Positivity bounds on Minimal Flavor Violation

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Based on: Q. Bonnefoy, E.G., C. Grojean: 2011.12855

Motivation and Outline

Motivation:

- Connect Positivity bounds with phenomenology
- Test unitarity and analyticity of the Minimal flavor Violation assumption in the Standard Model Effective Field Theory
- Investigate bounds on Flavor Violating couplings

Outline:

- Introduction: SMEFT, Minimal Flavor Violation and Positivity Bounds on dimension-8 4-Fermi operators
- Positivity bounds under MFV: analysis of the tensor structure, difference between $N_f = 2,3$, how to disentangle external states
- Results
- Summary

The SMEFT

The Standard Model is generally intended as the renormalizable part of a larger description, that includes the effects from heavy resonances that cannot be produced on-shell

$$\mathcal{L}_{SMEFT} = \mathcal{L}_{SM}^{(4)} + \sum_{n\geq 5} \frac{c_n}{\Lambda^{n-4}} \mathcal{O}^{(n)}$$

The operators are gauge invariants built with SM fields, and the coefficients are in principle arbitrary, and get fixed only when specifying the UV completion or by measurements.

Minimal Flavor Violation

D'Ambrosio, Giudice, Isidori, Strumia hep-ph/0207036

The Minimal Flavor Violation ansatz comes from the observation that sending the Yukawa couplings to 0

$$Y_{u,d,e} \longrightarrow 0$$

the SM Lagrangian enjoys a global $U(3)^5$ flavor symmetry, that can be extended to the Yukawa sector by promoting $Y_{u,d,e}$ to spurions

	$SU(3)_Q$	$SU(3)_u$	$SU(3)_d$	$SU(3)_L$	$SU(3)_e$
Y_u	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
Y_d	$\mathbf{3}$	$\mathbf{1}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$
Y_e	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	$\bar{\mathbf{3}}$

MFV requires the whole SMEFT Lagrangian to be a singlet of the flavor symmetry.

This takes care of the flavor structure of the higher dimensional operators coefficients, e.g.:

$$\mathcal{O}_{Hu} = \frac{C^{ij}_{Hu}}{\Lambda^2}\left(H^\dagger \overleftrightarrow{D}^\mu H\right)\bar{u}_i\gamma^\mu u_j \xrightarrow{MFV} \frac{C_{Hu}}{\Lambda^2}(Y_u^\dagger Y_u)^i_j\left(H^\dagger \overleftrightarrow{D}^\mu H\right)\bar{u}_i\gamma^\mu u^j$$

and prevents large FCNCs.

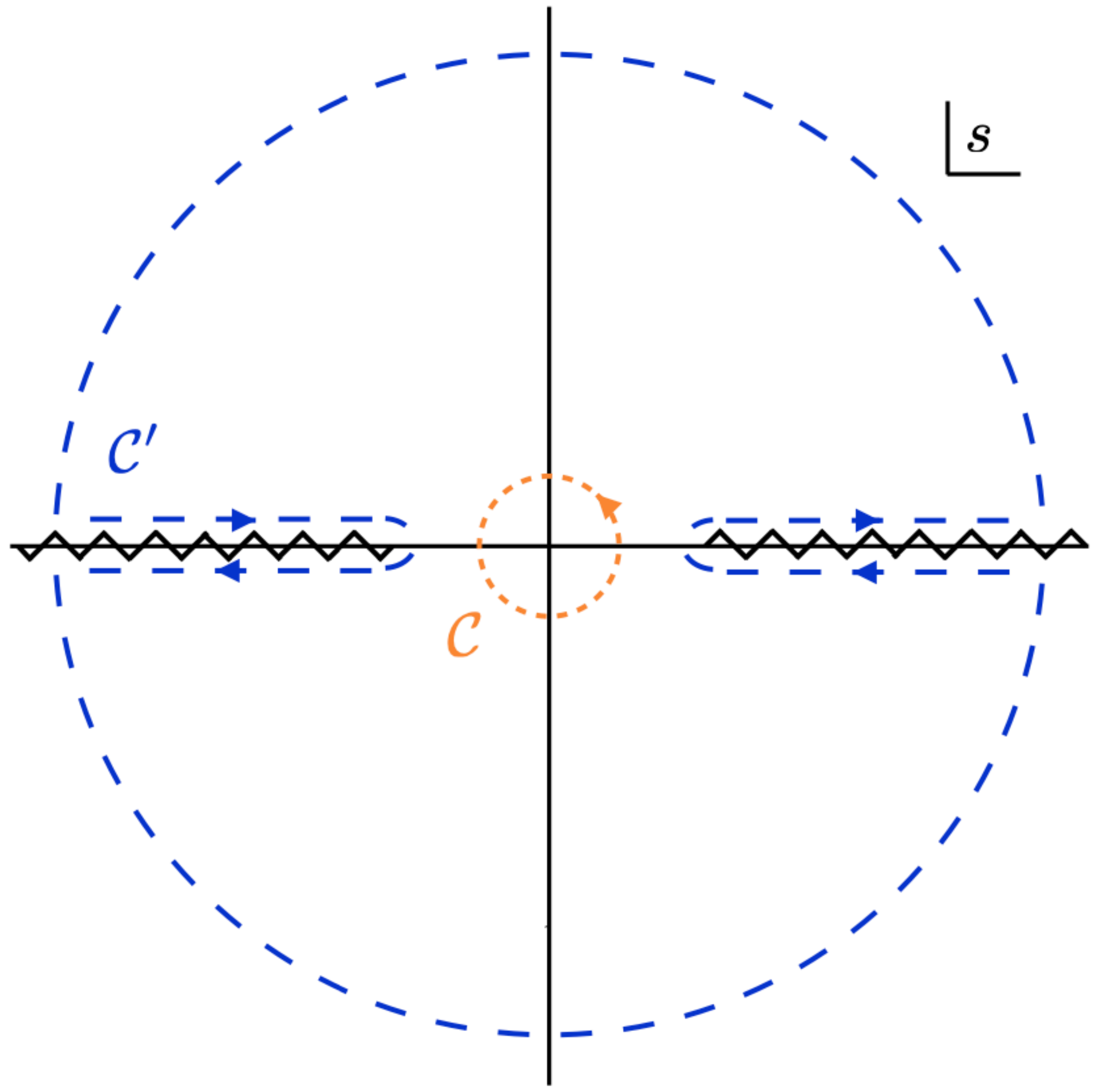
Positivity bounds

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi
hep-th/0602178

The coefficients appearing in front of EFT higher dimensional operators are not arbitrary!

Assuming analyticity and unitarity in the UV requires them to respect some bounds:

Scalar + shift symmetry: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{c}{M^4}(\partial_\mu \phi)^4$ → $2 \to 2$ scattering $t \to 0$ → $\boxed{\mathcal{A}(s) \equiv \mathcal{M}(s, t=0) = \frac{4cs^2}{M^4}}$



Picture from arXiv:1908.09845

Expand $\mathcal{A}(s) = \sum_{n=0}^{\infty} \lambda_n s^n$ with $\lambda_2 = \frac{1}{2\pi i} \oint_C \frac{\mathcal{A}(s)}{s^3} ds = \frac{4c}{M^4}$

By deforming the contour

$$\lambda_2 = C_R + \frac{1}{\pi i} \int_{s_d}^{+\infty} [\mathcal{A}(s + i\varepsilon) - \mathcal{A}(s - i\varepsilon)]\, ds$$

$$= \frac{2}{\pi} \int_{s_d}^{+\infty} \frac{ds}{s^2} \sigma(s) > 0$$

→ $\boxed{c > 0}$

Dimension 8 4-Fermi operators

Remmen, Rodd
2004.02885

The set of operators we focus on is:

Type	Content	Operator	Symmetry
self-quartic	(4-u)	$\mathcal{O}_1[u] = c^{u,1}_{mnpq}\partial_\mu(\bar{u}_m\gamma_\nu u_n)\partial^\mu(\bar{u}_p\gamma^\nu u_q)$	$c_{mnpq} = c_{pqmn}$, $c_{mnpq} = c^*_{nmqp}$
		$\mathcal{O}_3[u] = c^{u,3}_{mnpq}\partial_\mu(\bar{u}_m T^a\gamma_\nu u_n)\partial^\mu(\bar{u}_p T^a\gamma^\nu u_q)$	
	(4-Q)	$\mathcal{O}_1[Q] = c^{Q,1}_{mnpq}\partial_\mu(\bar{Q}_m\gamma_\nu Q_n)\partial^\mu(\bar{Q}_p\gamma^\nu Q_q)$	
		$\mathcal{O}_2[Q] = c^{Q,2}_{mnpq}\partial_\mu(\bar{Q}_m\tau^I\gamma_\nu Q_n)\partial^\mu(\bar{Q}_p\tau^I\gamma^\nu Q_q)$	
		$\mathcal{O}_3[Q] = c^{Q,3}_{mnpq}\partial_\mu(\bar{Q}_m T^a\gamma_\nu Q_n)\partial^\mu(\bar{Q}_p T^a\gamma^\nu Q_q)$	
		$\mathcal{O}_4[Q] = c^{Q,4}_{mnpq}\partial_\mu(\bar{Q}_m T^a\tau^I\gamma_\nu Q_n)\partial^\mu(\bar{Q}_p T^a\tau^I\gamma^\nu Q_q)$	
	(4-d)	$\mathcal{O}_1[d] = c^{d,1}_{mnpq}\partial_\mu(\bar{d}_m\gamma_\nu d_n)\partial^\mu(\bar{d}_p\gamma^\nu d_q)$	
		$\mathcal{O}_3[d] = c^{d,3}_{mnpq}\partial_\mu(\bar{d}_m T^a\gamma_\nu d_n)\partial^\mu(\bar{d}_p T^a\gamma^\nu d_q)$	
cross-quartic	(2-u)(2-Q)	$\mathcal{O}_{K1}[u,Q] = -a^{uQ,1}_{mnpq}\left(\bar{u}_m\gamma_\mu\partial_\nu u_q\right)\left(\bar{Q}_n\gamma^\nu\partial^\mu Q_p\right)$	$a^{\psi\chi}_{mnpq} = a^{\chi\psi}_{nmqp}$, $a_{mnpq} = a^*_{qpnm}$
		$\mathcal{O}_{K3}[u,Q] = -a^{uQ,3}_{mnpq}\left(\bar{u}_m T^a\gamma_\mu\partial_\nu u_q\right)\left(\bar{Q}_n T^a\gamma^\nu\partial^\mu Q_p\right)$	
	(2-d)(2-Q)	$\mathcal{O}_{K1}[d,Q] = -a^{dQ,1}_{mnpq}\left(\bar{d}_m\gamma_\mu\partial_\nu d_q\right)\left(\bar{Q}_n\gamma^\nu\partial^\mu Q_p\right)$	
		$\mathcal{O}_{K3}[d,Q] = -a^{dQ,3}_{mnpq}\left(\bar{d}_m T^a\gamma_\mu\partial_\nu d_q\right)\left(\bar{Q}_n T^a\gamma^\nu\partial^\mu Q_p\right)$	
	(2-d)(2-u)	$\mathcal{O}_{K1}[d,u] = -a^{du,1}_{mnpq}\left(\bar{d}_m\gamma_\mu\partial_\nu d_q\right)\left(\bar{u}_n\gamma^\nu\partial^\mu u_p\right)$	
		$\mathcal{O}_{K3}[d,u] = -a^{du,3}_{mnpq}\left(\bar{d}_m T^a\gamma_\mu\partial_\nu d_q\right)\left(\bar{u}_n T^a\gamma^\nu\partial^\mu u_p\right)$	

There is another set of dimension 8, 4-fermions operators $\mathcal{O} = \partial_\mu(\bar{\psi}_m\gamma_\nu\psi_n)\partial^\mu(\bar{\chi}_p\gamma^\nu\chi_q), \quad \psi \neq \chi$
but their contribution to the amplitude vanishes as $t \to 0$ so we cannot bound them

Dimension 8 operators under MFV

We now have to impose the MFV andata on this set of operators to see:

how do the bounds depend on the Yukawa entries?

Approximation:

$$N_f = 2 \;\to\; Y_u = \begin{pmatrix} \cancel{y_u} & 0 \\ 0 & y_c \end{pmatrix} \;;\; Y_d \to 0$$

$$N_f = 3 \;\to\; Y_u = \begin{pmatrix} \cancel{y_u} & 0 & 0 \\ 0 & \cancel{y_c} & 0 \\ 0 & 0 & y_t \end{pmatrix} \;;\; Y_d \to 0$$

Define:

$$M \equiv Y_u Y_u^\dagger \in \mathbf{3}_Q \otimes \bar{\mathbf{3}}_Q = \mathbf{1}_Q \oplus \mathbf{8}_Q$$
$$\tilde{M} \equiv Y_u^\dagger Y_u \in \mathbf{3}_u \otimes \bar{\mathbf{3}}_u = \mathbf{1}_u \oplus \mathbf{8}_u$$

Example: operators with 4 up fields ⟶

$$\mathcal{O}_1[u] = c^{u,1}_{mnpq} \partial_\mu (\bar{u}_m \gamma_\nu u_n) \partial^\mu (\bar{u}_p \gamma^\nu u_q)$$
$$\mathcal{O}_3[u] = c^{u,3}_{mnpq} \partial_\mu (\bar{u}_m T^a \gamma_\nu u_n) \partial^\mu (\bar{u}_p T^a \gamma^\nu u_q),$$

$$\boxed{c^{u,i}_{mnpq} = \rho^{u,i}_1 (\delta_{mn}\delta_{pq}) + \rho^{u,i}_2 (\tilde{M}_{mn}\delta_{pq} + \delta_{mn}\tilde{M}_{pq}) + \rho^{u,i}_3 (\delta_{mq}\delta_{pn}) + \rho^{u,i}_4 (\tilde{M}_{mq}\delta_{pn} + \delta_{mq}\tilde{M}_{pn})}$$

up to $\mathcal{O}(Y_u^2)$ and with the ρ expected to be $\mathcal{O}(1)$

Positivity bounds: dimension 8, 4−Fermi operators

To get the bounds:

- scatter flavor superpositions

$$|\psi_1\rangle = \alpha_{mi}\,|\bar{u}_{mi}\rangle\,, \qquad |\psi_2\rangle = \beta_{mi}\,|u_{mi}\rangle$$
$$|\psi_3\rangle = \beta^*_{mi}\,|\bar{u}_{mi}\rangle\,, \qquad |\psi_4\rangle = \alpha^*_{mi}\,|u_{mi}\rangle$$

- obtain the amplitude

$$\mathcal{A} = 4s^2\left[\left(c^{u,1}_{mnpq} - \frac{1}{6}c^{u,3}_{mnpq}\right)\alpha^*_{mi}\beta_{ni}\beta^*_{pj}\alpha_{qj} + \frac{1}{2}c^{u,3}_{mnpq}\alpha^*_{mi}\beta_{nj}\beta^*_{pj}\alpha_{qi}\right]$$

- marginalize over gauge indices

$$\alpha_m\alpha^*_q\beta_n\beta^*_p\left(c^{u,1}_{mnpq} + \frac{1}{3}c^{u,3}_{mnpq}\right) > 0\,,$$
$$\alpha_m\alpha^*_q\beta_n\beta^*_p c^{u,3}_{mnpq} > 0$$

To simplify them a bit

- perform linear redefinition

$$\xi^{u,1}_k \equiv \rho^{u,1}_k + \frac{1}{3}\rho^{u,3}_k \quad \text{and} \quad \xi^{u,3}_k \equiv \rho^{u,3}_k \quad \text{for} \quad k = 1,2,3,4,$$

- turn to the study of

$$c(\xi)^{u,i}_{mnpq} = \xi^{u,i}_1(\delta_{mn}\delta_{pq}) + \xi^{u,i}_2(\tilde{M}_{mn}\delta_{pq} + \delta_{mn}\tilde{M}_{pq}) + \xi^{u,i}_3(\delta_{mq}\delta_{pn}) + \xi^{u,i}_4(\tilde{M}_{mq}\delta_{pn} + \delta_{mq}\tilde{M}_{pn})$$

Finally:

$$\boxed{\alpha_m\alpha^*_q\beta_n\beta^*_p c(\xi)^{u,i}_{mnpq} > 0} \qquad i = 1,3$$

Disentangling the external states

First a simple check: are the bounds either trivial or empty?

NO: $$\xi_1^i = \xi_2^i = 0 \Rightarrow \xi_3^i > -\xi_4^i \left(\frac{\alpha_m \tilde{M}_{mq} \alpha_q^*}{|\alpha|^2} + \frac{\beta_p^* \tilde{M}_{pn} \beta_n}{|\beta|^2} \right)$$

Now, we want bounds on the coefficients alone, so we need to get rid of the external states α_n, β_n

Define $C(\beta)_{mq} \equiv c_{mnpq}(\xi)\beta_n\beta_p^*$ with eigenvalues $r(\beta)_I$, $I = 1,\ldots,N_f$

Then we can trade the positivity bounds for

$$\begin{cases} c_{mnpq}\beta_n\beta_p^*\alpha_m\alpha_q^* > 0 \\ \forall \alpha, \beta \text{ with } \|\alpha\| = \|\beta\| = 1 \end{cases} \Longleftrightarrow \begin{cases} r(\beta)_I > 0 \quad I = 1,\ldots,N_f \\ \forall \beta \text{ with } \|\beta\| = 1 \end{cases}$$

and deal with β alone!

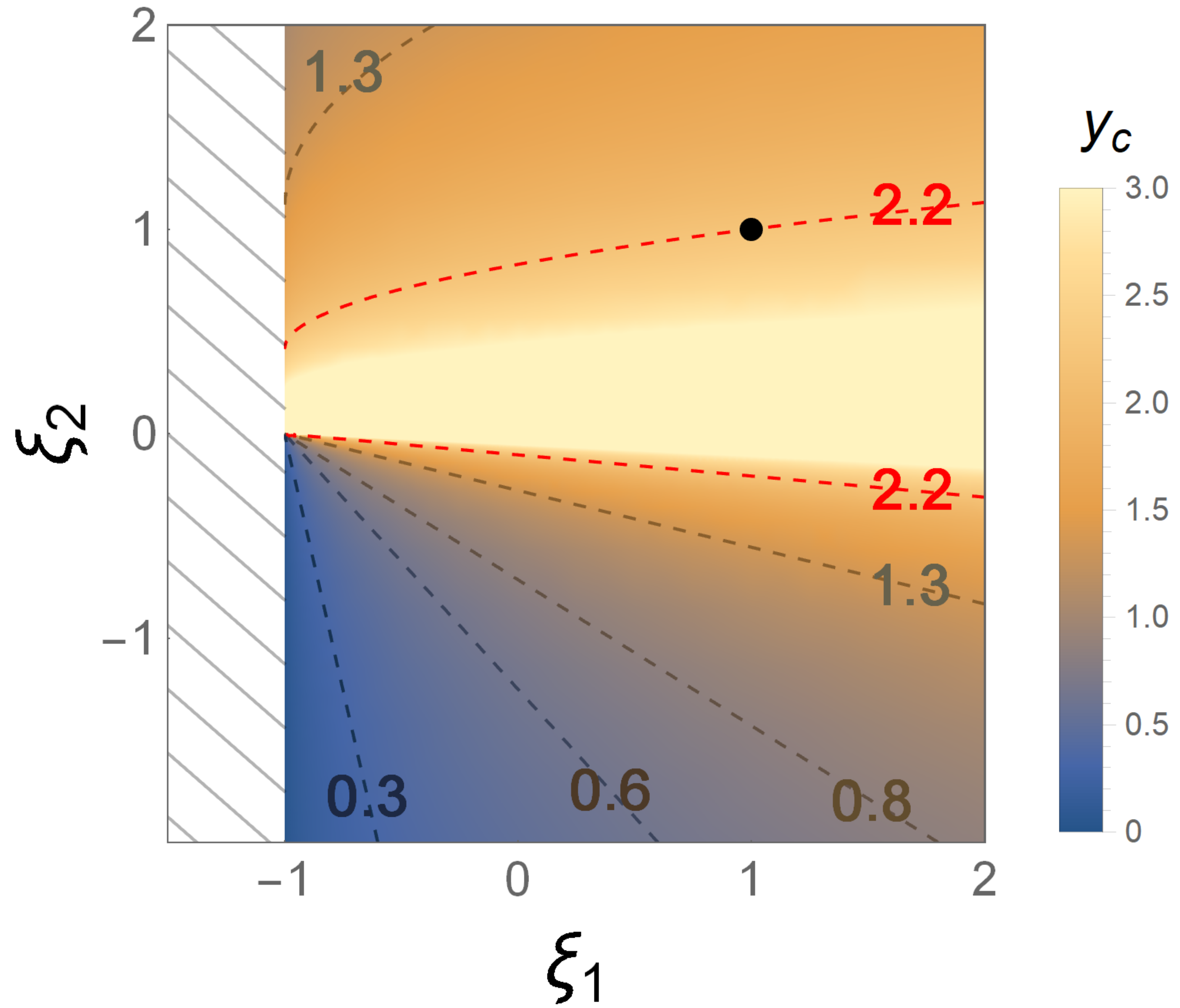
Results for $N_f = 2$

With two flavors, the only independent parameter is y_c, and we can remove ξ_4 because it is redundant in $SU(2)$.

Marginalizing over β we get

$$\begin{cases} \cancel{\xi_4^{u,i}} {y_c}^2 + \xi_3^{u,i} > 0, \\ 2{y_c}^2 \left(\xi_2^{u,i} + \cancel{\xi_4^{u,i}}\right) + \xi_1^{u,i} + \xi_3^{u,i} > 0, \\ \xi_1^{u,i} + \xi_3^{u,i} > 0, \\ {y_c}^4 (\cancel{\xi_4^{u,i}} - \xi_2^{u,i}) \left(\xi_2^{u,i} + 3\cancel{\xi_4^{u,i}}\right) + 8\xi_3^{u,i}\cancel{\xi_4^{u,i}}{y_c}^2 + 4\left(\xi_3^{u,i}\right)^2 > 0 \quad \text{or} \\ \quad \left(-4y_c^2 \left(\xi_1^{u,i}\cancel{\xi_4^{u,i}} + \xi_2^{u,i}\xi_3^{u,i}\right) + {y_c}^4 \left(\xi_2^{u,i} - \xi_4^{u,i}\right)^2 - 4\xi_1^{u,i}\xi_3^{u,i}\right) < 0. \end{cases}$$

We can visualise them as



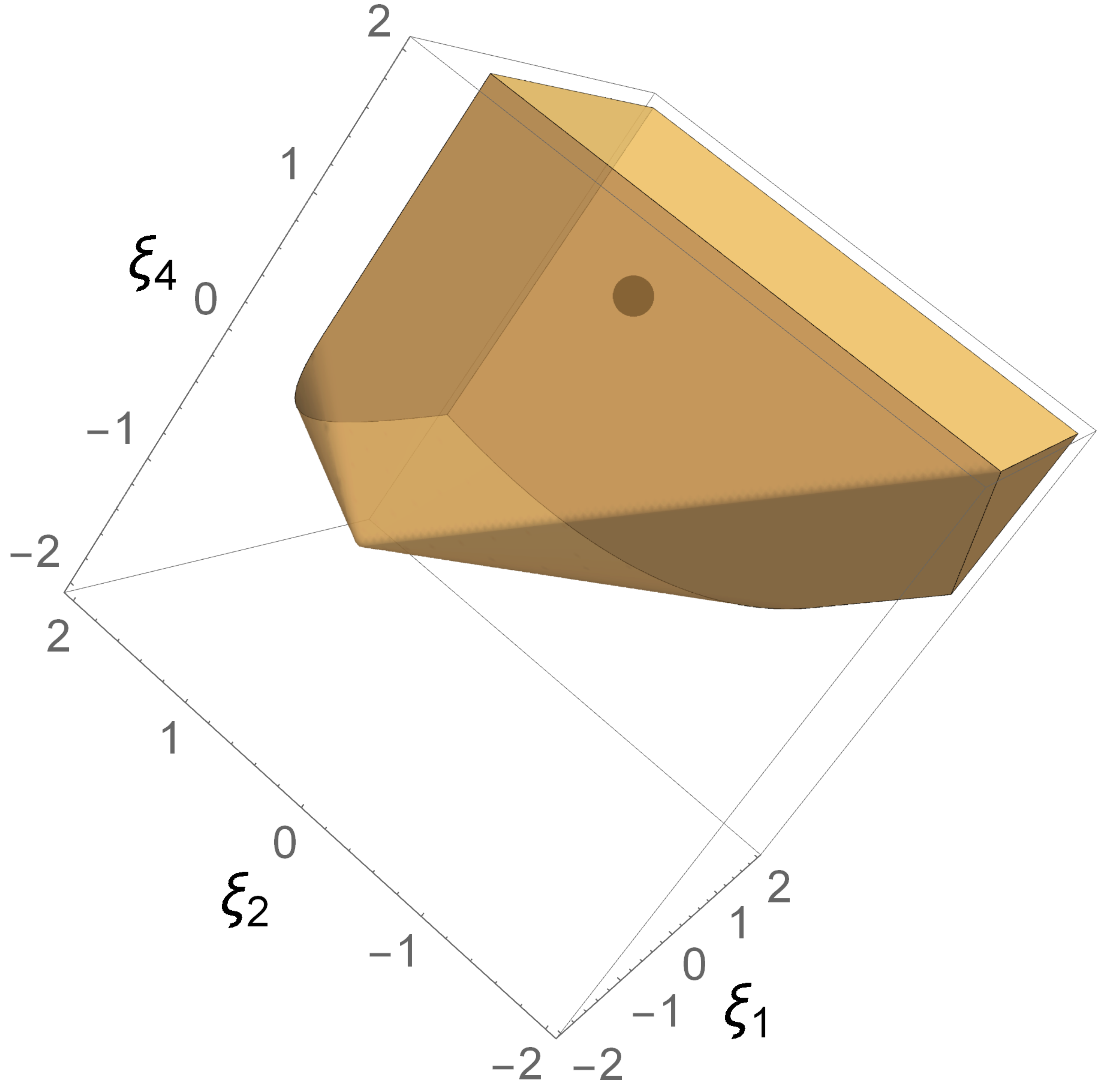
The allowed region (dashed line) shrinks for growing y_c.

The degenerate point $\xi_1 = \xi_2 = \xi_3 = 1$ is allowed up to $y_c \lesssim 2.2$

Results for $N_f = 3$

The large value of y_t requires us to resum the series in powers of Y_u and to set $y_t \to 1$ in the end.

Thus the bounds are a fixed region, with no Yukawa dependence. We can rescale $\xi_3 \to 1$ and visualise the bounds:



The degenerate point $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 1$ is well inside this region!

Where is the CKM?

We could expect the bounds to depend not only on the Yukawa couplings, but on the CKM matrix entries, too.

Why doesn't this happen?

Let us look again at the tensor structure at this order:

$$c^{u,i}_{mnpq} = \rho^{u,i}_1(\delta_{mn}\delta_{pq}) + \rho^{u,i}_2(\tilde{M}_{mn}\delta_{pq} + \delta_{mn}\tilde{M}_{pq}) + \rho^{u,i}_3(\delta_{mq}\delta_{pn}) + \rho^{u,i}_4(\tilde{M}_{mq}\delta_{pn} + \delta_{mq}\tilde{M}_{pn})$$

- As we only kept Y_u (remember $\tilde{M} = Y_u^\dagger Y_u$) we can always pick a basis where V_{CKM} is entirely in Y_d
- This can be done in the Lagrangian, or with a unitary rotation on α_n, β_n

There are two ways we can force the CKM to appear in the bounds:

- Project $N_f = 3$ onto $N_f = 2$ $\quad M_{ij} = \left(Y_u Y_u^\dagger\right)_{ij} \sim (V_{CKM})_{3i}(V^*_{CKM})_{3j} \qquad i,j = 1,2$
  - We obtain the same results for the true $N_f = 2$, but now the parameter is $y_c \to \sigma \equiv A^2\lambda^4$
- Go further in the expansion
  - Including $\mathcal{O}(Y_d^2)$ one cannot get rid of the CKM anymore.
  - Appearing at higer orders, it has conversely subleading effects on the bounds

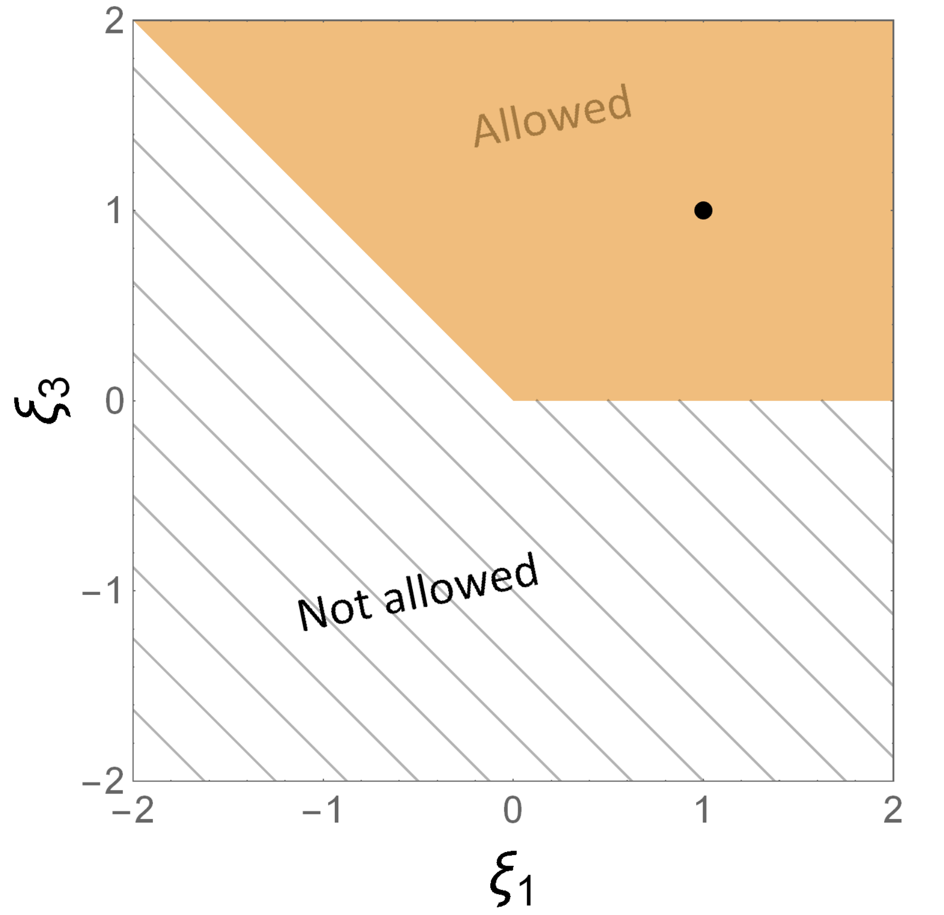
Other operators

Following the same procedure as for the operators with 4 up fields, we get bounds on the others.

Self-quartic operators

- 4-Q operators are obtained by sending $\tilde{M} \to M$ and produce the same bounds
- 4-d operators are obtained by sending $\tilde{M} \to 0$

$$\begin{cases} \xi_3^{d,i} > 0, \\ \xi_1^{d,i} + \xi_3^{d,i} > 0. \end{cases}$$

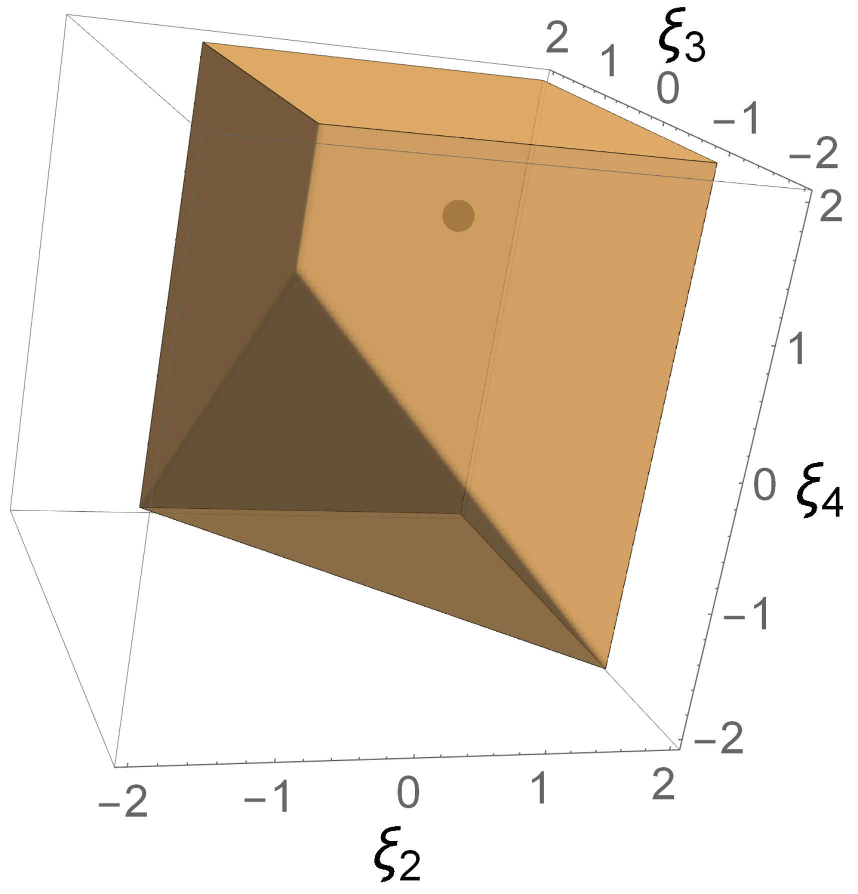


Degenerate point is inside!

Cross-quartic operators

Those are easier to deal with, because of the less rich tensor structure. For the (2u)(2Q) operators we get

$$\begin{cases} \xi_1^{uQ,i} > 0 \\ \xi_1^{uQ,i} + \xi_2^{uQ,i} > 0 \\ \xi_1^{uQ,i} + \xi_3^{uQ,i} > 0 \\ \xi_1^{uQ,i} + \xi_2^{uQ,i} + \xi_3^{uQ,i} + \xi_4^{uQ,i} > 0 \end{cases}$$



Degenerate point is inside!

Summary

- The Minimal Flavor Violation assumption can be made consistent with positivity bounds in the Standard Model Effective Field Theory

- The bounds for the flavor-blind coefficients are consistently controlled by the largest Yukawas

- A method to disentangle the arbitrary external states of the elastic scattering has been shown

Thank you!

Backup I

Explicit extraction of the bounds for $N_f = 2$:

Parametrize $\quad \beta = \begin{pmatrix} x e^{i\theta_x} \\ y e^{i\theta_y} \end{pmatrix} \equiv \cancel{e^{i\theta_y}} \begin{pmatrix} x e^{i\tilde{\theta}_x} \\ y \end{pmatrix}, \qquad x^2 + y^2 = 1$

get

$$
\begin{aligned}
\mathrm{Tr}[C(\beta)] =& 2x^2 \left(\xi_2^{Q,i} + \xi_4^{Q,i}\right) y_c^2 + \xi_4^{Q,i} y_c^2 + \xi_1^{Q,i} + 2\xi_3^{Q,i} > 0 \\
\det[C(\beta)] =& x^4 \left(\xi_2^{Q,i} + \xi_4^{Q,i}\right)^2 y_c^4 - x^2 y_c^2 \left(\xi_2^{Q,i} + \xi_4^{Q,i}\right) \left(y_c^2 \left(\xi_2^{Q,i} - \xi_4^{Q,i}\right) - 2\xi_3^{Q,i}\right) + \\
& + \left(\xi_4^{Q,i} y_c^2 + \xi_3^{Q,i}\right) \left(\xi_1^{Q,i} + \xi_3^{Q,i}\right) > 0
\end{aligned}
$$

The trace is linear in $x^2 \in [0,1]$, so it is positive everywhere iff it is positive at the boundaries.

The determinant is quadratic in x^2, it is positive iff:

- It is positive at the boundaries
- At least one of the following is met: $\quad \Delta < 0 \text{ or } a < 0 \text{ or } b(b+2a) > 0, \text{ with } \det[C(\beta)] \equiv a x^4 + b x^2 + c \quad \longrightarrow$

$\longrightarrow$

$$
\begin{cases}
\xi_4^{Q,i} y_c^2 + \xi_3^{Q,i} > 0, \\
2y_c^2 \left(\xi_2^{Q,i} + \xi_4^{Q,i}\right) + \xi_1^{Q,i} + \xi_3^{Q,i} > 0, \\
\xi_1^{Q,i} + \xi_3^{Q,i} > 0, \\
y_c^4 (\xi_4^{Q,i} - \xi_2^{Q,i}) \left(\xi_2^{Q,i} + 3\xi_4^{Q,i}\right) + 8\xi_3^{Q,i}\xi_4^{Q,i} y_c^2 + 4\left(\xi_3^{Q,i}\right)^2 > 0 \quad \text{or} \\
\quad \left(-4y_c^2 \left(\xi_1^{Q,i}\xi_4^{Q,i} + \xi_2^{Q,i}\xi_3^{Q,i}\right) + y_c^4 \left(\xi_2^{Q,i} - \xi_4^{Q,i}\right)^2 - 4\xi_1^{Q,i}\xi_3^{Q,i}\right) < 0.
\end{cases}
$$

Backup II

Proof that $\bar{u}_m u_n \bar{u}_p u_q (\tilde{M}_{mq}\delta_{pn} + \delta_{mq}\tilde{M}_{pn})$ is redundant in $N_f = 2$.

Define $X_{mn} = \bar{u}_m u_n$ to get
$$X_{ij}X_{kl}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in}\tilde{M}_{nm} = \left(X_{\{ij\}} + \frac{1}{2}X_{ab}\varepsilon_{ab}\varepsilon_{ij}\right)\left(X_{\{kl\}} + \frac{1}{2}X_{cd}\varepsilon_{cd}\varepsilon_{kl}\right)\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in}\tilde{M}_{nm} =$$
$$= X_{\{ij\}}X_{\{kl\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in}\tilde{M}_{nm} + X_{ab}\varepsilon_{ab}\varepsilon_{nk}\varepsilon_{lm}X_{\{kl\}}\tilde{M}_{nm} - \frac{1}{4}\varepsilon_{ab}X_{ab}\varepsilon_{cd}X_{cd}\varepsilon_{nm}\tilde{M}_{nm}$$

One sees that the second term can be reabsorbed in ρ_1 and the third in ρ_2. The remaining one can be split as:

$$X_{\{ij\}}X_{\{kl\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in}\tilde{M}_{nm} = X_{\{ij\}}X_{\{kl\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in}\left(\tilde{M}_{\{nm\}} + \frac{1}{2}\varepsilon_{nm}\varepsilon_{ef}\tilde{M}_{ef}\right) =$$
$$= X_{\{ij\}}X_{\{kl\}}\tilde{M}_{\{nm\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in} + \frac{1}{2}X_{\{ij\}}X_{\{kl\}}\varepsilon_{jk}\varepsilon_{il}\varepsilon_{ef}M_{ef}$$

whose second term can be reabsorbed in ρ_3, while the first one vanishes

$$X_{\{ij\}}X_{\{kl\}}M_{\{nm\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in} \overset{j\leftrightarrow k}{=} -X_{\{ij\}}X_{\{kl\}}M_{\{nm\}}\varepsilon_{kj}\varepsilon_{lm}\varepsilon_{in} \overset{(ij)\leftrightarrow(kl)}{=}$$
$$= -X_{\{ij\}}X_{\{kl\}}M_{\{nm\}}\varepsilon_{il}\varepsilon_{jm}\varepsilon_{kn} \overset{m\leftrightarrow n}{=}$$
$$= -X_{\{ij\}}X_{\{kl\}}M_{\{nm\}}\varepsilon_{il}\varepsilon_{jn}\varepsilon_{km} \overset{i\leftrightarrow j}{=}$$
$$= -X_{\{ij\}}X_{\{kl\}}M_{\{nm\}}\varepsilon_{jl}\varepsilon_{in}\varepsilon_{km} \overset{l\leftrightarrow k}{=}$$
$$= -X_{\{ij\}}X_{\{kl\}}M_{\{nm\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in}.$$