Inverse Problem:

A novel method for calculating the non-perturbation quantities

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In collaboration with Hsiang-nan Li, Hiroyuki Umeeda, Fanrong Xu
Ao-Sheng Xiong, Ting Wei

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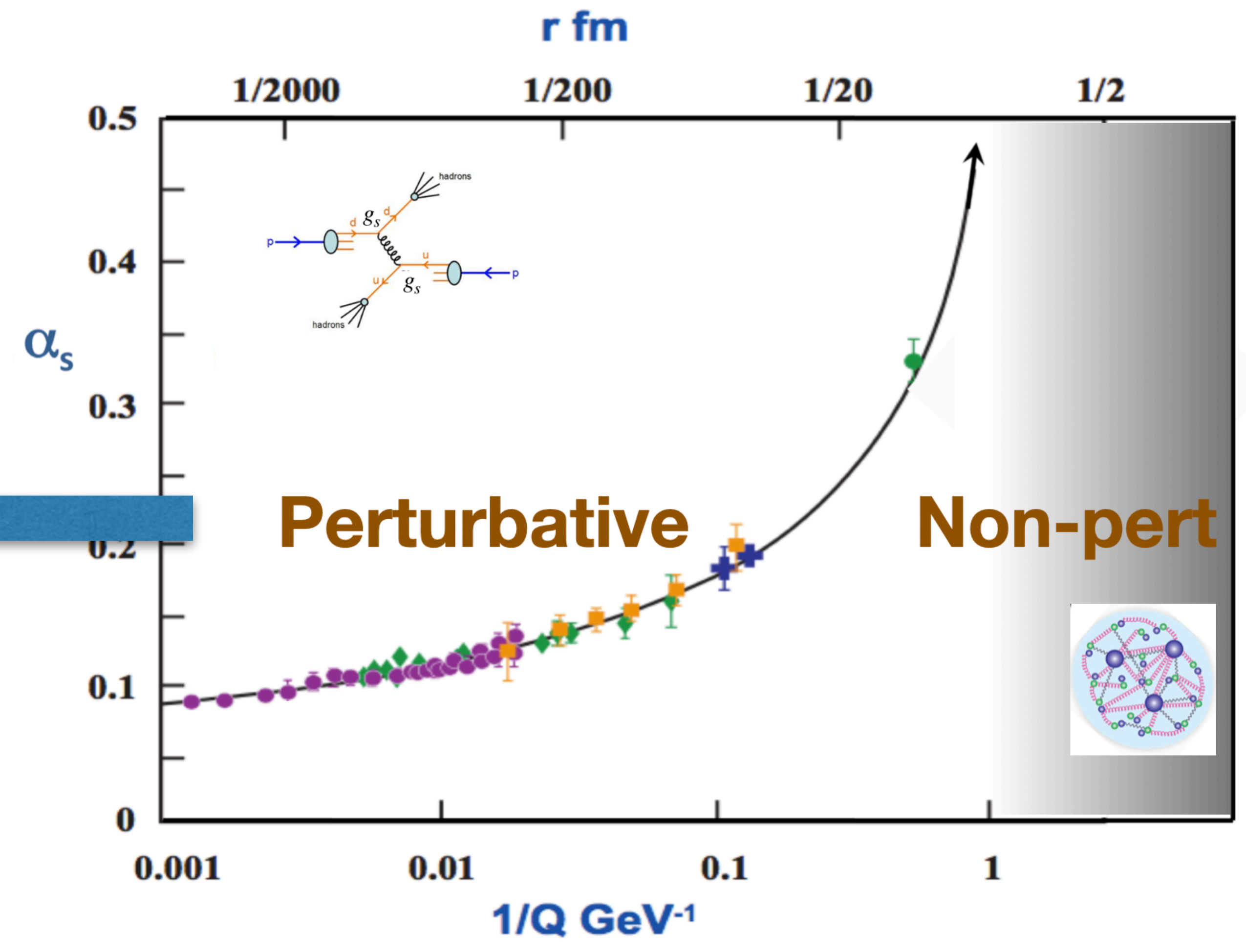
Outline

- What is the **Inverse Problem.** How for non-perturbation
- **Proposals**: D0-D0bar mixing, muon g-2, QCD sum rules
- **Development**: Proof of the ill-posedness
- Outlook and Summary

Motivation: Problems in HEP

- Within the Standard Model: **To calculate the non-perturbative quantities**

Asymptotic freedom



Quark confinement

未解之谜

Motivation: Problems in HEP

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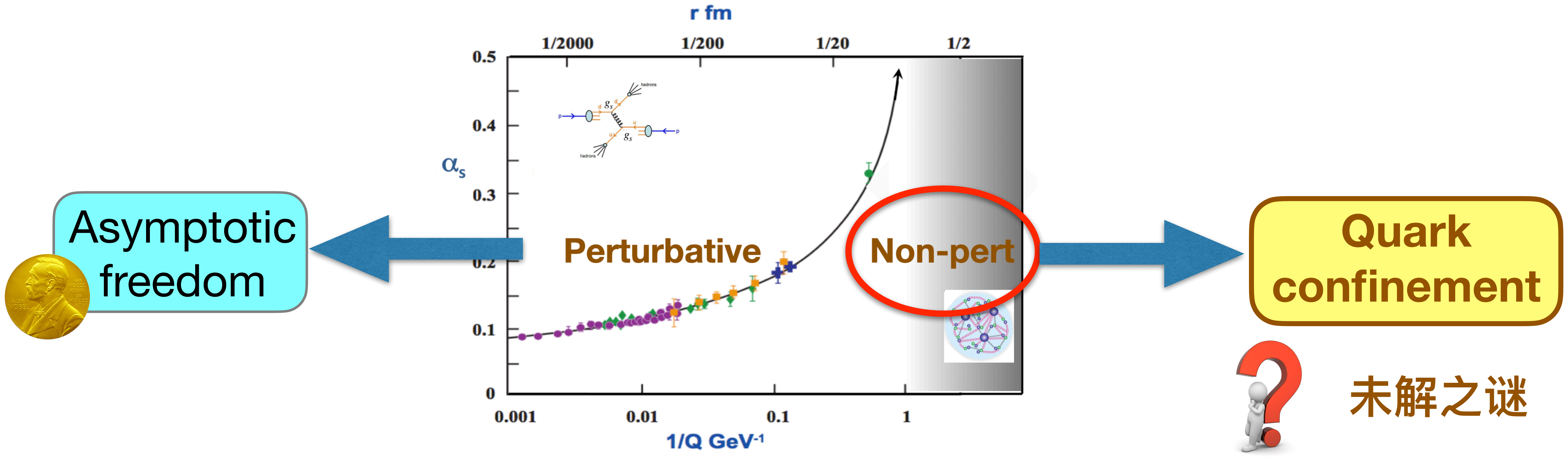

Lattice QCD, QCD sum rules, Dyson-Schwinger Equation, Chiral PT,…

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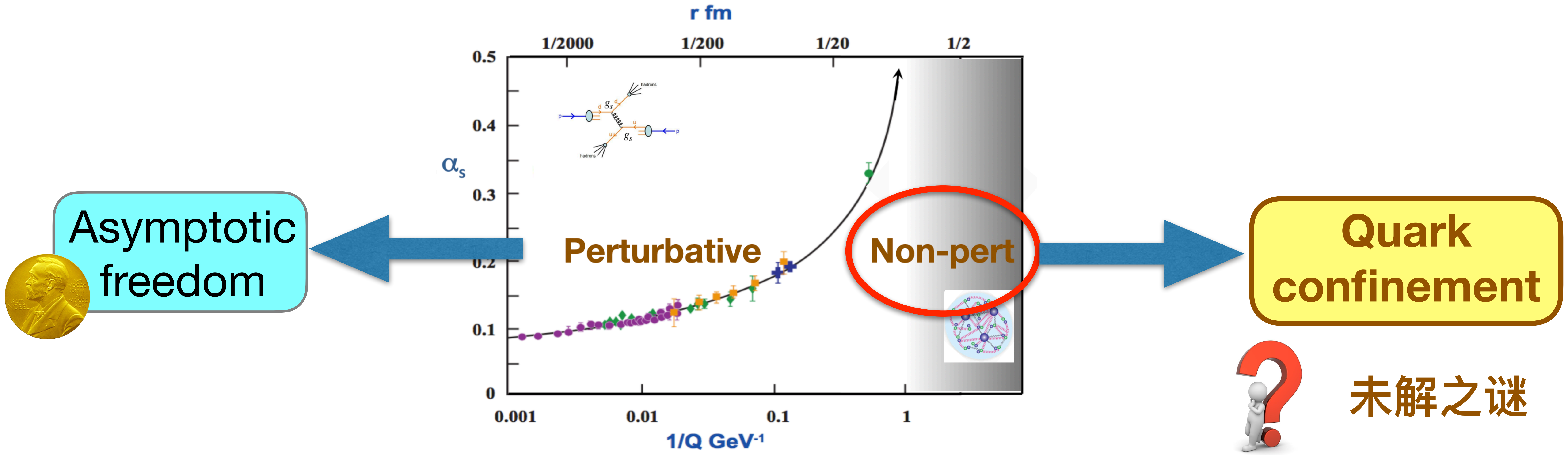

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- Beyond the Standard Model: **To search for the new physics**

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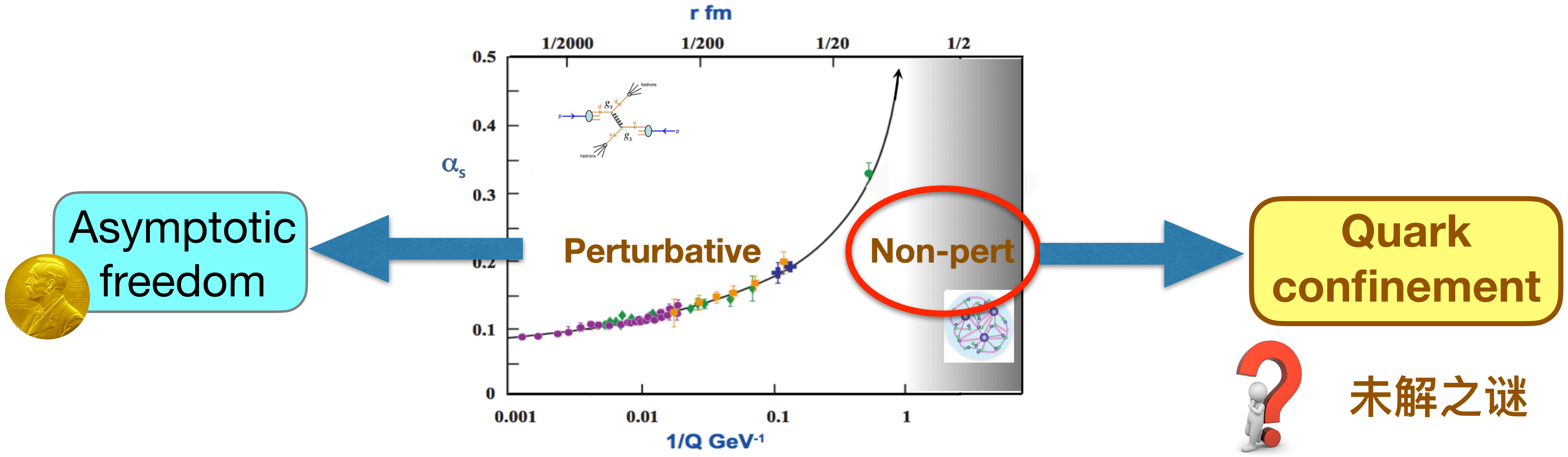


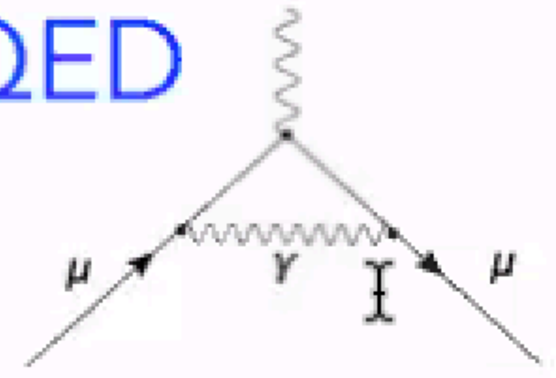
Lattice QCD, QCD sum rules, Dyson-Schwinger Equation, Chiral PT,…

- Beyond the Standard Model: **To search for the new physics**
  - In the present days of indirect searches, non-perturbative strong interaction play a significant role
  - High precision required for hadronic quantities: e.g. Hadronic Vacuum Polarization of muon g-2

Muon g-2: SM contributions

$$a_\mu = a_\mu(\text{QED}) + a_\mu(\text{Weak}) + a_\mu(\text{Hadronic})$$

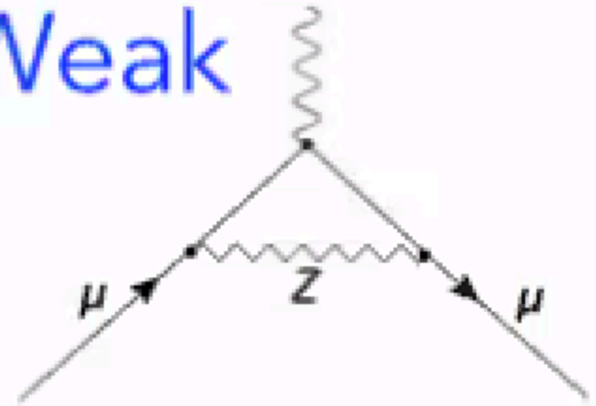
QED



$+\ldots$

$$116\,584\,718.9\,(1) \times 10^{-11}$$

Weak



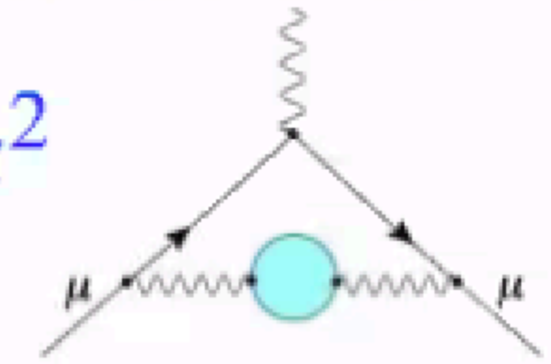
$+\ldots$

$$153.6\,(1.0) \times 10^{-11}$$

Hadronic…

…Vacuum Polarization (HVP)

α^2

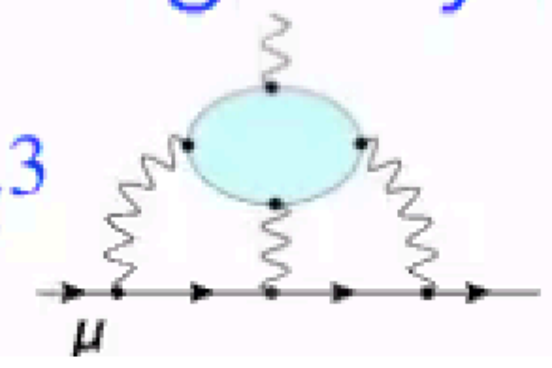


$+\ldots$

$$6845\,(40) \times 10^{-11}$$

…Light-by-Light (HLbL)

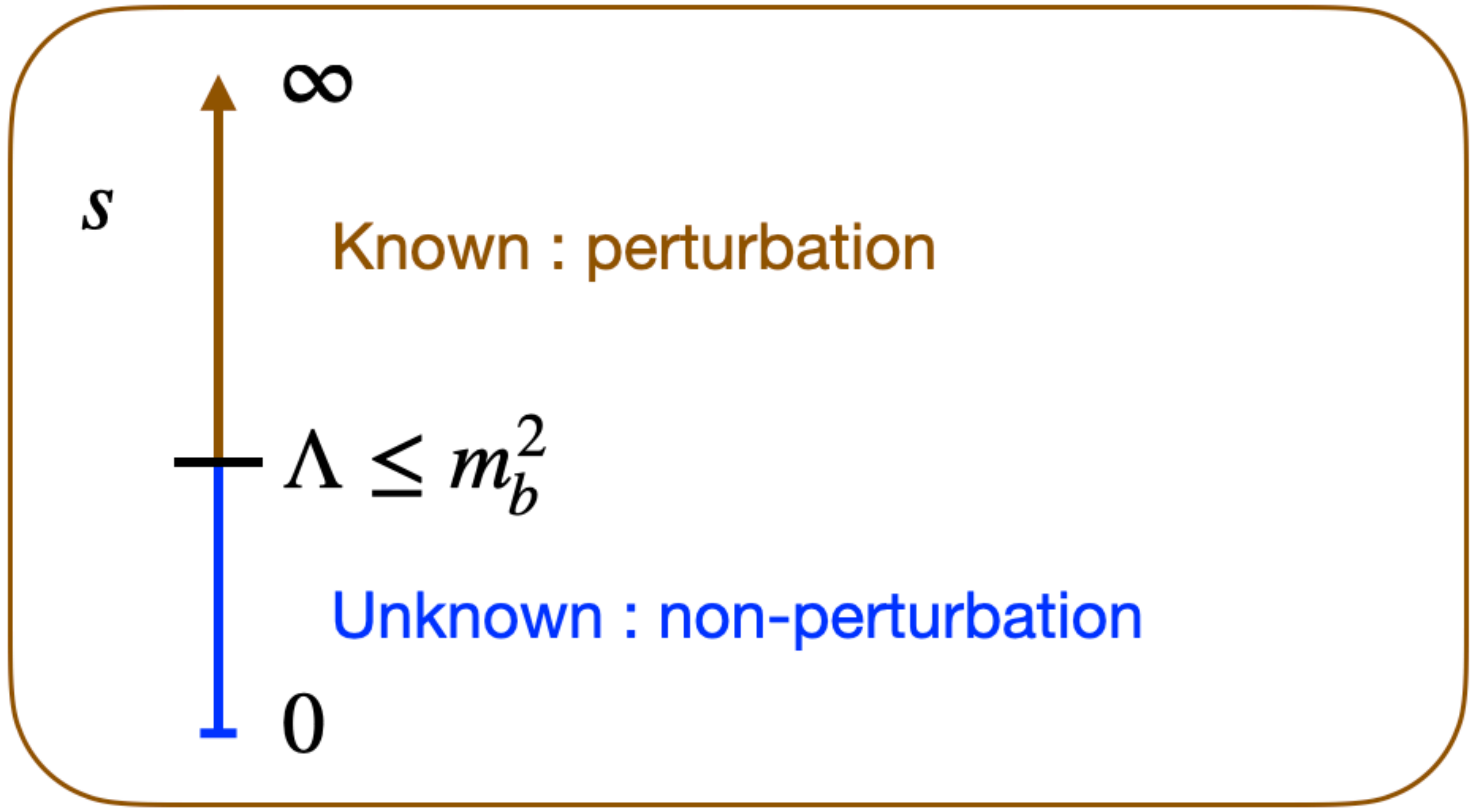
α^3



$+\ldots$

$$92\,(18) \times 10^{-11}$$

What is the Inverse Problem?



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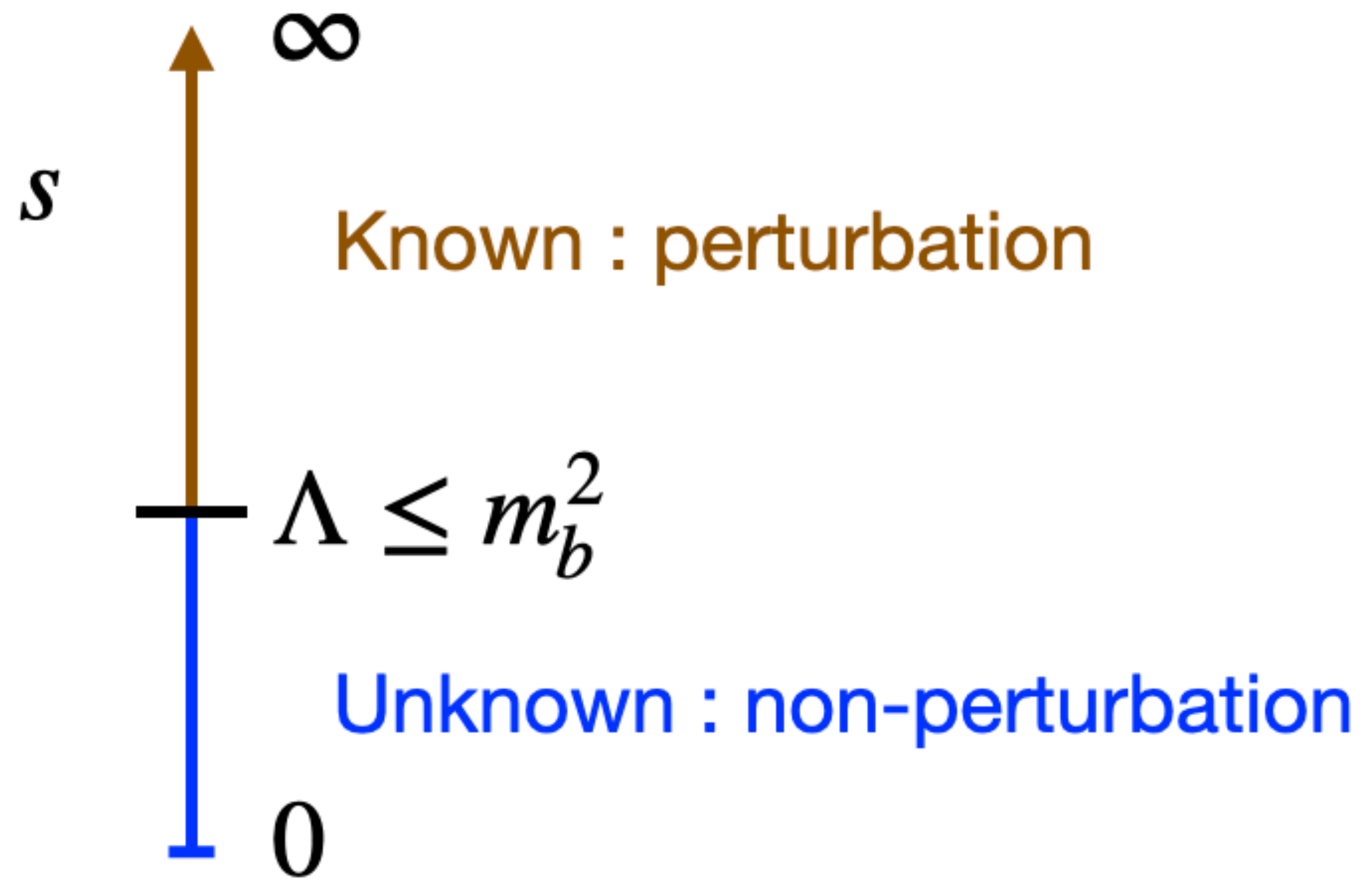

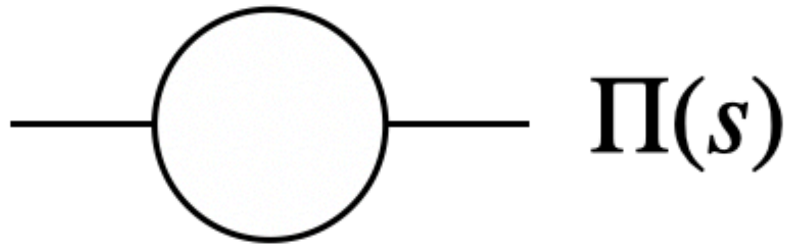

Dispersion Relation:

$$\mathcal{R}e[\Pi(s)] = \frac{1}{\pi}\mathcal{P}\int_0^\infty \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$$

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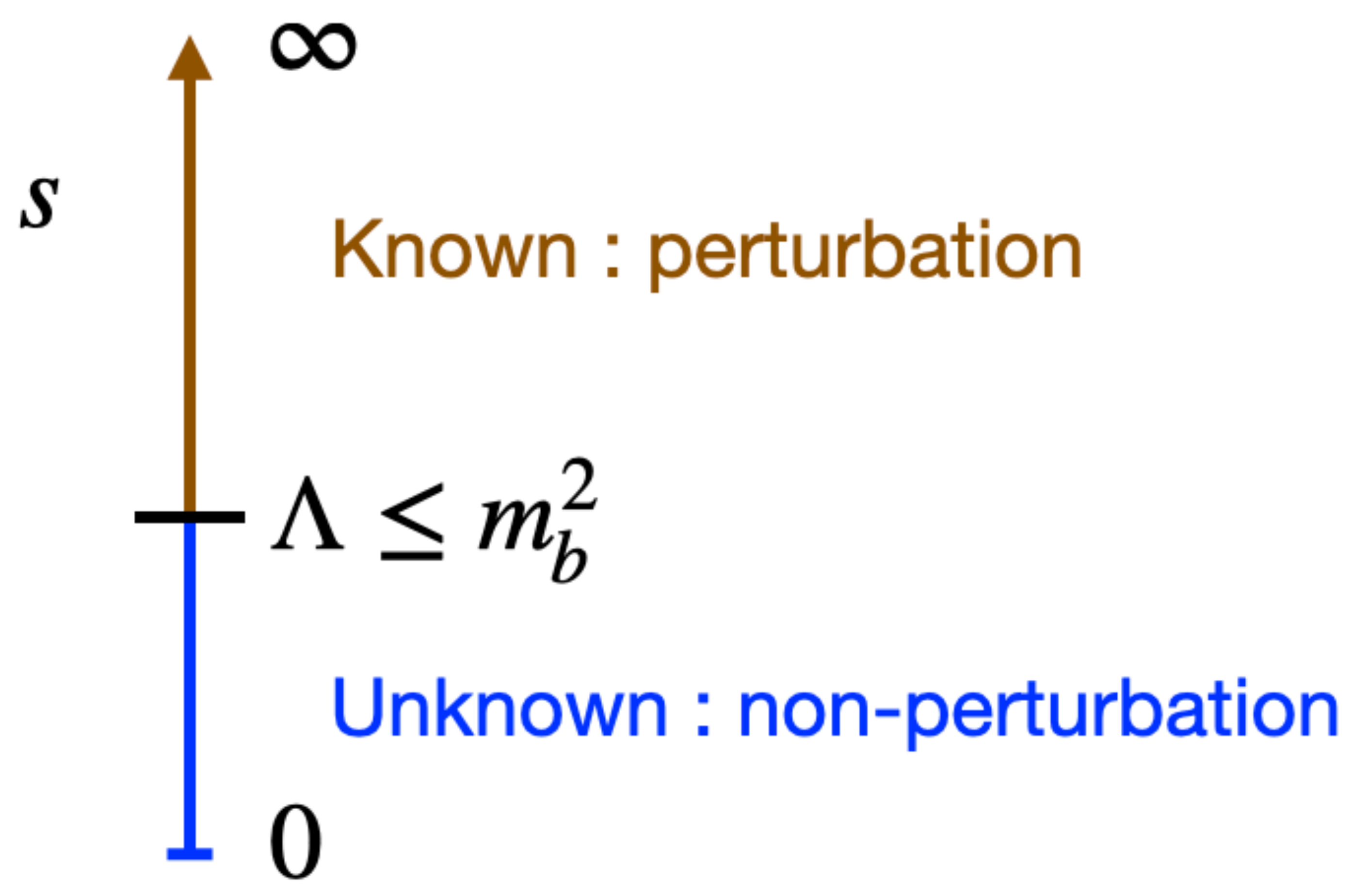

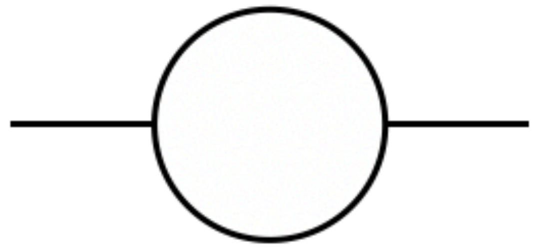

$\Pi(s)$

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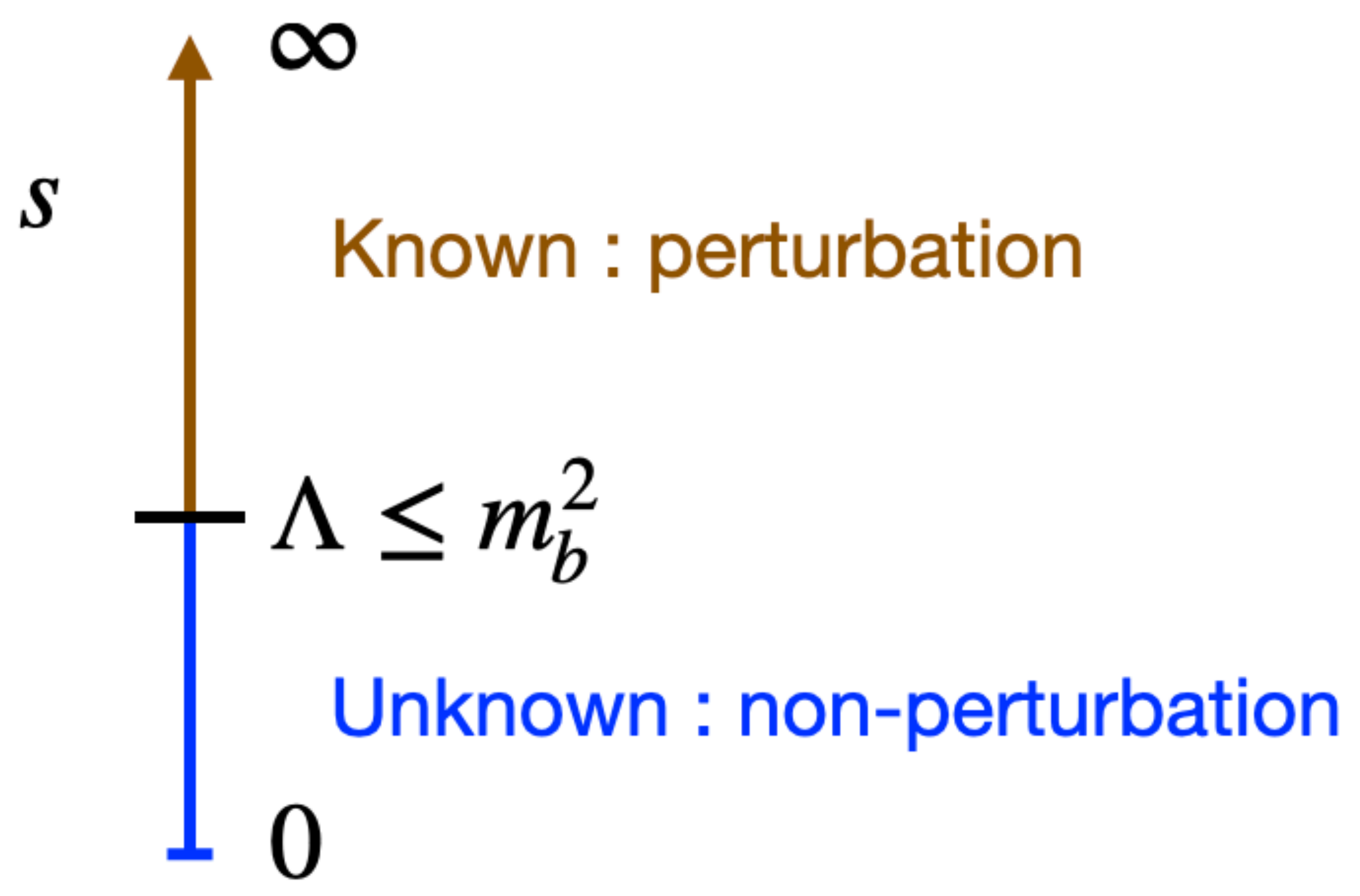

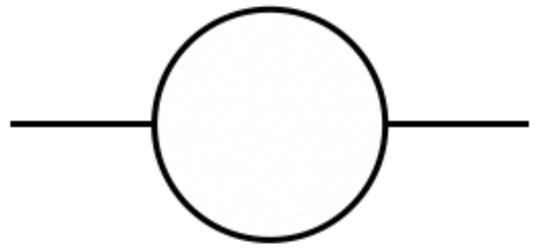

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To be solved

calculable

$D^0 - \overline{D}^0$ Mixing

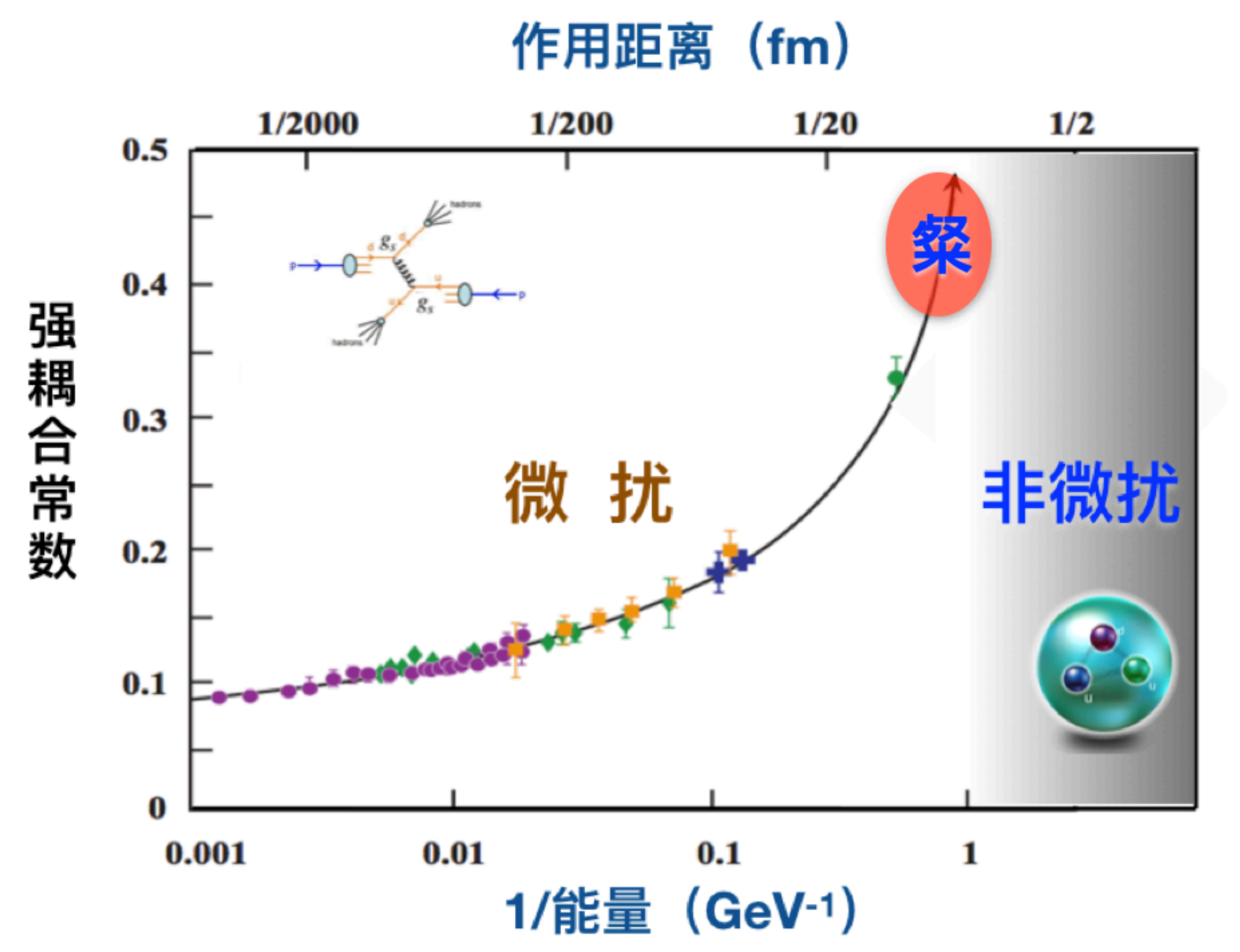
- The time evolution

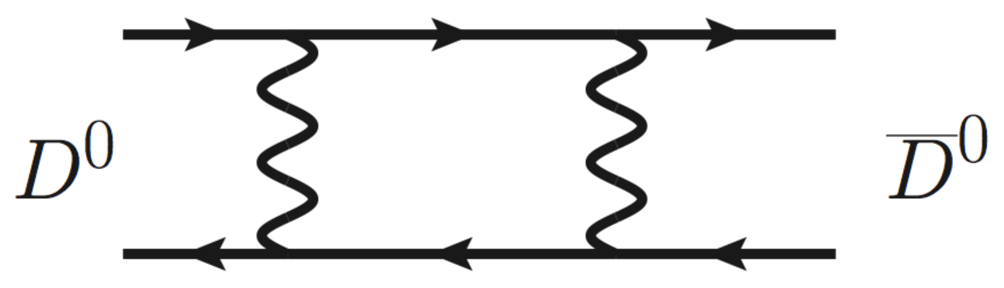
$$i\frac{\partial}{\partial t}\begin{pmatrix} D^0(t) \\ \overline{D}^0(t) \end{pmatrix} = \left(\mathbf{M} - \frac{i}{2}\mathbf{\Gamma}\right)\begin{pmatrix} D^0(t) \\ \overline{D}^0(t) \end{pmatrix}$$

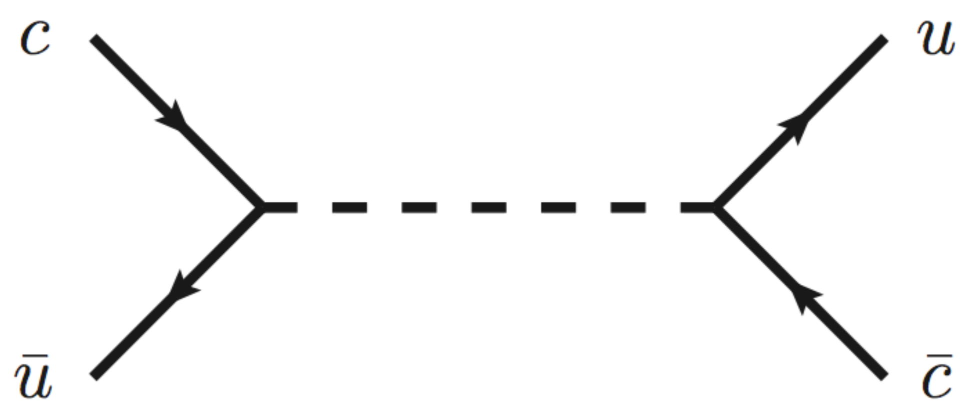
- Mixing parameters: Mass and Width differences

$$x \equiv \frac{\Delta m}{\Gamma} = \frac{m_1 - m_2}{\Gamma} \qquad y \equiv \frac{\Delta\Gamma}{2\Gamma} = \frac{\Gamma_1 - \Gamma_2}{2\Gamma}$$

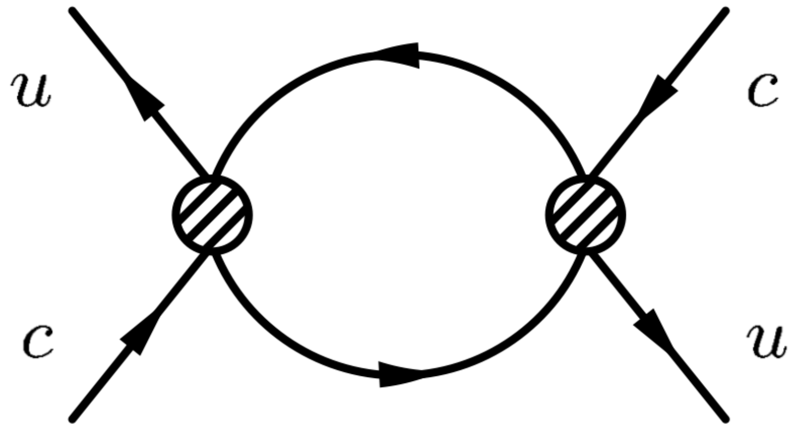
- Useful to search for new physics,
- but less understood in the Standard Model







Inclusive Approach



quark level

Short-distance

Theory / Exp. comparison (for inclusive)

D meson

Hagelin 1981, Cheng 1982
Buras, Slominski and Steger 1984
NLO QCD Golowich and Petrov 2005

SM $\begin{cases} x \simeq 6 \times 10^{-7} \\ y \simeq 6 \times 10^{-7} \end{cases}$

Suppressed by GIM

HFLAV at Moriond2019

Exp. $\begin{cases} x = (3.9^{+1.1}_{-1.2}) \times 10^{-3} \\ y = (6.51^{+0.63}_{-0.69}) \times 10^{-3} \end{cases}$

B_s meson

Artuso, Borissov and Lenz, 2016

SM $\begin{cases} \Delta M_s = (18.3 \pm 2.7)\text{ps}^{-1} \\ \Delta\Gamma_s = (0.088 \pm 0.020)\text{ps}^{-1} \end{cases}$

HFLAV

Exp. $\begin{cases} \Delta M_s = (17.757 \pm 0.021)\ \text{ps}^{-1} \\ \Delta\Gamma_s = (0.082 \pm 0.006)\ \text{ps}^{-1} \end{cases}$

B_d meson

Artuso, Borissov and Lenz, 2016

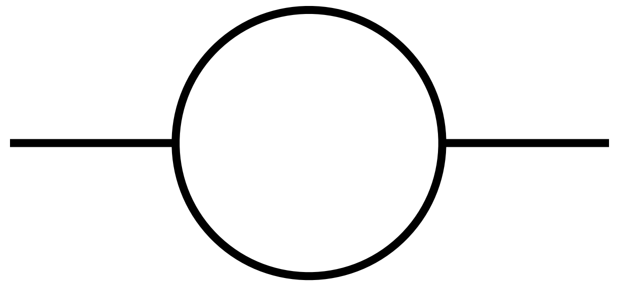
SM $\begin{cases} \Delta M_d = (0.528 \pm 0.078)\ \text{ps}^{-1} \\ \Delta\Gamma_d = (2.61 \pm 0.59) \cdot 10^{-3}\ \text{ps}^{-1} \end{cases}$

HFLAV

Exp. $\begin{cases} \Delta M_d = (0.5055 \pm 0.0020)\ \text{ps}^{-1} \\ \Delta\Gamma_d = 0.66(1 \pm 10) \cdot 10^{-3}\ \text{ps}^{-1} \end{cases}$

- For B_s, B_d mesons, the data are reproduced within 1σ.
- For D meson, the order of magnitude is not reproduced within leading-power.

Dispersion Relation



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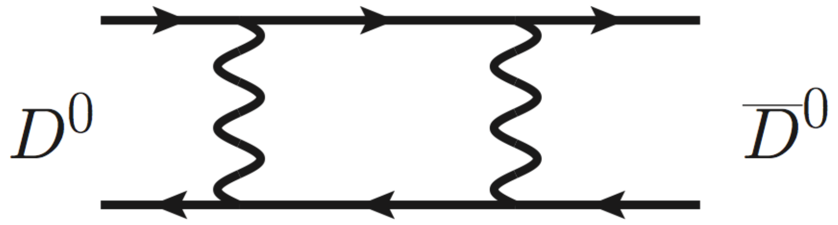
$D^0 - \overline{D}^0$ mixing:

$M_{12} =$ 

$$Re[M_{12}(s)] = \frac{1}{\pi}\int_0^\infty \frac{Im[M_{12}(s')]}{s-s'}ds'$$

$$x(s) = \frac{1}{\pi}\int_0^\infty \frac{y(s')}{s-s'}ds'$$

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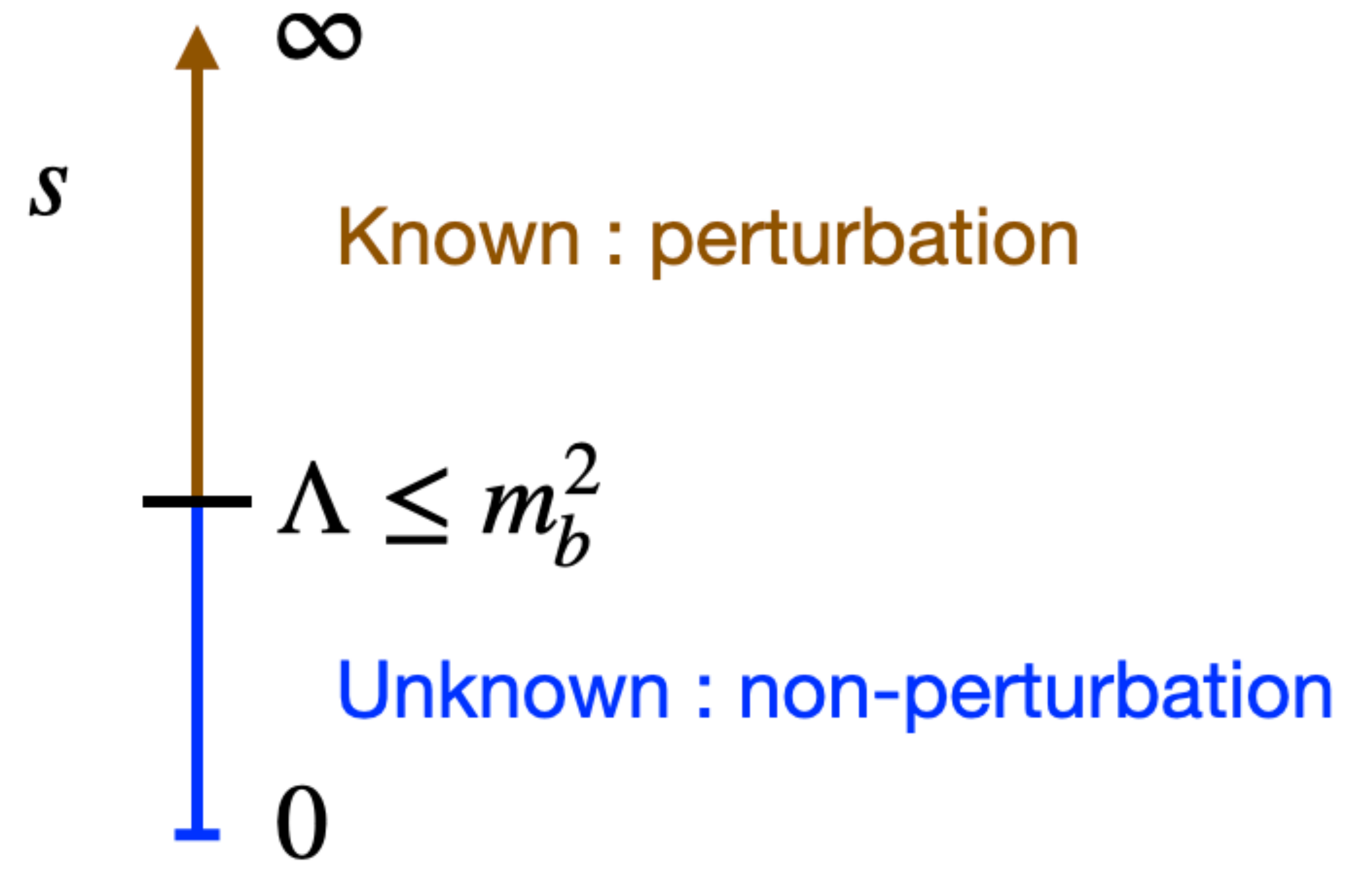

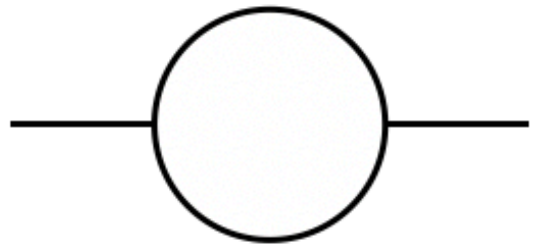

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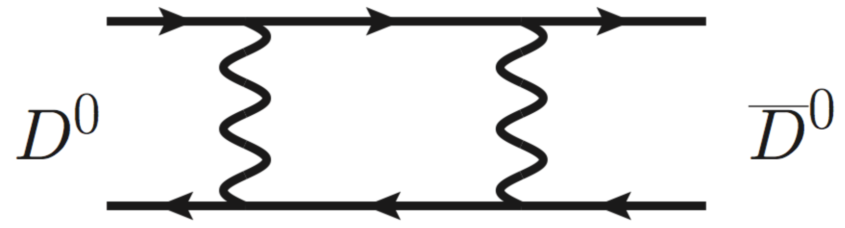

$$\int_0^{\Lambda} ds' \frac{y(s')}{s-s'} = \pi x(s) - \int_{\Lambda}^{\infty} ds' \frac{y(s')}{s-s'} \equiv \omega(s)$$

parametrization:

$$y(s) = \frac{Ns[b_0 + b_1(s-m^2) + b_2(s-m^2)^2]}{[(s-m^2)^2 + d^2]^2}$$

Li, Umeeda, Xu, **FSY**, PLB(2020)

Inverse Problem

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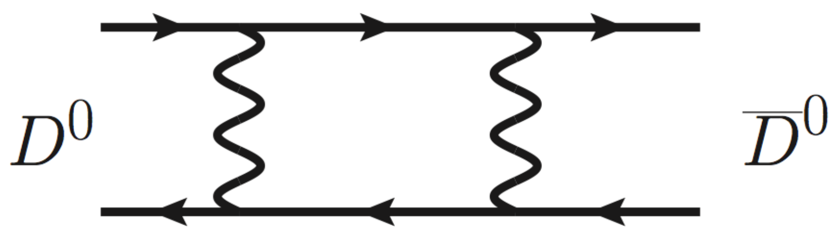

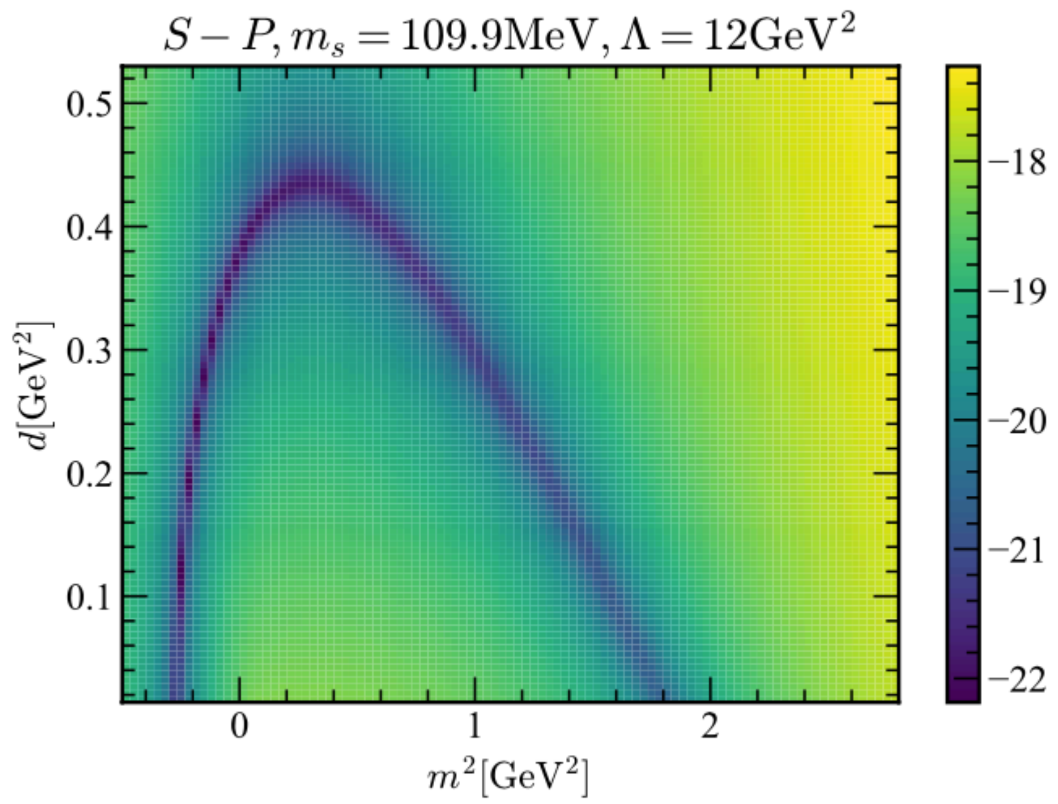

unstable solutions

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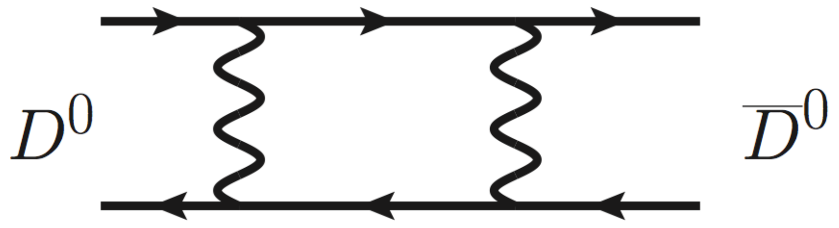

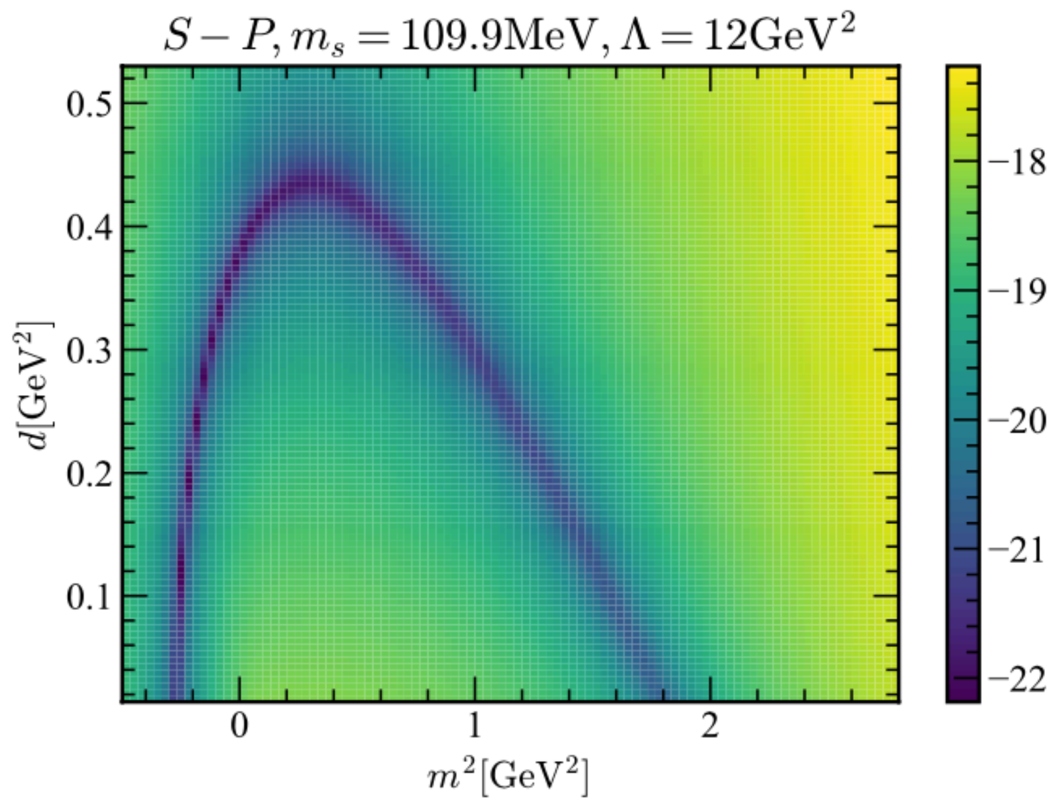

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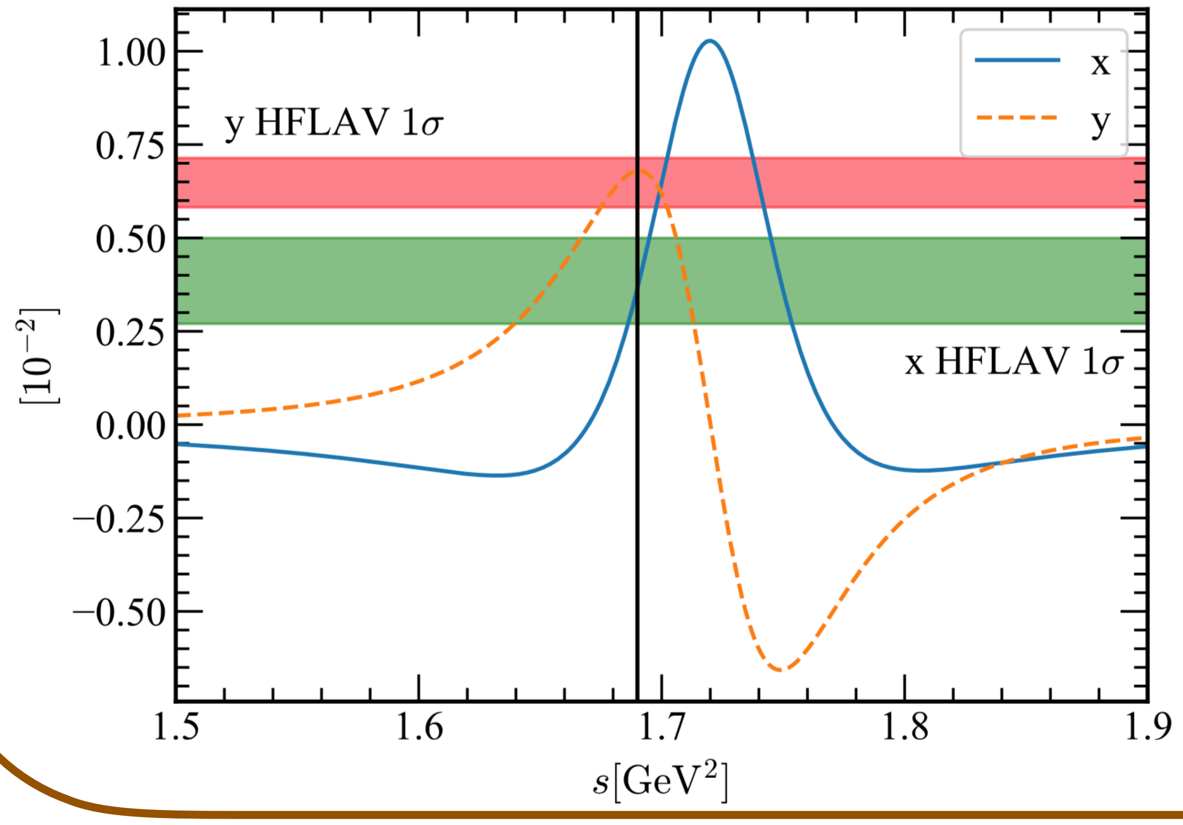

Additional conditions:
data of x and y as inputs

Li, Umeeda, Xu, **FSY**, PLB(2020)

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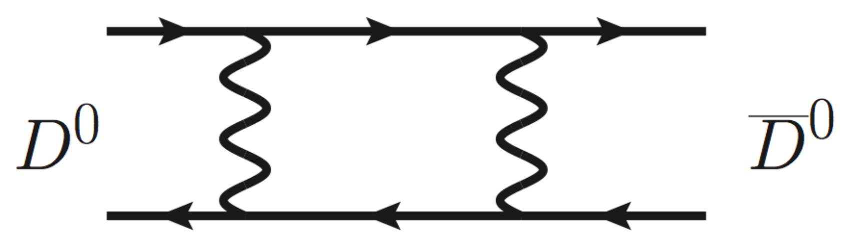

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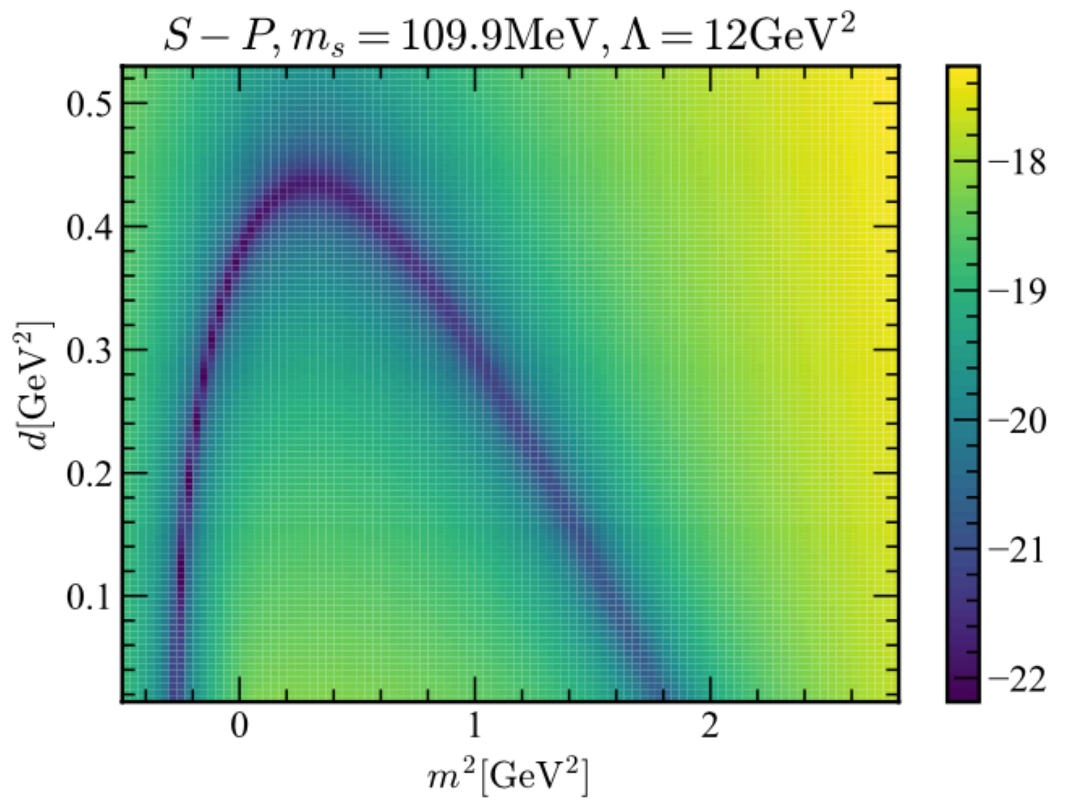

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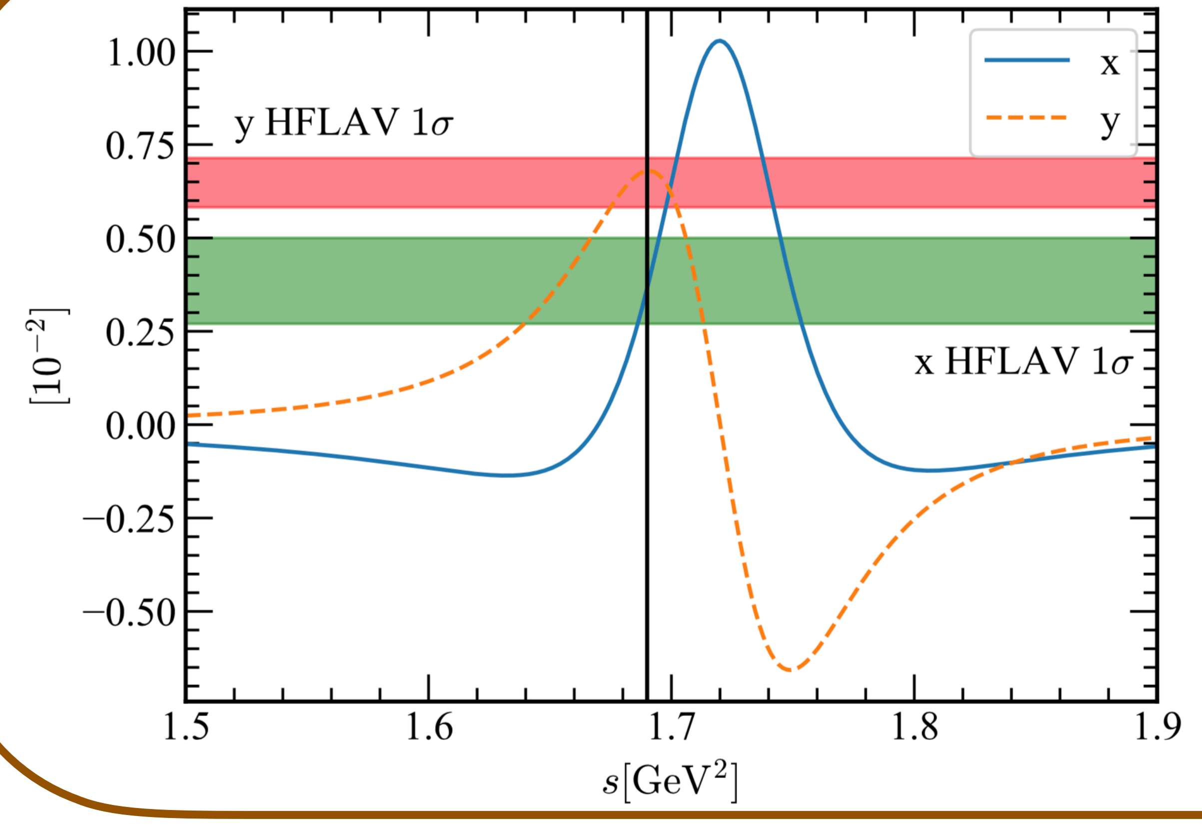


Additional conditions:
data of x and y as inputs

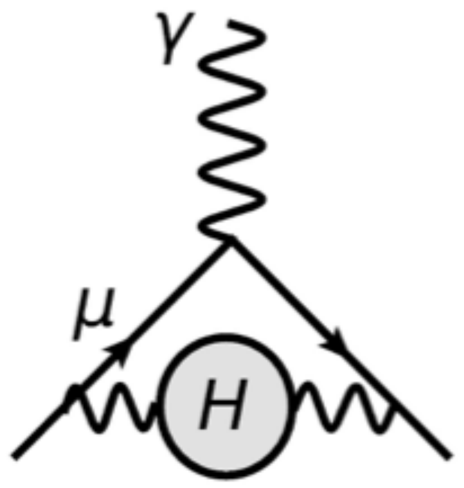
Predict indirect CPV

$q/p = 1.0002e^{i0.006^\circ}$

consistent with data

$q/p = (0.969^{+0.050}_{-0.045})e^{i(-3.9^{+4.5}_{-4.6})^\circ}$

Applications of the Inverse Problem: muon g-2



- Muon g-2: 4.2σ deviation from the SM   Muon g-2, PRL(2021)
- Dominate uncertainty of the SM prediction: hadronic vacuum polarization (HVP)

Aoyama, et al, Phys.Rept(2020)

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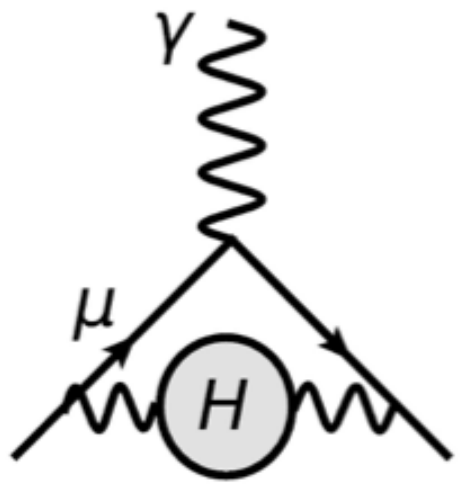

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$$\int_{\lambda_r}^{\Lambda_r} ds' \frac{\mathrm{Im}\Pi_r(s')}{s'(s'+s)} - \pi \frac{\Pi_r(0)}{s} = -\pi \frac{\Pi_r(-s)}{s} - \int_{\Lambda_r}^{\infty} ds' \frac{\mathrm{Im}\Pi_r(s')}{s'(s'+s)}$$

$$r = \rho,\ \omega,\ \phi$$

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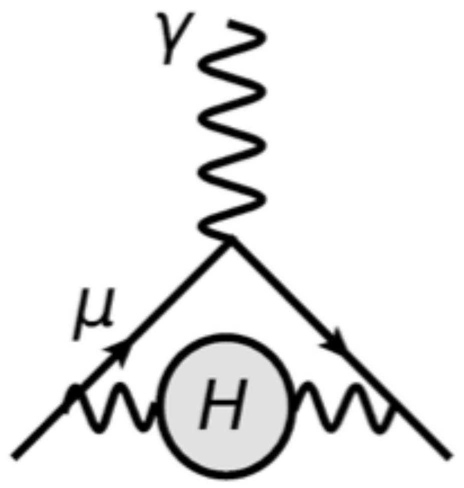

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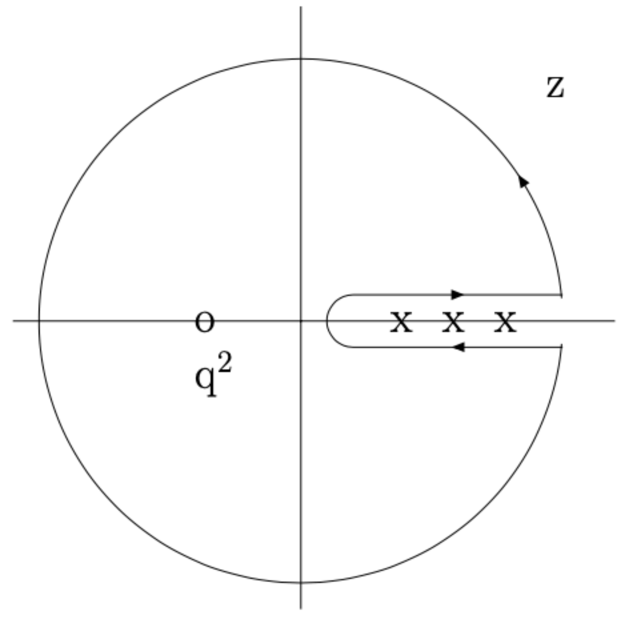
- Result: Inverse problem: $a_\mu^{\mathrm{HVP}} = (641^{+65}_{-63}) \times 10^{-10}$ — H.n.Li, Umeeda, '20

Data driven: $a_\mu^{\mathrm{HVP}} = (693.9 \pm 4.0) \times 10^{-10}$ — Davier, Hoecker, Malaescu, Zhang, '20

Lattice QCD: $a_\mu^{\mathrm{HVP}} = (654 \pm 32^{+21}_{-23}) \times 10^{-10}$ — Della Morte et al, '17

Applications of the Inverse Problem: QCD sum rules



- Conventional QCD sum rules $\Pi_{\mu\nu}(q^2) = i\int d^4x e^{iq\cdot x}\langle 0|T[J_\mu(x)J_\nu(0)]|0\rangle$

Dispersion relation: $$\Pi(q^2) = \frac{1}{2\pi i}\oint ds \frac{\Pi(s)}{s-q^2} = \frac{1}{\pi}\int_{t_{min}}^{\infty} ds \frac{\mathrm{Im}\,\Pi(s)}{s-q^2-i\epsilon}$$

$$\mathrm{Im}\Pi(q^2) = \pi f_V^2 \delta(q^2 - m_V^2) + \pi\rho^h(q^2)\theta(q^2 - s_h)$$

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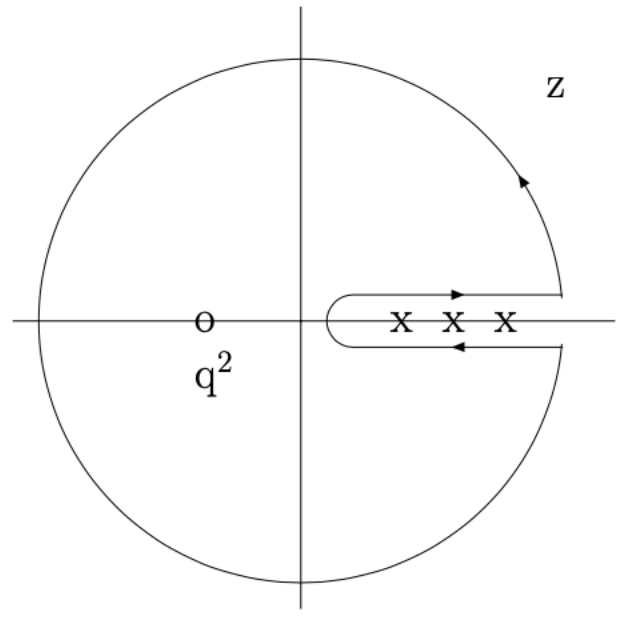

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Quark-hadron duality: $$\rho^h(s) = \frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{pert}}(s)\theta(s-s_0)$$

$$\int_{s_h}^{\infty} ds\frac{\rho^h(s)}{s-q^2} = \frac{1}{\pi}\int_{s_0}^{\infty} ds\frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2}$$

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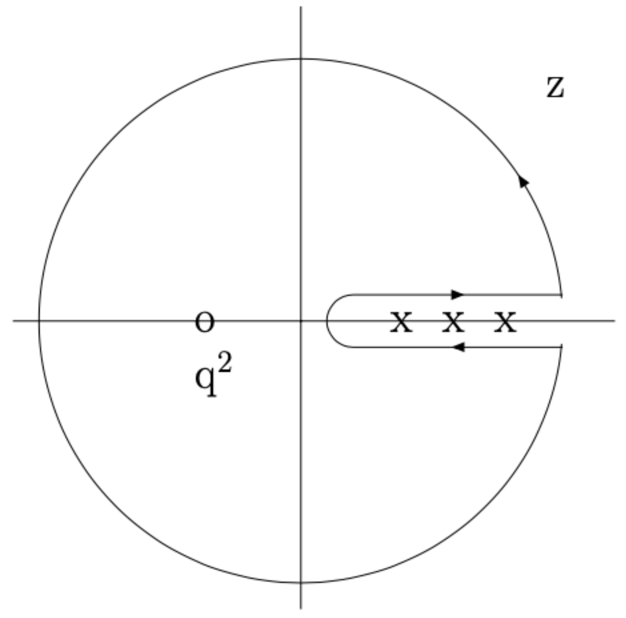

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Quark-hadron duality: $\rho^h(s) = \frac{1}{\pi}\text{Im}\Pi^{\text{pert}}(s)\theta(s-s_0)$

$$\int_{s_h}^{\infty} ds \frac{\rho^h(s)}{s-q^2} = \frac{1}{\pi}\int_{s_0}^{\infty} ds \frac{\text{Im}\Pi^{\text{pert}}(s)}{s-q^2}$$

Uncertainty sources: quark-hadron duality and Borel transformation

Applications of the Inverse Problem: QCD sum rules

- **Inverse-Problem QCD sum rules**

$$\frac{1}{2\pi i}\oint ds\,\frac{\Pi(s)}{s-q^2} = \frac{1}{\pi}\int_{s_i}^{\Lambda} ds\,\frac{\mathrm{Im}\Pi(s)}{s-q^2} + \frac{1}{\pi}\int_{\Lambda}^{R} ds\,\frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{2\pi i}\int_C ds\,\frac{\Pi^{\mathrm{pert}}(s)}{s-q^2}$$

H.n.Li, Umeeda, '20

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Involving excited states and parameterization:

$$\begin{aligned}\mathrm{Im}\Pi(q^2) &= \pi f_\rho^2 \delta(q^2 - m_\rho^2) + \pi f_{\rho(1450)}^2 \delta(q^2 - m_{\rho(1450)}^2) + \pi f_{\rho(1700)}^2 \delta(q^2 - m_{\rho(1700)}^2) \\ &\quad + \pi f_V^2 \delta(q^2 - m_V^2) + \pi \rho^h(q^2),\end{aligned}$$

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$$\rho^h(y) = b_0 P_0(2y-1) + b_1 P_1(2y-1) + b_2 P_2(2y-1) + b_3 P_3(2y-1) + \cdots$$

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$$m_{\rho(770)}(m_{\rho(1450)}, m_{\rho(1700)}, m_{\rho(1900)}) \approx 0.78\ (1.46,\ 1.70,\ 1.90)\ \mathrm{GeV}$$

$$f_{\rho(770)}(f_{\rho(1450)}, f_{\rho(1700)}, f_{\rho(1900)}) \approx 0.22\ (0.19,\ 0.14,\ 0.14)\ \mathrm{GeV}$$

H.n.Li, Umeeda, ’20

Non-perturbative properties can be revealed from perturbative QCD by solving an inverse problem.

It deserves to develop this novel method from mathematics

反问题是上世纪60、70年代才开始发展的数学新分支

反问题的不适定性

- 算子$K : X \to Y$，映射X到Y，X和Y均为度量空间，　$Kx = y, \quad x \in X, \quad y \in Y$

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- 适定性(well-posedness)：上述三个条件都满足，K^{-1}在整个空间Y上存在、单值且连续，则 $x = K^{-1}y$

色散关系的反问题

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y),\quad y \in [c,d],\quad c > b,\quad a > 0$$

1) 解的存在性：物理要求

2) 解的唯一性：证明

已知算子$K : L^2(a,b) \to L^2(c,d)$

若$f_1(x)$, $f_2(x)$都是方程$K(x,y)f(x) = g(y)$的解，则有 $Kf_1 = Kf_2 = g$，即 $K(f_1 - f_2) = 0$。

唯一性：证明$f_1 = f_2$。只需证明 $Kf = 0$ 只有零解。

$$\int_a^b \frac{f(x)}{y-x}dx = \int_a^b \frac{1}{y}\sum_{k=0}^{\infty}\left(\frac{x}{y}\right)^k f(x)dx, y \in [c,d], c > b, a > 0,$$

控制收敛定理：
$$y\int_a^b \frac{f(x)}{y-x}dx = \sum_{k=0}^{\infty}\frac{1}{y^k}\int_a^b x^k f(x)dx = 0, \forall y \in [c,d].$$

Ao-Sheng Xiong, Ting Wei, **FSY**, in progress

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当$d=+\infty$，令$y\to+\infty$，$\int_a^b f(x)dx=0$ 。两边同乘 y，$\int_a^b x^k f(x)dx = 0, k=0,1,2,3\ldots.$

由Weierstras定理，存在多项式 $\int_a^b f(x)P_n(x)dx = 0.$

由Cauchy不等式 $\|f\|^2_{L^2(a,b)} = \int_a^b f^2(x)dx = \int_a^b [f^2(x) - f(x)P_n(x)]dx \le \int_a^b |f(x)||f(x)-P_n(x)|dx$

$$\le \left[\int_a^b f^2(x)dx\right]^{\frac{1}{2}}\left\{\int_a^b [f(x)-P_n(x)]^2dx\right\}^{\frac{1}{2}} \le \|f\|_{L^2(a,b)}(\varepsilon+\sqrt{b-a}\varepsilon),$$

即 $\|f\|_{L^2(a,b)}\le\varepsilon+\sqrt{b-a}\varepsilon$。由于 ε 的任意性，令 $\varepsilon\to 0$，则有 $\|f\|_{L^2(a,b)}=0$，即 $f(x)=0$.

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$$\|g_1(y)-g_2(y)\|_{L^2}^2=\left\{\int_c^a\left[\int_a^b\frac{1}{y-x}\sin(wx)dx\right]^2dy\right\}\to 0, w\to\infty,$$

即 $\|g_1(y)-g_2(y)\|_{L^2}^2$ 在 $w\to\infty$ 时可以任意小。然而 $\|f_1(x)-f_2(x)\|_{L^2}$ 却不能任意小

$$\|f_1(x)-f_2(x)\|_{L^2}=\left\{\int_a^b\sin^2(wx)dx\right\}^{\frac{1}{2}}=\sqrt{\frac{b-a}{2}-\frac{1}{2w}\sin w(b-a)\cos w(b+a)},$$

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Ao-Sheng Xiong, Ting Wei, **FSY**, in progress

误差分析：正则化方法

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y),\quad y \in [c,d],\quad c > b,\quad a > 0$$

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Ao-Sheng Xiong, Ting Wei, **FSY**, in progress

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构造正则化参数 $\lambda = \lambda(\delta_y)$ 要与原始数据 y 的误差水平 δ_y 相匹配，从而使得求解的x 的误差$\delta_x \leq f(\lambda, \delta_y)$

Ao-Sheng Xiong, Ting Wei, **FSY**, in progress

总结与展望

1. 反问题方法：计算非微扰物理量的新方法——用微扰QCD反解非微扰物理量
2. 在$D^0-\overline{D}^0$ mixing、muon g-2、QCD sum rules得到检验和应用
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2) 理论发展：误差分析—正则化方法
3) 应用扩展：原则上低能非微扰都可以，首先努力在其它方法不太成功的地方，如粲物理能标、激发态和连续谱

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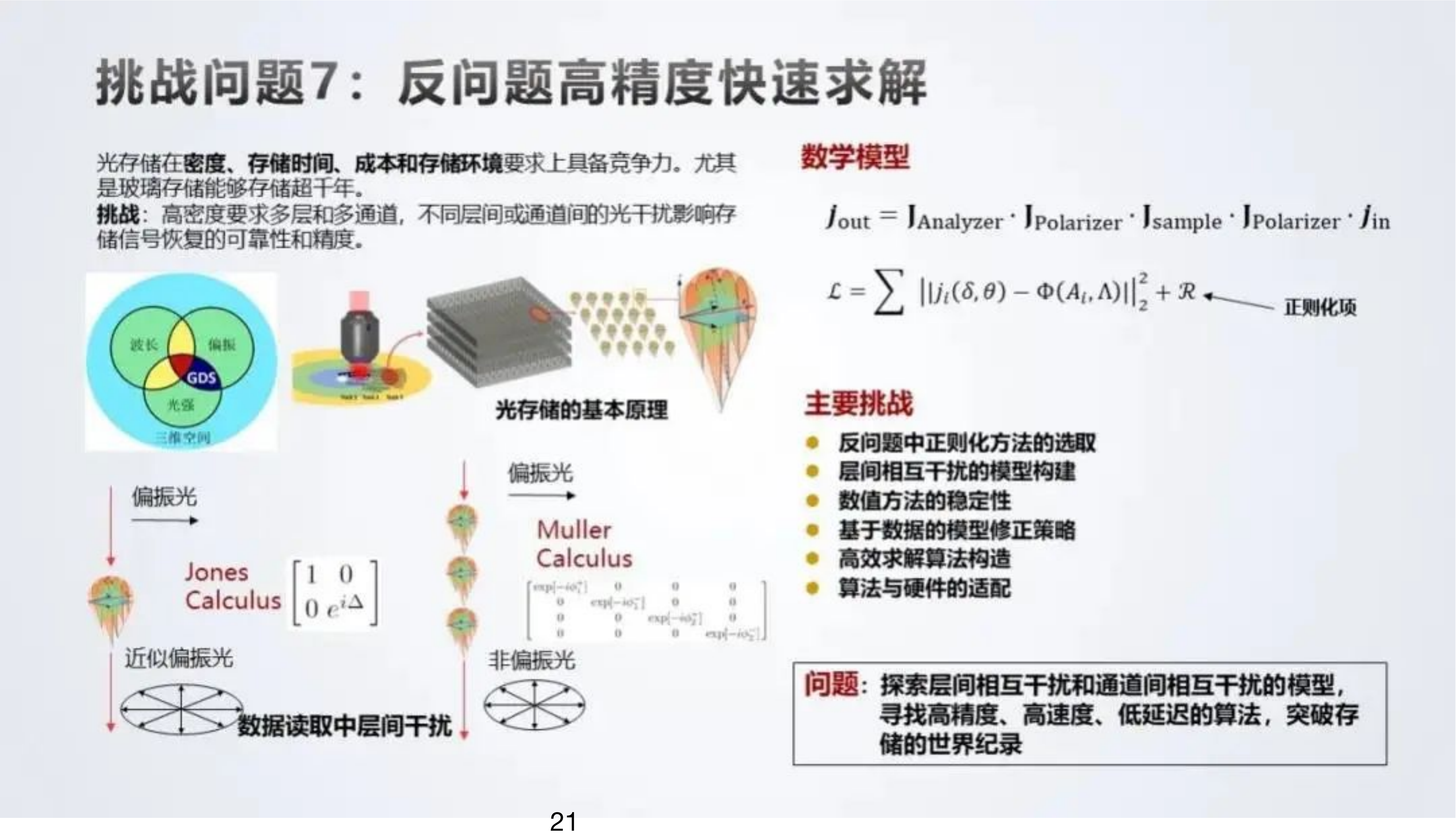
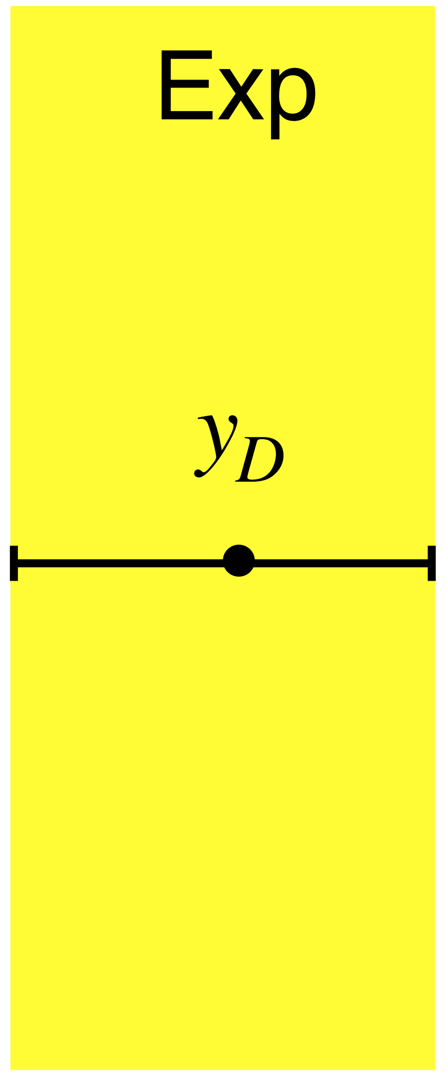
Thanks!

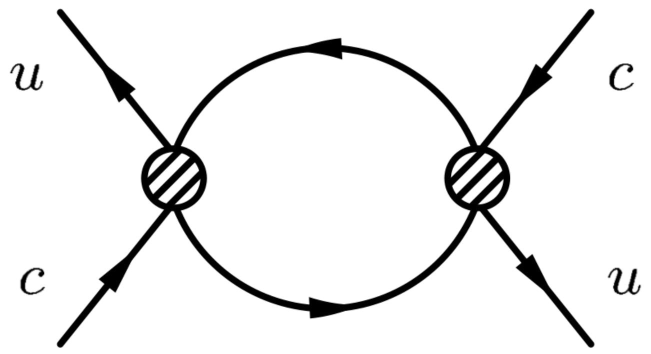

后香农时代，数学决定未来发展的边界
徐文伟
华为董事、战略研究院院长
HUAWEI


挑战问题7：反问题高精度快速求解
光存储在密度、存储时间、成本和存储环境要求上具备竞争力。尤其是玻璃存储能够存储超千年。
挑战：高密度要求多层和多通道，不同层间或通道间的光干扰影响存储信号恢复的可靠性和精度。
波长
偏振
GDS
光强
三维空间
光存储的基本原理
偏振光
近似偏振光
Jones Calculus
$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\Delta} \end{bmatrix}$
数据读取中层间干扰
偏振光
Muller Calculus
非偏振光
数学模型
$j_{out} = J_{Analyzer} \cdot J_{Polarizer} \cdot J_{sample} \cdot J_{Polarizer} \cdot j_{in}$
$\mathcal{L} = \sum \left\| j_i(\delta, \theta) - \Phi(A_i, \Lambda) \right\|_2^2 + \mathcal{R}$
正则化项
主要挑战
反问题中正则化方法的选取
层间相互干扰的模型构建
数值方法的稳定性
基于数据的模型修正策略
高效求解算法构造
算法与硬件的适配
问题：探索层间相互干扰和通道间相互干扰的模型，寻找高精度、高速度、低延迟的算法，突破存储的世界纪录



$y_{\text{exp}} = (6.1 \pm 0.8) \times 10^{-3}$

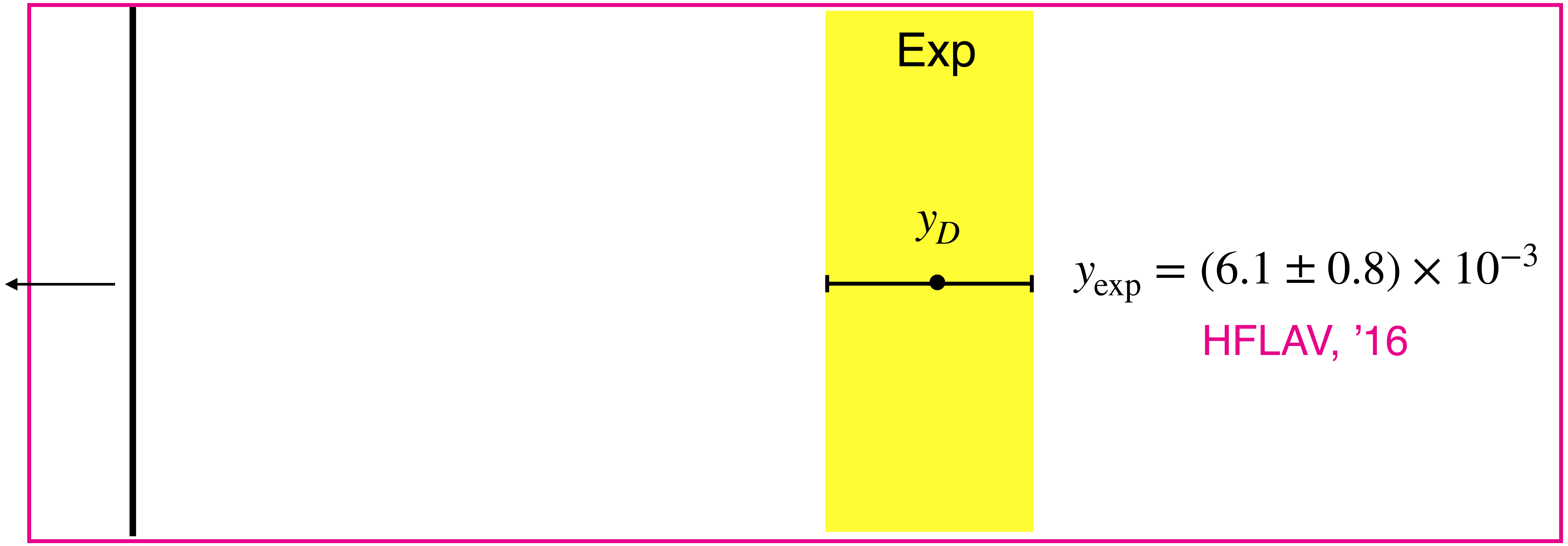
HFLAV, ’16



Inclusive approach doesn't work

$y_{\mathrm{incl}} \sim 10^{-7}$

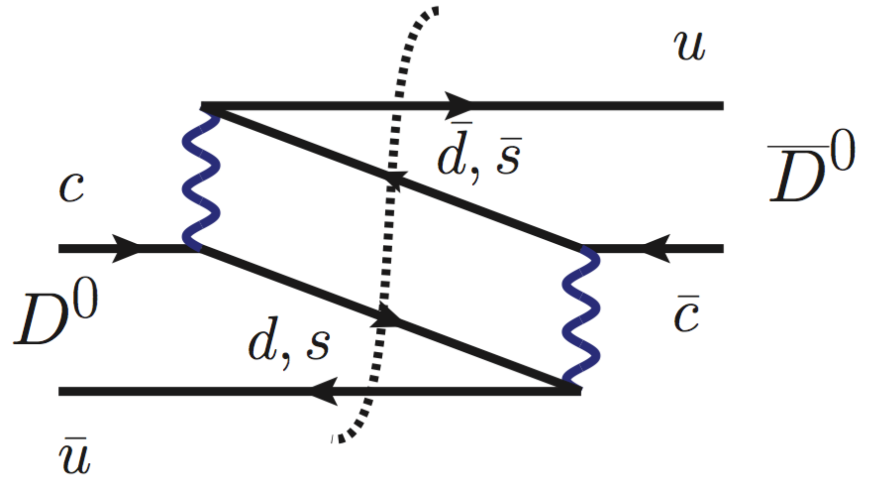
Lenz, *et al*, '12

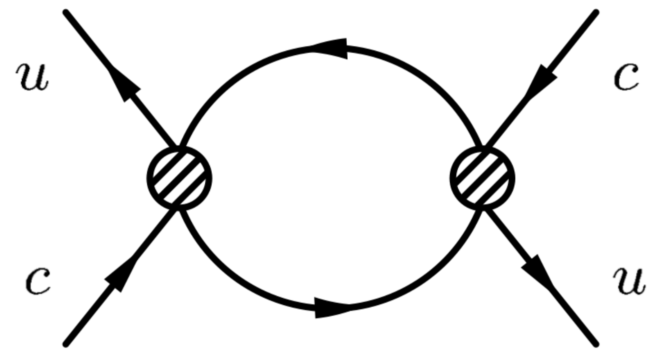


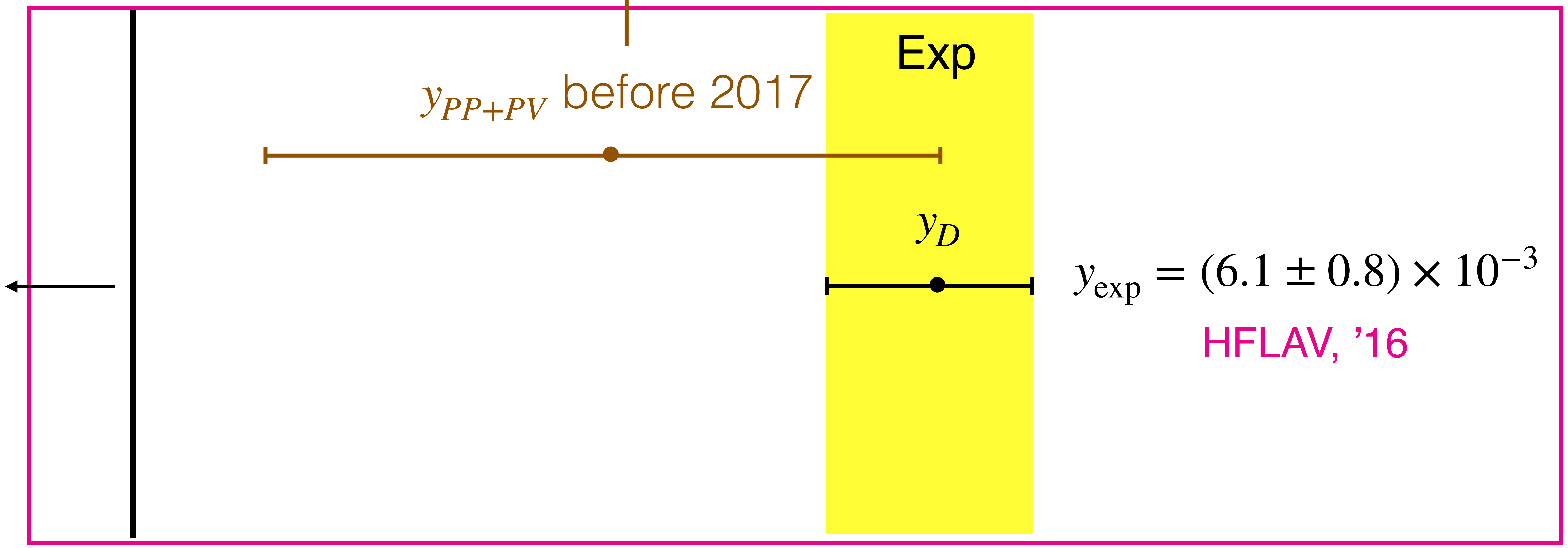
Falk, et al, ’02; Cheng, Chiang, ’10

$y_{\mathrm{PP+PV}} = (3.6 \pm 2.6) \times 10^{-3}$

- **Before 2017, exclusive approach is hopeful**







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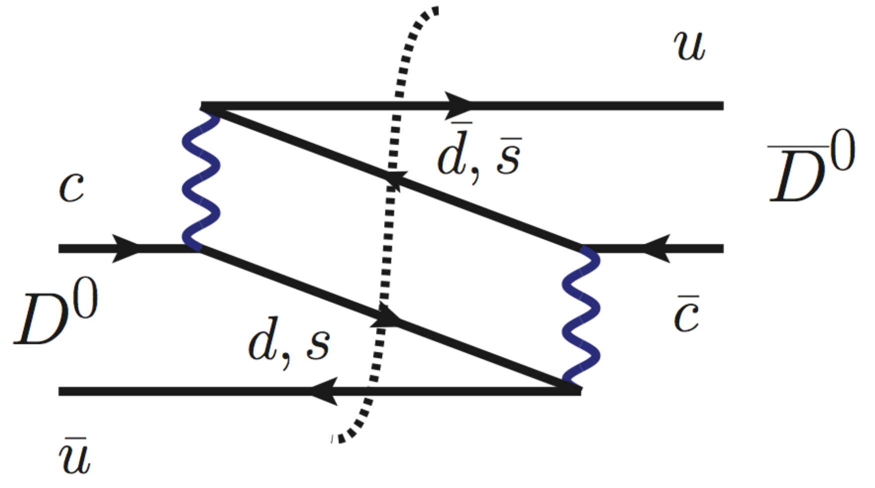

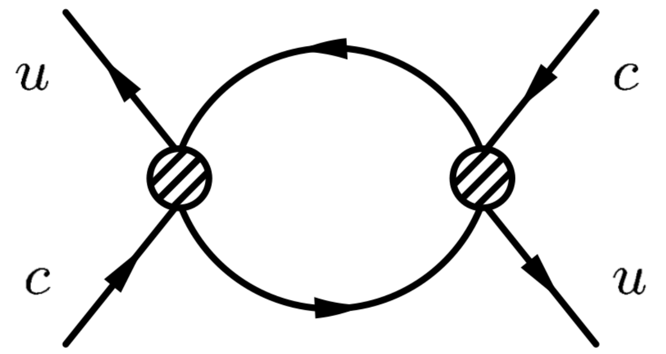

Inclusive approach doesn’t work

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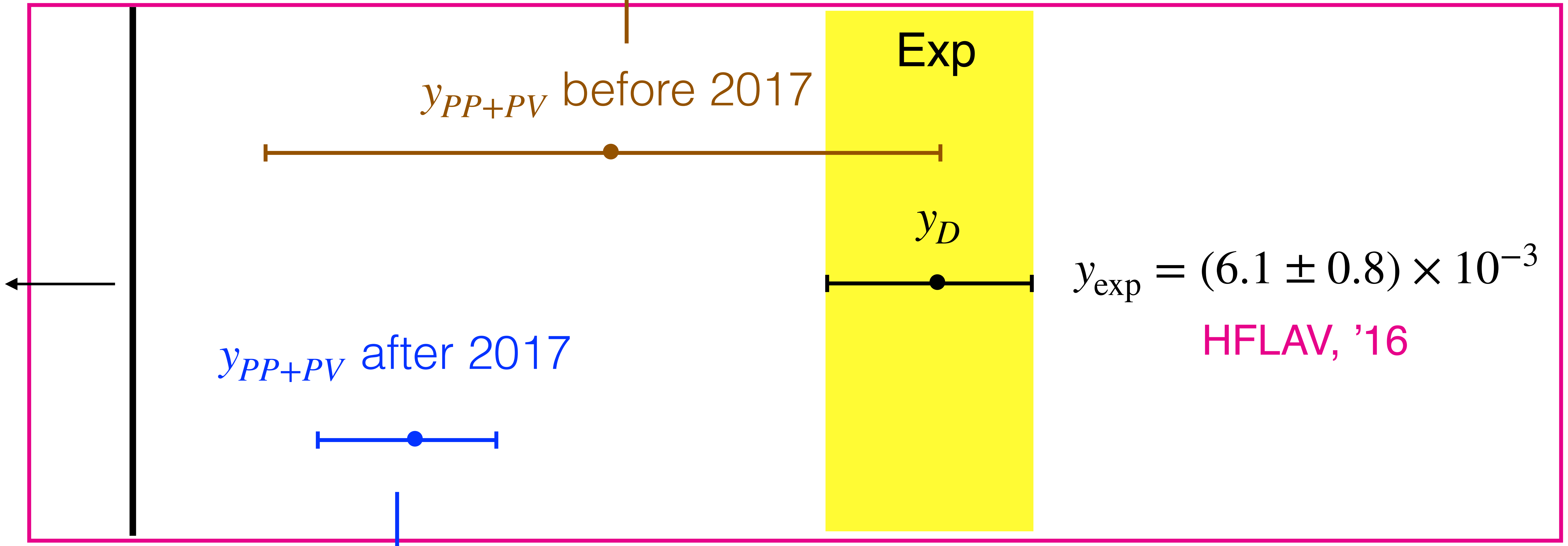
Lenz, *et al*, ’12



- **After 2017, exclusive approach is dying**

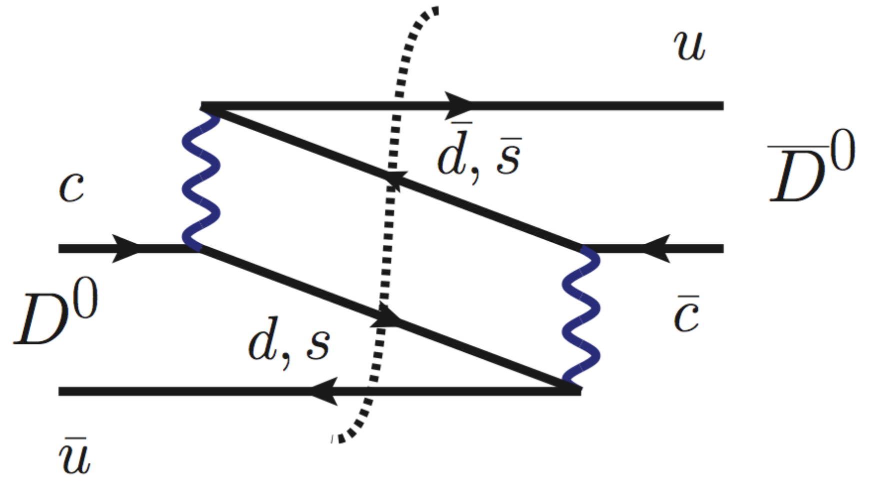
$y_{\mathrm{PP+PV}} = (2.1 \pm 0.7) \times 10^{-3}$

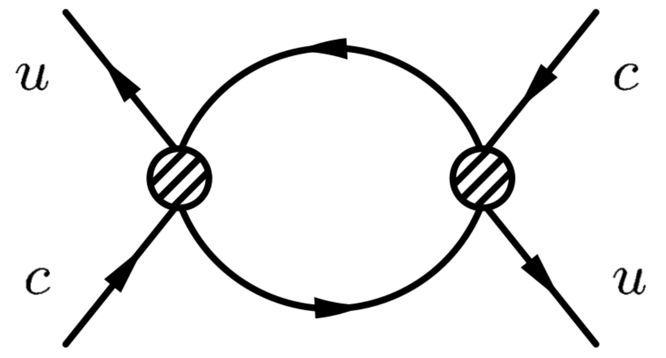
Jiang, **FSY**, Qin, Li, Lü, ’17

Falk, et al, '02; Cheng, Chiang, '10

$y_{\mathrm{PP+PV}} = (3.6 \pm 2.6) \times 10^{-3}$

- **Before 2017, exclusive approach is hopeful**





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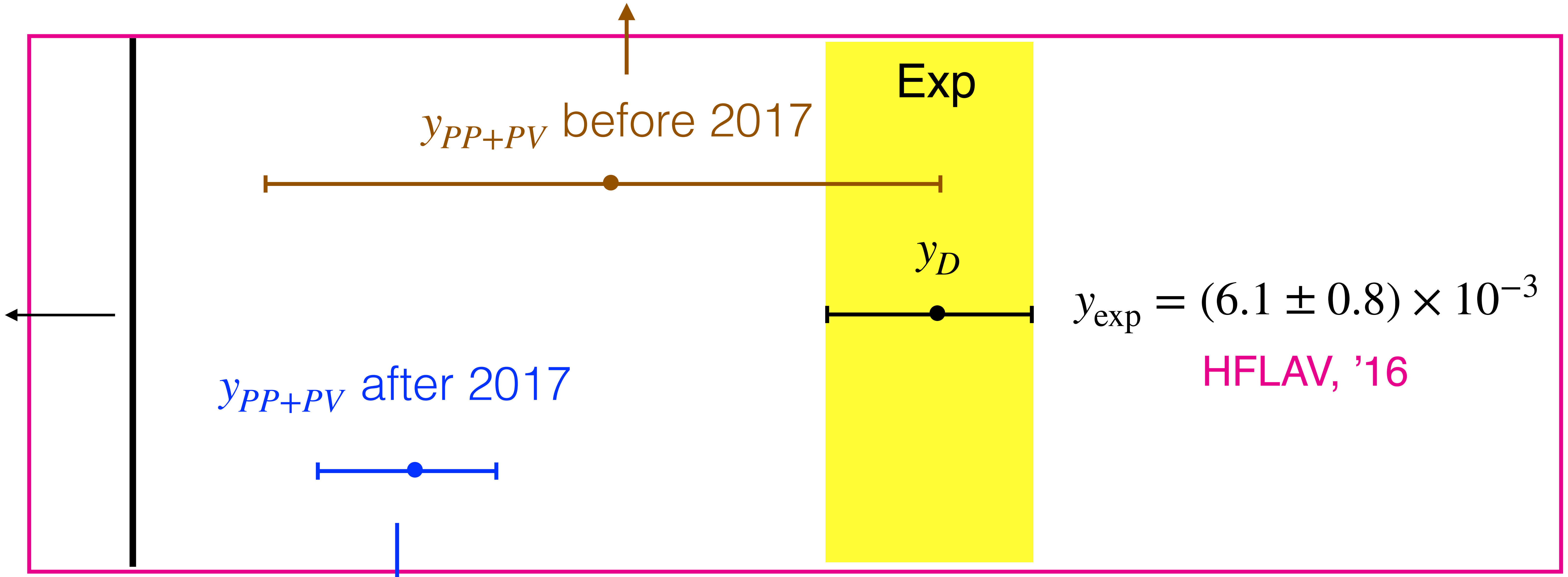
Lenz, *et al*, '12



$y_{\mathrm{exp}} = (6.1 \pm 0.8) \times 10^{-3}$

HFLAV, '16

- **After 2017, exclusive approach is dying**

$y_{\mathrm{PP+PV}} = (2.1 \pm 0.7) \times 10^{-3}$

Jiang, **FSY**, Qin, Li, Lü, '17

No theoretical methods work for D0 mixing

No theoretical predictions for indirect CP violation