Exotic double-charm molecules with hidden or open strangeness and around 4.5~4.7 GeV

Fu-Lai Wang (王福来)
Lanzhou University

Based on [**Phys. Rev. D 102, 094006 (2020); arXiv:2103.03127**]
Collaborators: **Xin-Dian Yang, Zhan-Wei Liu, and Xiang Liu**

第七届XYZ粒子研讨会@青岛
14-18 May, 2021

Outline

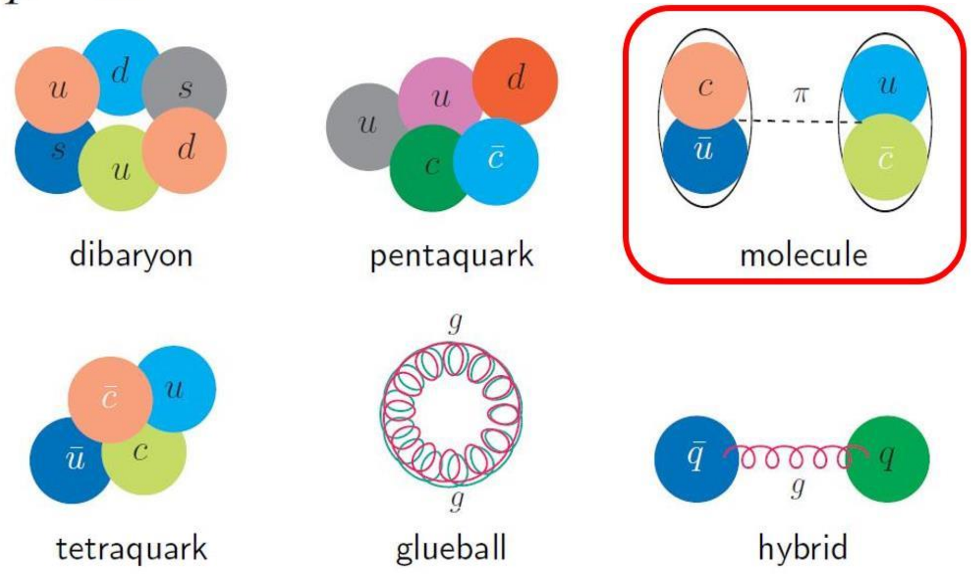
- Background
- Interactions involved in the Hs$\overline{\mathrm{T}}$s and HsTs systems
- Possible double-charm molecular candidates with hidden or open strangeness and around 4.5~4.7 GeV
- Newly observed $X(4630)$: charmoniumlike molecule
- Summary

Hadron Physics —— Hadron states (quark & gluon)

Hadron physics is mainly focused on hadron spectra, structures, interactions, etc.

In general, **the hadron configurations** include

Hadron
- Conventional states
  - Meson : $q\bar{q}$, $Q\bar{q}$, $Q\bar{Q}$
  - Baryon : qqq, Qqq, QQq, ...
- Exotic states
  - Molecular state
  - Hybrid meson
  - Glueball
  - Tetraquark
  - Pentaquark
  - ...

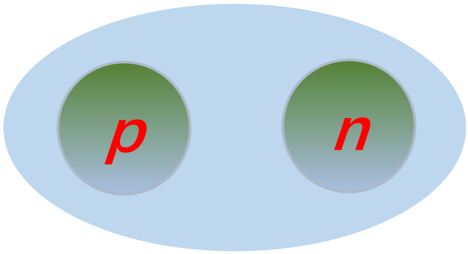


H. X. Chen, W. Chen, X. Liu and S. L. Zhu, Phys. Rept. 639, 1 (2016)

Hadronic molecule

H. X. Chen, W. Chen, X. Liu and S. L. Zhu, Phys. Rept. 639, 1 (2016)

Hadronic molecule is a **loosely bound state** formed by two or more hadrons.



The deuteron is a typical example of the molecular state, which is a loosely bound state of **a proton and a neutron** with a binding energy around 2.225 MeV.

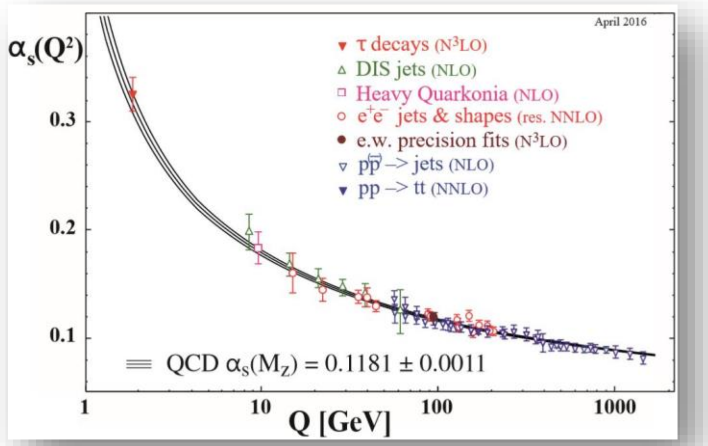
PRC 63, 024001 (2001)

Hadron-hadron interactions are important for the hadronic molecules.

Different models and approaches to probe the hadronic molecules:

- **Meson-exchange model;**
- QCD sum rules ;
- Lattice QCD simulation;
- Chiral perturbation theory;
- And so on.

H. X. Chen, et al., Phys. Rept. 639, 1 (2016)
F. K. Guo, et al., Rev. Mod.Phys.90, 015004 (2018).
Y. R. Liu, et al., Prog. Part. Nucl. Phys. 107, 237 (2019).



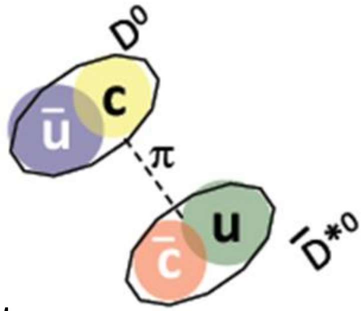
Interactions cannot be perturbatively solved at low energy.

Situation has been changed since 2003

Early study: The possible two charm-meson molecular states. M. B. Voloshin, L. B. Okun, JETP Lett. 23, 333 (**1976**)

$\Psi(4040)$: P−wave $D^*\bar{D}^*$ molecule. A.D. Rujula, H. Georgi, and S. L. Glashow, PRL 38, 317(**1977**)

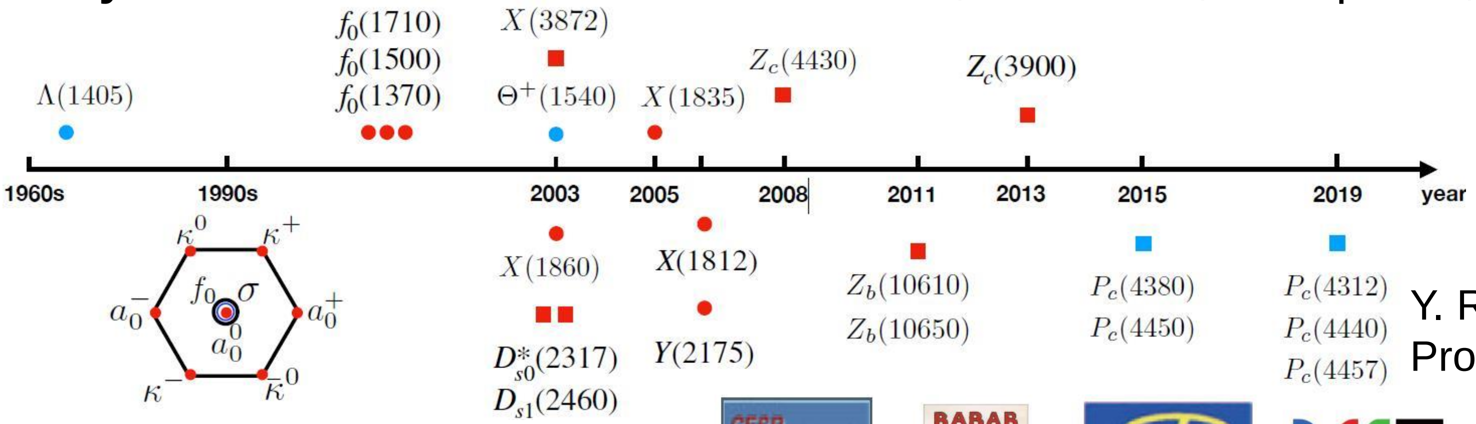
Unfortunately, the early-stage theoretical efforts on the molecular states were not supported by experiments.



Breakthrough in multiquarks: **X(3872)** was first observed by Belle Collaboration, the study of multiquark states has entered upon a new era.

[Belle] PRL 91, 262001 (2003)

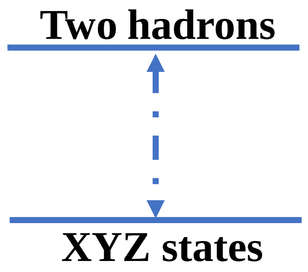
Plenty of XYZ states: traditional charmonium, molecules, tetraquarks, hybrids, kinetic effects, ...



Y. R. Liu, H. X. Chen, W. Chen, X. Liu and S. L. Zhu, Prog. Part. Nucl. Phys. 107 (2019) 237

How these XYZ states were decoded as the hidden-charm molecules ?

The masses of several XYZ states are close to the thresholds of two hadrons, hadronic molecular state has aroused heated discussion.



X(3872) $D\bar{D}^*$ **molecule**
[Belle] PRL 91,262001 (2003)
Z(4430) $D^*\bar{D}_1^{(\prime)}$ **molecule**
[Belle] PRL 100, 142001 (2008)
Y(3930) $D^*\bar{D}^*$ **molecule**
[Belle] PRL 96,082003 (2006)
Y(4220) $D\bar{D}_1$ **molecule**
[BABAR] PRL 95, 142001 (2005)

Y(4140) $D_s^*\bar{D}_s^*$ **molecule**
[CDF] PRL 102, 242002 (2009)
Y(4274) $D_s\bar{D}_{s0}^*$ **molecule**
[LHCb] PRD 95, 012002 (2017)
……

H. X. Chen, et al., Phys. Rept. 639, 1 (2016)
F. K. Guo, et al., Rev. Mod.Phys.90, 015004 (2018).
Y. R. Liu, et al., Prog. Part. Nucl. Phys. 107, 237 (2019).

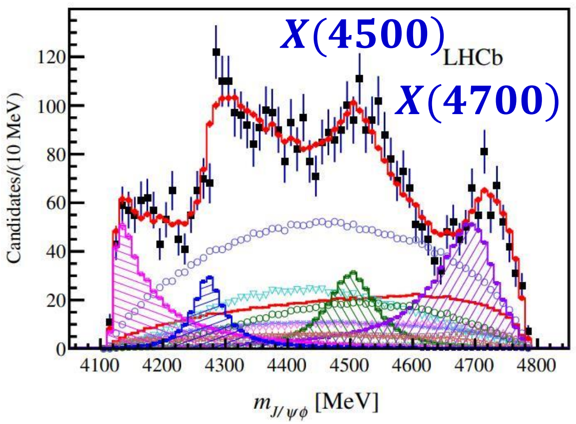
These observed XYZ states also result in several systematic theoretical calculations of the interaction between charmed mesons and anti-charmed mesons.

- ✓ L. L. Shen, X. L. Chen, Z. G. Luo, P. Z. Huang, S. L. Zhu, P. F. Yu, and X. Liu, EPJC 70, 183 (2010);
- ✓ B. Hu, X. L. Chen, Z. G. Luo, P. Z. Huang, S. L. Zhu, P. F. Yu, and X. Liu, CPC 35, 113 (2011);
- ✓ Z. F. Sun, Z. G. Luo, J. He, X. Liu, and S. L. Zhu, CPC 36, 194 (2012);
- ✓ Z. F. Sun, X. Liu, M. Nielsen, and S. L. Zhu, PRD 85, 094008 (2012);
- ✓ R. Chen, X. Liu, Y. R. Liu, and S. L. Zhu, EPJC 76, 319 (2016) .
- ✓ X. K. Dong, F. K. Guo, and B. S. Zou, Progr. Phys. 41, 65 (2021) . ……

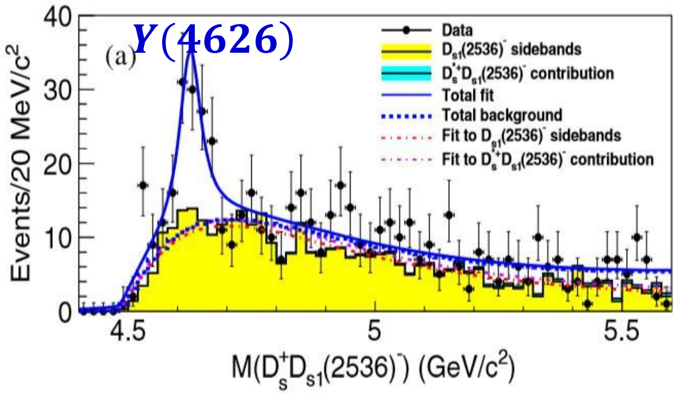
These studies enlarged our knowledge of hidden-charm molecular states with mass below 4.5 GeV. However, **our knowledge of hidden-charm molecular states above 4.5 GeV is still not enough.**

More XYZ states with higher mass and White paper of the BESIII

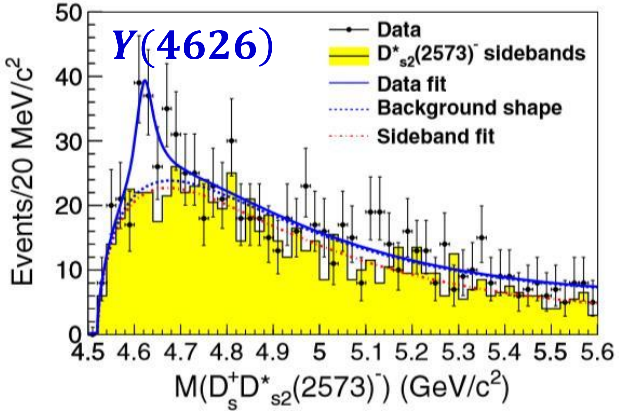
In the past years, more XYZ states with higher mass were announced.



[LHCb] PRL 118, 022003 (2017)



[Belle] PRD 100, 111103(2019)



[Belle] PRD 101, 091101 (2020)

In 2019, the BESIII announced a white paper, they plan to perform a detailed scan of cross sections between 4.0 to 4.6 GeV and take more data above 4.6 GeV, and these measurements provide us good opportunity to study XYZ states above 4.5 GeV.

[BESIII] CPC 44, 040001 (2020)

We propose to explore double-charm molecules with hidden or open strangeness existing in mass range around 4.5 ~ 4.7 GeV, which are relevant to the interactions between charmed-strange meson in H-doublet and (anti-)charmed-strange meson in T-doublet.

$$H = \left(0^-, 1^-\right)$$

$[D_S, D_S^*]$

$$T = \left(1^+, 2^+\right)$$

$[D_{S1}(2536), D_{S2}^*(2573)]$

The hidden-charm and hidden-strange molecular systems $(c\bar{s})(\bar{c}s)$

$D_s\bar{D}_{s1}$ $D_s\bar{D}^*_{s2}$ $D^*_s\bar{D}_{s1}$ $D^*_s\bar{D}^*_{s2}$

The open-charm and open-strange molecular systems $(c\bar{s})(c\bar{s})$

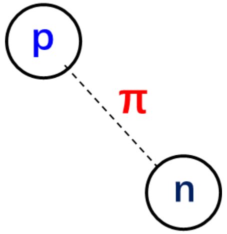
D_sD_{s1} $D_sD^*_{s2}$ $D^*_sD_{s1}$ $D^*_sD^*_{s2}$

Interactions involved in the $Hs\overline{T}s$ and HsTs systems

Meson-exchange model

Talks by 朱俊韬、陈锐

The molecular state is bound by the meson exchange force at the hadronic level.



Nucleon-nucleon interaction is mediated through the exchange of pion meson, which contributes to the long-range part of the nuclear force.

H. Yukawa, Proc.Phys. Math. Sco. Jap. 17 (1935)

In the **one boson exchange (OBE) model**, the potential between two hadrons is generated by the exchange of a series of mesons, which includes the σ, π, η, ρ, and ω (plus a few extra mesons in more sophisticated versions).

$$\sigma,\quad \mathbb{P} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}, \quad \mathbb{V} = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix}.$$

中程: $0.8\ \text{fm} \le r \le 2.0\ \text{fm}$

长程: $r \ge 2.0\ \text{fm}$

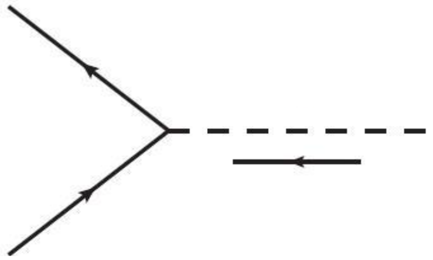
短程: $r \le 0.8\ \text{fm}$

The OBE model is a powerful model to depict nuclear force and explain observed X/Y/Z/Pc /Pcs states.

H. X. Chen, W. Chen, X. Liu and S. L. Zhu, Phys. Rept. 639, 1 (2016)

The interaction with light mesons $\mathcal{V}(\mathrm{r})$

1. We first obtain **the scattering amplitude** with the effective Lagrangian approach.



2. With the help of the Breit approximation, a relation between **the effective potentials in momentum space** and the scattering amplitudes is obtained,

$$\mathcal{V}_E(q) = -\frac{\mathcal{M}(h_1h_2 \to h_3h_4)}{\sqrt{\prod_i 2M_i \prod_f 2M_f}}$$

G. Breit, Phys. Rev. 34, 553 (1929)
G. Breit, Phys. Rev. 36, 383 (1930)

3. **The effective potential in the coordinate space $\mathcal{V}(\mathrm{r})$** is obtained by performing Fourier transformation,

$$\mathcal{V}_E(r) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r}\mathcal{V}_E(q)\mathcal{F}^2(q^2)$$

Due to the discussed hadrons are not point-like particles, we adopt the form factor $\mathcal{F}_M(q^2)$,

$$\mathcal{F}_M(q^2) = \frac{\Lambda^2 - m^2}{\Lambda^2 - q^2}$$

N. A. Tornqvist, Z. Phys. C 61, 525 (1994)
N. A. Tornqvist, Nuovo Cim. A 107, 2471 (1994)

Here, Λ is the phenomenological parameter.

One free parameter

The Lagrangians involved in the Hs$\bar{T}$s and HsTs systems

With the heavy quark symmetry, the chiral symmetry, and the hidden local symmetry, **the compact effective Lagrangians** depicting the interactions between the (anti-)charmed-strange mesons and light mesons can be written as

G. J. Ding, PRD 79, 014001 (2009)

$$
\begin{aligned}
\mathcal{L} = {} & ig\left\langle H_b^{(Q)} \mathcal{A}_{ba}\gamma_5 \overline{H}_a^{(Q)}\right\rangle + ig\left\langle \overline{H}_a^{(\overline{Q})} \mathcal{A}_{ab}\gamma_5 H_b^{(\overline{Q})}\right\rangle + ik\left\langle T_b^{(Q)\mu} \mathcal{A}_{ba}\gamma_5 \overline{T}_{a\mu}^{(Q)}\right\rangle + ik\left\langle \overline{T}_a^{(\overline{Q})\mu} \mathcal{A}_{ab}\gamma_5 T_{b\mu}^{(\overline{Q})}\right\rangle \\
& + \left[i\left\langle T_b^{(Q)\mu}\left(\frac{h_1}{\Lambda_\chi} D_\mu \not{\mathcal{A}} + \frac{h_2}{\Lambda_\chi} \not{D}\mathcal{A}_\mu\right)_{ba} \gamma_5 \overline{H}_a^{(Q)}\right\rangle + H.c.\right] + \left[i\left\langle \overline{H}_a^{(\overline{Q})}\left(\frac{h_1}{\Lambda_\chi} \not{\mathcal{A}}\, \overleftarrow{D}'_\mu + \frac{h_2}{\Lambda_\chi} \mathcal{A}_\mu \overleftarrow{\not{D}}'\right)_{ab} \gamma_5 T_b^{(\overline{Q})\mu}\right\rangle + H.c.\right] \\
& + \left\langle iH_b^{(Q)}\left(\beta v^\mu(\mathcal{V}_\mu - \rho_\mu) + \lambda\sigma^{\mu\nu}F_{\mu\nu}(\rho)\right)_{ba} \overline{H}_a^{(Q)}\right\rangle - \left\langle i\overline{H}_a^{(\overline{Q})}\left(\beta v^\mu(\mathcal{V}_\mu - \rho_\mu) - \lambda\sigma^{\mu\nu}F_{\mu\nu}(\rho)\right)_{ab} H_b^{(\overline{Q})}\right\rangle \\
& + \left\langle iT_{b\lambda}^{(Q)}\left(\beta'' v^\mu(\mathcal{V}_\mu - \rho_\mu) + \lambda''\sigma^{\mu\nu}F_{\mu\nu}(\rho)\right)_{ba} \overline{T}_a^{(Q)\lambda}\right\rangle - \left\langle i\overline{T}_{a\lambda}^{(\overline{Q})}\left(\beta'' v^\mu(\mathcal{V}_\mu - \rho_\mu) - \lambda''\sigma^{\mu\nu}F_{\mu\nu}(\rho)\right)_{ab} T_b^{(\overline{Q})\lambda}\right\rangle \\
& + \left[\left\langle T_b^{(Q)\mu}\left(i\zeta_1(\mathcal{V}_\mu - \rho_\mu) + \mu_1\gamma^\nu F_{\mu\nu}(\rho)\right)_{ba} \overline{H}_a^{(Q)}\right\rangle + H.c.\right] - \left[\left\langle \overline{H}_a^{(\overline{Q})}\left(i\zeta_1(\mathcal{V}_\mu - \rho_\mu) - \mu_1\gamma^\nu F_{\mu\nu}(\rho)\right)_{ab} T_b^{(\overline{Q})\mu}\right\rangle + H.c.\right],
\end{aligned}
$$

Super-field

Axial current

Vector current

Vector meson field

Vector meson field strength tensor

FLW and X. Liu, PRD 102, 094006 (2020)

The coupling constants involved in $H_s\bar{T}_s$ and H_sT_s systems

Determination of the coupling constants:

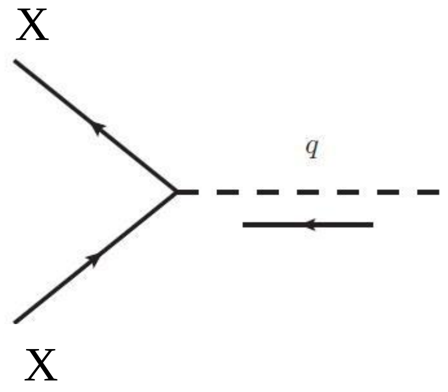
①fit to experimental data;
②phenomenological models.

The phase factors are fixed with the quark model.

Quark model

Nucl. Phys. A 679, 577 (2001)

All of the coupling constants are related to calculate the coupling vertex in the quark level and hadronic level, and to equate them.



Coupling constants	(g, k)	$\lvert h'\rvert$	f_π	(β, β'')
	(0.59, 0.59)	0.55	0.132	(−0.90, 0.90)
	(λ, λ'')	$\lvert\zeta_1\rvert$	μ_1	g_V
	(−0.56, 0.56)	0.20	0	5.83

FLW and X. Liu, PRD 102, 094006 (2020)

The interactions involved in the $H_s\bar{T}_s$ and H_sT_s systems

For the $H_s\bar{T}_s$ systems, we need distinguish **the charge parity quantum numbers**, and these systems have the charge conjugate transformation invariance.

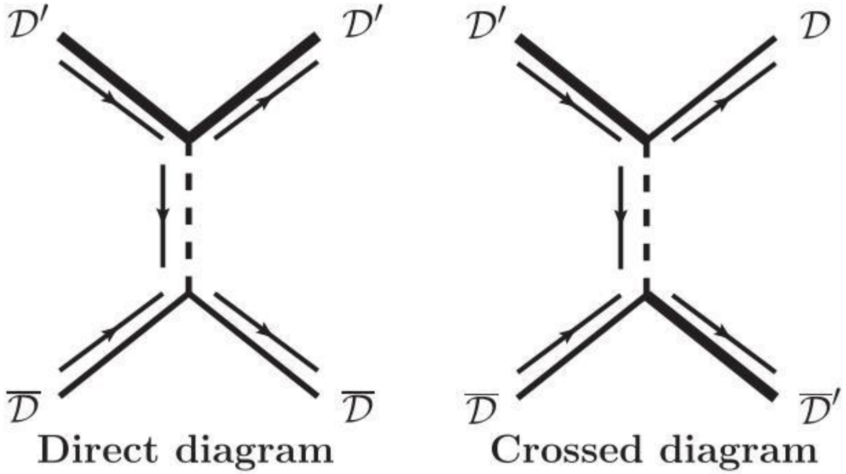


FIG. 1: The direct channel and crossed channel Feynman diagrams for the $\mathcal{D}'\bar{\mathcal{D}}$ systems. The thick (thin) lines stand for the charmed-strange mesons $\mathcal{D}'(\mathcal{D})$, while the dashed lines represent the exchanged light mesons.

The total effective potentials:

$$\mathcal{V}_{Total}(\boldsymbol{r}) = \mathcal{V}_D(\boldsymbol{r}) + c\,\mathcal{V}_C(\boldsymbol{r})$$

where $\mathcal{V}_D(\boldsymbol{r})$ and $\mathcal{V}_C(\boldsymbol{r})$ are the effective potentials corresponding to the direct channel and crossed channel, respectively.

The flavor wave function：

$$|0,0\rangle = \frac{|\mathcal{D}'^+\mathcal{D}^-\rangle + c|\mathcal{D}^+\mathcal{D}'^-\rangle}{\sqrt{2}}$$

Here, we want to emphasize that there exists a relation of c and the C parity, i.e., $C = -c\cdot(-1)^{J_1-J_2-J}$, where the notations J, J_1, and J_2 stand for the total angular momentum quantum numbers of the $\mathcal{D}'\bar{\mathcal{D}}$, the charmed-strange mesons $\mathcal{D}'$, and the charmed-strange mesons $\mathcal{D}$, respectively.

Y. R. Liu, X. Liu, W. Z. Deng, and S. L. Zhu, EPJC 56, 63 (2008)
X. Liu, Z. G. Luo, Y. R. Liu, and S. L. Zhu, EPJC 61, 411 (2009)
Z. F. Sun, X. Liu, M. Nielsen, and S. L. Zhu, PRD 85, 094008 (2012)

FLW and X. Liu, PRD 102, 094006 (2020)

The effective potentials

Single channel analysis !

- $D_s\bar{D}_{s1} \to D_s\bar{D}_{s1}$ process:

$$\mathcal{V}_D = \frac{C}{2}\frac{\mathcal{A}_1+\mathcal{A}_1'}{2}Y_\phi,$$
$$\mathcal{V}_C = \frac{E}{3}\frac{\mathcal{A}_2+\mathcal{A}_2'}{2}Y_{\phi 0}.$$

- $D_s\bar{D}^*_{s2} \to D_s\bar{D}^*_{s2}$ process:

$$\mathcal{V}_D = \frac{C}{2}\frac{\mathcal{A}_3+\mathcal{A}_3'}{2}Y_\phi,$$
$$\mathcal{V}_C = \frac{2B}{3}\left[\frac{\mathcal{A}_4+\mathcal{A}_4'}{2}\mathcal{Z}\mathcal{Z}+\frac{\mathcal{A}_5+\mathcal{A}_5'}{2}\mathcal{T}\mathcal{T}+\frac{\mathcal{A}_6+\mathcal{A}_6'}{2}\{\mathcal{T},\mathcal{Z}\}\right]Y_{\eta 1}.$$

- $D^*_s\bar{D}_{s1} \to D^*_s\bar{D}_{s1}$ process:

$$\mathcal{V}_D = \frac{5A}{27}[\mathcal{A}_8\mathcal{Z}+\mathcal{A}_9\mathcal{T}]Y_\eta+\left[\frac{C}{2}\mathcal{A}_7+\frac{5D}{9}(\mathcal{A}_9\mathcal{T}-2\mathcal{A}_8\mathcal{Z})\right]Y_\phi,$$
$$\mathcal{V}_C = \frac{B}{9}[\mathcal{A}_{10}\mathcal{Z}\mathcal{Z}+\mathcal{A}_{11}\mathcal{T}\mathcal{T}+\mathcal{A}_{12}\{\mathcal{T},\mathcal{Z}\}]Y_{\eta 2}+\frac{E}{12}\mathcal{A}_8 Y_{\phi 2}.$$

- $D^*_s\bar{D}^*_{s2} \to D^*_s\bar{D}^*_{s2}$ process:

$$\mathcal{V}_D = \frac{2A}{9}\left[\frac{\mathcal{A}_{14}+\mathcal{A}_{14}'}{2}\mathcal{Z}+\frac{\mathcal{A}_{15}+\mathcal{A}_{15}'}{2}\mathcal{T}\right]Y_\eta+\left[\frac{2D}{3}\left(\frac{\mathcal{A}_{15}+\mathcal{A}_{15}'}{2}\mathcal{T}-2\frac{\mathcal{A}_{14}+\mathcal{A}_{14}'}{2}\mathcal{Z}\right)+\frac{C}{2}\frac{\mathcal{A}_{13}+\mathcal{A}_{13}'}{2}\right]Y_\phi$$
$$\mathcal{V}_C = \frac{2B}{3}\left[\frac{\mathcal{A}_{16}+\mathcal{A}_{16}'}{2}\mathcal{Z}\mathcal{Z}+\frac{\mathcal{A}_{17}+\mathcal{A}_{17}'}{2}\mathcal{T}\mathcal{T}+\frac{\mathcal{A}_{18}+\mathcal{A}_{18}'}{2}\{\mathcal{T},\mathcal{Z}\}\right]Y_{\eta 3}+\frac{E}{2}\frac{\mathcal{A}_{19}+\mathcal{A}_{19}'}{2}Y_{\phi 3}.$$

The coupling constants：

$$A = gk/f_\pi^2,\ B = h'^2/f_\pi^2,\ C = \beta\beta'' g_V^2,\ D = \lambda\lambda'' g_V^2,\ E = \zeta_1^2 g_V^2.$$

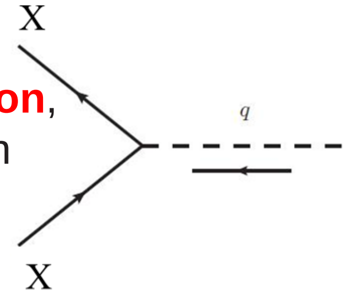
The function and operators:

$$Y_i \equiv Y(\Lambda_i, m_i, r) = \frac{e^{-m_i r}-e^{-\Lambda_i r}}{4\pi r}-\frac{\Lambda_i^2-m_i^2}{8\pi\Lambda_i}e^{-\Lambda_i r}$$
$$m_i = \sqrt{m^2-q_i^2},\ \Lambda_i = \sqrt{\Lambda^2-q_i^2}.$$
$$q_0 = m_{D_{s1}}-m_{D_s},\ q_1 = m_{D^*_{s2}}-m_{D_s},\ q_2 = m_{D_{s1}}-m_{D^*_s},\ q_3 = m_{D^*_{s2}}-m_{D^*_s}.$$
$$\mathcal{Z} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r},\ \mathcal{T} = r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r},\ \{\mathcal{T},\mathcal{Z}\} = \mathcal{T}\mathcal{Z}+\mathcal{Z}\mathcal{T}.$$

The operators:

$\mathcal{A}_k^{(\prime)}$

According to **the spin-parity conservation**, we may determine exchanged particles in the OBE model.



FLW and X. Liu, PRD 102, 094006 (2020)

Possible double-charm molecular candidates with hidden or open strangeness and around 4.5~4.7 GeV

FLW and X. Liu, PRD 102, 094006 (2020)

Before discussing the bound state properties, we need to remark hadronic molecules:

a). For the molecular state, **the binding energy should be tens of MeV or less, and the typical size should be larger than the size of all the included hadrons**.

R. Chen, A. Hosaka, and X. Liu, PRD 97, 036016 (2018)

b). **The S-wave bound states** should first appear since there exists repulsive centrifugal potential for higher partial wave states. Thus, we are mainly interested in the S-wave Hs$\bar{T}$s and HsTs systems.

$$-\frac{1}{2\mu}\left(\nabla^2-\frac{\ell(\ell+1)}{r^2}\right)\psi(r)+V(r)\psi(r)=E\psi(r)$$

Interactions —— **Schrödinger equation** ——→ **Bound state solutions**

Single channel analysis
Coupled channel analysis

The hidden-charm and hidden-strange molecular systems $(c\bar{s})(\bar{c}s)$

FLW and X. Liu, PRD 102, 094006 (2020)

The single channel analysis

TABLE III: Bound state solutions for the S-wave $D_s\bar{D}_{s1}$ system.

J^{PC}	Λ	E	r_{RMS}	Λ	E	r_{RMS}	$P(^3\mathbb{S}_1/^3\mathbb{D}_1)$
1^{--}	4.56	−0.30	4.73	4.56	−0.30	4.73	**100.00**/$o(0)$
	4.78	−0.67	3.63	4.78	−0.67	3.63	**100.00**/$o(0)$
	5.00	−1.15	2.89	5.00	−1.15	2.89	**100.00**/$o(0)$
1^{-+}	3.85	−0.31	4.72	3.85	−0.31	4.72	**100.00**/$o(0)$
	4.43	−2.26	2.12	4.43	−2.26	2.12	**100.00**/$o(0)$
	5.00	−5.07	1.46	5.00	−5.07	1.46	**100.00**/$o(0)$

TABLE IV: Bound state solutions for the S-wave $D_s\bar{D}^*_{s2}$ system.

J^{PC}	Λ	E	r_{RMS}	Λ	E	r_{RMS}	$P(^5\mathbb{S}_2/^5\mathbb{D}_2)$
2^{--}	×	×	×	4.70	−0.22	5.00	**99.87**/0.13
	×	×	×	4.76	−2.87	1.81	**99.53**/0.47
	×	×	×	4.82	−9.76	1.00	**99.07**/0.93
2^{-+}	2.44	−0.31	4.69	2.41	−0.34	4.58	**99.99**/0.01
	2.52	−2.95	1.87	2.49	−3.35	1.76	**99.97**/0.03
	2.60	−10.16	1.06	2.56	−10.26	1.05	**99.93**/0.07

Such cutoffs are unusual and deviate from the reasonable range. Thus, these states as the candidates of the hadronic molecular states are no priority.

TABLE V: Bound state solutions for the S-wave $D_s^*\bar{D}_{s1}$ system.

J^{PC}	Λ	E	$r_{\rm RMS}$	Λ	E	$r_{\rm RMS}$	$P({}^1\mathbb{S}_0/{}^5\mathbb{D}_0)$
	1.68	−0.41	4.14	1.68	−0.42	4.09	**100.00**/$o(0)$
0^{--}	1.72	−4.59	1.38	1.72	−4.63	1.38	**100.00**/$o(0)$
	1.75	−10.27	0.95	1.75	−10.32	0.95	**100.00**/$o(0)$
	1.55	−0.22	4.98	1.55	−0.35	4.48	**99.95**/0.05
0^{-+}	1.59	−3.91	1.57	1.59	−4.28	1.51	**99.89**/0.11
	1.62	−9.75	1.03	1.62	−10.26	1.01	**99.87**/0.13
J^{PC}	Λ	E	$r_{\rm RMS}$	Λ	E	$r_{\rm RMS}$	$P({}^3\mathbb{S}_1/{}^3\mathbb{D}_1/{}^5\mathbb{D}_1)$
	1.83	−0.30	4.63	1.82	−0.23	4.95	**99.96**/0.04/$o(0)$
1^{-+}	1.89	−3.84	1.59	1.88	−3.54	1.66	**99.89**/0.11/$o(0)$
	1.94	−10.06	1.03	1.93	−9.53	1.05	**99.86**/0.13/0.01
	2.00	−0.32	4.51	1.99	−0.23	4.90	**100.00**/$o(0)$/$o(0)$
1^{--}	2.07	−4.20	1.47	2.06	−4.01	1.51	**99.99**/$o(0)$/0.01
	2.13	−10.82	0.95	2.12	−10.71	0.95	**99.97**/0.01/0.02
J^{PC}	Λ	E	$r_{\rm RMS}$	Λ	E	$r_{\rm RMS}$	$P({}^5\mathbb{S}_2/{}^1\mathbb{D}_2/{}^3\mathbb{D}_2/{}^5\mathbb{D}_2)$
	3.313	−0.33	4.46	3.119	−0.29	4.67	**99.55**/0.40/$o(0)$/0.05
2^{--}	3.316	−5.60	1.23	3.125	−3.81	1.55	**98.28**/1.59/$o(0)$/0.13
	3.318	−13.00	0.78	3.130	−10.75	0.92	**97.03**/2.81/$o(0)$/0.16
	2.96	−0.29	4.84	2.86	−0.31	4.76	**99.97**/0.02/$o(0)$/0.01
2^{-+}	3.20	−3.27	1.87	3.04	−3.07	1.92	**99.85**/0.09/$o(0)$/0.06
	3.43	−10.00	1.17	3.22	−10.20	1.15	**99.59**/0.21/$o(0)$/0.20

TABLE VI: Bound state solutions for the S-wave $D_s^*\bar{D}_{s2}^*$ system.

J^{PC}	Λ	E	$r_{\rm RMS}$	Λ	E	$r_{\rm RMS}$	$P({}^3\mathbb{S}_1/{}^3\mathbb{D}_1/{}^5\mathbb{D}_1/{}^7\mathbb{D}_1)$
	1.70	−0.60	3.65	1.69	−0.50	3.90	**99.97**/0.02/$o(0)$/0.01
1^{--}	1.74	−4.15	1.49	1.73	−4.01	1.51	**99.94**/0.05/$o(0)$/0.01
	1.78	−10.68	0.96	1.77	−10.69	0.96	**99.91**/0.08/$o(0)$/0.01
	1.61	−0.56	3.76	1.60	−0.28	4.67	**99.99**/$o(0)$/$o(0)$/0.01
1^{-+}	1.65	−5.12	1.36	1.64	−4.28	1.48	**99.96**/0.01/$o(0)$/0.03
	1.68	−11.56	0.93	1.67	−10.37	0.98	**99.94**/0.02/$o(0)$/0.04
J^{PC}	Λ	E	$r_{\rm RMS}$	Λ	E	$r_{\rm RMS}$	$P({}^5\mathbb{S}_2/{}^3\mathbb{D}_2/{}^5\mathbb{D}_2/{}^7\mathbb{D}_2)$
	2.67	−0.28	4.69	2.63	−0.26	4.77	**99.99**/$o(0)$/0.01/$o(0)$
2^{-+}	2.86	−3.58	1.60	2.80	−3.68	1.58	**99.96**/$o(0)$/0.04/$o(0)$
	3.05	−10.08	0.98	2.96	−10.43	0.96	**99.91**/$o(0)$/0.09/$o(0)$
	1.91	−0.47	4.06	1.90	−0.26	4.82	**99.99**/$o(0)$/0.01/$o(0)$
2^{--}	1.96	−3.57	1.65	1.96	−3.81	1.60	**99.97**/$o(0)$/0.03/$o(0)$
	2.01	−10.20	1.01	2.01	−10.62	1.00	**99.96**/$o(0)$/0.04/$o(0)$
J^{PC}	Λ	E	$r_{\rm RMS}$	Λ	E	$r_{\rm RMS}$	$P({}^7\mathbb{S}_3/{}^3\mathbb{D}_3/{}^5\mathbb{D}_3/{}^7\mathbb{D}_3)$
	4.18	−0.28	4.87	3.87	−0.29	4.80	**99.96**/0.03/$o(0)$/0.01
3^{--}	4.59	−2.56	2.07	4.04	−1.97	2.32	**99.83**/0.13/$o(0)$/0.04
	5.00	−6.36	1.40	4.20	−9.89	1.15	**98.87**/0.97/$o(0)$/0.16

The S-D wave mixing effect **plays a minor role**.

If cutoff smaller than **2.00 GeV** is a reasonable input parameter.

Possible hidden-charm and hidden-strange molecular candidates.

Exotic quantum number!

FLW and X. Liu, PRD 102, 094006 (2020)

The coupled-channel analysis

TABLE VII. Bound state solutions for the S-wave $H_s\bar{T}_s$ coupled systems.

J^{PC}	Threshold	Λ	E	$r_{\rm RMS}$	$P(D_s\bar{D}_{s1}/D_s\bar{D}^*_{s2}/D^*_s\bar{D}_{s1}/D^*_s\bar{D}^*_{s2})$
1^{--}	$D_s\bar{D}_{s1}$	1.92	−0.32	4.42	**91.19**/⋯/$o(0)$/8.18
		1.93	−3.38	1.47	**72.37**/⋯/$o(0)$/27.63
		1.94	−9.24	0.83	**57.50**/⋯/$o(0)$/**42.50**
	$D^*_s\bar{D}_{s1}$	1.86	−2.17	0.56	⋯/⋯/$o(0)$/**100.00**
		1.87	−6.01	0.54	⋯/⋯/$o(0)$/**100.00**
		1.88	−10.06	0.52	⋯/⋯/$o(0)$/**100.00**
1^{-+}	$D_s\bar{D}_{s1}$	1.87	−0.50	3.36	**69.27**/⋯/$o(0)$/30.73
		1.88	−8.88	0.61	25.86/⋯/$o(0)$/**74.14**
		1.89	−20.56	0.39	16.13/⋯/$o(0)$/**83.87**
	$D^*_s\bar{D}_{s1}$	1.74	−0.43	0.58	⋯/⋯/$o(0)$/**100.00**
		1.75	−5.27	0.55	⋯/⋯/$o(0)$/**100.00**
		1.76	−10.46	0.52	⋯/⋯/$o(0)$/**100.00**
2^{--}	$D_s\bar{D}^*_{s2}$	2.27	−3.24	0.28	⋯/$o(0)$/$o(0)$/**100.00**
		2.28	−13.95	0.27	⋯/$o(0)$/$o(0)$/**100.00**
		2.29	−25.24	0.26	⋯/$o(0)$/$o(0)$/**100.00**
	$D^*_s\bar{D}_{s1}$	2.11	−3.89	0.54	⋯/⋯/$o(0)$/**100.00**
		2.12	−8.00	0.51	⋯/⋯/$o(0)$/**100.00**
		2.13	−12.41	0.49	⋯/⋯/$o(0)$/**100.00**
2^{-+}	$D_s\bar{D}^*_{s2}$	2.40	−0.33	4.61	⋯/**99.94**/0.06/$o(0)$
		2.49	−3.47	1.72	⋯/**99.81**/0.19/$o(0)$
		2.57	−10.76	1.01	⋯/**99.64**/0.36/$o(0)$
	$D^*_s\bar{D}_{s1}$	2.86	−0.31	4.74	⋯/⋯/**100.00**/$o(0)$
		3.04	−3.07	1.90	⋯/⋯/**100.00**/$o(0)$
		3.22	−10.18	1.13	⋯/⋯/**100.00**/$o(0)$

The cutoffs in coupled-channel analysis are **smaller** than that in single channel analysis with same binding energy in many cases.

The coupled-channel effect to the S-wave $H_s\bar{T}_s$ systems is not obvious.

- We can predict **the existence of several possible hidden-charm and hidden-strange molecular candidates**, and they can decay into a charmonium plus a light meson, and a charmed-strange meson plus an anticharmed-strange meson if kinetically allowed.

- **The BESIII** should focus on the vector hidden-charm and hidden-strange molecule **from e^+e^- collision,** and the remaining hidden-charm and hidden-strange molecules can be accessible at **the LHCb and Belle II by B meson decays**.

Hidden-charm and hidden-strange molecular systems (Hs$\bar{\text{T}}$s) → **Open-charm and open-strange molecular systems (HsTs)**

The open-charm and open-strange molecular systems $(c\bar{s})(c\bar{s})$

The single channel analysis

TABLE VIII: Bound state solutions for the S-wave H_sT_s systems.

$D_s^*D_{s1}$ ($J^P=2^-$)						
Λ	E	$r_{\rm RMS}$	Λ	E	$r_{\rm RMS}$	$P(^5\mathbb{S}_2/^1\mathbb{D}_2/^3\mathbb{D}_2/^5\mathbb{D}_2)$
4.69	−1.73	1.90	3.32	−0.64	3.49	**97.11**/0.13/o(0)/2.76
4.70	−5.74	0.98	3.36	−4.90	1.33	**93.64**/0.27/o(0)/6.09
4.71	−10.84	0.70	3.39	−10.27	0.93	**91.95**/0.33/o(0)/7.72
$D_s^*D_{s2}^*$ ($J^P=2^-$)						
Λ	E	$r_{\rm RMS}$	Λ	E	$r_{\rm RMS}$	$P(^5\mathbb{S}_2/^3\mathbb{D}_2/^5\mathbb{D}_2/^7\mathbb{D}_2)$
×	×	×	4.66	−0.25	4.85	**98.21**/0.49/0.98/0.33
×	×	×	4.76	−3.52	1.62	**94.49**/1.44/3.06/1.02
×	×	×	4.86	−10.32	0.99	**91.66**/2.09/4.69/1.56
$D_s^*D_{s2}^*$ ($J^P=3^-$)						
Λ	E	$r_{\rm RMS}$	Λ	E	$r_{\rm RMS}$	$P(^7\mathbb{S}_3/^3\mathbb{D}_3/^5\mathbb{D}_3/^7\mathbb{D}_3)$
4.20	−2.59	1.53	3.14	−0.65	3.46	**96.94**/0.09/0.31/3.86
4.21	−6.99	0.90	3.17	−3.89	1.48	**93.83**/0.13/0.44/5.60
4.22	−12.40	0.66	3.20	−9.38	0.97	**91.80**/0.16/0.58/7.47

The coupled-channel analysis

TABLE IX: Bound state solutions for the S-wave H_sT_s coupled-channel systems.

J^P	Th.	Λ	E	$r_{\rm RMS}$	$P(D_sD_{s1}/D_sD_{s2}^*/D_s^*D_{s1}/D_s^*D_{s2}^*)$
2^-	$D_sD_{s2}^*$	2.92	−1.75	1.62	⋯/**60.58**/34.02/5.40
		2.93	−7.86	0.67	⋯/**47.20**/**45.48**/7.32
		2.94	−15.15	0.47	⋯/**42.22**/**49.65**/8.13
	$D_s^*D_{s1}$	3.25	−0.30	4.51	⋯/⋯/**99.88**/0.12
		3.29	−4.45	1.36	⋯/⋯/**99.58**/0.42
		3.32	−10.19	0.91	⋯/⋯/**99.37**/0.63

The analysis does not support the existence of the open-charm and open-strange tetraquark molecular candidates with the S-wave HsTs systems.

FLW and X. Liu, PRD 102, 094006 (2020)

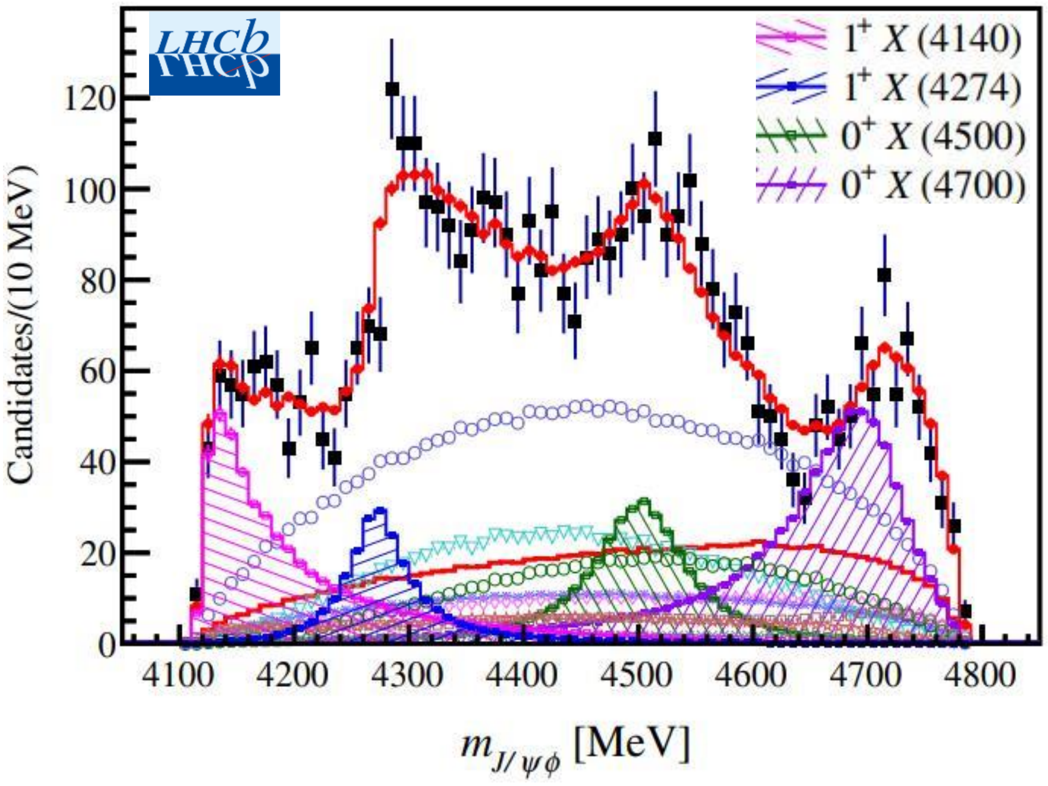
Newly observed $X(4630)$: charmoniumlike molecule

X. D. Yang, **FLW**, Z. W. Liu, and X.Liu, arXiv:2103.03127

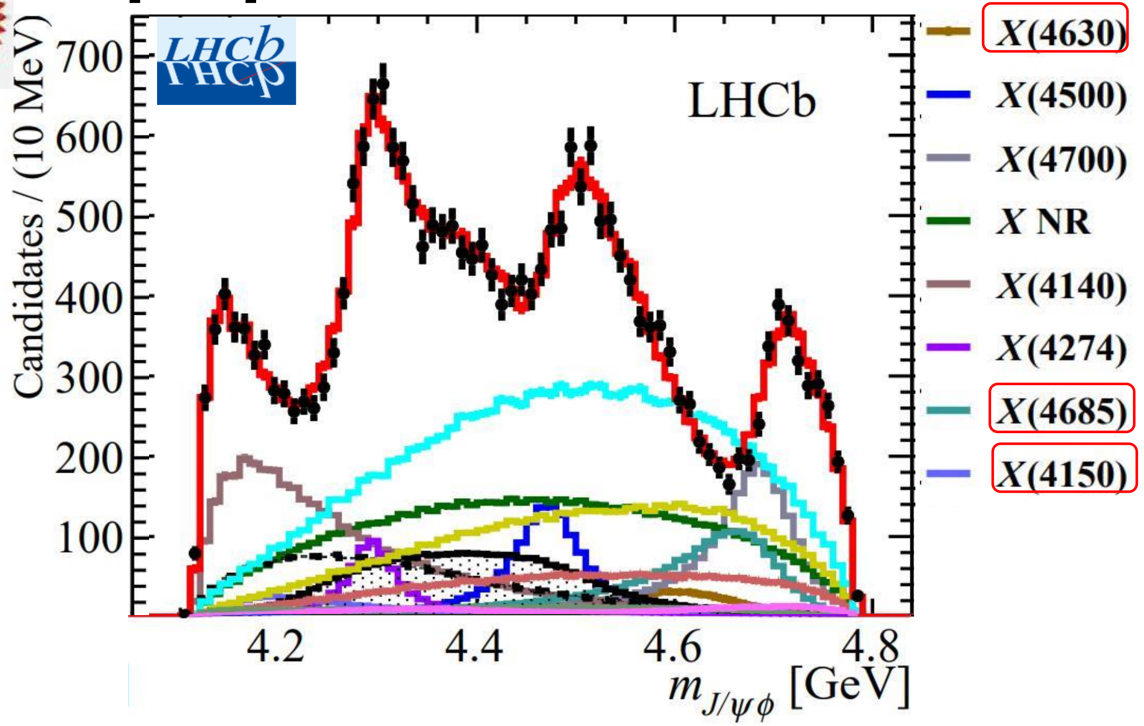
$B \to J/\psi\phi K$ decays from LHCb

Talk by 许泽华

[LHCb] PRL 118, 022003 (2017)



[LHCb] arXiv:2103.01803

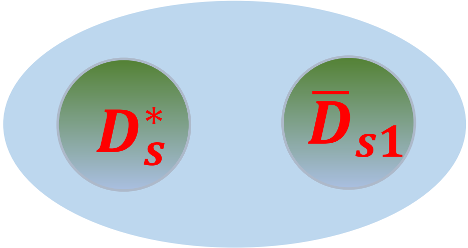


Observation of rich structure in $J/\psi\phi$ invariant mass spectrum

[LHCb] arXiv:2103.01803

Contribution	Significance $[\times\sigma]$	M_0 [MeV]	Γ_0 [MeV]
$X(2^-)$			
$X(4150)$	4.8 (8.7)	$4146 \pm 18 \pm 33$	$135 \pm 28\,^{+59}_{-30}$
$X(1^-)$			
$X(4630)$	5.5 (5.7)	$4626 \pm 16\,^{+18}_{-110}$	$174 \pm 27\,^{+134}_{-73}$
All $X(0^+)$			
$X(4500)$	20 (20)	$4474 \pm 3 \pm 3$	$77 \pm 6\,^{+10}_{-8}$
$X(4700)$	17 (18)	$4694 \pm 4\,^{+16}_{-3}$	$87 \pm 8\,^{+16}_{-6}$
$\mathrm{NR}_{J/\psi\phi}$	4.8 (5.7)		
All $X(1^+)$			
$X(4140)$	13 (16)	$4118 \pm 11\,^{+19}_{-36}$	$162 \pm 21\,^{+24}_{-49}$
$X(4274)$	18 (18)	$4294 \pm 4\,^{+3}_{-6}$	$53 \pm 5 \pm 5$
$X(4685)$	15 (15)	$4684 \pm 7\,^{+13}_{-16}$	$126 \pm 15\,^{+37}_{-41}$

The best hypothesis for the X(4630) state is $\mathbf{1}^-$.



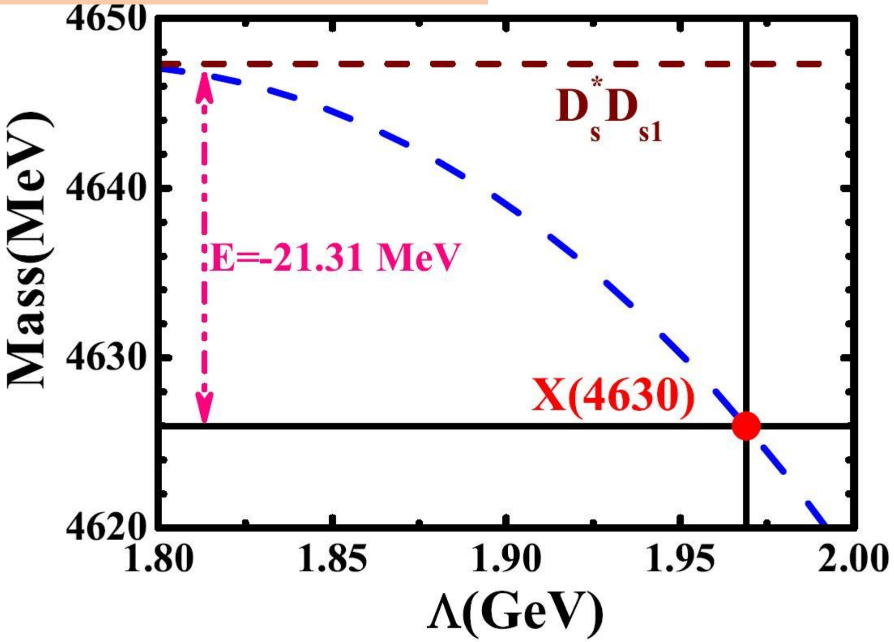
$\boldsymbol{X(4630)}$ $J^{PC} = \mathbf{1}^{-+}$; ⟶ The exotic state different from conventional meson.

$\boldsymbol{M = 4626 \pm 16^{+18}_{-110}}$ **MeV;** $\boldsymbol{\Gamma = 174 \pm 27^{+134}_{-73}}$ **MeV.** ⟶ Near the $D_s^*\, D_{s1}(2536)$ threshold.

The X(4630) can be decoded as the $D_s^*\bar{D}_{s1}(2536)$ molecule

X. D. Yang, **FLW**, Z. W. Liu, and X.Liu, arXiv:2103.03127

The mass spectrum:



The mass of the X(4630) can be reproduced under the $D_s^*\bar{D}_{s1}(2536)$ molecule with $J^{PC}=1^{-+}$ in the OBE model.

Two-body hidden-charmed decay behaviors:

Quark-interchange model

$$\Gamma(J/\psi\phi) : \Gamma(\eta_c(2S)\eta') : \Gamma(\eta_c(2S)\eta) : \Gamma(\chi_{c1}(1P)\eta') : \Gamma(\chi_{c1}(1P)\eta) = 1 : 0.56 : 0.43 : 0.09 : 0.04.$$

The X(4630) was firstly discovered by analyzing **J/ψφ decay channel**.

The X(4630) can be identified as the $D_s^*\bar{D}_{s1}(2536)$ molecule with $J^{PC}=1^{-+}$.

Also supported by Z. G. Wang, arXiv:2103.04236

Summary

- The observation of XYZ states with higher mass and the announced white paper by the BESIII show that **hunting charmoniumlike states with mass above 4.5 GeV become possible**.

- We study the interactions between charmed-strange meson in H-doublet and (anti-)charmed-strange meson in T-doublet with **the OBE model** by considering the S-D wave mixing and coupled channel effects.

- We can predict the existence of several possible hidden-charm and hidden-strange molecular candidates and around 4.5 ~ 4.7 GeV.

- However, **our calculation does not support the existence of such hadronic molecules for the HsTs systems**.

- **The X(4630)** can be decoded as **the** $\boldsymbol{D_s^*\bar{D}_{s1}}(2536)$ **molecule with** $\boldsymbol{J^{PC}=1^{-+}}$ **.**

Thank you for your attention!

Backup

Charmed−strange mesons in $H, \mathrm{T}-$ doublets

M. B. Wise, PRD 45, R2188(1992)

In the heavy quark limit, the charmed-strange mesons can be categorized into different doublets:

$$\boldsymbol{H=(0^-,1^-)},\ \boldsymbol{S=(0^+,1^+)},\ \boldsymbol{T=(1^+,2^+)}.$$

The angular momentum of the light degrees of freedom: $S_l \equiv S_q + L$

The total angular momentum: $J \equiv S_l + S_Q$

S-wave: $S_l \equiv S_q + L = \frac{1}{2} \otimes 0 = \frac{1}{2}$

$$J \equiv S_l + S_Q = \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

$H = \left(0^-, 1^-\right)$

$[\boldsymbol{D_S}, \boldsymbol{D_S^*}]$

P-wave: $S_l \equiv S_q + L = \frac{1}{2} \otimes 1 = \frac{1}{2} \oplus \frac{3}{2}$

$$J \equiv S_l + S_Q = \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

$S = \left(0^+, 1^+\right)$

$$J \equiv S_l + S_Q = \frac{3}{2} \otimes \frac{1}{2} = 1 \oplus 2$$

$T = \left(1^+, 2^+\right)$

$[\boldsymbol{D_{S1}(2536)}, \boldsymbol{D_{S2}^*(2573)}]$

Breit 近似

L. P. P. V. B. Berestetsky, E. M. Lifshitz, Quantum electrodynamics,1982
叶元滔. 奇特类奇异强子分子态[D]. 兰州大学. 2013

对于一般的散射过程如图 3.1 所示。根据一般的散射理论，我们可以得到归一化后的 S 矩阵的矩阵元

$$S_{fi} = \delta_{fi} - i2\pi\delta(E_f - E_i)\langle CD|t|AB\rangle, \quad (3.1)$$

其中 V 和 H_0 分别为体系 Hamilton 量中的相互作用部分和自由部分。t 则满足

$$t = V + V\frac{t}{E - H_0 + i\epsilon},$$

取入射波和出射波为平面波，则可得

$$\langle CD|t|AB\rangle = \frac{1}{(2\pi)^6}\int\int d^3r_1 d^3r_2 e^{-i(\vec{P}_C\cdot\vec{r}_1 + \vec{P}_D\cdot\vec{r}_2)} V(\vec{r}) e^{i(\vec{P}_A\cdot\vec{r}_1 + \vec{P}_B\cdot\vec{r}_2)},$$

其中我们定义 $V(\vec{r})$ 为有效势，$\vec{r} = \vec{r}_1 - \vec{r}_2$，$\vec{r}_1$ 和 $\vec{r}_1$ 分别为 A、B 的坐标。定义 T 矩阵为

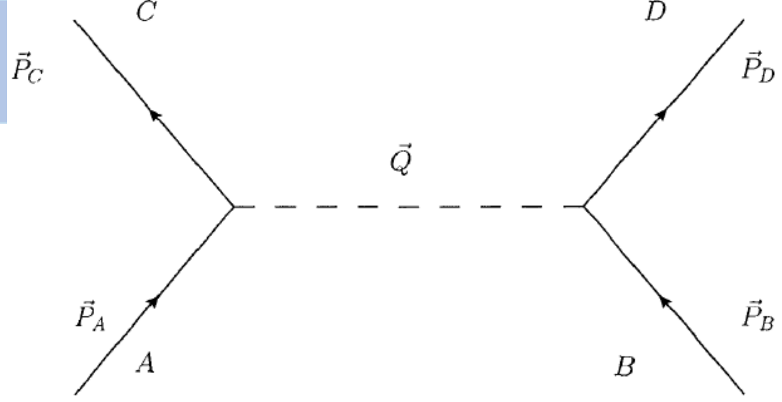
$$\langle CD|t|AB\rangle = \frac{1}{(2\pi)^3} T_{fi}\delta^{(3)}(\vec{P}_A + \vec{P}_B - \vec{P}_C - \vec{P}_D),$$

引进变量 $\vec{P}_1$ 、$\vec{P}_2$ 和 $\vec{Q}$

$$\begin{aligned} \vec{P}_1 &= (\vec{P}_A + \vec{P}_C)/2, \\ \vec{P}_2 &= (\vec{P}_B + \vec{P}_D)/2, \\ \vec{Q} &= \vec{P}_C - \vec{P}_A. \end{aligned}$$

将 T_{fi} 按照 $\vec{P}_1$、$\vec{P}_2$ 展开得

$$T_{fi} = T^{(0)}(Q) + T_i^{(1,0)}(Q)P_{1i} + T_i^{(0,1)}(Q)P_{2i} + \ldots,$$



我们只保留到 T_{fi} 展开式的第一阶，那么有

$$T^{(0)}(Q) = \frac{1}{(2\pi)^6}\int\int d^3r_1 d^3r_2 e^{-i(\vec{P}_C\cdot\vec{r}_1 + \vec{P}_D\cdot\vec{r}_2)} V(\vec{r}) e^{i(\vec{P}_A\cdot\vec{r}_1 + \vec{P}_B\cdot\vec{r}_2)},$$

由此可得到有效势

$$V(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3Q e^{i\vec{Q}\cdot\vec{r}} T^{(0)}(\vec{Q}). \quad (3.2)$$

将 $T^{(0)}(\vec{Q})$ 计为 $V(\vec{Q})$，其实这就是动量空间中的有效势。在量子场论中，散射振幅定义为 [50]

$$S = I + i(2\pi)^4\delta^{(4)}(P_f - P_i)\mathcal{M}, \quad (3.3)$$

其中 P_i, P_f 分别为初态和末态的四动量之和。考虑到量子态要归一化到 $(2\pi)^3 2E$，我们补偿这因子再将 (3.3) 式与 (3.1) 式对比可得

$$i2\pi\delta(E_f - E_i)\frac{1}{(2\pi)^3} T_{fi}\delta^{(3)}(\vec{P}_f - \vec{P}_i),$$

等于

$$i(2\pi)^4\delta^{(4)}(P_f - P_i)\frac{\mathcal{M}}{(2\pi)^{3/2}\sqrt{2E_A}(2\pi)^{3/2}\sqrt{2E_B}(2\pi)^{3/2}\sqrt{2E_C}(2\pi)^{3/2}\sqrt{2E_D}},$$

由此可得

$$T_{fi} = -\frac{\mathcal{M}}{\sqrt{2E_A}\sqrt{2E_B}\sqrt{2E_C}\sqrt{2E_D}},$$

The open-charm and open-strange molecular systems

Hidden-charm and hidden-strange molecular systems ($H_s\bar{T}_s$)	The G−parity rule →	Open-charm and open-strange molecular systems (HsTs)

The G−parity rule

$$V^{H_sT_s\to H_sT_s}(\mathbf{r}) = \sum_i G_i V_i^{H_s\bar{T}_s\to H_s\bar{T}_s}(\mathbf{r})$$

where G_i is the G−parity of the exchanged meson i.

E. Klempt, F. Bradamante, A. Martin, and J. M. Richard, Phys. Rep. 368, 119 (2002)

We only need to consider the direct diagram contribution to the HsTs systems since the charge conjugate transformation invariance for the HsTs systems does not exist.

Quark-interchange model

Scattering amplitude

$$A(c\bar{q}) + B(\bar{c}q) \to C(c\bar{c}) + D(q\bar{q}),$$

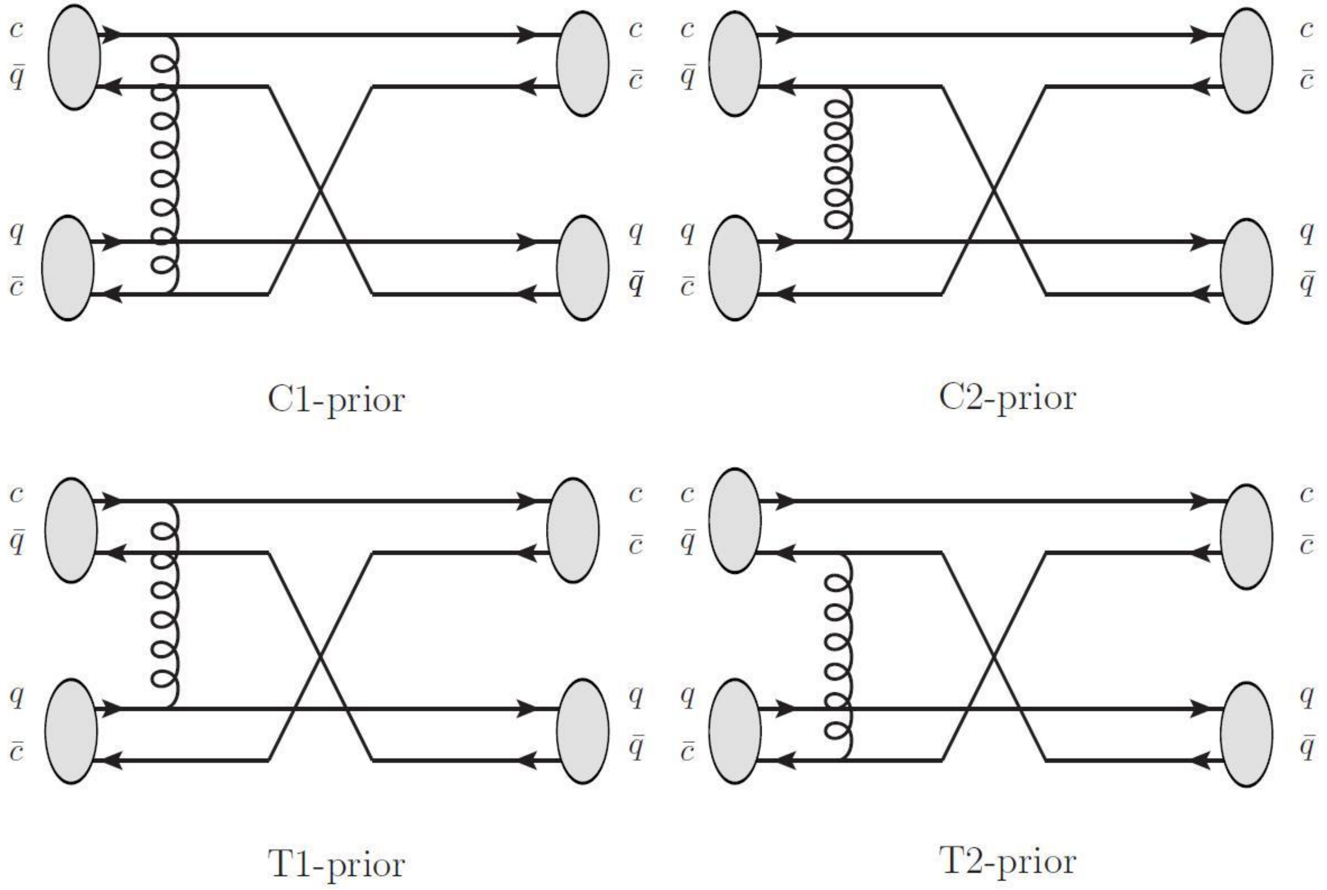
The interaction in the initial state

$$H = V_A^0 + V_B^0 + V_{AB},$$

Non-relativistic potential

$$V_{ij}(q^2) = \frac{\lambda_i}{2} \cdot \frac{\lambda_j}{2} \left(\frac{4\pi\alpha_s}{q^2} + \frac{6\pi b}{q^4} - \frac{8\pi\alpha_s}{3m_i m_j} e^{-\frac{q^2}{4\sigma^2}} \mathbf{s}_i \cdot \mathbf{s}_j \right)$$

$$\alpha_s(Q^2) = \frac{12\pi}{(32 - 2n_f)\ln(A + Q^2/B^2)}.$$



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Wave functions:

$$\psi = \omega_{\text{color}}\chi_{\text{flavor}}\chi_{\text{spin}}\phi(\mathbf{p}).$$

Effective potential:

$$V_{\text{eff}} = I_{\text{color}} I_{\text{flavor}} I_{\text{spin}} I_{\text{space}},$$

Two body decay:

$$\begin{aligned} T &= \frac{1}{(2\pi)^3}\int d\mathbf{P}_A \int d\mathbf{k}\delta^3(\mathbf{k}-\mathbf{P}_C)V_{\text{eff}}(\mathbf{P}_A,\mathbf{k})\psi_{AB}(\mathbf{P}_A) \\ &= \frac{1}{(2\pi)^3}\int d\mathbf{P}_A V_{\text{eff}}(\mathbf{P}_A,\mathbf{P}_C)R_l(\mathbf{P}_A)Y_{lm}(\mathbf{P}_A) \\ &= \frac{1}{(2\pi)^2}M_l, \end{aligned}$$

$$M_l = \int d\theta \int dP_A P_A^2 \sin(\theta) V_{\text{eff}}(\mathbf{P}_A,\mathbf{P}_C,\theta)R_l(P_A)Y_{lm}(\mathbf{P}_A).$$

$$\Gamma = |\mathbf{P}_C|\frac{E_C E_D}{(2\pi)^3 M}|M_l|^2.$$

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