symmetries

- Left-handed quarks with full flavor symmetry:

$$(\mathbf{2},\mathbf{3},\mathbf{3}) \sim SU(2)\times SU(3)\times SU(3)$$

$$(M_{(\mathbf{1},\bar{\mathbf{3}},\bar{\mathbf{3}})}, M_{(\mathbf{3},\bar{\mathbf{3}},\bar{\mathbf{3}})}, M_{(\mathbf{1},\mathbf{6},\bar{\mathbf{3}})}, M_{(\mathbf{3},\mathbf{6},\bar{\mathbf{3}})}, M_{(\mathbf{1},\bar{\mathbf{3}},\mathbf{6})}, M_{(\mathbf{3},\bar{\mathbf{3}},\mathbf{6})}, M_{(\mathbf{1},\mathbf{6},\mathbf{6})}, M_{(\mathbf{3},\mathbf{6},\mathbf{6})})$$

0	1	0	1	0	1	0	1	1	0	0	3	1	0	0	3
1	1	1	1	1	1	1	1	4	0	1	9	4	0	1	9
1	3	2	0	2	0	0	6	8	0	3	15	8	0	0	24
5	9	8	0	8	0	2	18	0	0	0	0	4	0	1	9
11	15	16	0	16	0	0	48	4	0	0	8	4	0	3	11
1	0	1	0	0	3	0	3	0	0	0	0	1	0	0	3
4	0	4	0	1	9	1	9	1	0	1	0	6	0	0	18
8	0	8	0	3	15	0	24	0	3	1	0	5	0	0	15
1	1	0	0	0	0	2	2	0	1	0	1	3	0	0	9
8	0	4	0	0	12	5	21	0	0	1	1	0	0	1	1
0	1	0	0	0	0	0	6	4	0	4	0	0	8	3	11
7	0	2	0	2	0	0	36	0	0	4	0	0	0	1	9
2	0	0	0	0	0	1	9	0	0	1	0	0	0	0	3
4	0	0	0	0	4	5	13	1	0	6	0	1	0	0	18
8	0	0	12	4	0	5	21	0	1	3	0	0	1	0	9
3	0	0	5	0	5	0	13	0	3	5	0	1	0	0	15
8	0	0	4	0	4	7	23	0	0	0	0	0	0	0	1
4	0	0	4	0	0	5	13	0	0	0	0	0	0	1	1
0	1	0	0	0	0	0	2	0	4	8	0	8	0	7	23
0	0	0	0	1	1	1	1	0	0	0	1	0	0	0	1
0	0	0	0	0	1	0	1	0	3	2	1	1	0	0	7
0	3	1	0	2	1	0	7	0	5	3	0	3	0	0	13

- More degrees of freedom

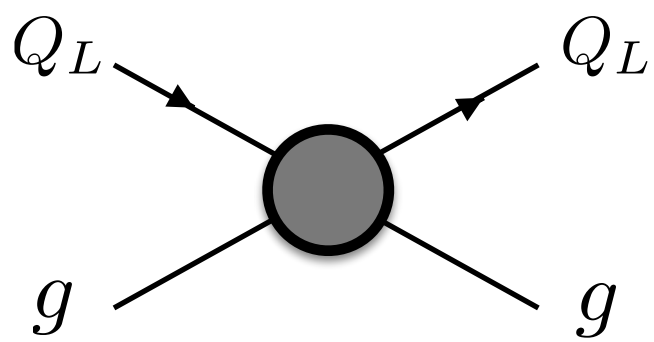
$\Rightarrow$ many more constraints.

- 8 constraints from pure state scattering, 36 additional from entangled states.

More examples, similar results:
Parity symmetric W bosons **Zhang, Zhou 2005.03047**
Parity symmetric gluons
Li, Yang, et.al. 2101.01191

Internal Symmetries

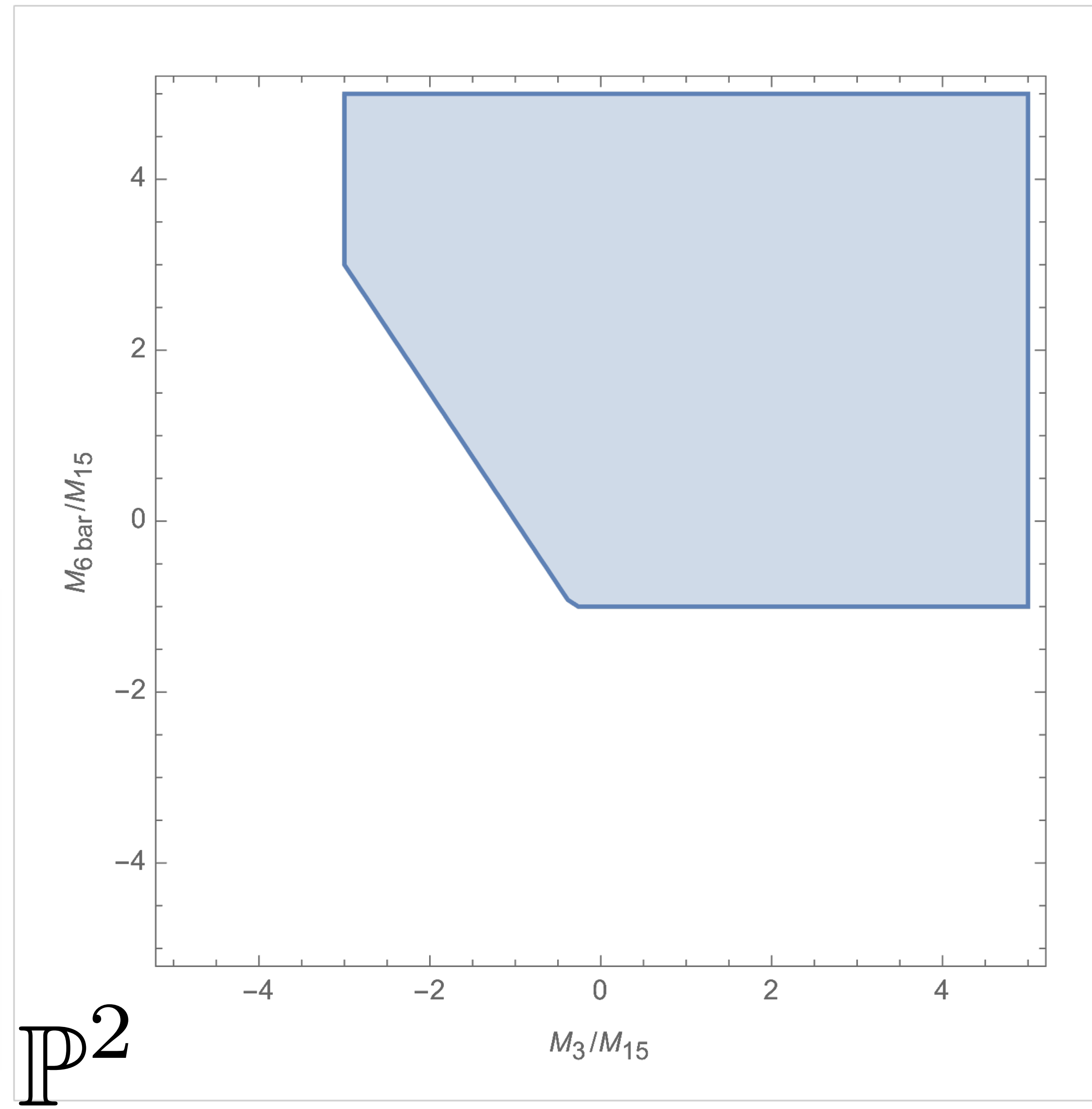
- Quark-gluon scattering: $SU(3)$ $\mathbf{3} \otimes \mathbf{8} = \mathbf{3} \oplus \overline{\mathbf{6}} \oplus \mathbf{15}$



Extremal Rays become vertices in projective space:

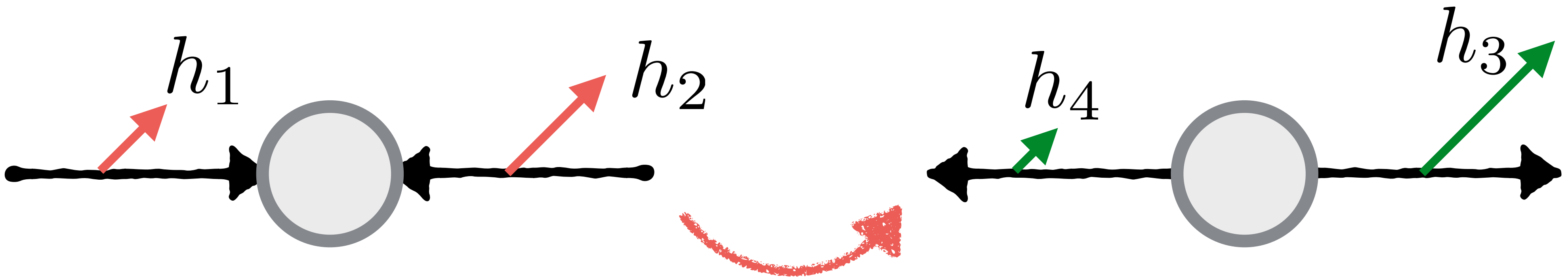
$$\{(1,0,0), (0,1,0), (-1,-3,3), (-3,3,1)\}$$

$$M_{\mathbf{3}} + 3M_{\mathbf{15}} > 0$$
$$3M_{\mathbf{3}} + 2M_{\overline{\mathbf{6}}} + 3M_{\mathbf{15}} > 0$$
$$M_{\mathbf{15}} > 0$$
$$M_{\overline{\mathbf{6}}} + M_{\mathbf{15}} > 0$$



Spin

- Total spin projection along beam axis must be conserved, but otherwise this just a conserved $U(1)$ charge $h_1 - h_2$



- For identical particles, have two degenerate spin singlet configurations. Can define parity eigenstates:

$$|A\rangle = |+\rangle|+\rangle + |-\rangle|-\rangle \qquad \text{P even spin singlet}$$

$$|B\rangle = |+\rangle|+\rangle - |-\rangle|-\rangle \qquad \text{P odd spin singlet}$$

- Positivity of simple spinning elastic amplitudes derived in **Bellazzini 1605.06111**.

$$M^{\gamma^+\phi\gamma^+\phi},\ M^{\psi^+\psi^+\psi^+\psi^+},\ M^{\psi^+\phi\psi^+\phi},\ M^{\gamma^+\psi^+\gamma^+\psi^+} > 0$$

Spin

- Complete the constraints on minimal theories of spinning particles, with and without parity. Simple inelastic bounds sufficient.

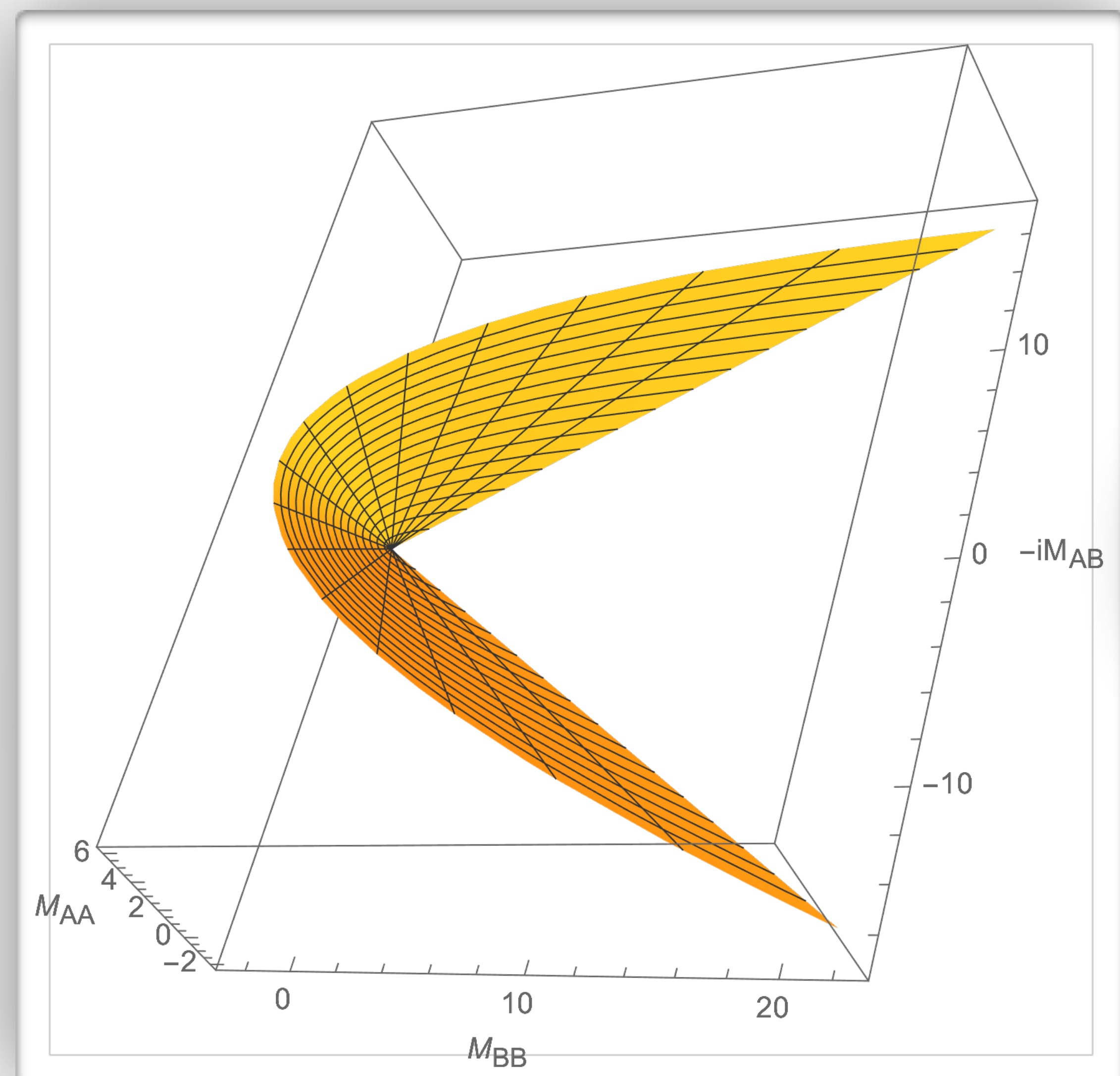
$$|M^{\phi\psi^-\gamma^+\psi^+}|, |M^{\phi\psi^+\gamma^-\psi^-}| \leq \sqrt{2M^{\phi\psi^+\phi\psi^+}M^{\psi^+\gamma^+\psi^+\gamma^+}}$$

$$|M^{\phi\phi\gamma^+\gamma^+}|, |M^{\phi\phi\gamma^-\gamma^-}| \leq \frac{1}{2}M^{\phi\gamma^+\phi\gamma^+} + \sqrt{\frac{1}{2}M^{\phi\phi\phi\phi}M^{\gamma^+\gamma^-\gamma^+\gamma^-}}$$

- Can obtain by scattering factorized superpositions of photons and scalars.

Simple Example: Identical Spinning Particles

Simplest example of non-polyhedral cone arising as a result of permitted transitions between symmetry-degenerate states. Geometerise the cone of allowed amplitudes. Bounds are well-known:



$$M^{ijkl} = \frac{1}{2}\left(3|\boldsymbol{m}_A|^2 - |\boldsymbol{m}_B|^2\right) P^{ijkl}_{AA}$$

$$+ \frac{1}{2}\left(-|\boldsymbol{m}_A|^2 + 3|\boldsymbol{m}_B|^2\right) P^{ijkl}_{BB} + 2\boldsymbol{m}_A \cdot \boldsymbol{m}_B \left(P^{ijkl}_{AB} - P^{ijkl}_{BA}\right)$$

Extremal rays: $r = -im_{BB}/m_{AA}$

$$(M_{AA}, M_{BB}, M_{AB}) = (3 - r^2, -1 + 3r^2, 4r)$$

$$|M_{AB}| < \frac{1}{2}\sqrt{(3M_{AA} + M_{BB})(M_{AA} + 3M_{BB})}$$

$$\frac{c}{16\Lambda^4}\left(\left(F^2\right)^2 + (F\tilde{F})^2\right) + \frac{d}{32\Lambda^4}\left(\left(F^2\right)^2 - (F\tilde{F})^2\right) + \frac{e}{16\Lambda^4}F^2\left(F\tilde{F}\right)$$

See e.g. **Remmen, Rodd 1908.09845**

$$c > \frac{1}{2}\sqrt{d^2 + e^2}$$

Applications to electroweak bosons in **Yamashita, Zhang, Zhou 2009.04490**

Supersymmetry

- The known (minimal) elastic positivity theorems unify, different components of the same constraint:

Mostly established in **Bellazzini 1605.06111**

$\mathcal{N} = 1$

c_{Φ^4} $\quad$ $c_{\Phi^2 V^2}$ $\quad$ c_{V^4}

$$\frac{1}{4}\phi\phi(\partial^2)^2(\phi^*\phi^*)$$
$$\frac{1}{2}\psi\psi\partial^2(\psi^\dagger\psi^\dagger)$$
$$2i\partial_\mu\phi^\dagger\partial^\nu\phi\partial_\nu\psi\sigma^\mu\psi^\dagger$$

$$2\mathrm{tr}\left(F_L\sigma^\mu F_R\overline{\sigma}^\nu\right)\partial_\mu\phi^\dagger\partial_\nu\phi$$
$$-\partial_\mu\psi\lambda\psi^\dagger\partial^\mu\lambda^\dagger$$
$$-2i\lambda^\dagger\overline{\sigma}^\mu\partial_\nu\lambda\partial_\mu\phi^\dagger\partial^\nu\phi$$

$$\frac{1}{2}\lambda\lambda\partial^2(\lambda^\dagger\lambda^\dagger)$$
$$2i\lambda F_L\sigma^\mu F_R\partial_\mu\lambda^\dagger$$
$$\frac{1}{16}\left((F^2)^2+(F\tilde{F})^2\right)$$

$$-\sqrt{2}i\partial_\mu\psi\lambda F^{\mu\nu}\partial_\nu\phi^\dagger$$

$\mathcal{N} = 2, 4$

See **Liu, You 2011.11299** for interpretation with orthogonal polynomials

Supersymmetry

- "Inelastic" operators are less supersymmetric to varying degrees.

$$\left.\begin{array}{l}\frac{1}{4}\psi\psi\partial^2(\lambda\lambda)\\ \frac{1}{8}\left(F^2+iF\tilde{F}\right)\partial^2\phi^2\\ 2\sqrt{2}i\partial_\mu\phi\lambda F_L\partial^\mu\psi\end{array}\right\} d_{\Phi^2V^2}$$

$\mathcal{N}=1,2$

$\neq 4$

$$|\Re,\Im\, d_{\Phi^2V^2}| < c_{\Phi^2V^2}+\sqrt{c_{\Phi^4}c_{V^4}}$$

$$\left.\begin{array}{llll}\frac{1}{32}\left(\left(F^2\right)^2-(F\tilde{F})^2\right) & \frac{1}{16}F^2\left(F\tilde{F}\right) & \psi\lambda\partial^2\left(\psi^\dagger\lambda^\dagger\right) & \\ \frac{1}{2}\psi\psi\partial^2(\psi\psi) & \frac{1}{2}\lambda\lambda\partial^2(\lambda\lambda) & \psi\psi\partial^2\left(\lambda^\dagger\lambda^\dagger\right) & \psi\lambda\partial^2(\psi\lambda)\end{array}\right\}$$

$\mathcal{N}=0$

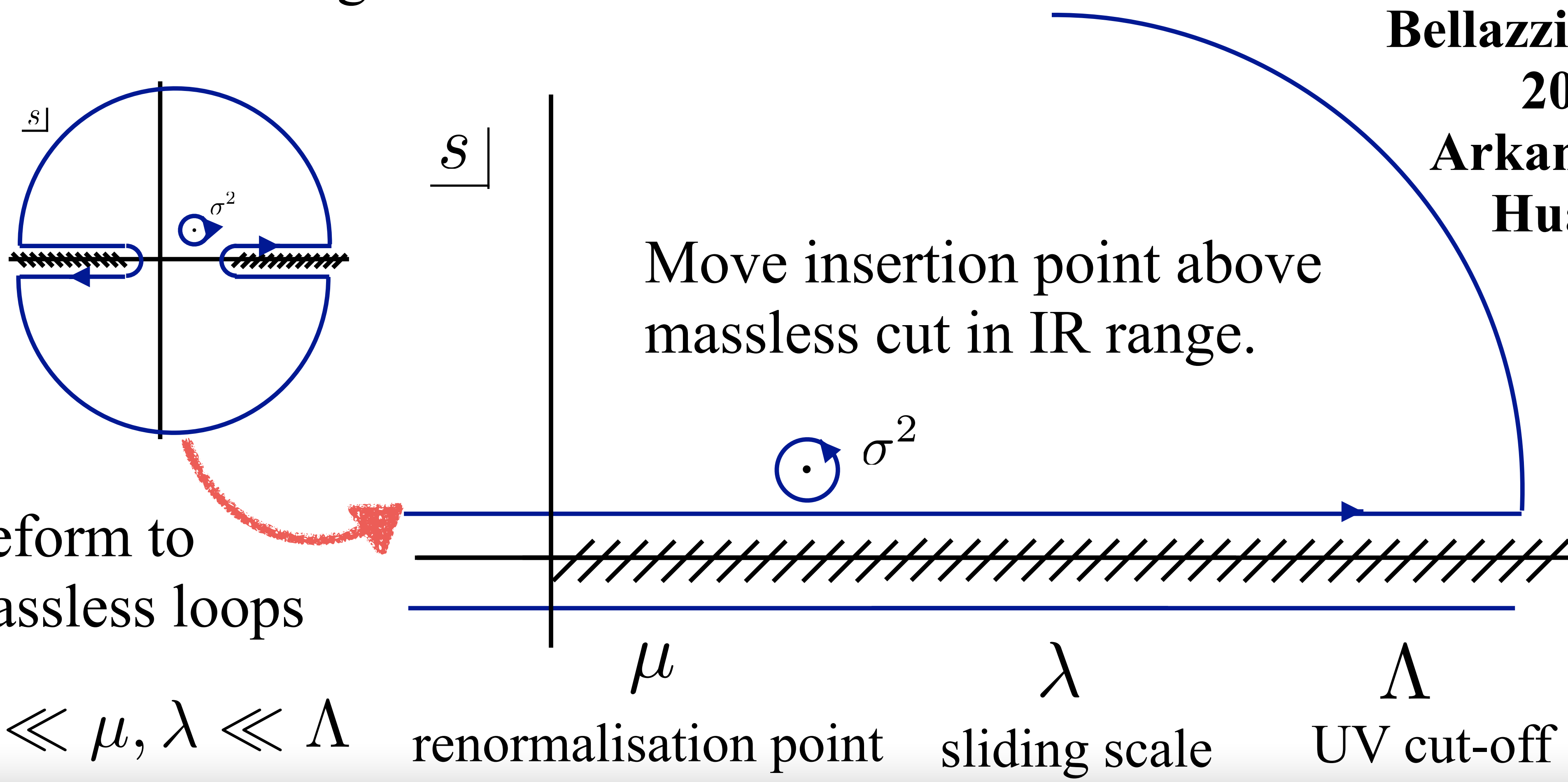
Other inelastic operators bounded in usual way.

The existence of these operators raises the lower bounds on the elastic, more supersymmetry-compatible operators beyond mere positivity.

Loops

Insert at energies above branch cut:

See similar formulation in **Bellazzini, Elias Miro, et.al. 2011.00037**, also **Arkani-Hamed, Huang, Huang 2012.15849**



Deform to massless loops

$\sigma \ll \mu, \lambda \ll \Lambda$

Move IR segment of dispersion integral to IR side of sum rule.

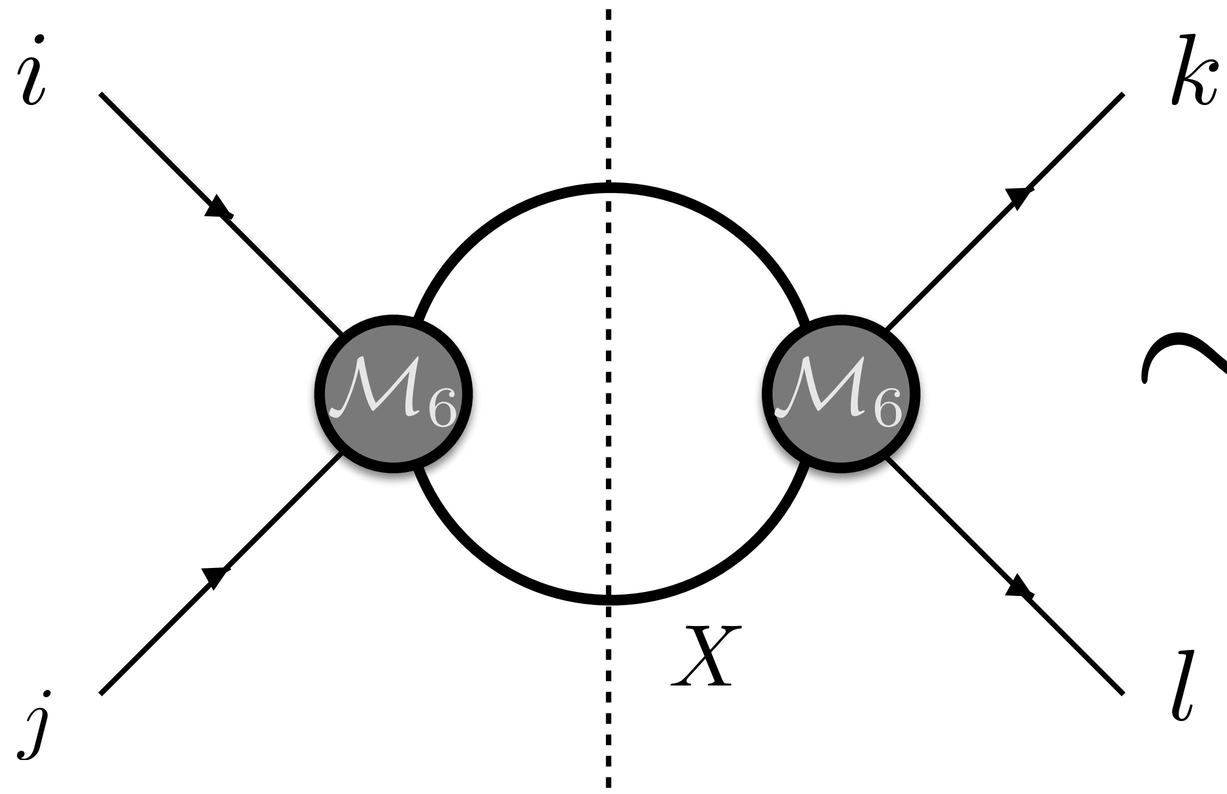
$$M^{ijkl}(\sigma^2,\lambda) = \frac{d^2}{ds^2}\mathcal{A}(s)$$

$$-\frac{1}{\pi}\int_0^{\lambda^2}\left(\frac{1}{(s-\sigma^2-i\delta)^3}\sum_{X\in IR}\mathcal{M}^{ij\to X}\left(\mathcal{M}^{kl\to X}\right)^* + \frac{1}{(s+\sigma^2)^3}\sum_{X\in IR}\mathcal{M}^{i\bar{l}\to X}\left(\mathcal{M}^{k\bar{j}\to X}\right)^*\right)ds$$

$$= \boldsymbol{m}^{kl}\cdot\boldsymbol{m}^{ij} + \boldsymbol{m}^{k\bar{j}}\cdot\boldsymbol{m}^{i\bar{l}}$$

Loops

- Have to address IR divergences (including forward limit).
- Special subset of IR finite loops: bubbles of higher-dimensional contact operators.

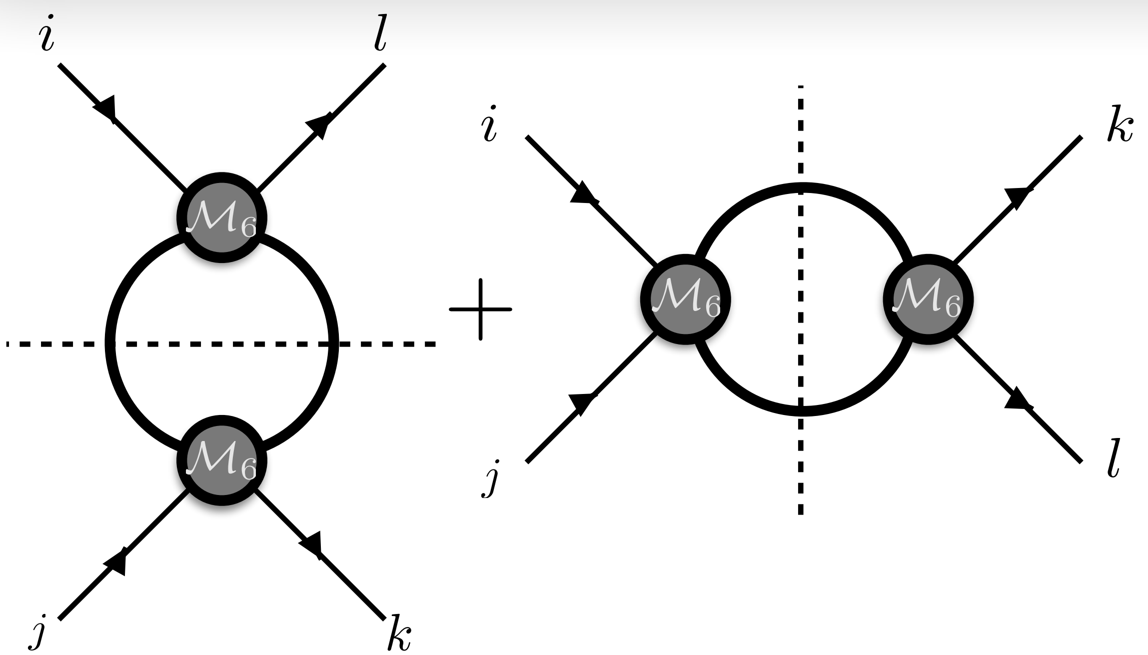


$$\sim \mathcal{M}_6^{kl} \cdot \mathcal{M}_6^{ij} \log\left(-\frac{\mu^2}{s}\right)$$

Forward-limit bubble cut is an inner product on space of cut amplitudes.

- Bubble cuts give RG functions for dim-8 operators (coefficients of logs).

Loops



$$\sim \mathcal{M}_6^{kl} \cdot \mathcal{M}_6^{ij} \log\left(-\frac{\mu^2}{s}\right) + \mathcal{M}_6^{k\bar{j}} \cdot \mathcal{M}_6^{i\bar{l}} \log\left(\frac{\mu^2}{s}\right)$$

Elastic forward scattering

$\Longrightarrow$ positive coefficient of log

$\Longrightarrow$ elastic Wilson coefficients increase into the IR.

$$(i,j) = (k,l)$$

$$|\mathcal{M}_6^{ij}|^2 + |\mathcal{M}_6^{i\bar{j}}|^2 \geq 0$$

Inelastic forward scattering

$\Longrightarrow$ coefficient of log bounded by elastic coefficients by Schwarz bound

$\Longrightarrow$ slower variation of Wilson coefficient into the IR.

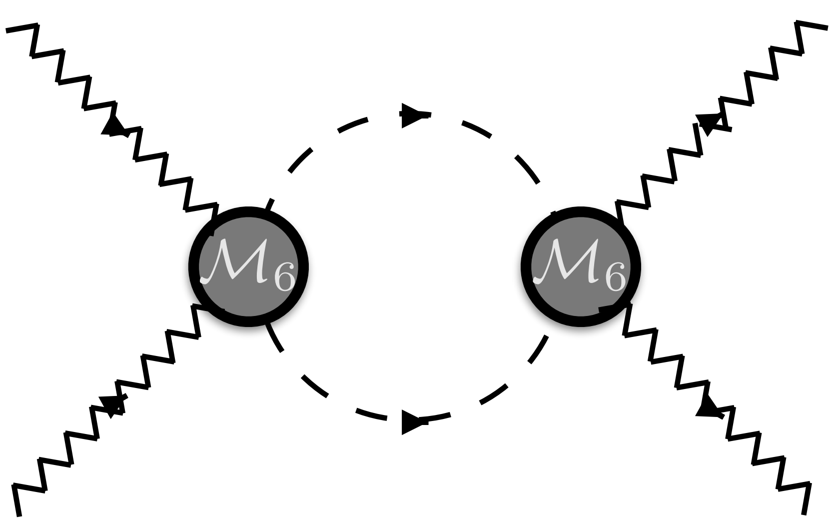
$$(i,j) \neq (k,l)$$

$$|\mathcal{M}_6^{kl} \cdot \mathcal{M}_6^{ij} + \mathcal{M}_6^{k\bar{j}} \cdot \mathcal{M}_6^{i\bar{l}}|$$

$$\leq \sqrt{|\mathcal{M}_6^{kl}||\mathcal{M}_6^{ij}|} + \sqrt{|\mathcal{M}_6^{k\bar{j}}||\mathcal{M}_6^{i\bar{l}}|}$$

Simple Example: Scalar Loop for Photon Amplitudes

EFT of photon and electrically neutral complex scalar: no renormalisable loops.



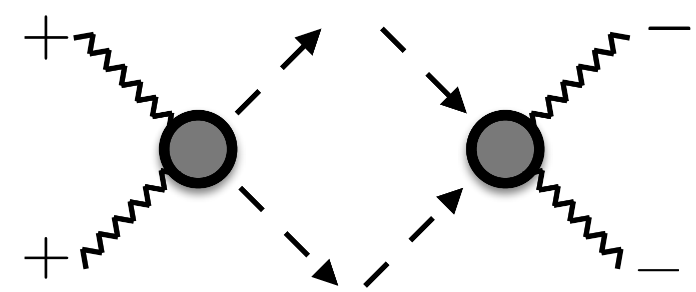
$$\mathcal{L}_{EFT_8} = \frac{c}{16\Lambda^4}\left(\left(F^2\right)^2 + (F\tilde{F})^2\right) + \frac{d}{32\Lambda^4}\left(\left(F^2\right)^2 - (F\tilde{F})^2\right) + \frac{e}{16\Lambda^4}F^2\left(F\tilde{F}\right)$$

$$\mathcal{L}_{EFT_6} = \frac{a}{\Lambda^2}F^2\phi^2 + \frac{b}{\Lambda^2}F\tilde{F}\phi^2 + \text{conj}$$

$$+ \frac{\tilde{a}}{\Lambda^2}F^2\phi\phi^\dagger + \frac{\tilde{b}}{\Lambda^2}F\tilde{F}\phi\phi^\dagger$$

E/M duality limit: Scalars assigned helicity charge

$$b = \pm ia \qquad \tilde{a}, \tilde{b} = 0$$



$$M^{+-+-} > \frac{1}{4}\sqrt{|M^{++--} - M^{--++}|^2 + |M^{++--} + M^{--++}|^2}$$

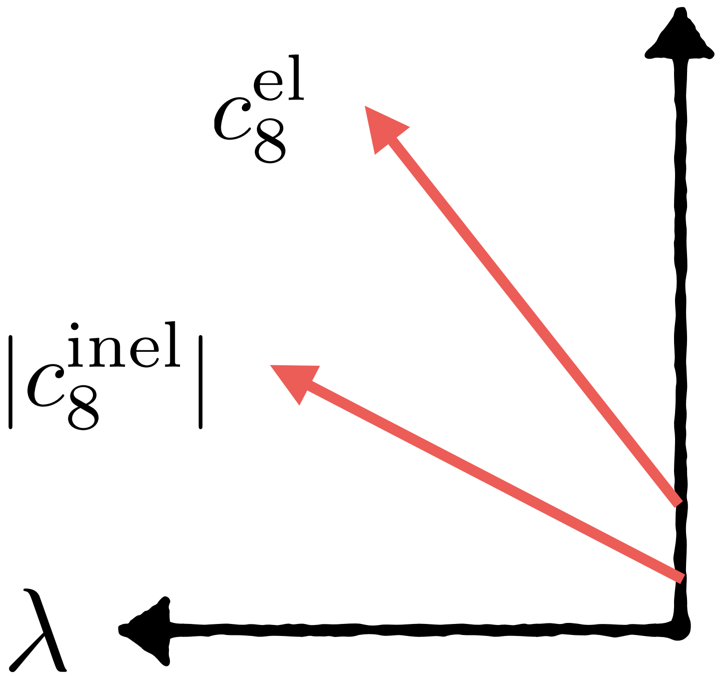
$$\Rightarrow \quad 2c + \frac{8}{(4\pi)^2}\left(\tilde{a}^2 + \tilde{b}^2 + 2|a - ib|^2 + 2|a + ib|^2\right)\left(2 + \log\left(\frac{\mu^2}{\lambda^2}\right)\right)$$

$$> \left| d + ie + \frac{8}{(4\pi)^2}\left((\tilde{a} - i\tilde{b})^2 + 4(a - ib)(a^* - ib^*)\right)\left(2 + \log\left(\frac{\mu^2}{\lambda^2}\right)\right)\right|$$

Loops

- If causality constraints satisfied in EFT, unitarity ensures preservation under RG flow into IR.
- But reversing, if constraints narrowly satisfied in deep IR, then cannot flow consistently far into UV (not without a higher order term(s) kicking-in).
- More evidence for fundamental inconsistency of EFT tuning (suppression of parameters).
- Discussion incomplete, need to include relevant/ marginal loops for realistic theories.

<u>Schematic RGE</u>



See more in **Bellazzini, Elias Miro, et.al. 2011.00037 Arkani-Hamed, Huang, Huang 2012.15849**

Open Questions and Directions

- Further extension to loops, full kinematical structure, IR divergences.
- Systematic method of computing constraints for multiple non-degenerate allowed transitions / non-polyhedral cones. Recent advance in **Li, Yang, et.al. 2101.01191** - reformulate constraints in dual space to amplitude, the entire space of scattering states (entangled, polarised). This “spectrahedron” is positive semi-definite. Next step is application to apply to multi field theories, like SMEFT.
- Applications to realistic EFTs, particle physics analyses (where relevant). Constraints are usually operative at subleading dim-8 order in EFT expansion, but lower dim operators still appear in loops. RG flow, exploration of UV completions independently of interest as well.
- Applications to (super)gravity.
- Higher order, departure from forward limit with spinning polynomials (as in **Arkani-Hamed, Huang, Huang 2012.15849**).