Developments in Hilbert series for EFT

Tom Melia, Kavli IPMU

HEFT 2021, Hefei

Based on work with

Pal 2010.08560

Cao, Herzog, Roosmale-Nepveu (to appear)

Graf, Henning, Lu, Murayama 2009.01239

& Jasper’s talk tomorrow @ HEFT

The use of Hilbert series (and more generally polynomial rings) in EFT

Henning, Lu, **TM**, Murayama, Comm Math Phys 347 no. 2 363-388 (2016)

Henning, Lu, **TM**, Murayama, JHEP 1708 (2017) 016

Henning, Lu, **TM**, Murayama, JHEP 1710 (2017) 199

See also Lehman, Martin '15 x2

Novel probe of EFT

Organize operator bases

The use of Hilbert series (and more generally polynomial rings) in EFT

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See also Lehman, Martin '15 x2

Novel probe of EFT

Asymptotic expansions

TM, Pal 2010.08560

Organize operator bases

Loop calculations

Cao, Herzog, TM, Roosmale-Nepveu (to appear)

Non-linear symmetries

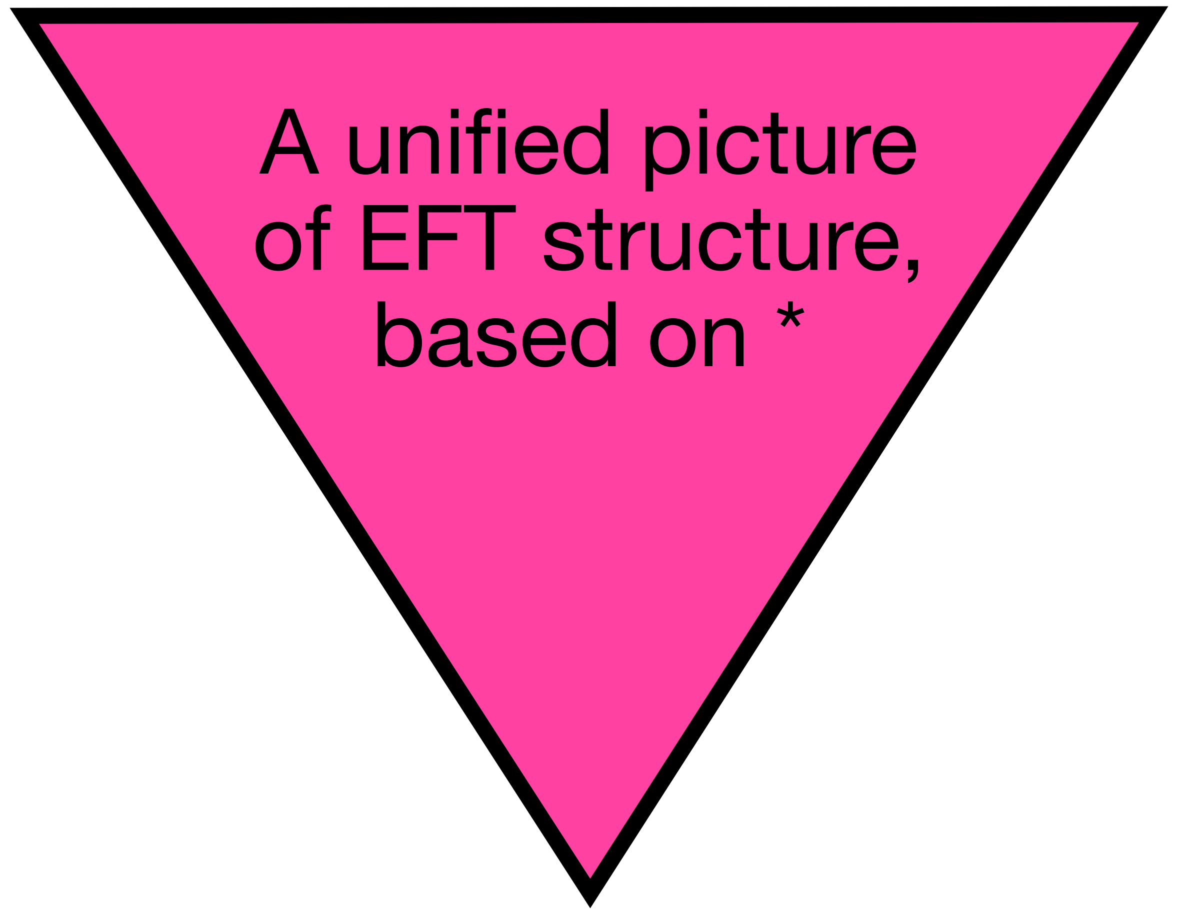
Graf, Henning, Lu, TM, Murayama 2009.01239

New Developments

Background & Outline

Systematic EFT Lagrangian construction

Partition function of (free) CFT



S-Matrix

* Henning, Lu, **TM**, Murayama, Comm Math Phys 347 no. 2 363-388 (2
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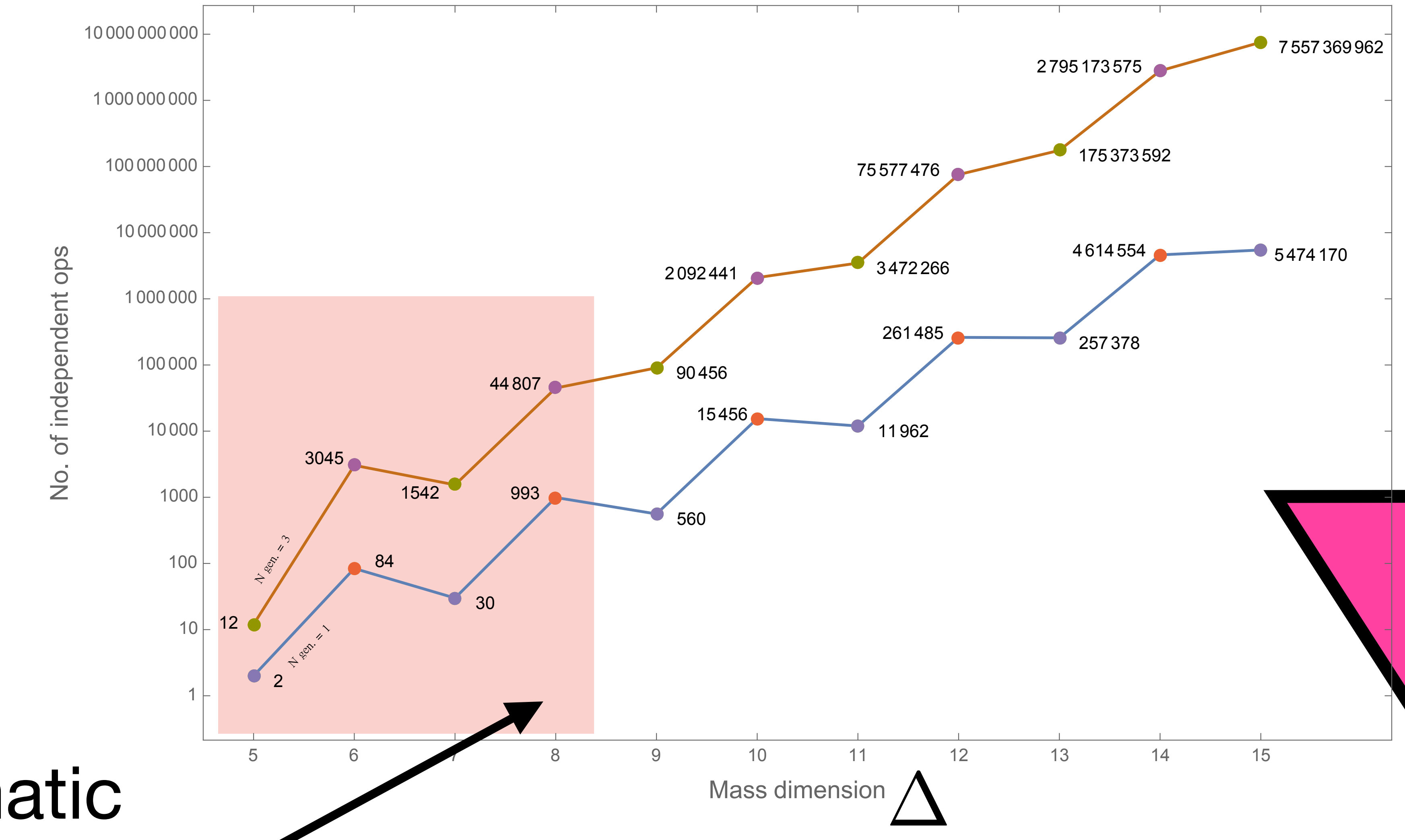
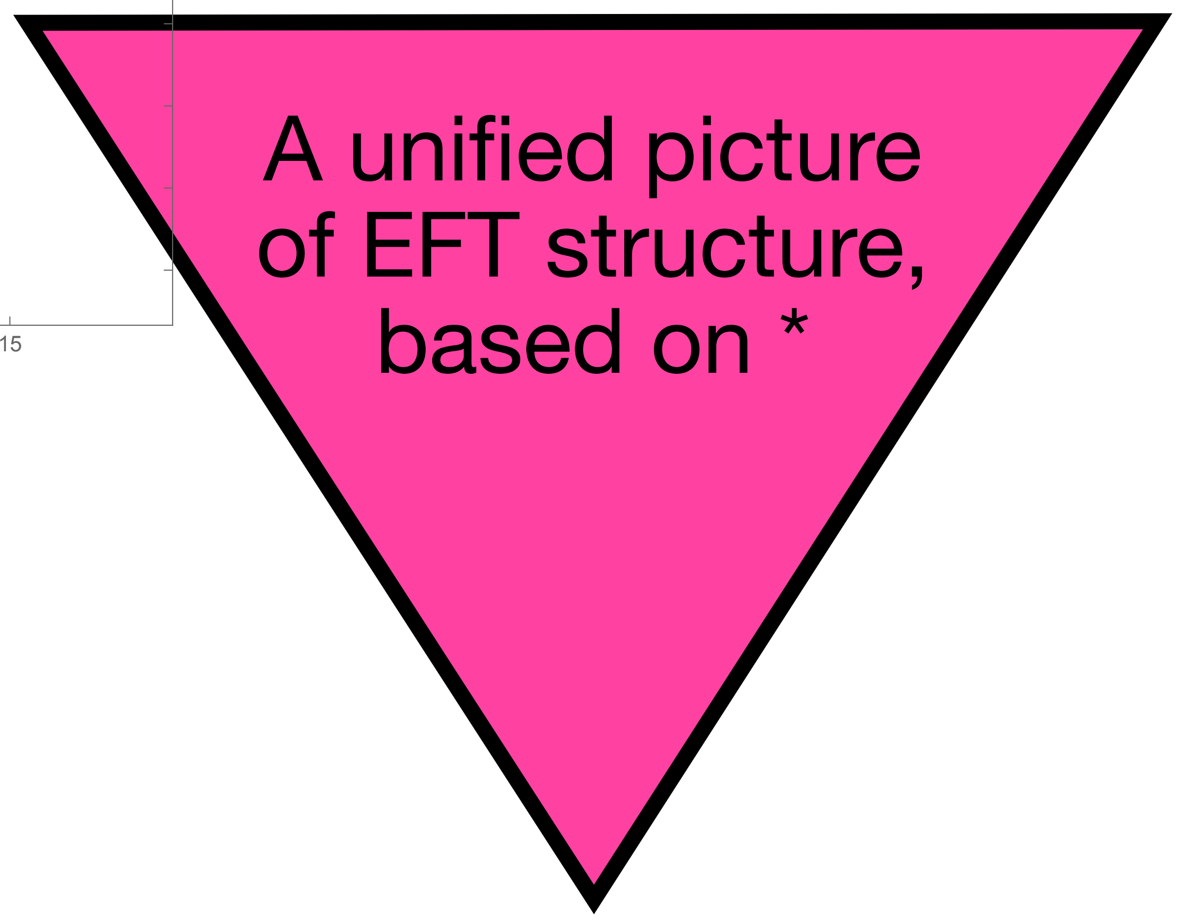
SM EFT



Systematic construction

(deal with field redefinition, integration by parts redundancies)

Partition function of (free) CFT



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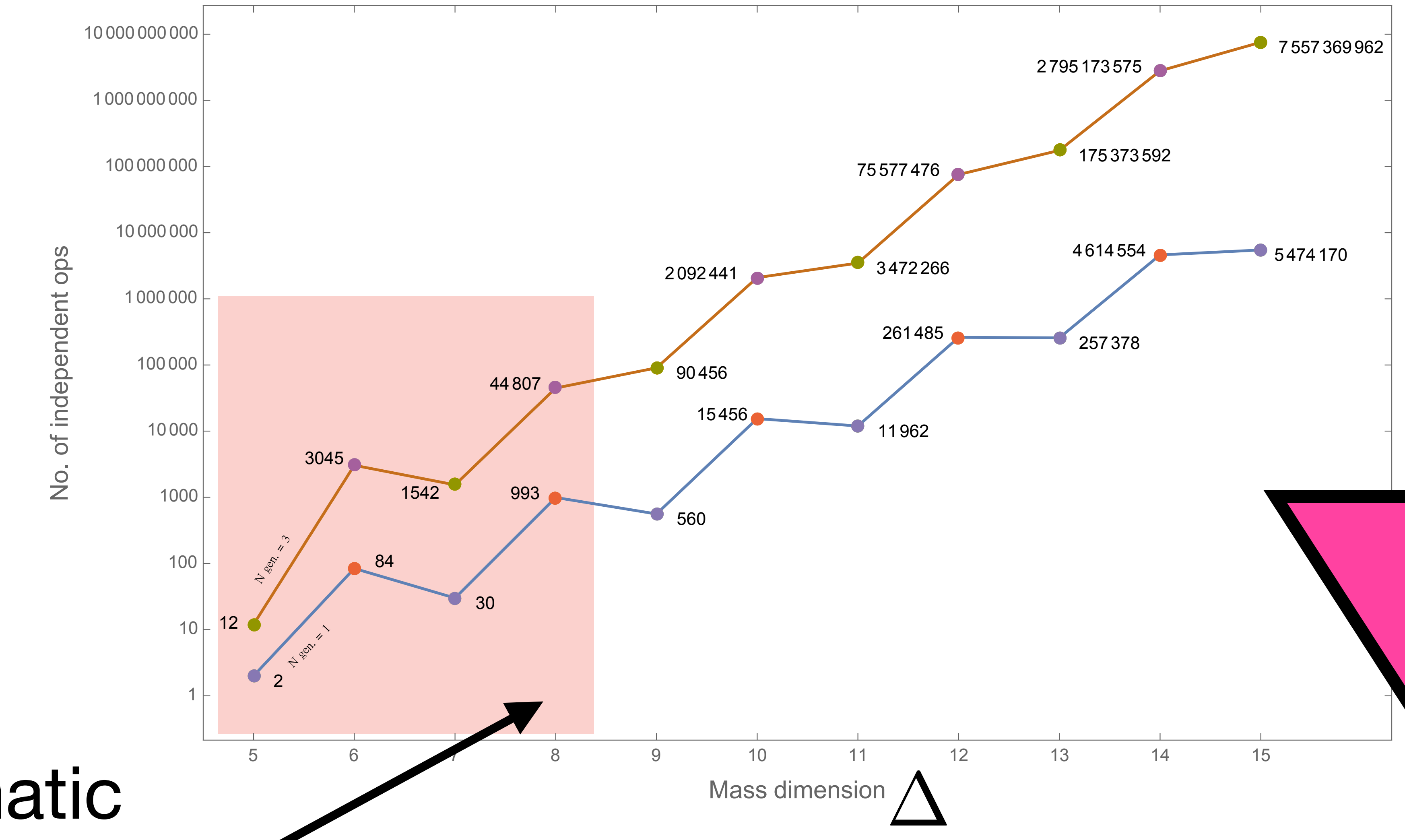

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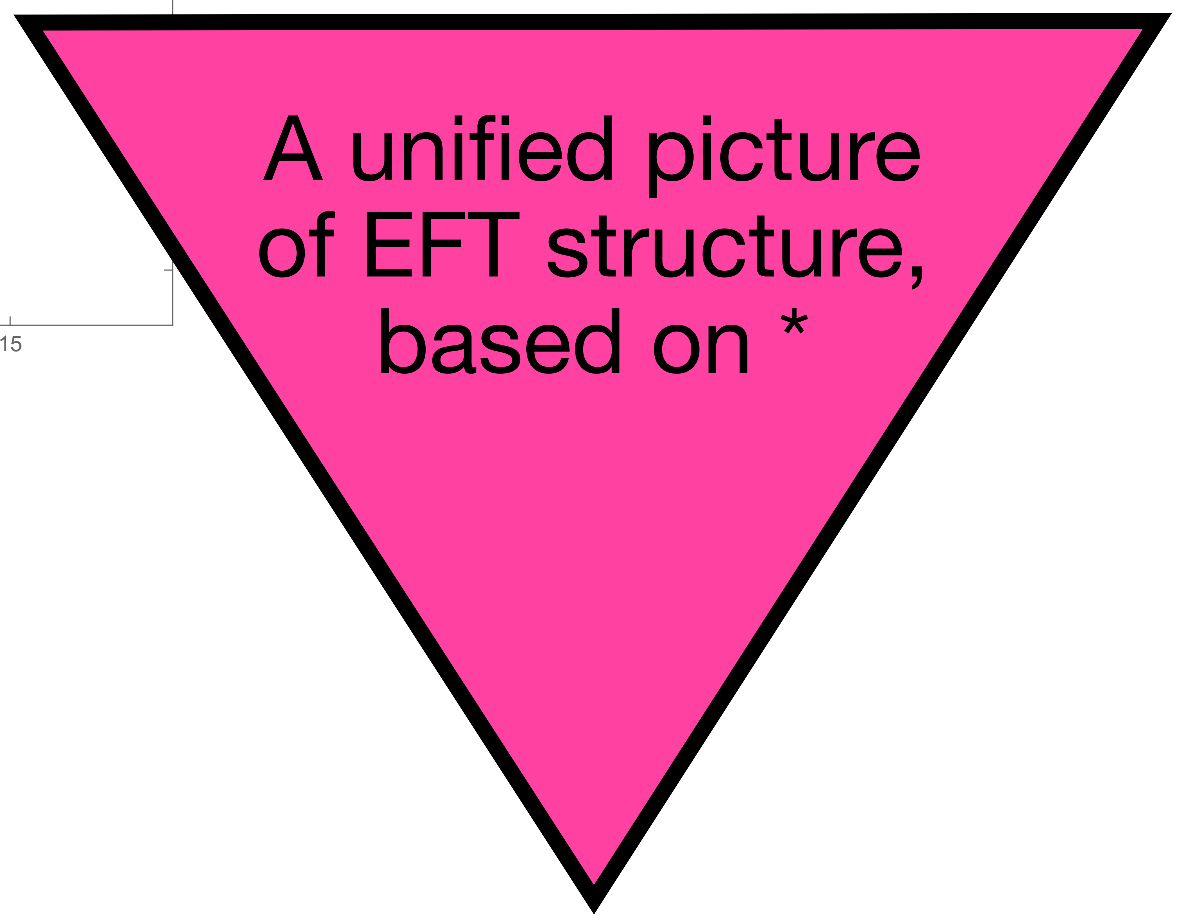
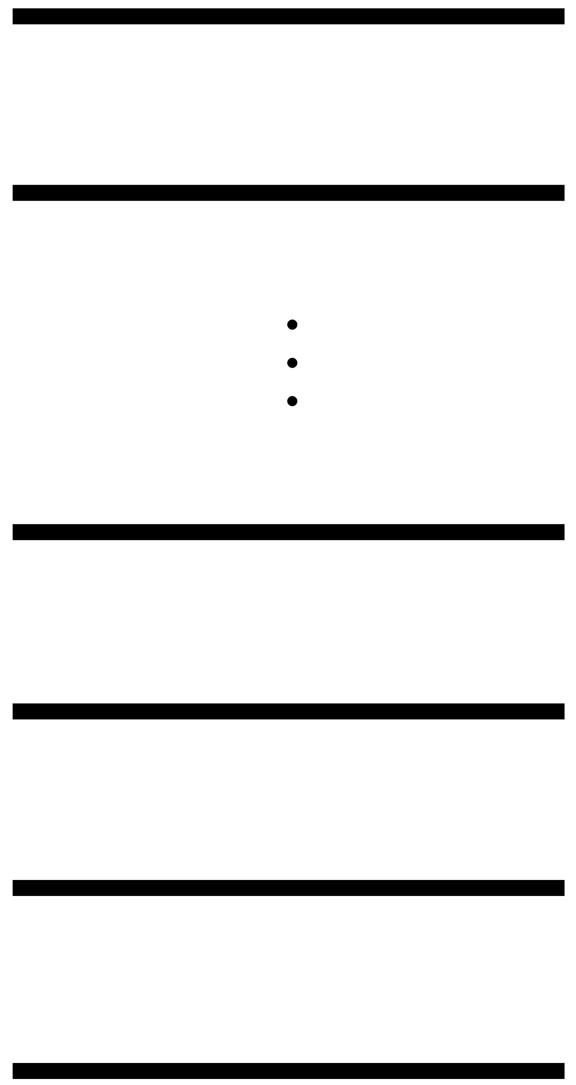


$Z_{\text{free SM}}$

State operator correspondence



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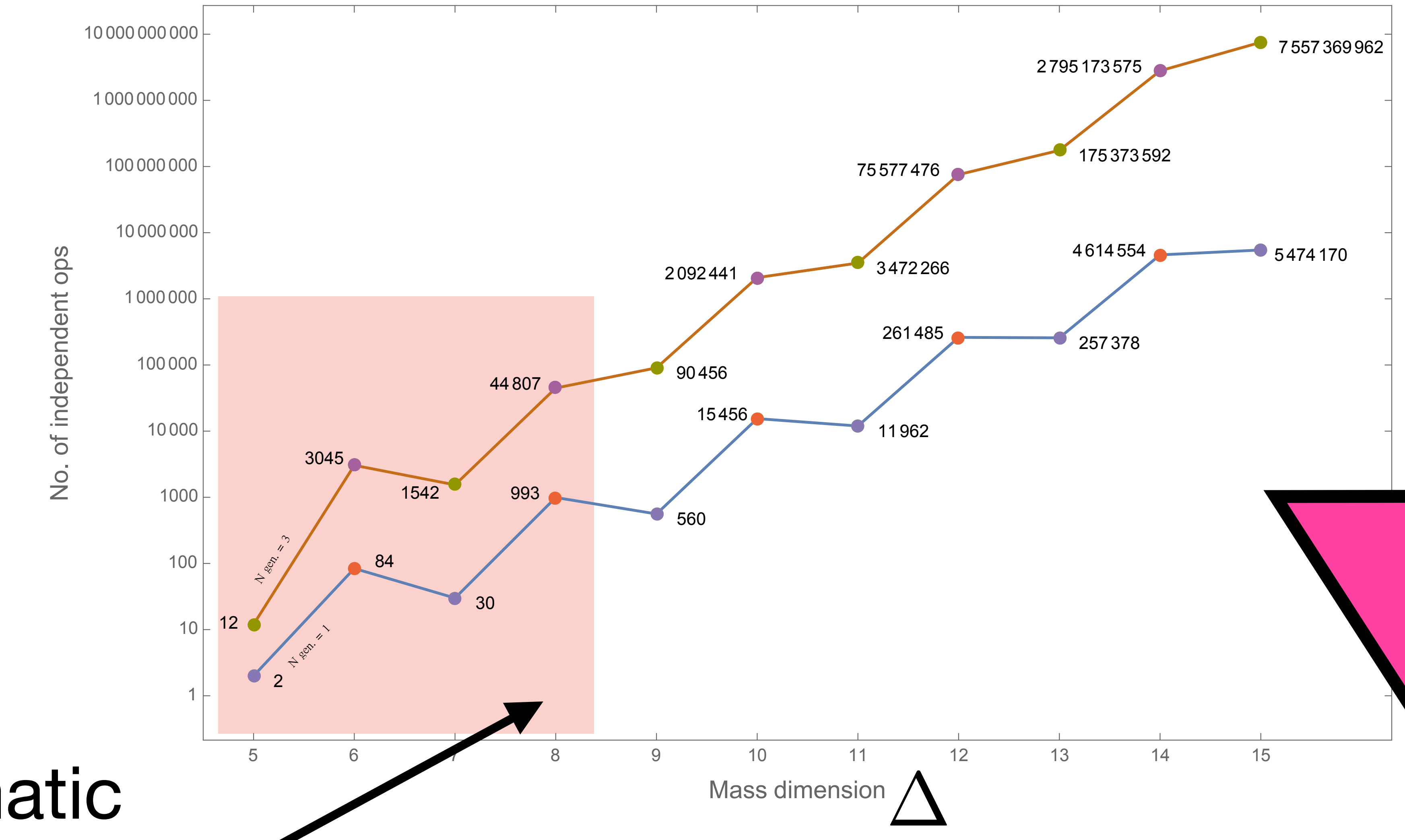


Systematic construction

(deal with field redefinition, integration by parts redundancies)

On-shell amplitudes

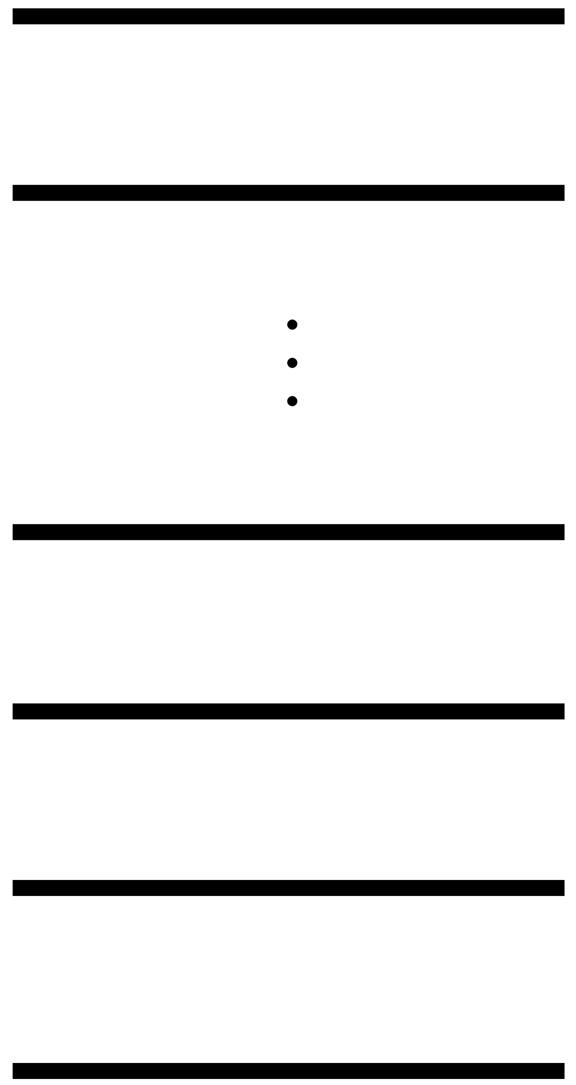
$$p_i^2 = 0$$

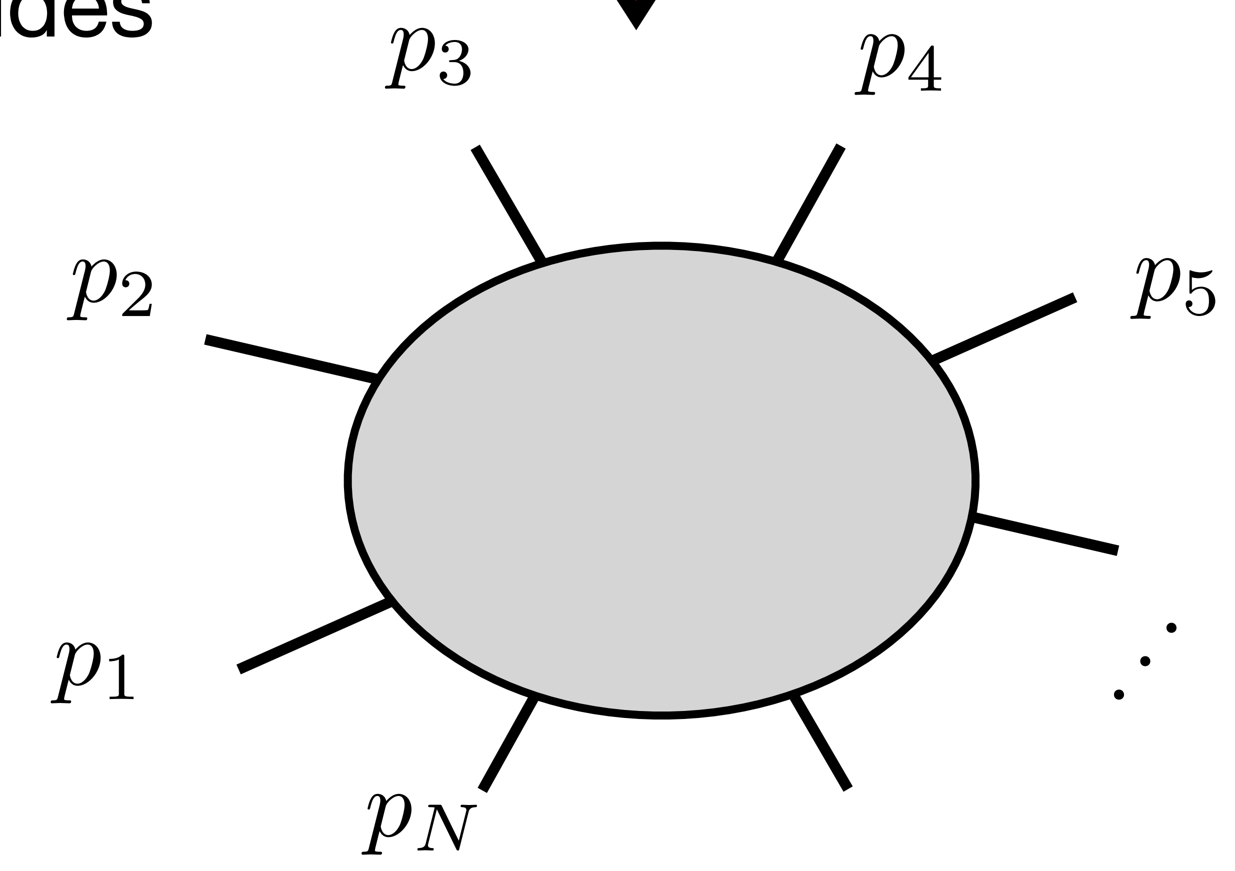
$$\sum_i p_i = 0$$

A unified picture of EFT structure, based on *

$Z_{\text{free SM}}$

State operator correspondence





$$\mathcal{A}(p_1, \ldots, p_N) = \sum_k c_k\, f_k(\{p_i\})$$

Henning, Lu, **TM**, Murayama, Comm Math Phys 347 no. 2 363-388 (2

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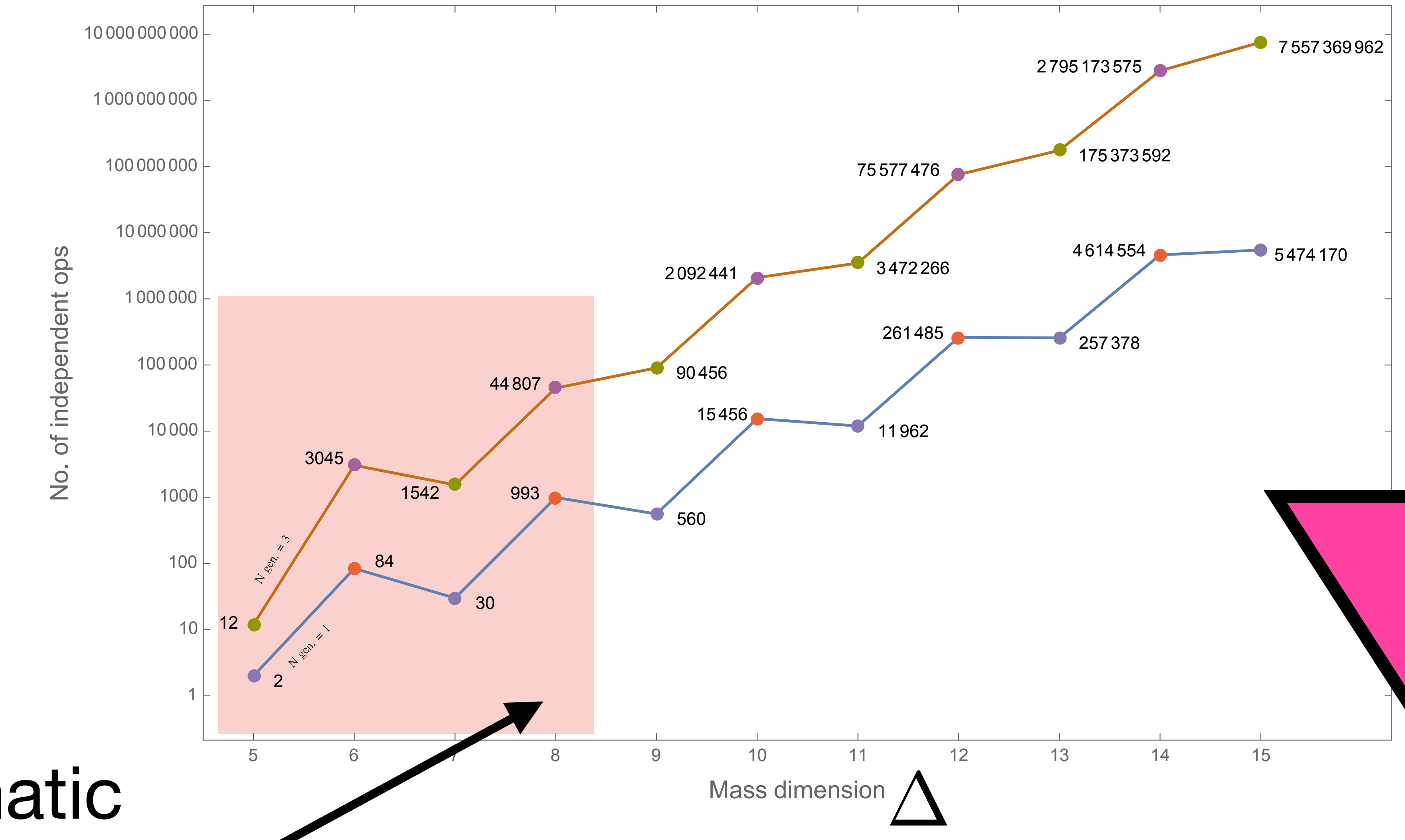

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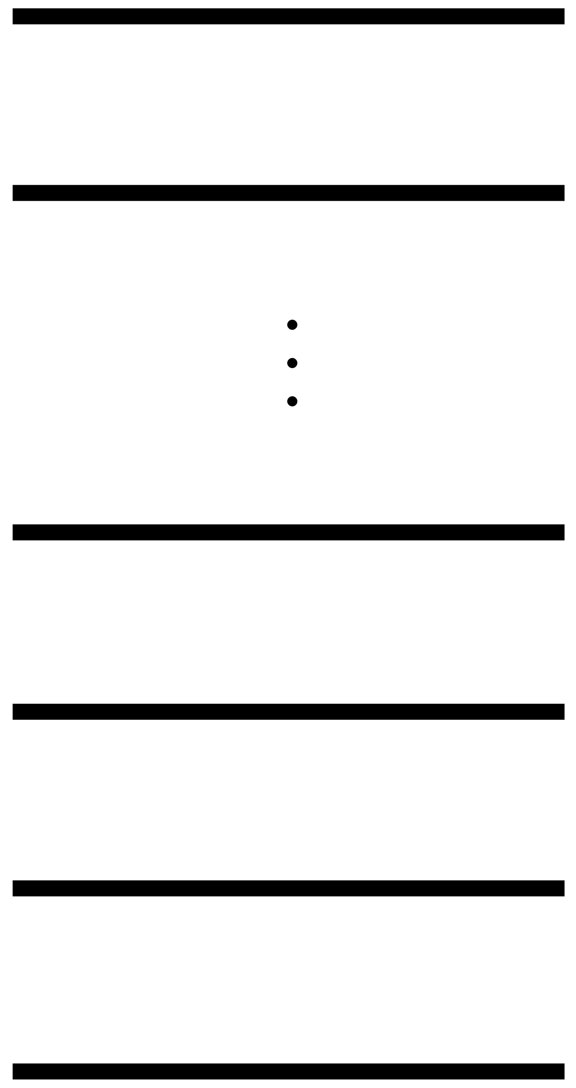


$Z_{\text{free SM}}$

State operator correspondence

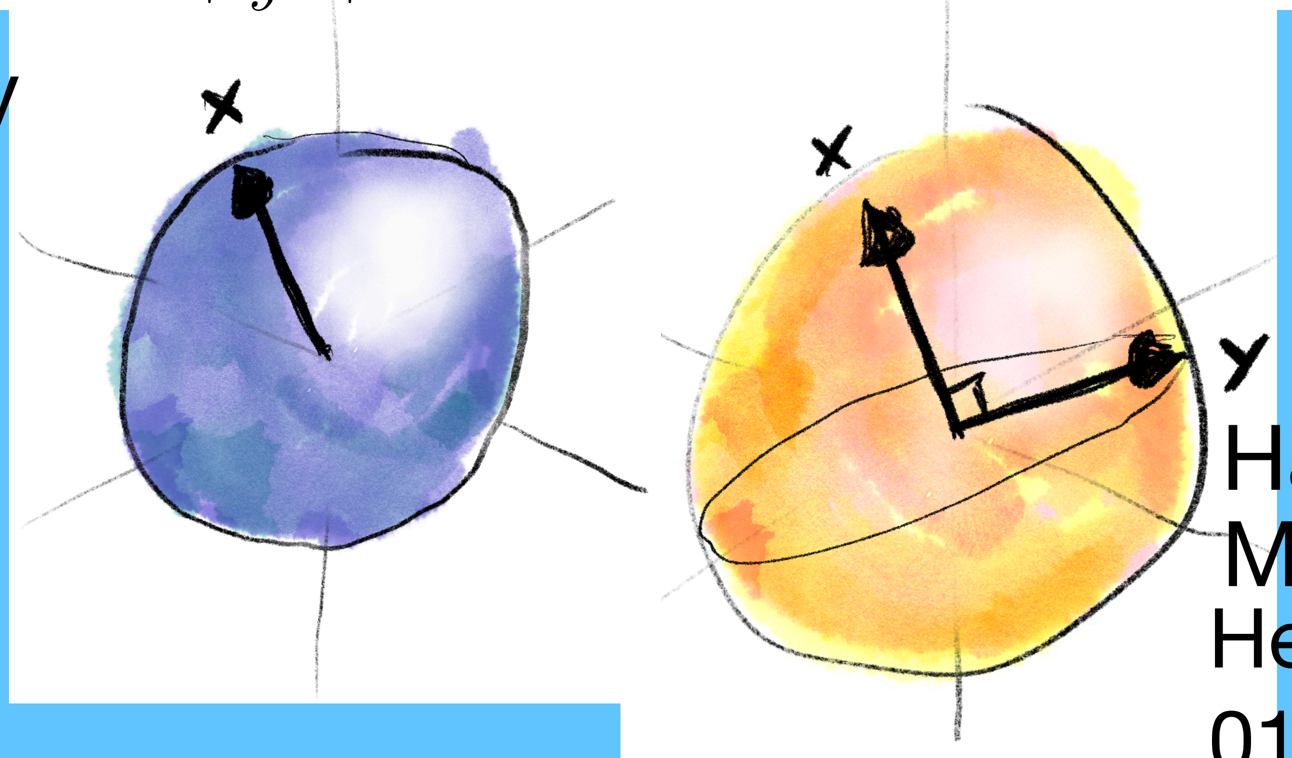
A unified picture of EFT structure, based on *

On-shell EFT construction methods with spinor helicity

Shadmi, Weiss (2018); Ma, Shu, Xiao (2019); Christensen, Field (2018); Durieux, Kitahara, Shadmi, Weiss (2019); Aoude, Machado (2019); Jiang, Shu, Xiao, Zheng (2020); Christensen, Field, Moore (2020); Durieux, Machado (2020); Li, Shu, Xiao, Yu (2020) …

Massive: Durieux, Kitahara, Machado, Shadmi, Weiss (2020); Dong, Ma, Shu (2021) …

$x^2 + y^2 + z^2 = 1$



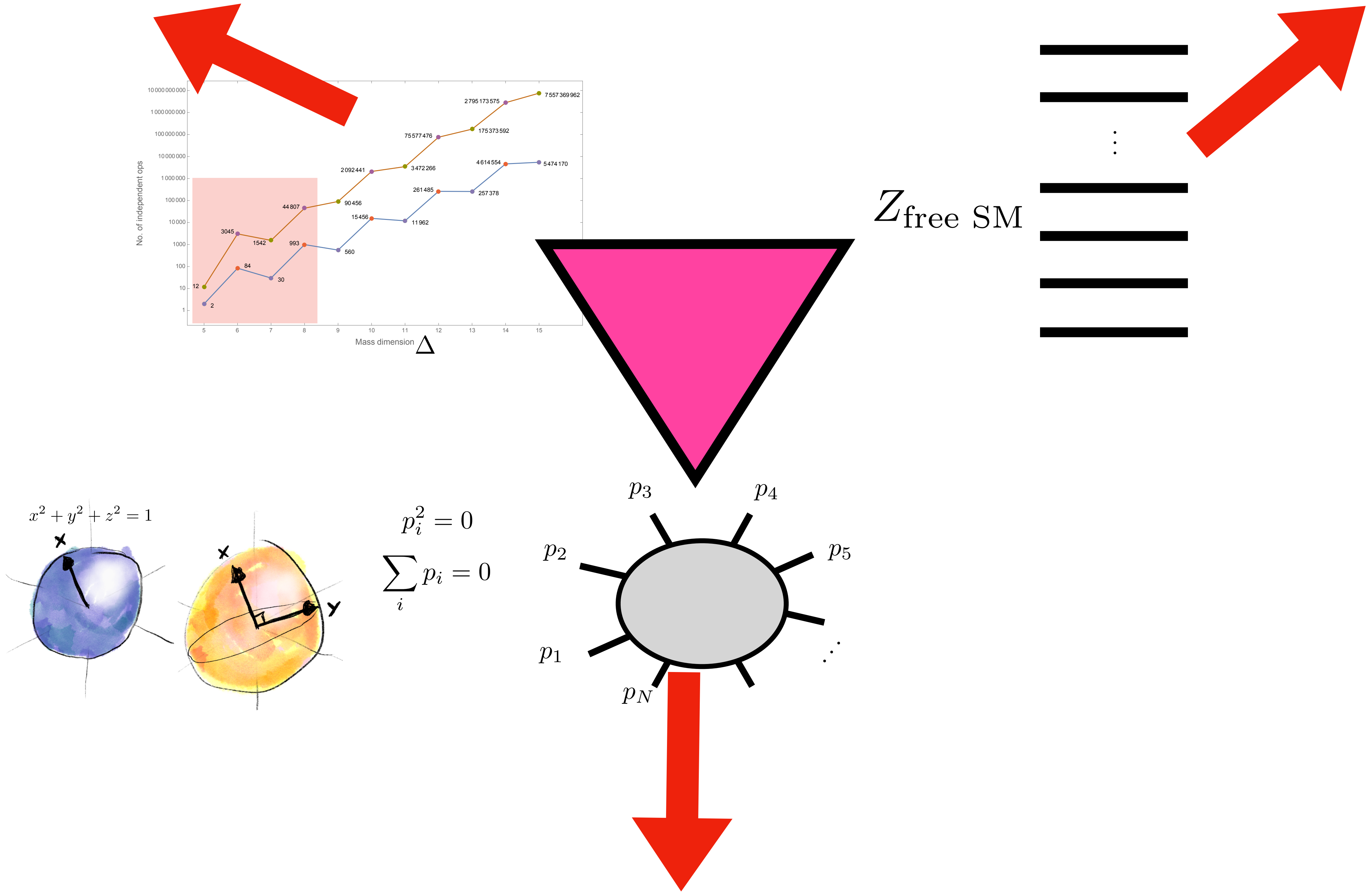
Harmonics of Stiefel Manifold

Henning, **TM**, Phys Rev D 016015 (2019); 1902.06747

Outline

3. Hilbert series for Chiral Lagrangians

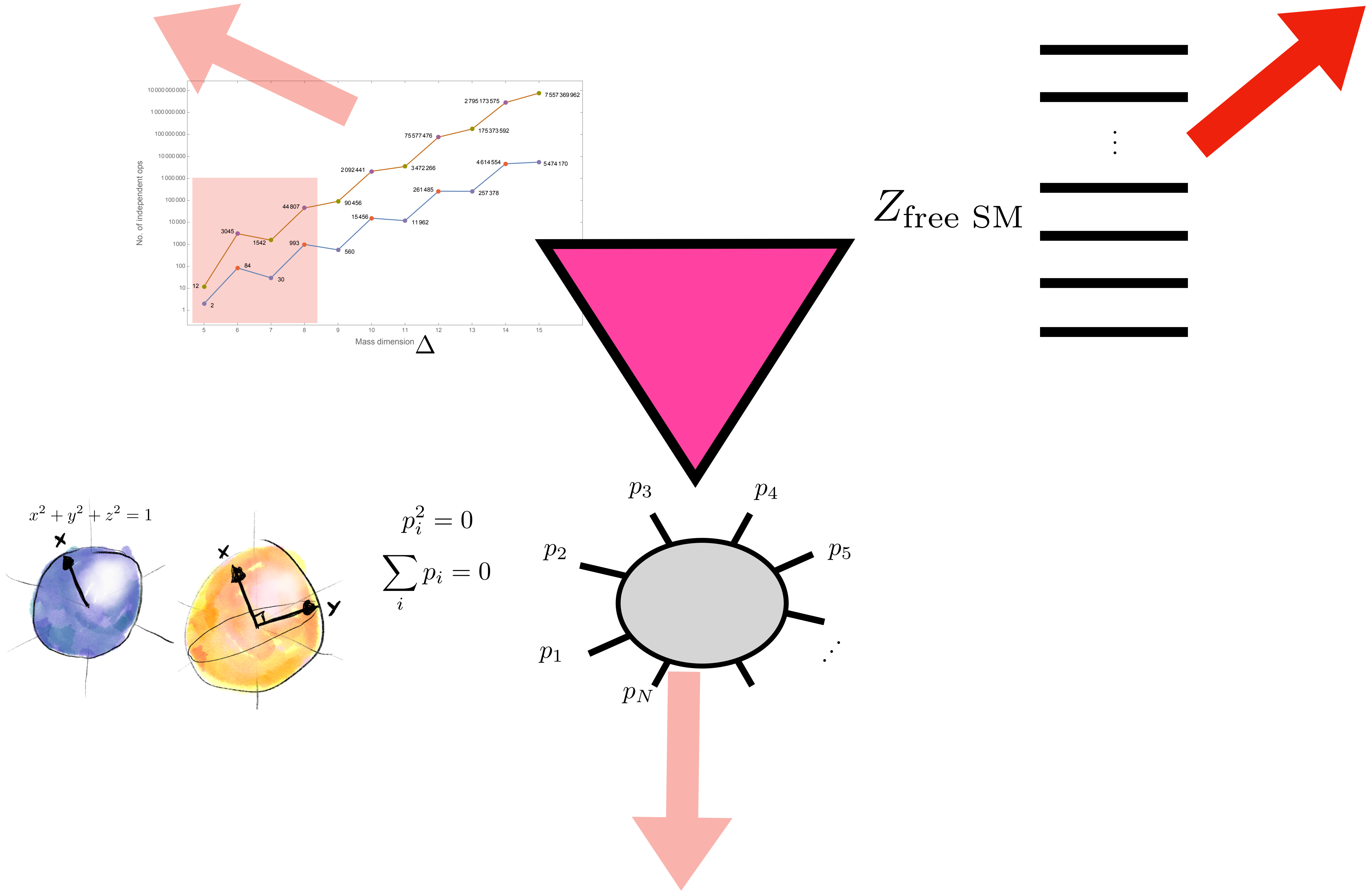
1. Asymptotic growth of the partitions of the Standard Model



2. Hilbert series for loop calculations in EFT

1. Asymptotic growth of the partitions of the Standard Model

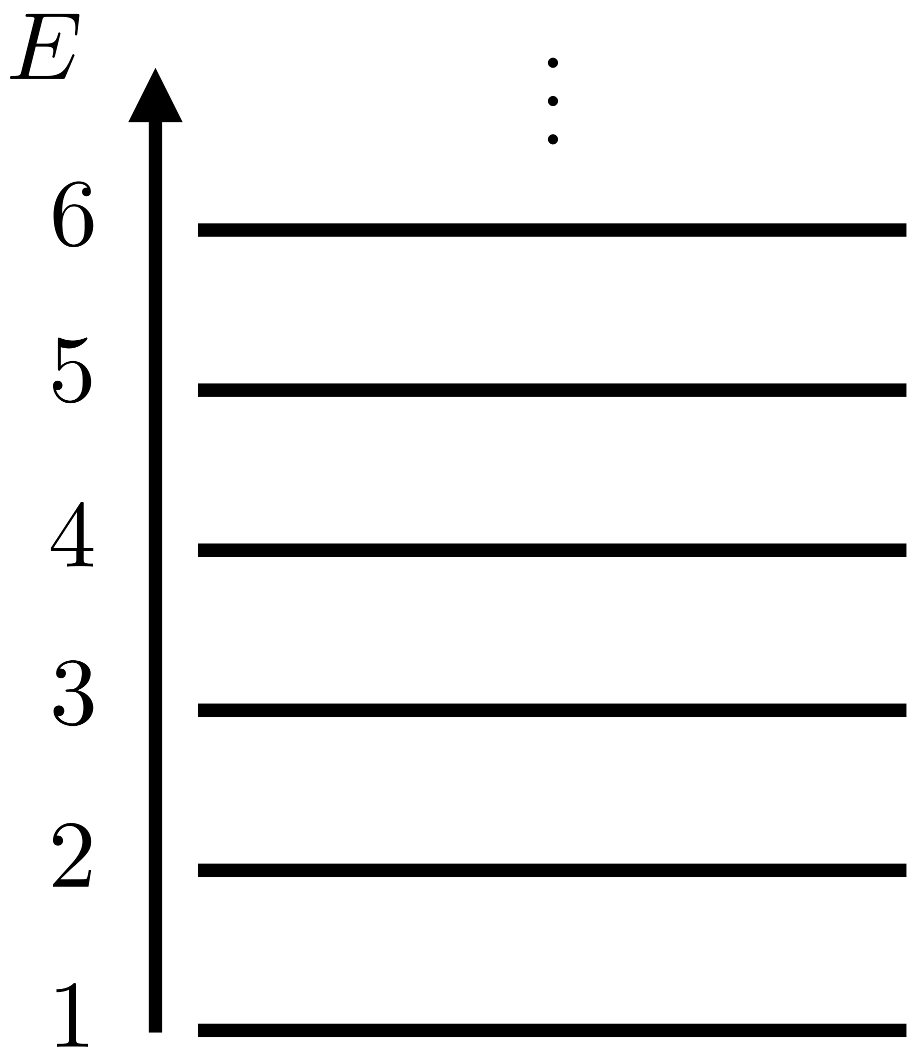
3. Hilbert series for Chiral Lagrangians



2. Hilbert series for loop calculations in EFT

Let me try and make the concept of Hilbert series a bit more physical..

Partition function for a system with some energy levels

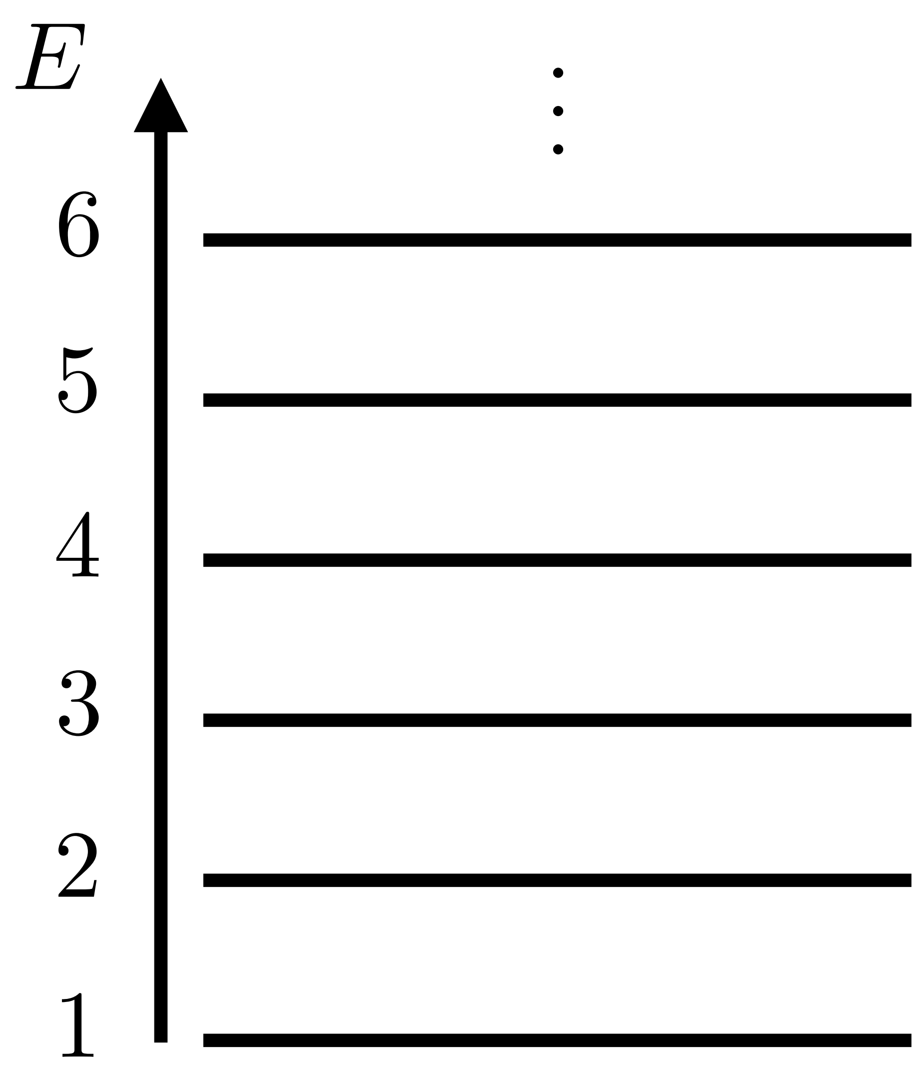


$$q = e^{\frac{1}{k_B T}}$$

$$Z(q) = \prod_n \frac{1}{1 - q^n} \quad \left(= \prod_n \frac{1}{1 - e^{E_n/kT}} \right)$$

$$E_n = n$$

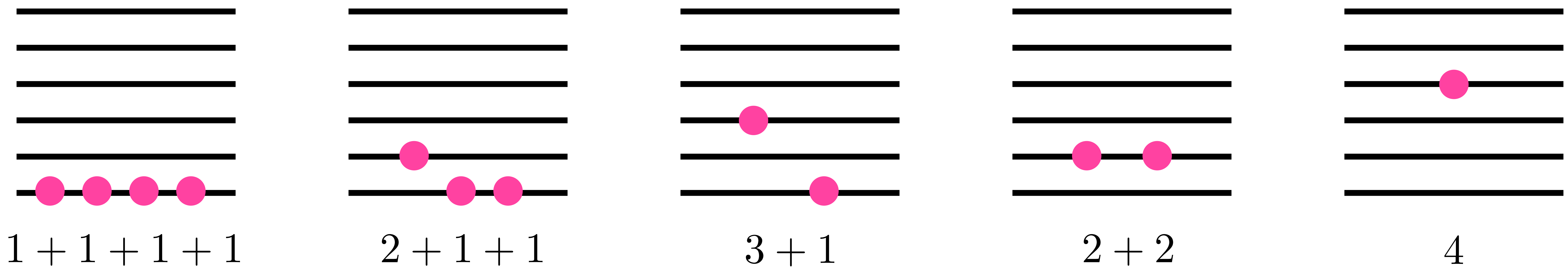
Partition function for a system with some energy levels



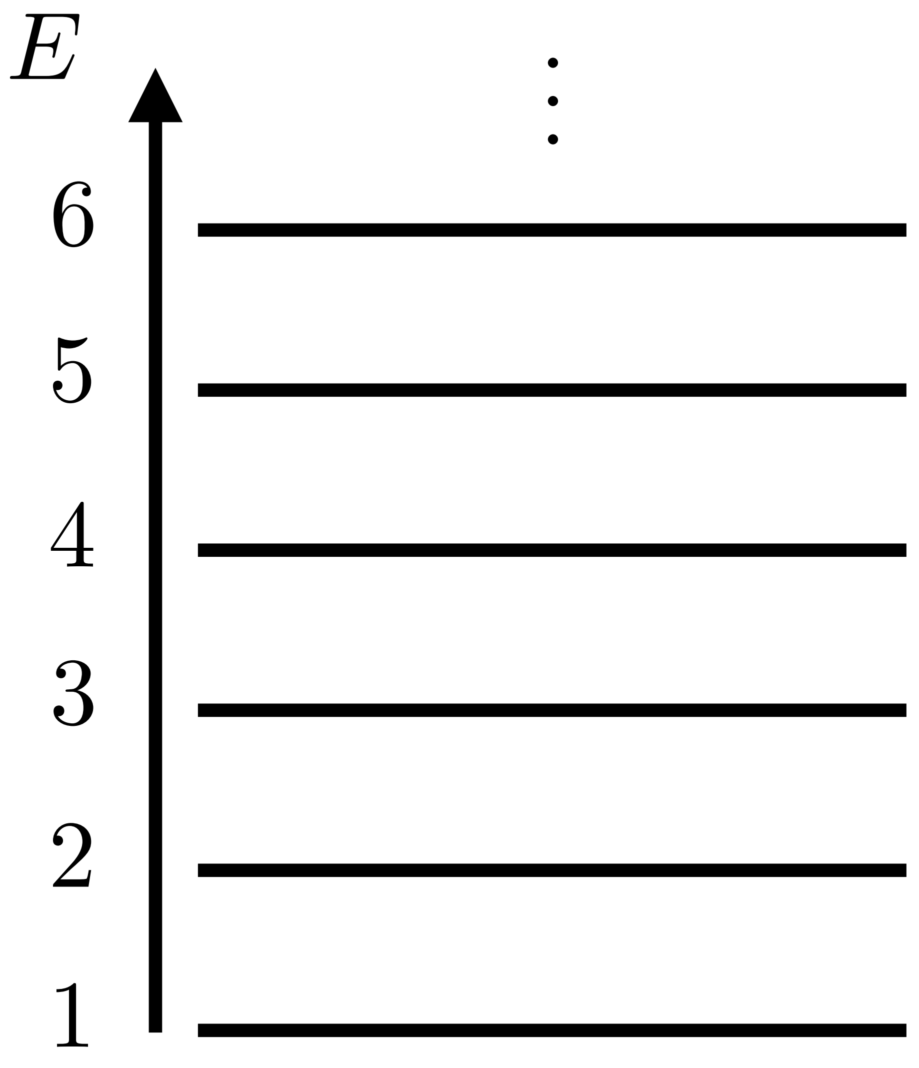
$$q = e^{\frac{1}{k_B T}}$$

$$Z(q) = \prod_n \frac{1}{1 - q^n}$$

e.g. how many configurations have energy = 4?



Partition function for a system with some energy levels



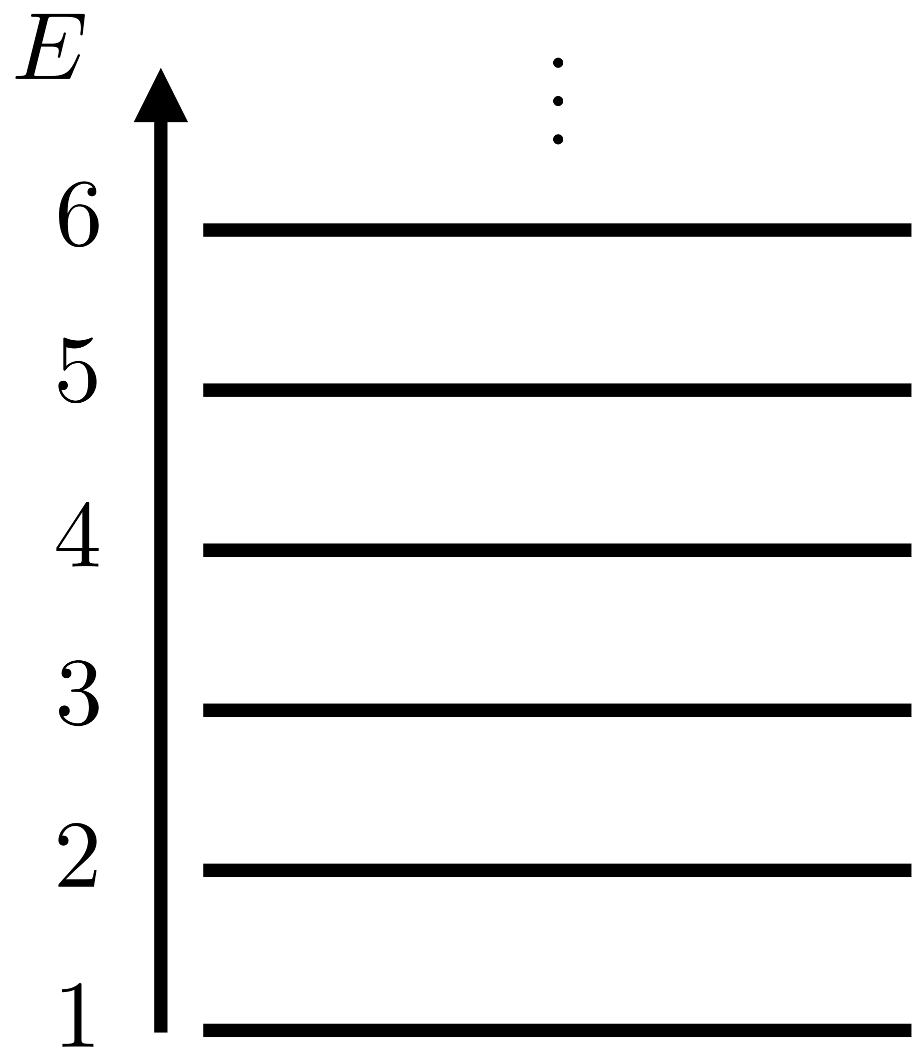
$$q = e^{\frac{1}{k_B T}}$$

$$Z(q) = \prod_n \frac{1}{1 - q^n}$$

$$= \sum_n p(n)\, q^n$$

Integer partitions of n

Partition function for a system with some energy levels



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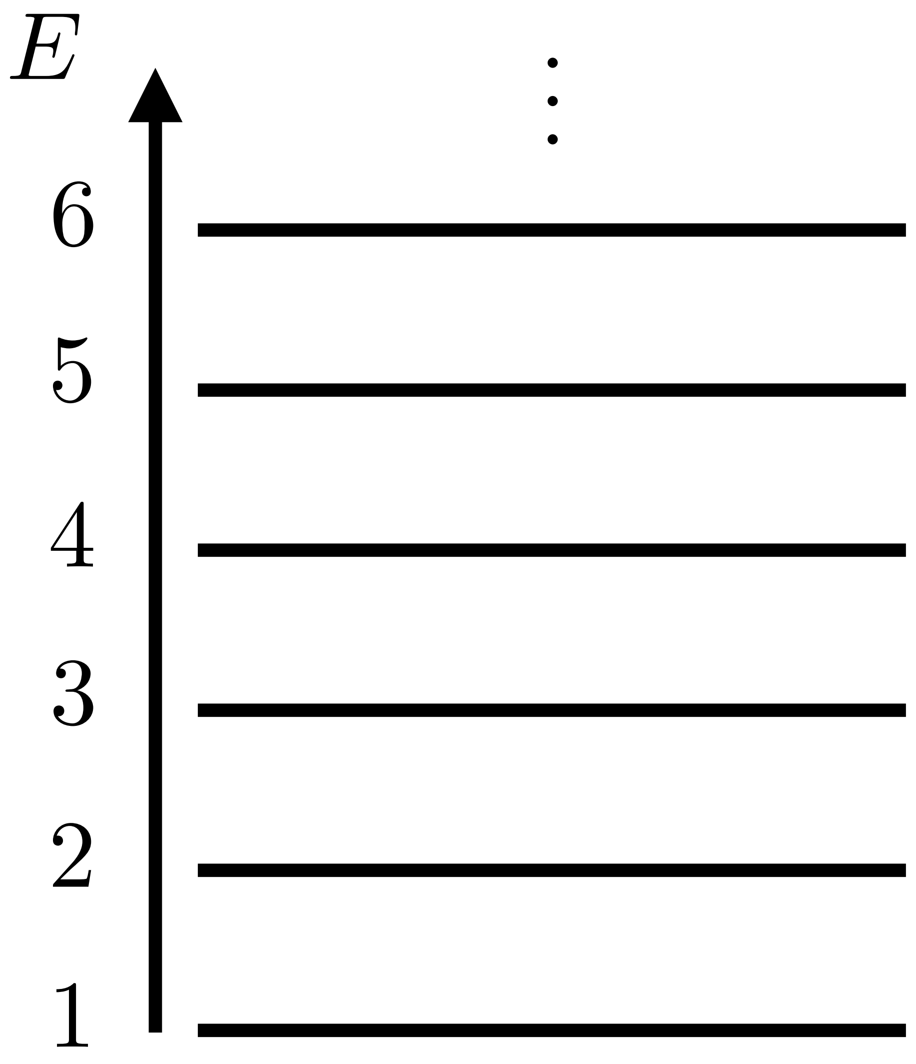
Integer partitions of n

Hardy-Ramanujan (1918)

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

Partition function for a system with some energy levels



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$$Z(q) = \prod_n \frac{1}{1 - q^n}$$

$$= \sum_n p(n)\, q^n$$

Integer partitions of n

"High temperature behaviour"

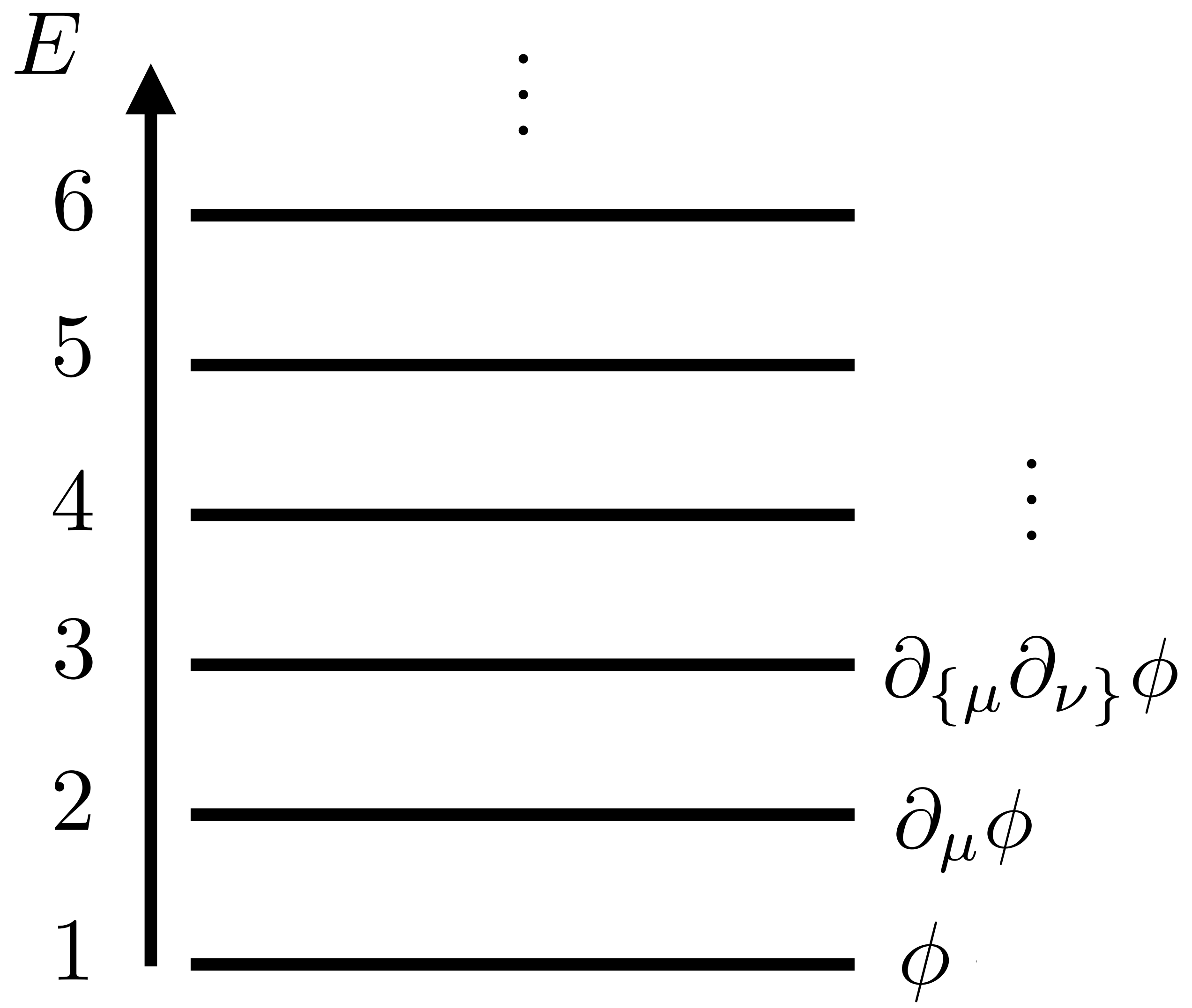
$q = e^{\frac{1}{k_B T}} = e^{\beta}$

$\beta \to 0, \quad q \to 1$

Hardy-Ramanujan (1918)

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

Partition function for a free scalar field



Labelled appealing to state operator correspondence

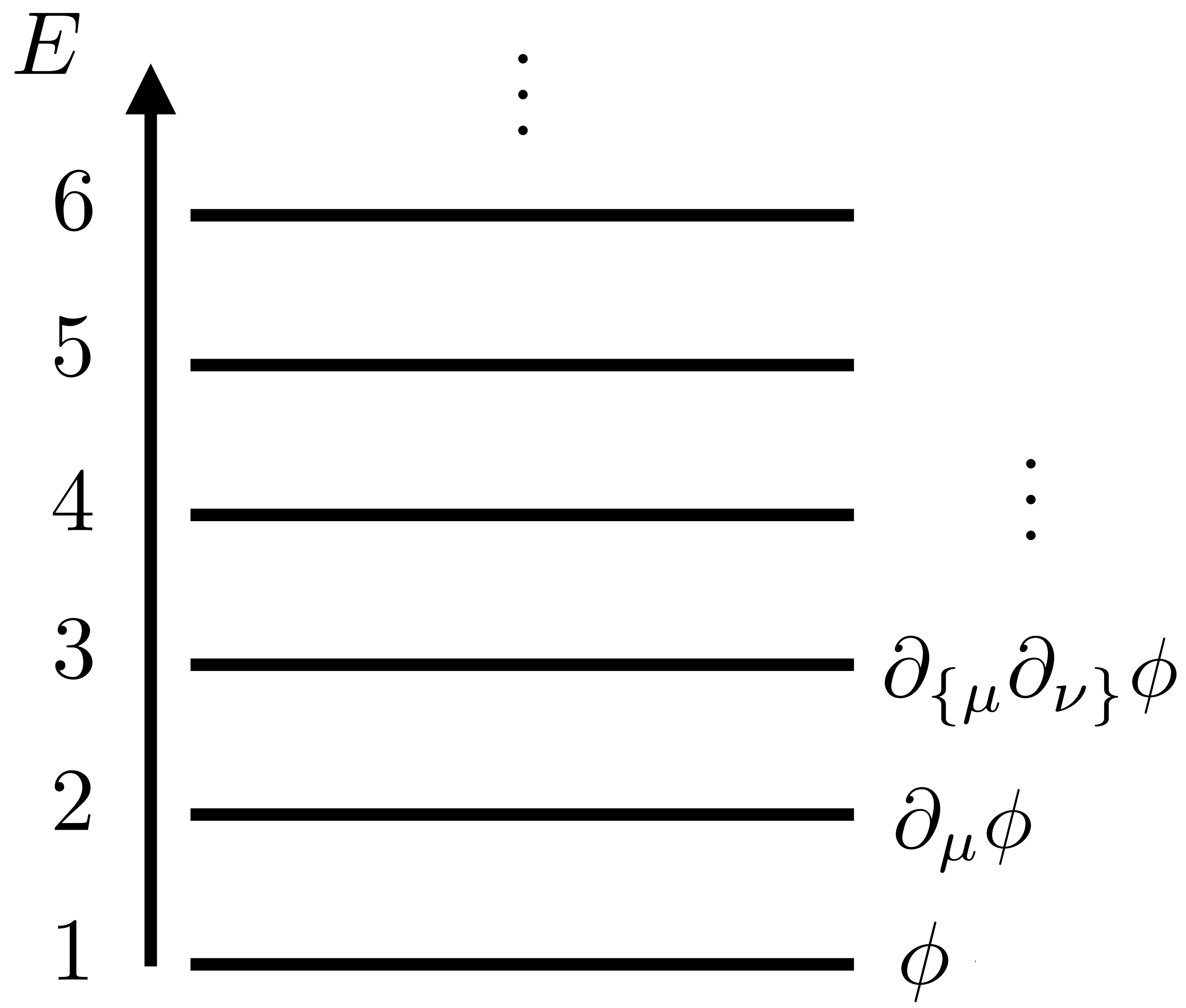
$q = e^{\frac{1}{k_B T}}$

$$Z(q) = \prod_n \frac{1}{(1-q^n)^{d_n}}$$

$$= \sum_\Delta c_\Delta \, q^\Delta$$

Level degeneracy from d derivatives in d-dimensional spacetime

Partition function for a free scalar field



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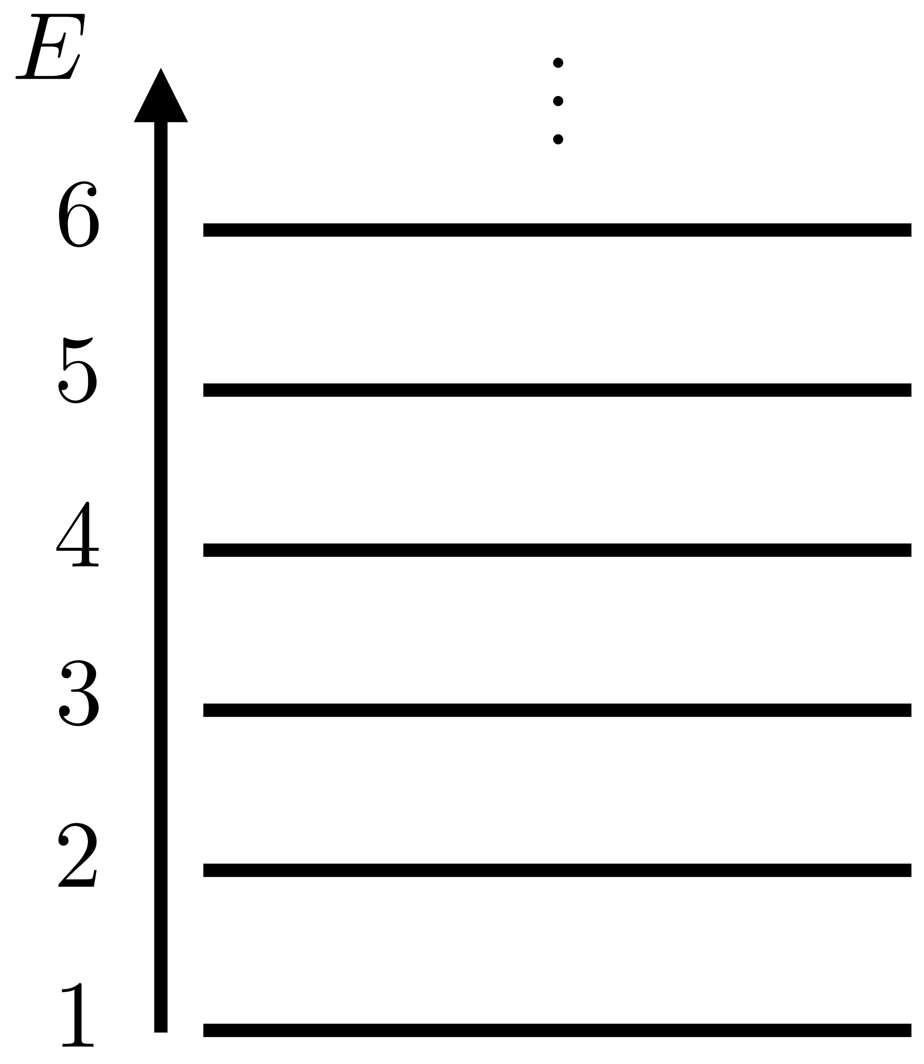
Level degeneracy from d derivatives in d-dimensional spacetime

$$= \sum_\Delta c_\Delta \, q^\Delta$$

This counts the number of operators at some given mass dimension. Essentially a Hilbert series (which counts only a subset which are the Lorentz scalars)

Partition function for a free scalar field



e.g. SMEFT fields

$q = e^{\frac{1}{k_B T}}$

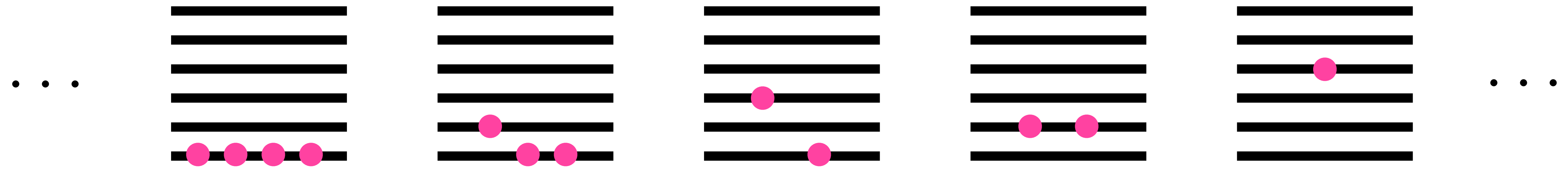
$$Z(q) = \prod_n \frac{1}{(1-q^n)^{d_n}}$$

Level degeneracy from d derivatives in d-dimensional spacetime

(After the projection to scalars)

$$= \ldots + 2\,q^5 + 84\,q^6 + 30\,q^7 + 993\,q^8 + \ldots$$

Dim 5 **Dim 6** **Dim 7** **Dim 8**



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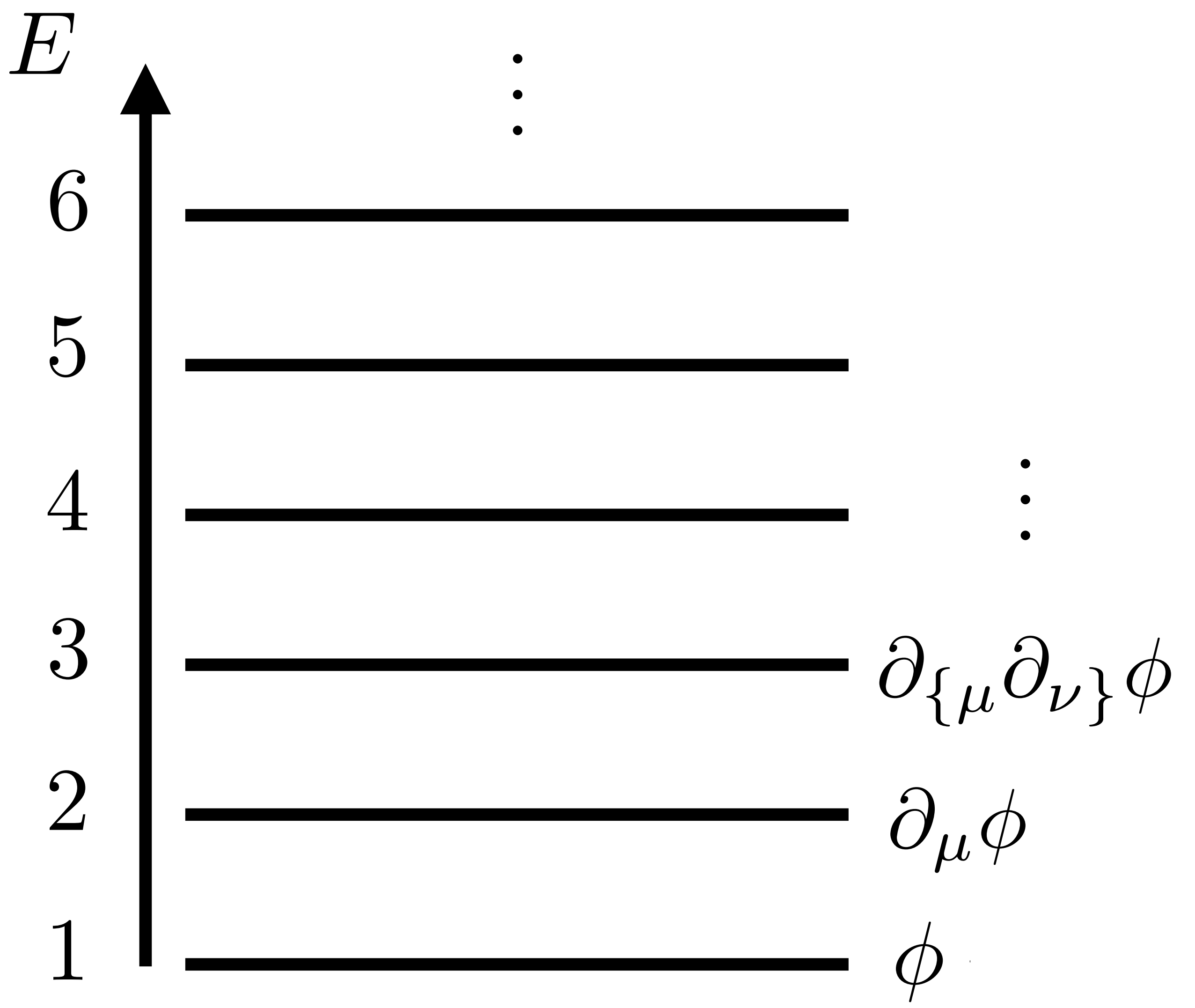

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Level degeneracy from d derivatives in d-dimensional spacetime

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"High temperature behaviour"

$$q = e^{\frac{1}{k_B T}} = e^\beta$$

$$\beta \to 0, \quad q \to 1$$

Cardy '91

Leading asymptotic behaviour of c for Hardy-Ramanujan-eqsue formula for the real scalar field

Why growth?

Analytic probes of the S-matrix

S-matrix theory of the 60s

On-shell, unitarity, locality etc. ideas in modern amplitudes

Many reviews, e.g. Ellis, Kunszt, Melnikov, Zanderighi '11

…these methods in EFT; soft theorems in EFT

E.g. Cohen, Elvang, Kiermaier '10; McGady, Rodina '14; Cheung, Kampf, Novotny, Shen, Trnka '15,'16; Caron-Huot, Wilhelm '16; Miro, Ingoldby, Riembau '20; Baratella, Fernandez, Pomarol '20…

Positivity constraints

E.g. Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi '06; Nicolis, Rattazzi, Trincherini '09; Bellazzini, Cheung, Remmen '15; de Rham, Melville, Tolley, Zhou '17; Remmen, Rodd '19,'20; Bellazzini, Miro, Rattazzi, Riembau, Riva '20 …

Large charge expansion

Hellerman, Orlando, Reffert, Watanabe '15; Monin, Pirtskhalava, Rattazzi, Seibold '16; Alvarez-Gaume, Loukas, Orlando, Reffert '17; Badel, Cuomo, Monin, Rattazzi '19, '20 …

Recent S-matrix bootstrap

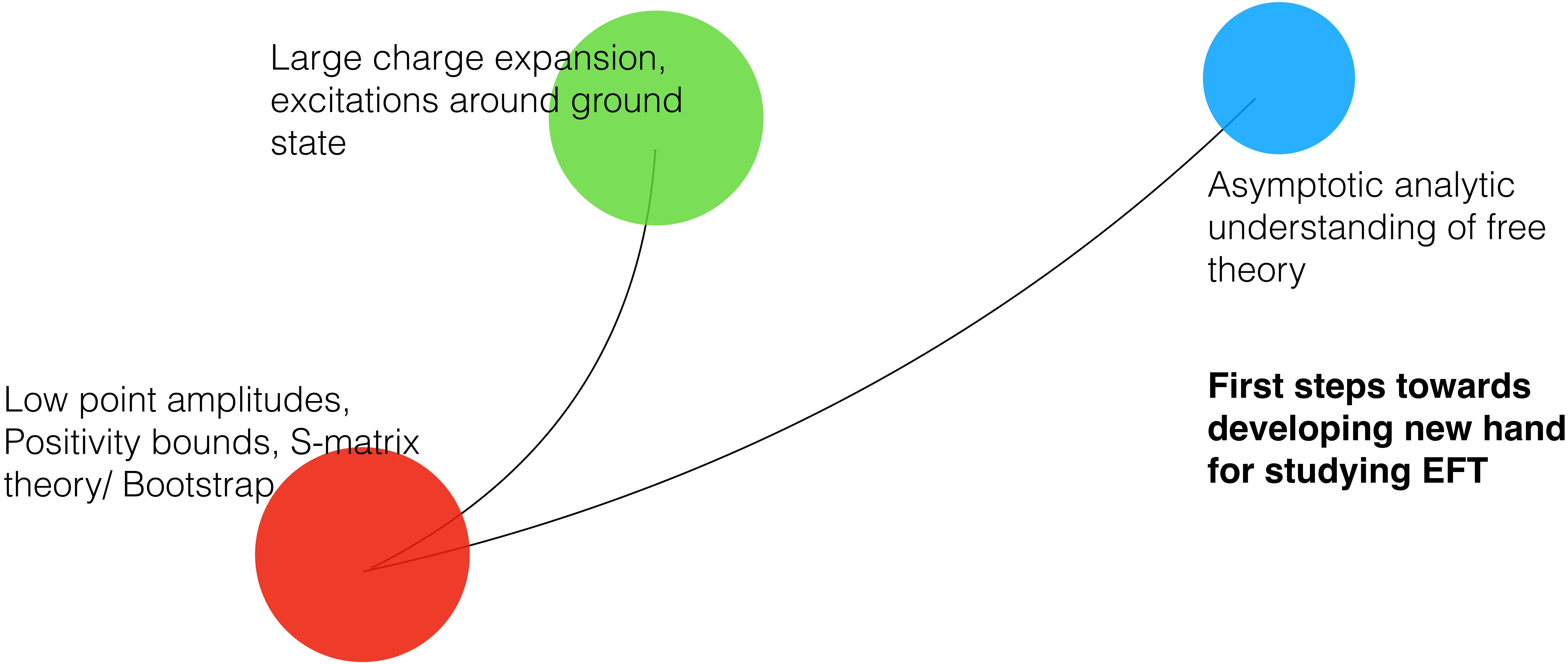
M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees, and P. Vieira, '16, '17

L. Cordova and P. Vieira, '18; D. Mazac and M. Paulos '18,'19; Cordova, He, Kruczenski, Vieira, '19; Karateev, Kuhn, Penedones '19; Correia, Sever, Zhiboedov, '20; Homrich, Penedones, Toledo, van Rees, Vieira, '20 …

Asymptotics is a study of the high temperature behaviour of a theory, in particular the entropy of high energy states.

Famous results in d=2, Cardy '86 ; much less known d>2 (leading behaviour of scalar theories Cardy '91)

Why growth?



First steps towards developing new handle for studying EFT

Main results

TM, Pal 2010.08560

Asymptotic formula for arbitrary d, arbitrary spin particles

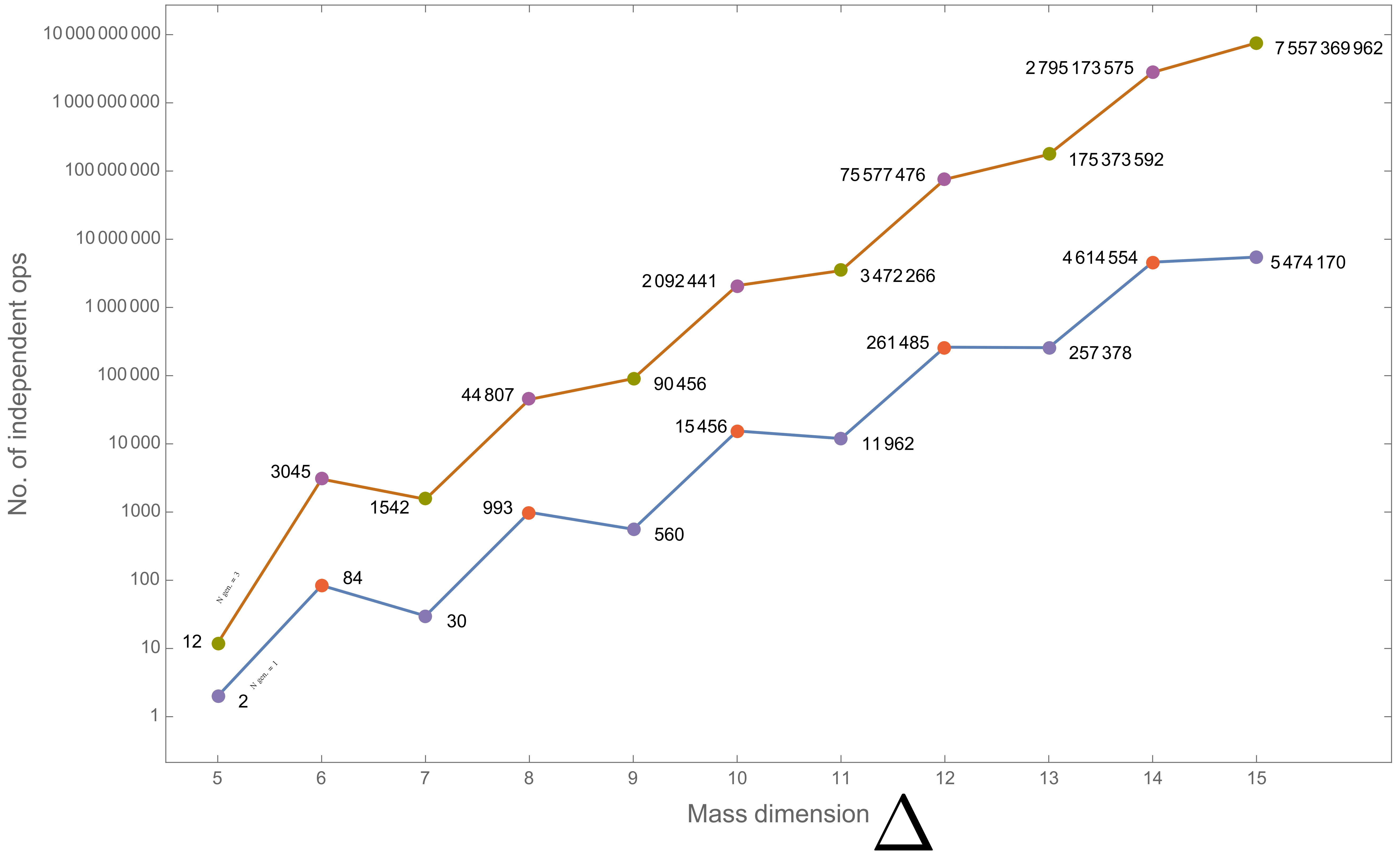
Also projections to subsets (Lorentz, internal symmetry singlets)

Procedure to fully capture sub-leading terms

E.g. the application to the SM EFT

TM, Pal 2010.08560

We generalised a theorem of Meinardus (1953) to obtain all sub-leading terms



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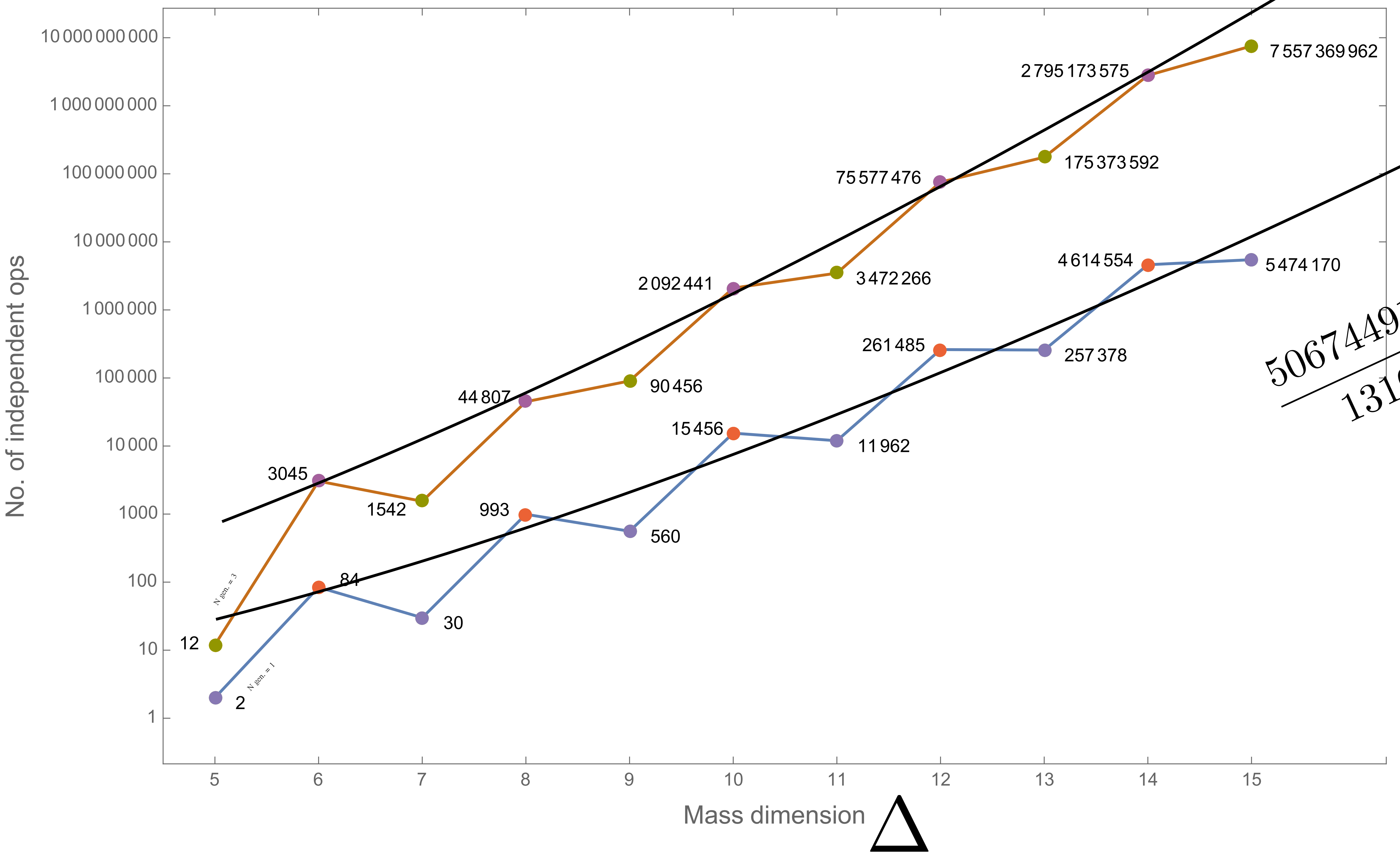
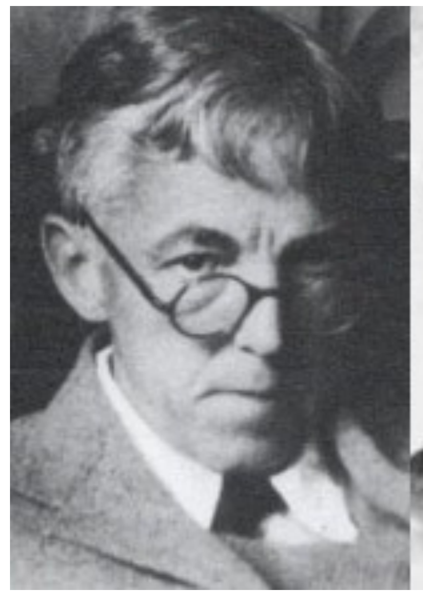


$$\frac{50674491\, 3^{5/8} \left(\frac{31}{5}\right)^{3/8} 7^{7/8} \pi^{10}}{131072000 \sqrt[4]{2}\sqrt{13}\,\Delta^{55/8}} \exp\left(\frac{2}{3}\sqrt{2}\sqrt[4]{\frac{217}{15}}\,\pi\Delta^{3/4} - \frac{37\sqrt[4]{\frac{15}{217}}\,\pi}{4\sqrt{2}}\Delta^{1/4} + 28\zeta'(-2)\right)$$



Hardy-Ramanujan-esque formula for the "integer partitions of the standard model"

Where $\frac{50674491\ 3^{5/8}\left(\frac{31}{5}\right)^{3/8} 7^{7/8}\pi^{10}}{131072000\sqrt[4]{2}\sqrt{13}\Delta^{55/8}} \exp\left(\frac{2}{3}\sqrt{2}\sqrt[4]{\frac{217}{15}}\pi\Delta^{3/4} - \frac{37\sqrt[4]{\frac{15}{217}}\pi}{4\sqrt{2}}\Delta^{1/4} + 28\zeta'(-2)\right)$ comes from…

Suppression by Internal symmetry group G singlets

Suppression by projecting to Lorentz scalars

Suppression keeping only primary operators

$$H(\beta) \underset{\beta\to 0}{\simeq} \left[K\beta^{\frac{3}{2}\dim\,\mathfrak{g}}\right] \left[\frac{91125\beta^{15}}{32\pi^{13}}\left(\dim_B + \frac{7}{8}\dim_f\right)^{-3}\right] \left[\beta^4\right]$$
$$\times \exp\left[A\beta^{-3} + B\beta^{-1} + C\zeta'(-2) + D\log\left(\frac{\beta}{2\pi}\right) + \text{higher spin}{>}1\right],$$

where

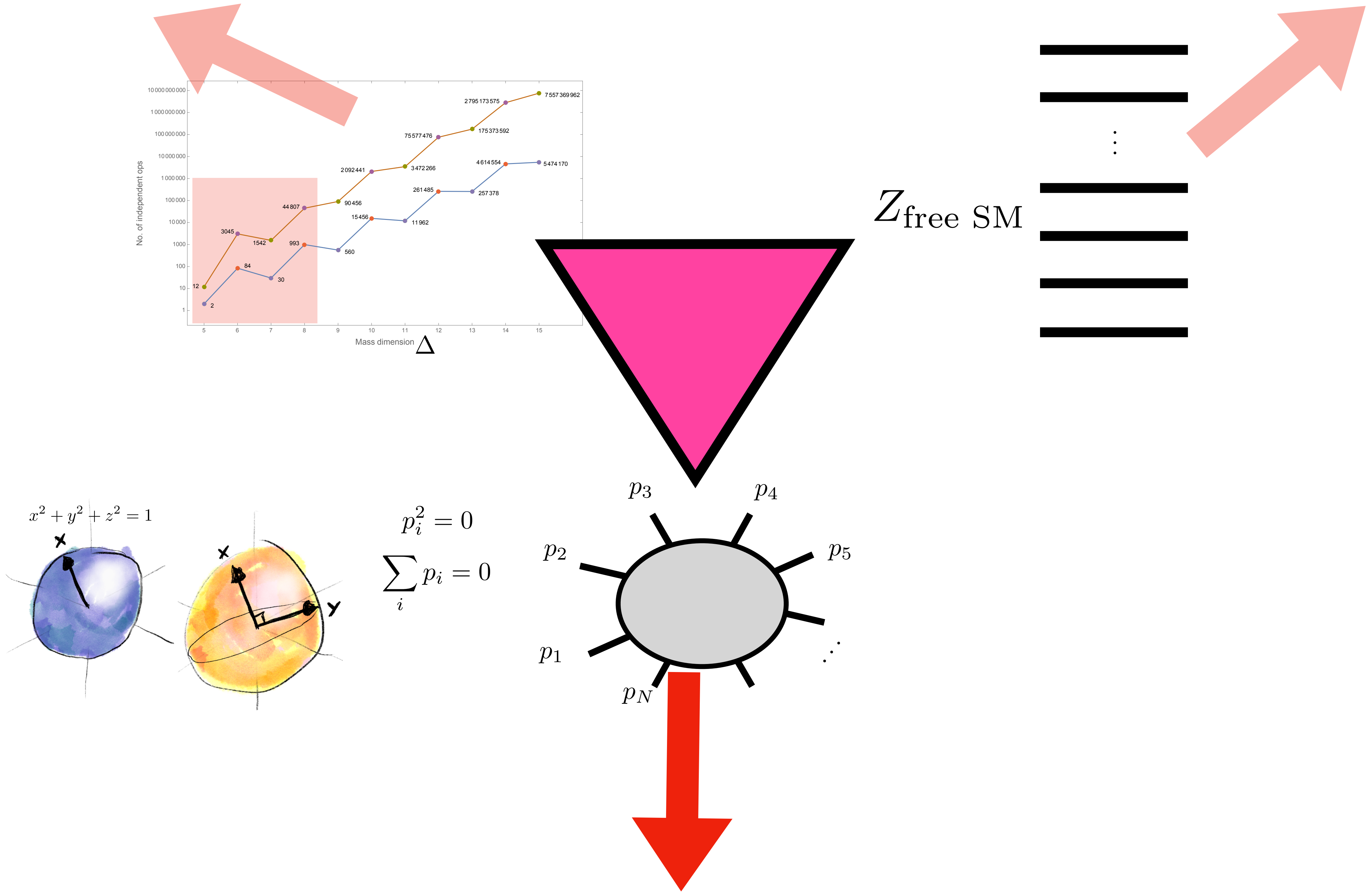
$$A = \left(\frac{\pi^4}{45}\dim_B + \frac{7\pi^4}{360}\dim_f\right),$$
$$B = -\left(\frac{\pi^2}{48}\dim_{1/2} + \frac{\pi^2}{6}\dim_1\right),$$
$$C = \dim_B,$$
$$D = -\frac{1}{2}\dim_1.$$

(& an inverse Laplace transform)

$$\rho(\Delta) \underset{\Delta\to\infty}{=} \frac{1}{2\pi i}\int_{\Gamma - i\infty}^{\Gamma + i\infty} d\beta\, Z(\beta\to 0)\, e^{\beta\Delta}$$

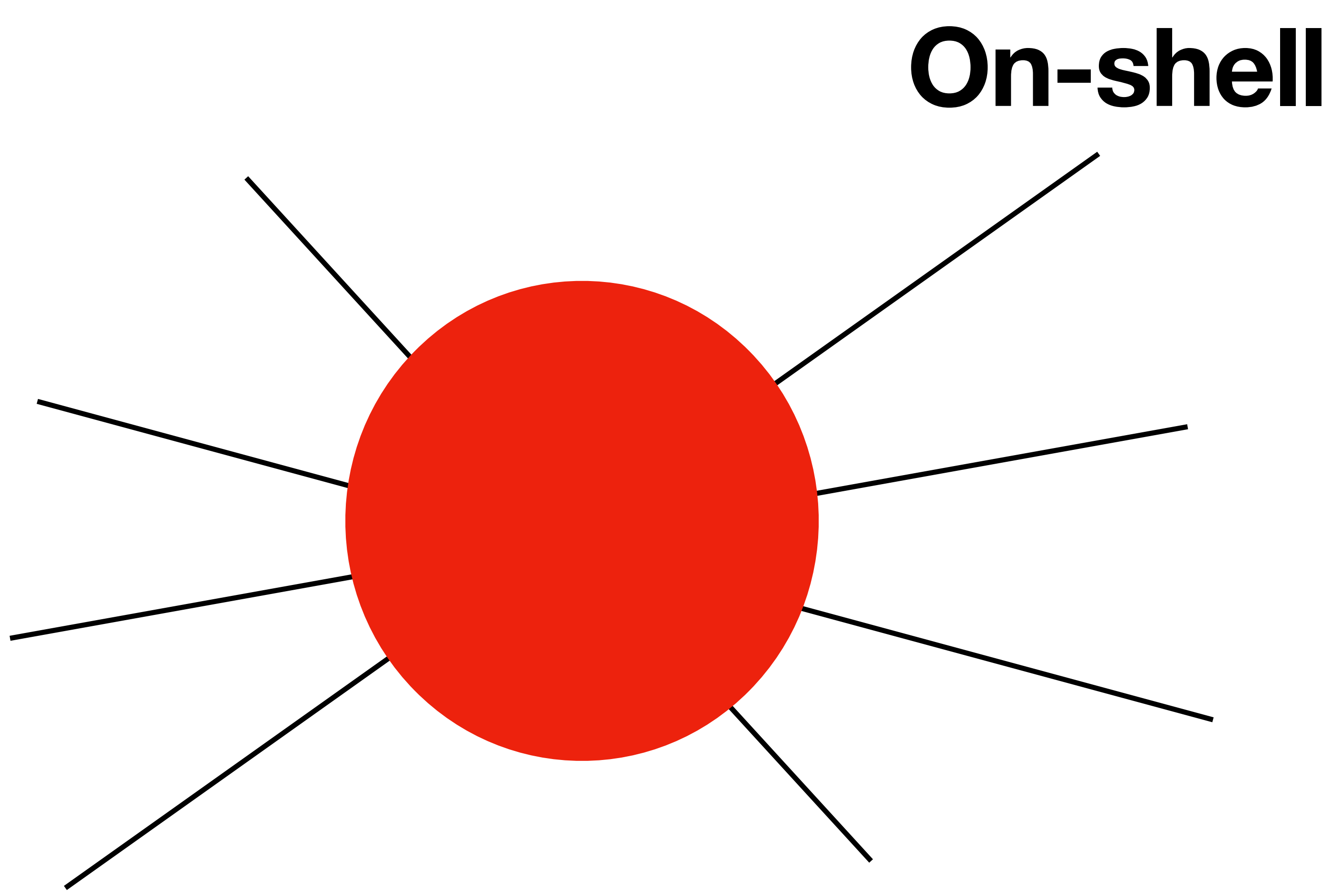
1. Asymptotic growth of the partitions of the Standard Model

3. Hilbert series for Chiral Lagrangians



2. Hilbert series for loop calculations in EFT

Hilbert series & polynomial rings for enumeration and construction of the elements of the S-matrix



One can work with a physical basis of operators & assured to capture all independent effects

(Or turn the logic around to find independent operators)

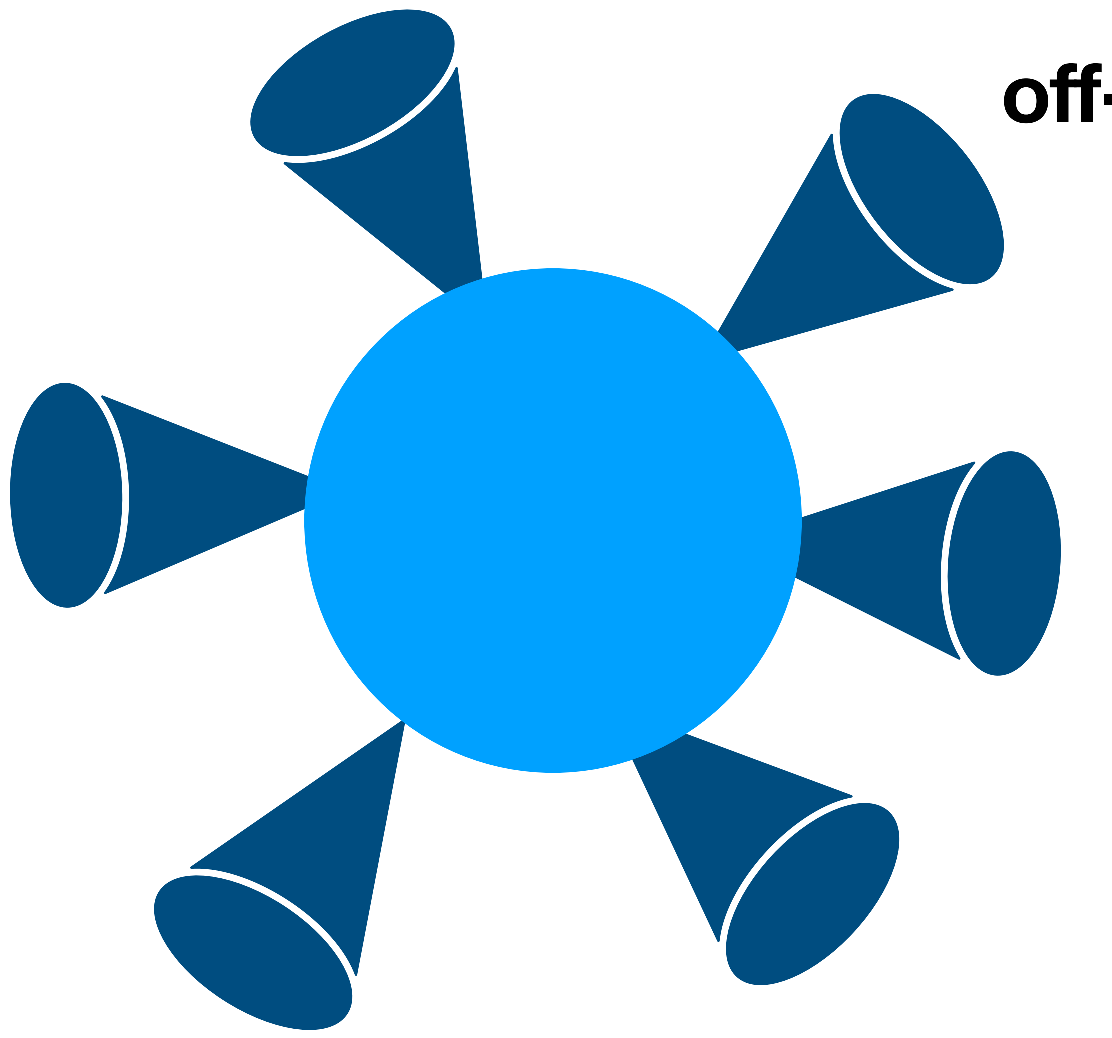
Construction (scalars)

$$R[\{p_i^2,\, p_i \cdot p_j\}]/\langle p_i^2 = 0,\; p_i \cdot \sum_j p_j \rangle \sim R[\{p_i \cdot p_j\}]/\langle p_i \cdot \sum_{j \neq i} p_j \rangle$$

On-shell Mom. con.

There are other scenarios where one relaxes the on-shell or momentum conservation requirements, typical for loops/renormalization

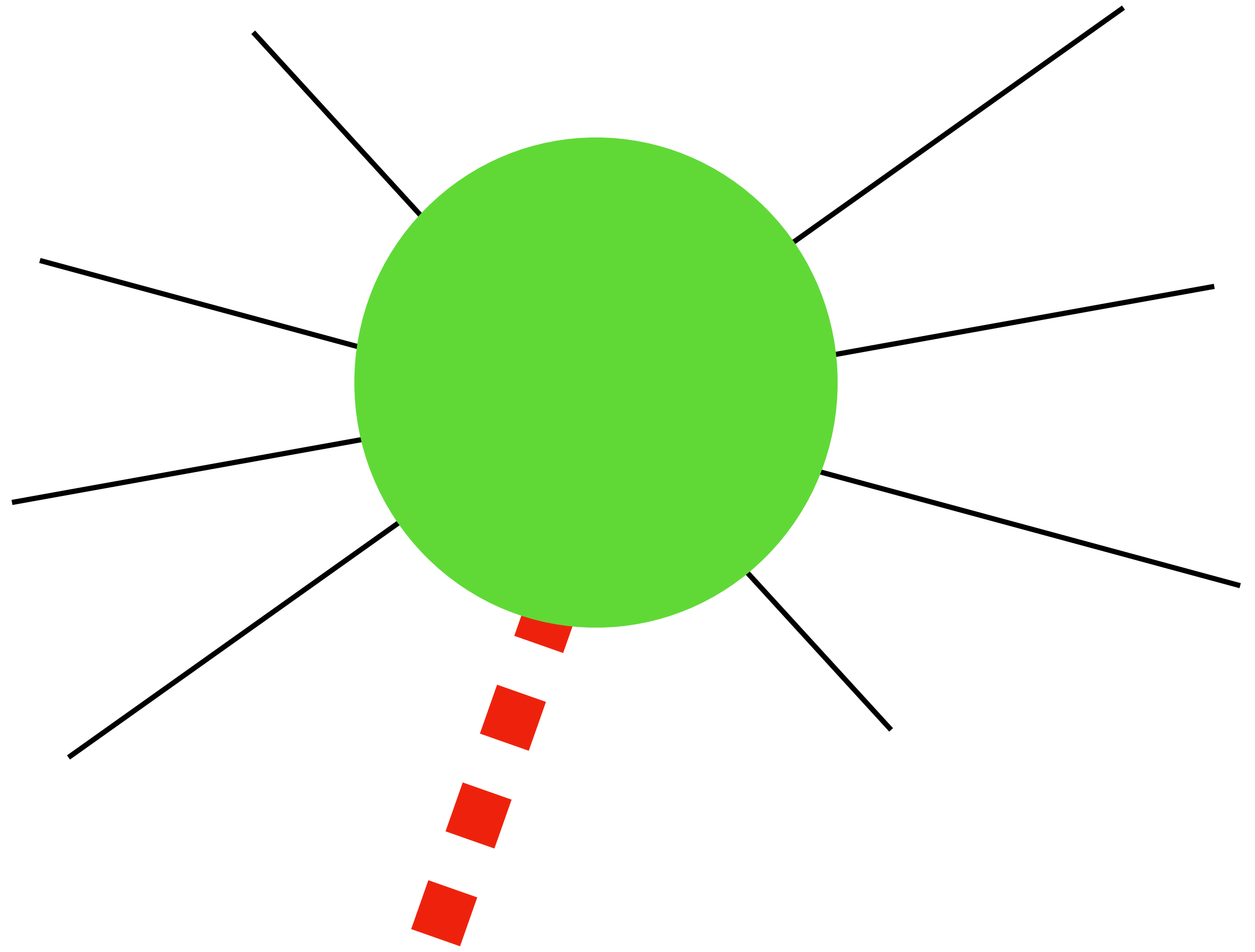
e.g. off-shell Green's function



$$R[\{p_i^2,\, p_i\cdot p_j\}]/\langle p_i \cdot \sum_j p_j\rangle \sim R[\{\, p_i \cdot p_j\}]$$

Mom. con.

e.g. form factor



$$P = \sum_i p_i$$

$$R[\{p_i^2,\, p_i\cdot p_j\}]/\langle p_i^2\rangle \sim R[\{\, p_i \cdot p_j\}]$$

On-shell

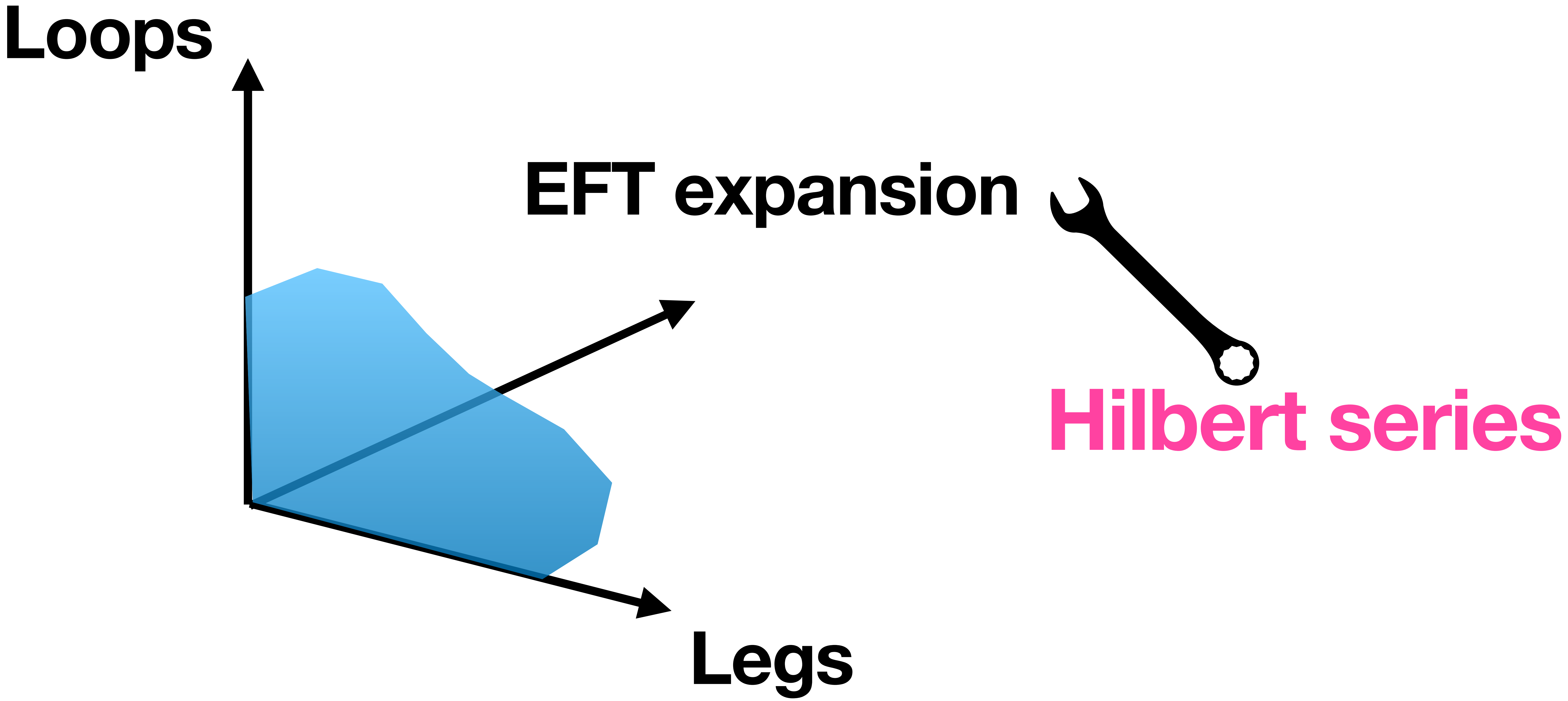
All the required knobs and switches were developed in "S-matrices, operator bases.." Henning, Lu, TM, Murayama 2017

On-shellness Mom. Cons. Parity Spacetime dim

Bottom line: all ideas for physical basis carry through to systematically enumerate, construct and relate bases useful for EFT loops

Hilbert series valuable in pushing in the “EFT” direction in precision calculations, along with “loops” and “legs”

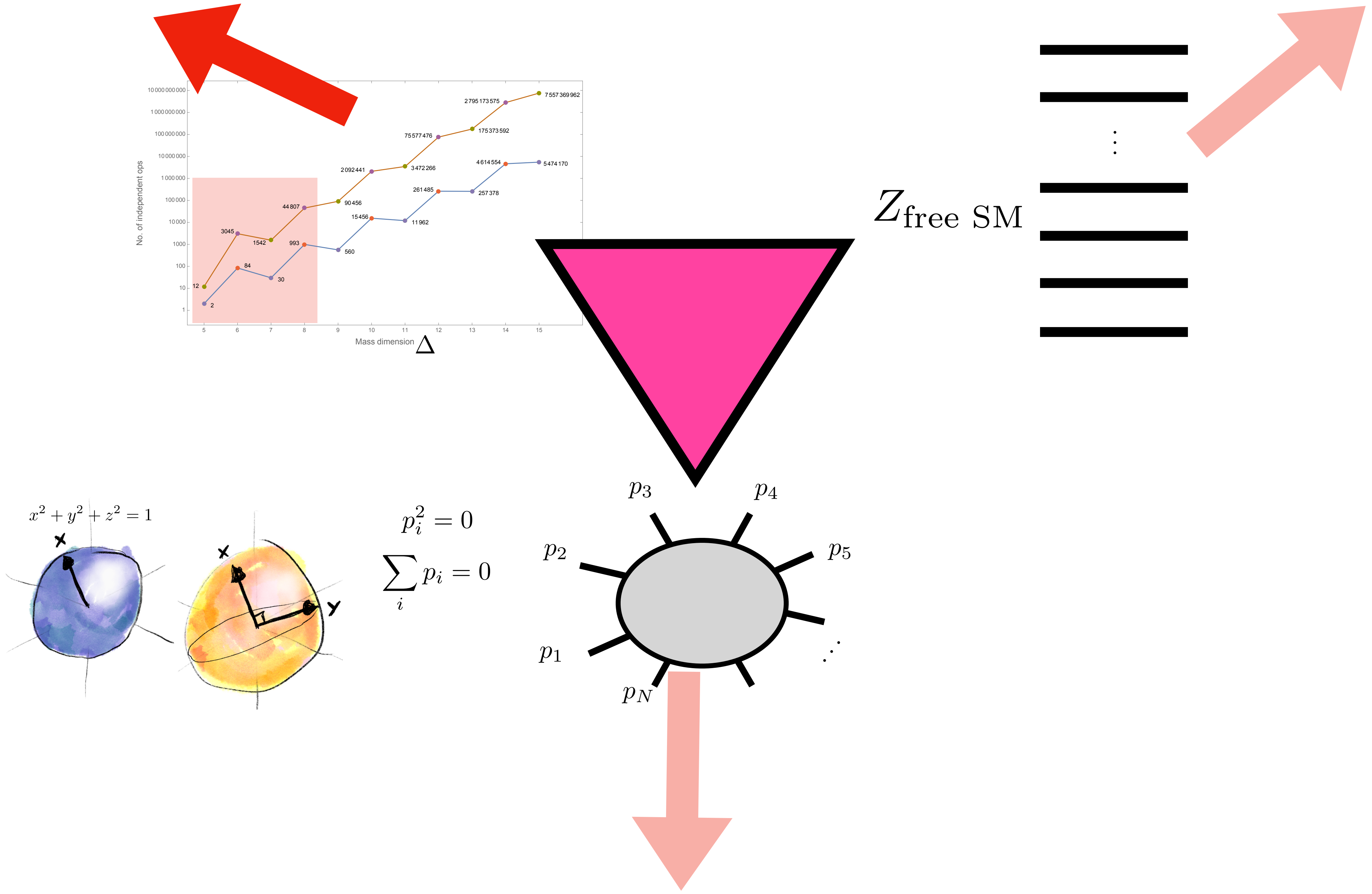
(See Jasper’s talk tomorrow; Cao, Herzog, TM, Roosmale-Nepveu, to appear)



Metric of difficulty

1. Asymptotic growth of the partitions of the Standard Model

3. Hilbert series for Chiral Lagrangians



2. Hilbert series for loop calculations in EFT

Systematic construction of nonlinear rep. Lagrangians

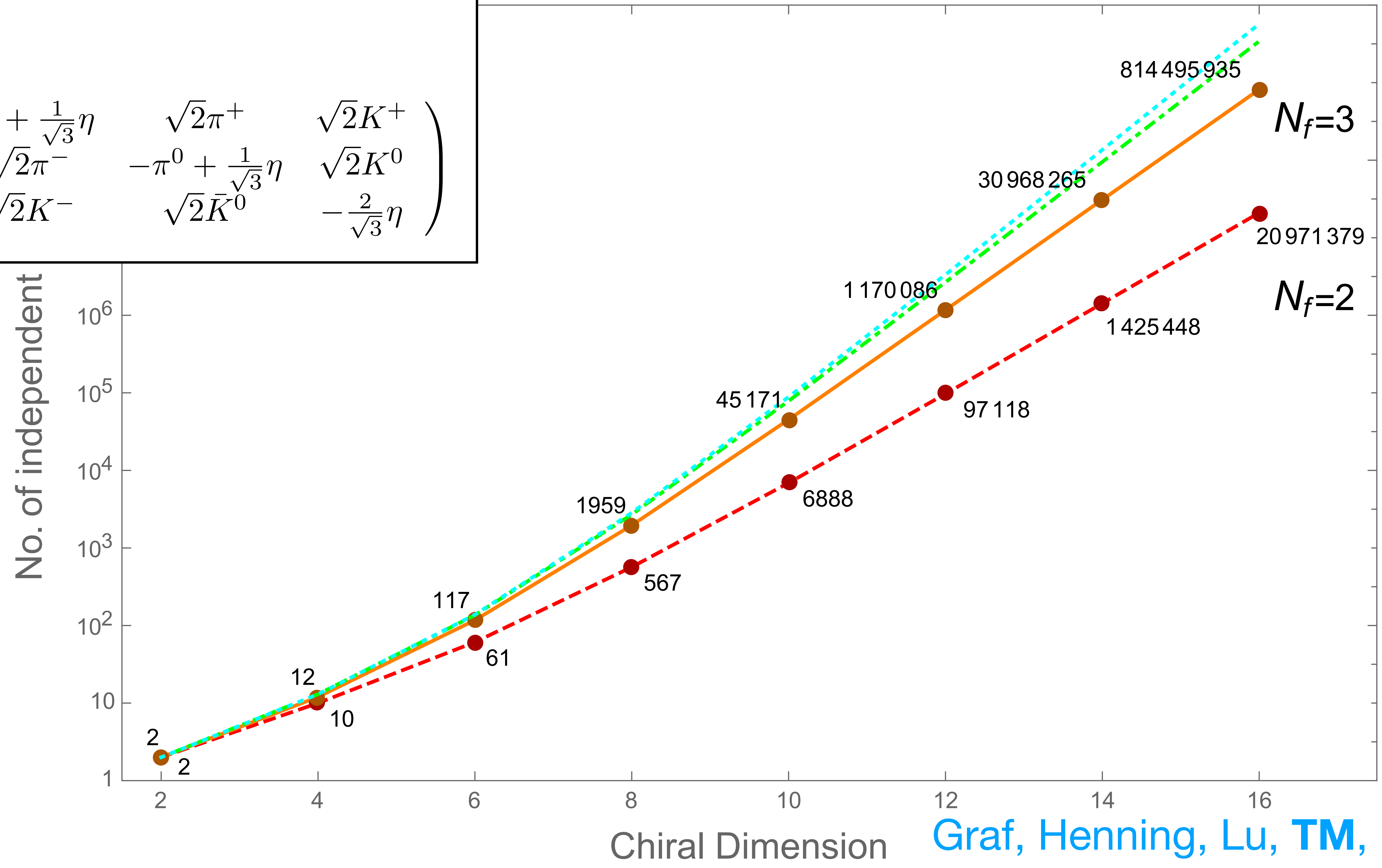
Given in “S-matrices, operator bases..” Henning, Lu, TM, Murayama

All bells and whistles (incl. external fields) new application to QCD Chiral Lagrangian

$$\mathcal{L} = \frac{F_0^2}{4} \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \ldots$$

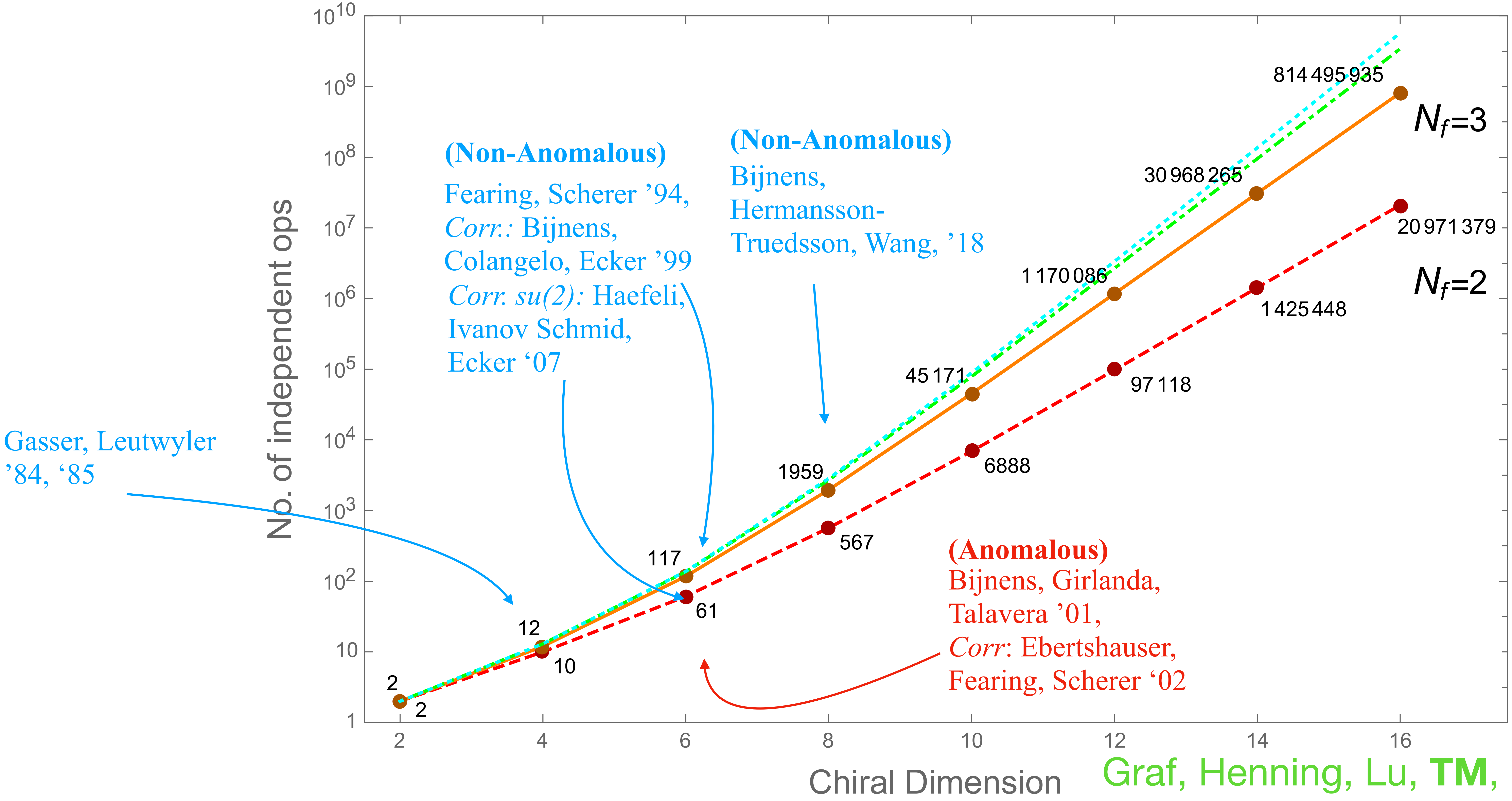
$$U = \exp\left(i\frac{\phi(x)}{F_0}\right)$$

$$\phi(x) = T^a \phi_a(x) = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$



Graf, Henning, Lu, **TM**, Murayama, 2020

QCD chiral Lagrangian



Graf, Henning, Lu, **TM**, Murayama, 2020

First identification of all p^8 so-called 'anomalous' terms

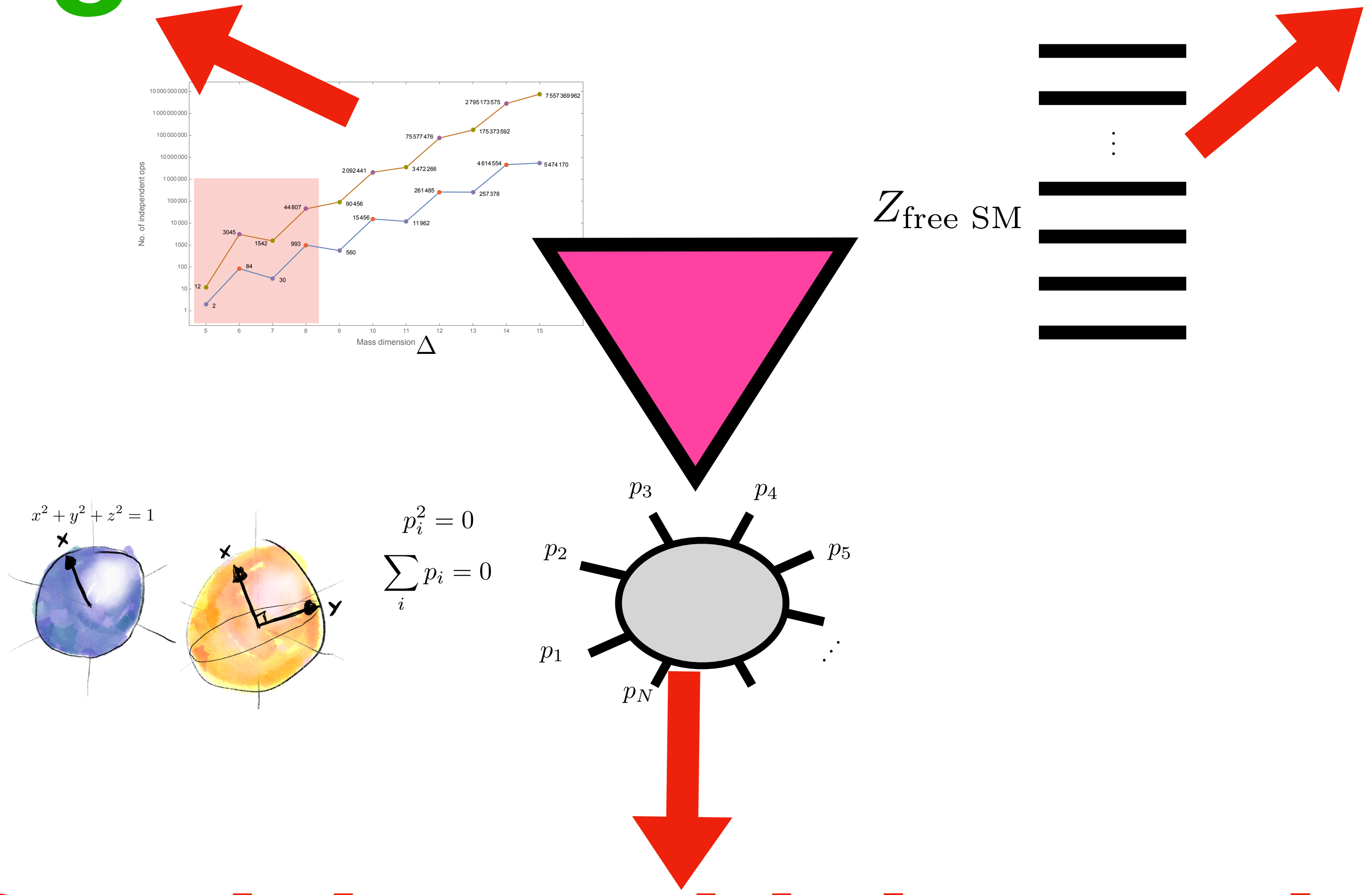
Developed systematic treatment of charge conjugation invariance, mirroring previous treatment of parity in Hilbert series

Many developing new directions

Ideas in the Hilbert series for EFT program are very general and can be widely utilised

Chiral Lagrangians

Asymptotics, analytic probe



Precision, with loops, legs

Thanks for listening!