Path integral

Functional Matching and STrEAM

Zhengkang Zhang (张正康)

zkzhang@caltech.edu

BURKE INSTITUTE

Caltech

Outline

1. Introduction.
2. Tree-level matching.
3. One-loop matching and STrEAM.
4. *Additional examples.

References: Timothy Cohen, Xiaochuan Lu, Zhengkang Zhang, *"Functional Prescription for EFT Matching,"* [2011.02484] + refs. therein; *"STrEAMlining EFT Matching,"* [2012.07851].

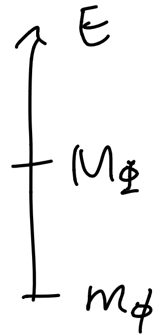
1. Introduction

Matching

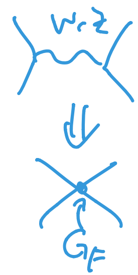
$$\mathcal{L}_{UV}[\Phi, \phi]$$

$$\Downarrow$$

$$\mathcal{L}_{EFT}[\phi] = ?$$



Examples



$$\mathrm{SM} + \Phi_{BSM}$$

$$\Downarrow$$

SMEFT / HEFT

RG

$$A \sim \alpha^{\#} \log^{\#} \frac{P}{\mu} \qquad P = \sqrt{s},\ \cancel{M_\Phi},\ m_\phi,\ \cdots$$

2. Tree-level matching

Path integral

$$Z_{UV} = \int \underline{\underline{D\Phi}}\, D\phi \; e^{i\int dx\, \mathcal{L}_{UV}[\Phi,\phi]}$$

"integrate out"

$$Z_{EFT} = \int D\phi \; e^{i\int dx\, \mathcal{L}_{EFT}[\phi]}$$

Toy model

$$\mathcal{L}_{UV}[\Phi, \phi] = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2$$
$$+ \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$
$$+ \frac{1}{2}\kappa\,\Phi\phi^2 + \frac{1}{4}\eta\,\Phi^2\phi^2$$

EOM solution

$$Z_{UV} = \int D\Phi\, D\phi\; e^{i\int dx\, \mathcal{L}_{UV}[\Phi,\phi]} \simeq \int D\phi\; e^{i\int dx\, \underbrace{\mathcal{L}_{UV}[\Phi_c[\phi],\phi]}_{\mathcal{L}_{EFT}^{(tree)}[\phi]}}$$

Dominated by stationary pt:

$$\frac{\delta \mathcal{L}_{UV}}{\delta \Phi} = 0 \Rightarrow \Phi_c[\phi] \quad \text{Classical EOM}$$

$$\mathcal{L}_{UV}[\Phi,\phi] = \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{1}{2}M^2\Phi^2 \qquad -\frac{1}{2}\Phi\,\partial^2\Phi$$

$$+\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

$$+\frac{1}{2}\kappa\,\Phi\phi^2 + \frac{1}{4}\eta\,\Phi^2\phi^2$$

$$\frac{\delta\mathcal{L}_{UV}}{\delta\Phi} = -\partial^2\Phi - M^2\Phi + \frac{1}{2}\kappa\phi^2 + \frac{1}{2}\eta\,\Phi\phi^2$$

$$= -\left(\partial^2 + M^2 - \frac{1}{2}\eta\phi^2\right)\Phi + \frac{1}{2}\kappa\phi^2 = 0$$

$$\Rightarrow \Phi_c[\phi] = \frac{1}{2}\kappa\,\frac{1}{\partial^2 + M^2 - \frac{1}{2}\eta\phi^2}\,\phi^2 \qquad \left(\text{expand in } \frac{1}{M^2}\right)$$

$$\mathcal{L}^{(\text{tree})}_{\text{EFT}}[\phi] = \mathcal{L}_{UV}\left[\Phi_c[\phi], \phi\right] \qquad \Phi_c[\phi] = \frac{1}{2}\kappa \frac{1}{\partial^2 + M^2 - \frac{1}{2}\eta\phi^2}\phi^2$$

$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

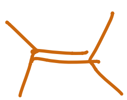
$$- \frac{1}{2}\Phi_c\left(\partial^2 + M^2 - \frac{1}{2}\eta\phi^2\right)\Phi_c + \frac{1}{2}\kappa\Phi_c\phi^2$$

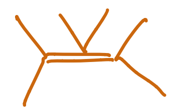
$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

$$+ \frac{1}{8}\kappa^2\phi^2 \underbrace{\frac{1}{\partial^2 + M^2 - \frac{1}{2}\eta\phi^2}}_{\longrightarrow\ \frac{1}{M^2} + \frac{1}{M^4}\left(\partial^2 - \frac{1}{2}\eta\phi^2\right) + \cdots}\phi^2$$

$$\lambda_{\text{eff}} = \lambda - \frac{3\kappa^2}{M^2}$$

$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$





$$+ \frac{\kappa^2}{8M^2}\phi^4 + \frac{\kappa^2}{8M^4}\left(\phi^2\partial^2\phi^2 - \frac{1}{2}\eta\phi^6\right) + \cdots$$

(threshold correction) dim-4 dim-6

3. One-loop Matching

$$\mathcal{L}_{UV}[\Phi,\phi] \simeq \mathcal{L}_{UV}[\Phi_c,\phi] + \underbrace{\cancel{\frac{\delta \mathcal{L}_{UV}}{\delta \Phi}}\Big|_{\Phi_c}}_{0} \underbrace{\Phi'}_{\Phi-\Phi_c} + \frac{1}{2}\frac{\delta^2 \mathcal{L}_{UV}}{\delta \Phi^2}\Big|_{\Phi_c} \Phi'^2 + \cdots$$

$$Z_{UV} = \int D\Phi' D\phi \; e^{i\int dx \left[\mathcal{L}_{UV}[\Phi_c] + \frac{1}{2}\frac{\delta^2 \mathcal{L}_{UV}}{\delta \Phi^2}\big|_{\Phi_c} \Phi'^2 + \cdots \right]}$$

$$\simeq \int D\phi \; e^{i\int dx\, \mathcal{L}_{UV}[\Phi_c]} \int D\Phi' \; e^{i\int dx \frac{1}{2}\frac{\delta^2 \mathcal{L}_{UV}}{\delta \Phi^2}\big|_{\Phi_c} \Phi'^2}$$

$$= \int D\phi \; e^{i\int dx\, \mathcal{L}_{UV}[\Phi_c]} \left[\det\left(- \frac{\delta^2 \mathcal{L}_{UV}}{\delta \Phi^2}\Big|_{\Phi_c} \right) \right]^{-\frac{1}{2}}$$

$$Z_{EFT} \simeq \int D\phi \; e^{i\int dx\, \mathcal{L}_{EFT}^{(tree)}} \; e^{i\int dx\, \mathcal{L}_{EFT}^{(1\text{-}loop)}}$$

(later)

$$\Rightarrow \int dx\, \mathcal{L}_{EFT}^{(1\text{-}loop)} = \frac{i}{2} \log \det \left(- \frac{\delta^2 \mathcal{L}_{UV}}{\delta \Phi^2}\Big|_{\Phi_c} \right)$$

Toy model

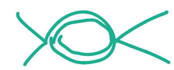
$$\mathcal{L}_{UV}[\Phi,\phi] = \underbrace{\tfrac{1}{2}(\partial_\mu \Phi)^2}_{} - \tfrac{1}{2}M^2\Phi^2 \qquad -\tfrac{1}{2}\Phi\partial^2\Phi$$

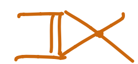
$$+ \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4$$

$$+ \tfrac{1}{2}\kappa\Phi\phi^2 + \tfrac{1}{4}\eta\Phi^2\phi^2$$

$$\frac{\delta^2 \mathcal{L}_{UV}}{\delta\Phi^2} = -\partial^2 - M^2 + \tfrac{1}{2}\eta\phi^2$$

$n=1$: 

$n=2$: 

 ?

$$\Rightarrow \int dx\, \mathcal{L}_{EFT}^{(1\text{-loop})} = \tfrac{i}{2}\log\det\left(\partial^2 + M^2 - \tfrac{1}{2}\eta\phi^2\right)$$

$$= \tfrac{i}{2}\,\mathrm{Tr}\log\left(\partial^2 + M^2 - \tfrac{1}{2}\eta\phi^2\right)$$

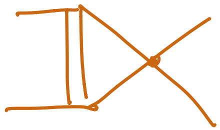
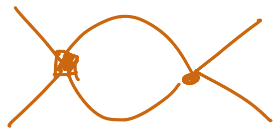
$$= \tfrac{i}{2}\,\mathrm{Tr}\left\{\log(\partial^2+M^2) + \log\left[1 - \frac{1}{\partial^2+M^2}\left(\tfrac{1}{2}\eta\phi^2\right)\right]\right\}$$

$$= \tfrac{i}{2}\underbrace{\mathrm{Tr}\log(\partial^2+M^2)} - \tfrac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}\underbrace{\mathrm{Tr}\left[\frac{1}{\partial^2+M^2}\left(\tfrac{1}{2}\eta\phi^2\right)\right]^n}$$

Functional traces $\rightarrow$ CDE (STrEAM)

(covariant derivative exp.)

Method of regions

 $\neq$ 

integrate → expand　　　　expand → integrate

LHS: $$\int dq \frac{1}{(q^2-M^2)(q^2-m^2)^2} = \frac{i}{16\pi^2}\left[\frac{1}{M^2-m^2}\left(1-\log\frac{M^2}{m^2}\right) - \frac{m^2}{(M^2-m^2)^2}\log\frac{M^2}{m^2}\right]$$

$$= \frac{i}{16\pi^2}\,\frac{1}{M^2}\left(1-\log\frac{M^2}{m^2}\right) + O\left(\frac{1}{M^4}\right)$$

$\frac{d^dq}{(2\pi)^d}$　　$d = 4-\epsilon$

RHS: $$\int dq \left(-\frac{1}{M^2}+\dots\right)\frac{1}{(q^2-m^2)^2} = \frac{i}{16\pi^2}\,\frac{1}{M^2}\left(-\frac{2}{\bar{\epsilon}} + \log\frac{m^2}{\mu^2}\right) + O\left(\frac{1}{M^4}\right)$$

$$\frac{2}{\bar{\epsilon}} \equiv \frac{2}{\epsilon} - \gamma + \log 4\pi$$

Method of regions: $\int dq = \int dq \Big|_{\text{hard}} + \int dq \Big|_{\text{soft}}$

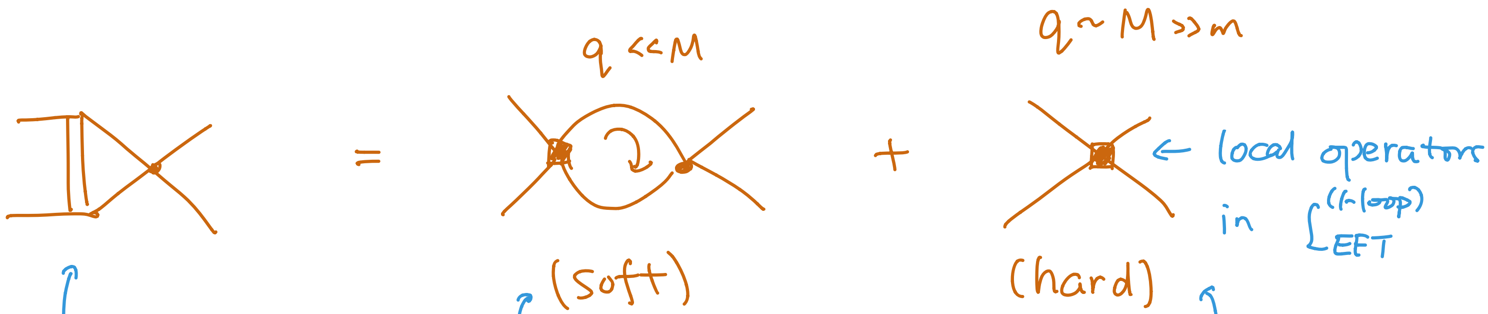
Hard: Expand for $q \sim M \gg m$, then integrate

Soft: " " $q \sim m \ll M$. " "

$$\int dq \frac{1}{(q^2-M^2)(q^2-m^2)^2} = \frac{i}{16\pi^2}\frac{1}{M^2}\left(1-\log\frac{M^2}{m^2}\right) + O\left(\frac{1}{M^4}\right)$$

Soft: $\displaystyle \int dq \left(-\frac{1}{M^2}+\cdots\right)\frac{1}{(q^2-m^2)^2} = \frac{i}{16\pi^2}\frac{1}{M^2}\left(-\frac{2}{\epsilon}+\log\frac{m^2}{\mu^2}\right) + O\left(\frac{1}{M^4}\right)$

Hard: $\displaystyle \int dq \frac{1}{q^2-M^2}\left(\frac{1}{q^4}+\cdots\right) = \frac{i}{16\pi^2}\frac{1}{M^2}\left(\frac{2}{\epsilon}+1-\log\frac{M^2}{\mu^2}\right) + O\left(\frac{1}{M^4}\right)$



$[A_{UV}^{(1\text{-loop})}]$ $\quad$ $[\mathcal{L}_{EFT}^{(\text{tree})}$ used in $A_{EFT}^{(1\text{-loop})}]$ $\quad$ $[\mathcal{L}_{EFT}^{(1\text{-loop})}$ used in $A_{EFT}^{(\text{tree})}]$

$$\boxed{\int dx\, \mathcal{L}_{EFT}^{(1\text{-loop})} = \frac{i}{2} \log \det \left(- \frac{\delta^2 \mathcal{L}_{UV}}{\delta \varphi^2} \Big|_{\Phi_c} \right) \Big|_{\text{hard}}}$$

$\varphi = \{\Phi, \phi\}$

Proved using 1PI effective action

(Legendre transform of the path integral).

[1610.00710]

Toy model

$$\mathcal{L}_{uv}[\Phi,\phi] = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \frac{1}{2}\kappa\Phi\phi^2 + \frac{1}{4}\eta\Phi^2\phi^2$$

$$\left.\frac{\delta^2\mathcal{L}_{uv}}{\delta\varphi^2}\right|_{\Phi_c} = \left.\begin{pmatrix} \frac{\delta^2\mathcal{L}_{uv}}{\delta\Phi^2} & \frac{\delta^2\mathcal{L}_{uv}}{\delta\Phi\,\delta\phi} \\ \frac{\delta^2\mathcal{L}_{uv}}{\delta\phi\,\delta\Phi} & \frac{\delta^2\mathcal{L}_{uv}}{\delta\phi^2} \end{pmatrix}\right|_{\Phi_c}$$

$$= \begin{pmatrix} -\partial^2 - M^2 + \frac{1}{2}\eta\phi^2 & \kappa\phi + \eta\Phi_c\phi \\ \kappa\phi + \eta\Phi_c\phi & -\partial^2 - m^2 - \frac{1}{2}\lambda\phi^2 + \kappa\Phi_c + \frac{1}{2}\eta\Phi_c^2 \end{pmatrix}$$

Define : $K_\Phi \equiv -\partial^2 - M^2$, $K_\phi \equiv -\partial^2 - m^2$. $\underline{\underline{K}} = \begin{pmatrix} K_\Phi & 0 \\ 0 & K_\phi \end{pmatrix}$

$$\underline{\underline{X}} \equiv \begin{pmatrix} X_{\Phi\Phi} & X_{\Phi\phi} \\ X_{\phi\Phi} & X_{\phi\phi} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\eta\phi^2 & -\kappa\phi - \eta\Phi_c\phi \\ -\kappa\phi - \eta\Phi_c\phi & \frac{1}{2}\lambda\phi^2 - \kappa\Phi_c - \frac{1}{2}\eta\Phi_c^2 \end{pmatrix}$$

$$\Rightarrow \left.\frac{\delta^2\mathcal{L}_{uv}}{\delta\varphi^2}\right|_{\Phi_c} = \underline{\underline{K}} - \underline{\underline{X}}$$

$$\int dx\, \mathcal{L}_{\mathrm{EFT}}^{(1\text{-loop})} = \frac{i}{2} \log \mathrm{Sdet} \left(- \frac{\delta^2 \mathcal{L}_{\mathrm{UV}}}{\delta \varphi^2} \bigg|_{\Phi_c} \right) \bigg|_{\mathrm{hard}}$$

$$= \frac{i}{2} \log \mathrm{Sdet} \left(\underline{\underline{K}} - \underline{\underline{X}} \right) \Big|_{\mathrm{hard}}$$

$$= \frac{i}{2} \mathrm{STr} \log \left(\underline{\underline{K}} - \underline{\underline{X}} \right) \Big|_{\mathrm{hard}}$$

$$= \frac{i}{2} \mathrm{STr} \left[\log \underline{\underline{K}} + \log \left(1 - \underline{\underline{K}}^{-1} \underline{\underline{X}} \right) \right] \Big|_{\mathrm{hard}}$$

$$= \frac{i}{2} \underbrace{\mathrm{STr} \log \underline{\underline{K}}}_{\text{log-type STr}} - \frac{i}{2} \sum_{n=1}^{\infty} \underbrace{\mathrm{STr} \left(\underline{\underline{K}}^{-1} \underline{\underline{X}} \right)^n}_{\text{power-type STr}} \Big|_{\mathrm{hard}}$$

Generally, $\varphi = (\varphi_1, \varphi_2, \cdots)$. $\underline{\underline{K}} = \mathrm{diag}(K_1, K_2, \cdots)$

$$K_i = \begin{cases} P^2 - m_i^2 & (\text{spin-}0) \\ \not{P} - m_i & (\text{spin-}\tfrac{1}{2}) \\ -\eta^{\mu\nu}(P^2 - m_i^2) + \cancel{\left(1 - \frac{1}{\xi}\right)}^{\to 0} P^\mu P^\nu & (\text{spin-}1) \end{cases}$$

$P_\mu \equiv i D_\mu$

$(P_\mu)^\dagger = P_\mu$

Toy model

$$\underline{\underline{K}} = \begin{pmatrix} K_\Phi & 0 \\ 0 & K_\phi \end{pmatrix} = \begin{pmatrix} P^2 - M^2 & 0 \\ 0 & P^2 - m^2 \end{pmatrix}$$

$P_\mu = iD_\mu = i\partial_\mu$

$[D_\mu, D_\nu] = -iF_{\mu\nu}$

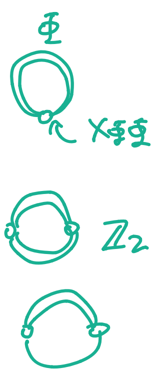
$F_{\mu\nu} = \sum_a g_a F^a_{\mu\nu} T^a_R$

$$\underline{\underline{X}} = \begin{pmatrix} X_{\Phi\Phi} & X_{\Phi\phi} \\ X_{\phi\Phi} & X_{\phi\phi} \end{pmatrix}$$

- Log-type : $\mathrm{Tr}\log \underline{\underline{K}}$ = const. (drop)
- Power-type : $-\frac{i}{2}\frac{1}{n}\mathrm{Tr}\left[(\underline{\underline{K}}^{-1}\underline{\underline{X}})^n\right] = -\frac{i}{2}\frac{1}{n}\sum_{i_1\cdots i_n}\mathrm{Tr}\left(K^{-1}_{i_1}X_{i_1 i_2}\cdots K^{-1}_{i_n}X_{i_n i_1}\right)$

$n=1$: $-\frac{i}{2}\mathrm{Tr}\left(K_\phi^{-1}X_{\phi\phi}\right) \xrightarrow{\text{hard}} 0$ (scaleless int. $\int dq\, q^\#$)

$-\frac{i}{2}\mathrm{Tr}\left(K_\Phi^{-1}X_{\Phi\Phi}\right) = -\frac{i}{2}\mathrm{Tr}\left(\frac{1}{P^2-M^2}X_{\Phi\Phi}\right)$



$n=2$: $-\frac{i}{2}\frac{1}{2}\mathrm{Tr}\left[(K_\Phi^{-1}X_{\Phi\Phi})^2\right] = -\frac{i}{2}\frac{1}{2}\mathrm{Tr}\left(\frac{1}{P^2-M^2}X_{\Phi\Phi}\frac{1}{P^2-M^2}X_{\Phi\Phi}\right)$

$-\frac{i}{2}\mathrm{Tr}\left[K_\Phi^{-1}X_{\Phi\phi}K_\phi^{-1}X_{\phi\Phi}\right] = -\frac{i}{2}\mathrm{Tr}\left(\frac{1}{P^2-M^2}X_{\Phi\phi}\frac{1}{P^2-m^2}X_{\phi\Phi}\right)$

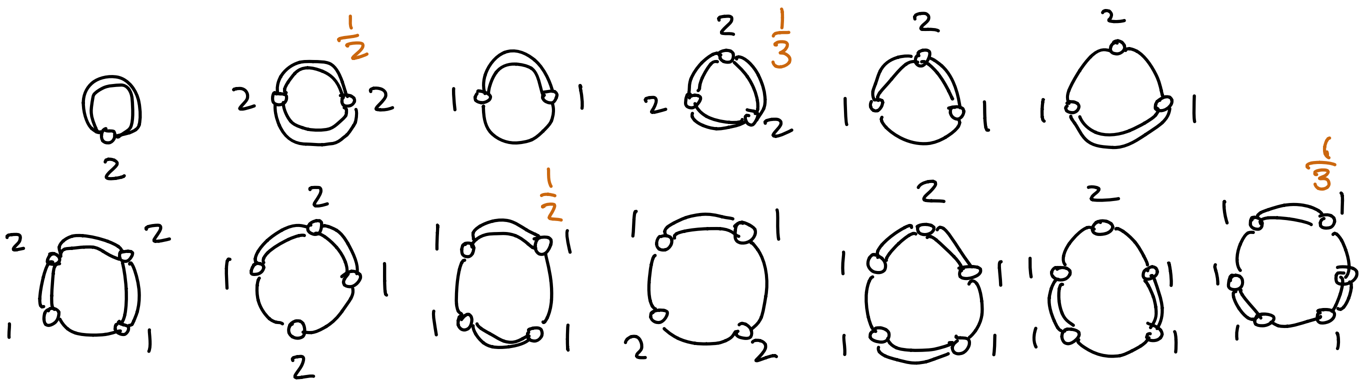
$\mathbb{Z}_r \Rightarrow \frac{1}{r}$

Enumerate (S)Tr

$$\begin{pmatrix} X_{\Phi\Phi} & X_{\Phi\phi} \\ X_{\phi\Phi} & X_{\phi\phi} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\eta\phi^2 & -\kappa\phi - \eta\underbrace{\Phi_c}_{\text{dim-2}}\phi \\ -\kappa\phi - \eta\Phi_c\phi & \frac{1}{2}\lambda\phi^2 - \kappa\Phi_c - \frac{1}{2}\eta\Phi_c^2 \end{pmatrix}$$

$$\text{dim}: \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Want LEFT up to dim-6.



Evaluate (S)Tr

CDE = Covariant Derivative Expansion

2012.07851
1912.08814
(App. B)

STrEAM = Super Trace Evaluation Automated for Matching

dim-2

$$-i\,\mathrm{STr}\left(\frac{1}{p^2-M^2}X^{[2]}_{\Phi\Phi}\right)\Big|_{\text{hard}} = \int dx\,\frac{1}{16\pi^2}\,\mathrm{tr}\Big\{\left(1-\log\frac{M^2}{\mu^2}\right)M^2\,X_{\Phi\Phi} + \frac{1}{12M^2}\cancelto{0}{F_{\mu\nu}F^{\mu\nu}}\,X_{\Phi\Phi}\Big\} + \dots$$

$$\Rightarrow\ -\int dx\,\frac{1}{16\pi^2}\,\frac{1}{2}\left(1-\log\frac{M^2}{\mu^2}\right)\eta\,M^2\underline{\phi^2} + \dots$$

Threshold correction

Hierarchy problem : $m^2_{EFT} \sim m^2_{UV} + \dfrac{M^2}{16\pi^2}$