QED factorization for the leptonic B meson decays

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in collaboration with Y. K. Huang, Y. L. Shen, Y. M. Wang, X. C. Zhao

The workshop of Heavy flavor Physics and QCD
2 May 2021

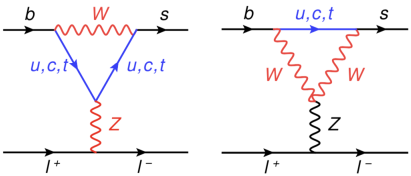
Outline

1. Introduction $B_{d/s} \to \ell^+ \ell^-$ in the SM
2. Motivation for QED effects in B decays
3. QED effects in $B_{d/s} \to \tau^+ \tau^-$ in SCET
4. Numerical results
5. Summary and outlook

$B_{d/s} \to \ell^+ \ell^-$ in the SM

In the SM the process is

- Loop suppressed (FCNC)
- Helicity suppresses (scalar meson decaying into energetic leptons through vector interactions)
- QCD contained in the meson decay constant f_B for purely leptonic final state, in the absence of QED.



$$\left\langle 0 \left| \bar{q}(0)\gamma^\mu \gamma_5 b(0) \right| \bar{B}_q(p) \right\rangle = i f_{B_q} p^\mu$$

$\rightarrow$ Highly suppressed in the SM and can be computed with a good precision!

Theoretical predictions of leptonic B−meson decays

- Lattice-QCD values of $f_{B_{u,d}}$ and f_{B_s} have reached the relative precision of about 0.7% and 0.5%, respectively. [A. Bazavov et al. Phys. Rev. D98 074512]
- Weak EFT (at the scale m_b) matching coefficients
  - $\rightarrow$ NNLO QCD [T. Hermann, M. Misiak, M. Steinhauser, 2013]
  - $\rightarrow$ NLO EW [C. Bobeth, M. Gorbahn, E. Stamou, 2014]
- QED and QCD corrections below the m_b scale not included nearly ($B \rightarrow \mu^+\mu^-$ [Beneke, Bobeth, Szafron '19])

QED corrections to $B_{d/s} \to \ell^+ \ell^-$

Why QED?

1. precision calculations: QCDF studied up to NNLO e.g. [Bell, Beneke, Huber, Li ’20]

2. Factorization theorems do not yet exist for QED

3. NP searches: Large logarithmic $\ln m_b^2/m_\ell^2$ enhancements can from Ultra-soft QED correction can mimic lepton-flavor universality violation e.g. [M. Bordone, G. Isidori, A. Pattori, 1605.07633]

4. theoretically interesting problem: power-enhancement $\frac{m_B}{\Lambda_{QCD}}$ and large logarithmic $\ln m_b\omega/m_\ell^2$ from jet function in $B_s \to \mu^+\mu^-$ [Beneke, Bobeth, Szafron ’19] , $B_s \to \tau^+\tau^-$

QED corrections to $B_{d/s} \to \ell^+ \ell^-$ at the scale $\lesssim m_b$

IR finite observable:

$\to$ must include ultrasoft photon radiation

$$\Gamma\left[B_q \to \ell^+\ell^-\right] + \Gamma\left[B_q \to \ell^+\ell^- + n\gamma\,(E_\gamma < \Delta E)\right]$$

ΔE-cut on photon energy (e.g. due to detector resolution)

$\to$ ultrosoft photon inclusive width below scale ΔE

$$\Gamma(\Delta E) \equiv \Gamma[\bar{B} \to \ell^+\ell^- X_s]\big|_{E_{X_s} \leq \Delta E}$$

$\Gamma(\Delta E)$ **factorizes** in non-radiative amplitude and ultrasoft function

$$\Gamma(\Delta E) = |\mathcal{A}(\bar{B} \to \ell^+\ell^-)|^2 \times \sum_{X_s} |\langle X_s|(\bar{S}_v^{(Q_B)} S_{v_1}^{\dagger(Q_1)} S_{v_2}^{\dagger(Q_2)})|0\rangle|^2\, \theta(\Delta E - E_{X_s})$$

Simple classification:

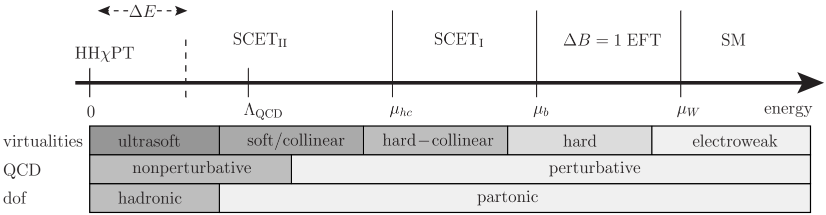
- ultrasoft photons with $\Delta E \ll \Lambda_{\text{QCD}}$ see pointlike mesons ("universal") $\to$ based on eikonal approximation. e.g. [A. von Manteuffel, R. M. Schabinger and H. X. Zhu '14]
- virtual photons with $E \gtrsim \Lambda_{\text{QCD}}$ can resolve partonic structure!

  Non-universal, structure dependent corrections (hard, hard-collinear, collinear, soft, etc.)

Scales and EFTs

Hierarchy of energy scales:

$M_W \sim 80\,\text{GeV} \gg m_b \sim 4.2\,\text{GeV} \gg \text{few times } \Lambda_{\text{QCD}} \gg \Delta E \sim 60\,\text{MeV}$



(figure from [Beneke, Bobeth, Szafron '19])

- ✓ short-distance QED at $\mu \gtrsim m_b \to$ Wilson coefficients of weak eff. Lagrangian
- ✓ Far IR (ultrasoft) region $\mu_{\text{us}} \ll \Lambda_{\text{QCD}}$ described by point-like mesons

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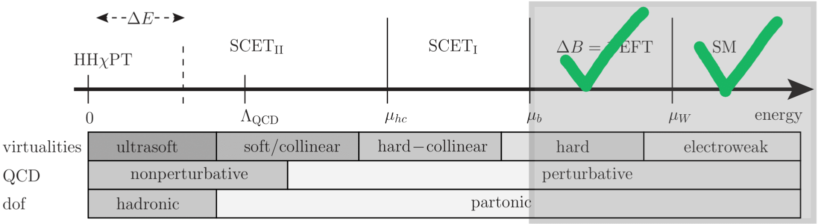

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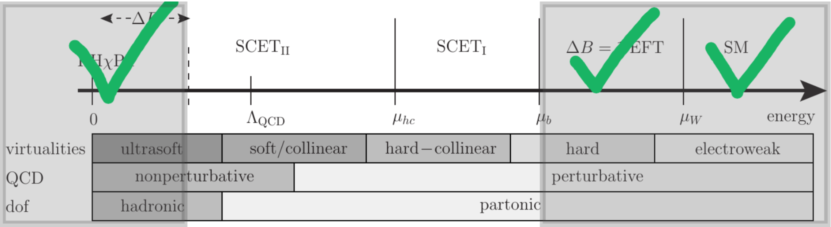

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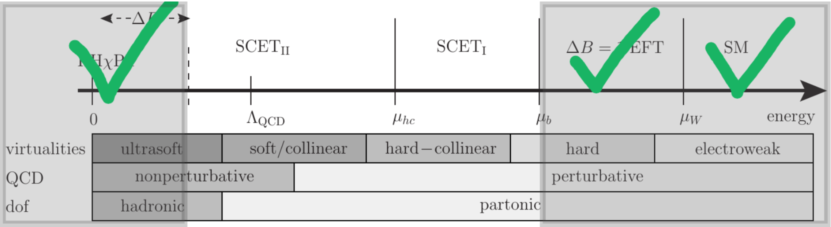

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Goal: theory for QED corrections between m_b and Λ_{QCD} ("structure dependent effects")

$\rightarrow$ Soft-Collinear Effective field Theory

SCET for $B_q \to \ell^+\ell^-$ with $\ell = \tau$

Kinematics of $B_q \to \tau^+\tau^-$ and power counting

The momentums of initial and final state for $b\,(p_b) + q\,(\ell_q) \to \ell\,(p_\ell) + \bar{\ell}\,(p_{\bar{\ell}})$ with $p^\mu = (n \cdot p, \bar{n} \cdot p,\, p_\perp)$ are

$$\left.\begin{aligned} p_b^\mu &= m_b v^\mu + \ell_b\,, \quad \ell_{b,q} \sim \Lambda_{\rm QCD} \\ p_\ell^\mu &\sim (m_b,\, \Lambda_{\rm QCD}, 0) \sim m_b(1,\, \Lambda_{\rm QCD}/m_b, 0) \\ p_{\bar{\ell}}^\mu &\sim (\Lambda_{\rm QCD}, m_b,\, 0) \sim m_b(\Lambda_{\rm QCD}/m_b, 1, 0) \end{aligned}\right\} \text{kinematic invarants}$$

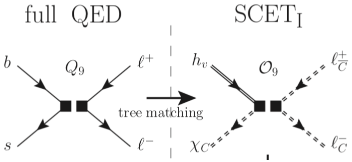
Different from $\ell = \mu$, $p_\ell^2 = m_\tau^2 = m_b \Lambda_{\rm QCD}$

Modes for virtual photon with momentum k: with scaling parameter $\lambda^2 = \frac{\Lambda_{\rm QCD}}{m_b}$

mode	relative scaling	absolute scaling	virtuality k^2
hard	(1,1,1)	$m_b(1,1,1)$	m_b^2
hard-collinear	$(1, \lambda^2, \lambda)$	$(m_b, \Lambda_{\rm QCD}, \sqrt{m_b \Lambda_{\rm QCD}})$	$m_b \Lambda_{\rm QCD}$
hard anti-collinear	$(\lambda^2, 1, \lambda)$	$(\Lambda_{\rm QCD}, m_b, \sqrt{m_b \Lambda_{\rm QCD}})$	$m_b \Lambda_{\rm QCD}$
soft	$(\lambda^2, \lambda^2, \lambda^2)$	$(\Lambda_{\rm QCD}, \Lambda_{\rm QCD}, \Lambda_{\rm QCD})$	$\Lambda_{\rm QCD}^2$

QED $\to$ $\rm SCET_I$ $\to$ HQET $\otimes$ $\rm SCET_I$

1. Integrate out hard scale m_b and match QED → SCET



- weak EFT operators

$$Q_9 = \frac{\alpha_{\rm em}}{4\pi}\,(\bar{q}\gamma^\mu P_L b)\sum_\ell \bar{\ell}\gamma_\mu \ell$$

$$Q_{10} = \frac{\alpha_{\rm em}}{4\pi}\,(\bar{q}\gamma^\mu P_L b)\sum_\ell \bar{\ell}\gamma_\mu\gamma_5 \ell$$

$$Q_7 = \frac{e}{(4\pi)^2}\,\bar{m}_b\,[\bar{q}\sigma^{\mu\nu}P_R b]\,F_{\mu\nu}$$

- Operators for a hard-collinear light quark in SCET in position space,

$$\widetilde{\mathcal{O}}_9(s,t) = g^\perp_{\mu\nu}\left[\bar{\chi}_C(sn_+)\gamma^\mu_\perp P_L h_v(0)\right]\left[\bar{\ell}_C(tn_+)\gamma^\nu_\perp \ell_{\bar{C}}(0)\right]$$

$$\widetilde{\mathcal{O}}_{10}(s,t) = i\varepsilon^\perp_{\mu\nu}\left[\bar{\chi}_C(sn_+)\gamma^\mu_\perp P_L h_v(0)\right]\left[\bar{\ell}_C(tn_+)\gamma^\nu_\perp \ell_{\bar{C}}(0)\right]$$

$n_+ \to n_-$ for anti-hard-collinear quark, $\widetilde{\mathcal{O}}_{\bar{9},\bar{10}}$

Q_7 can match to $\widetilde{\mathcal{O}}_9$ operator.

$$\widetilde{\mathcal{O}}_9(s,t) = \widetilde{\mathcal{O}}_{10}(s,t)$$
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actually, by using $\frac{\bar{n}\!\!\!/}{2}\gamma^\perp_\mu\gamma_5 = i\epsilon^\perp_{\mu\nu}\gamma^\perp_\nu$

$\mathcal{O}(\alpha_{\rm em})$ Matching Calculation

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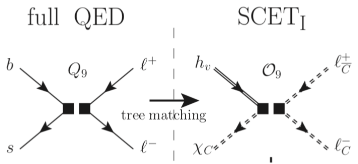

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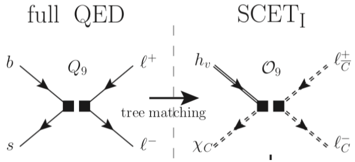

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The Fourier-transformed SCET operators are defined as

$$\mathcal{O}_i(u) = n_+ p_C \int \frac{dr}{2\pi} e^{-iur(n_+ p_C)} \widetilde{\mathcal{O}}_i(0, r) \,,$$

where $u = n \cdot p_\ell / n \cdot p_C$ carried by the lepton field.

- Matching equation in momentum space,

$$\sum_k C_k(\mu_b) Q_k = \sum_i \int du H_i(u, \mu_b) \mathcal{O}_i(u)$$

$\rightarrow$ At leading order,

$$H_9^{(0)}(u, \mu_b) = C_9^{\text{eff}}(u, \mu_b) \,, \quad H_7^{(0)} = -\frac{2\,Q_\ell}{u} C_7^{\text{eff}}(u, \mu_b) \,, \quad H_{10}^{(0)}(u, \mu_b) = C_{10}(u, \mu_b)$$

$$H_{\bar{i}}^{(0)} = H_i^{(0)} \qquad i = 7, 9, 10$$

$\rightarrow$ To NLO,

$$\sum_k C_k^{(1)}(\mu_b) \langle Q_k \rangle^{(0)} = \sum_i \int du H_i^{(0)}(u, \mu_b) \langle \mathcal{O}_i(u) \rangle^{(1)} + H_i^{(1)}(u, \mu_b) \langle \mathcal{O}_i(u) \rangle^{(0)}$$

$$\text{with} \quad \langle \mathcal{O}_i(u) \rangle^{(1)} = Z_{ij}^{(1)} \langle \mathcal{O}_j(u) \rangle^{(0)} \,, \quad \text{in Dimension regulation}$$

$$\text{For} \quad k = Q_9, \quad H_9^{(1)}(u, \mu_b) = C_9^{(1)}(\mu_b) - \int du H_9^{(0)}(u, \mu_b) Z_{99}^{(1)}$$

the second term in the r.h.s is IR subtraction.

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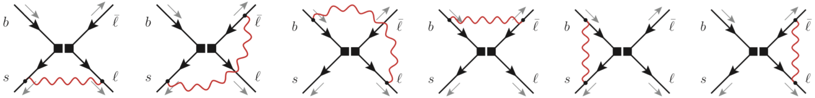
$C_k^{(1)}(\mu_b)$ will be can be calculated by one loop diagrams in the Weak EFT,



(a) (b) (c) (d) (e) (f)

one loop $H_i^{(1)} = -H_{\bar{i}}^{(1)}$ for Fig. (a)-(d), $H_i^{(1)} = H_{\bar{i}}^{(1)}$ for Fig. (e)-(f),

as $\langle \gamma \,|\, Q_9\,, \mathcal{L}_{QED} \,|\, \ell^+\ell^- \rangle \xrightarrow{\text{C and matching}} \langle 0 |\, \mathcal{O}_9 \,|\, \ell^+\ell^- \rangle \;\Rightarrow (-1)\,\gamma$

more general form, $H_i^{(n)} = (-1)^n H_{\bar{i}}^{(n)}$ for Fig. (a)-(d), $H_i^{(n)} = H_{\bar{i}}^{(n)}$ for Fig. (e)-(f),

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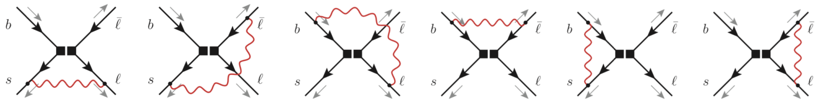

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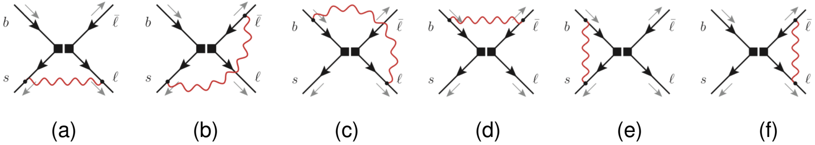

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$\Rightarrow$ Hard function to NLO,

$$H_{9/\bar{9}}^{(0)} = C_9^{\text{eff}}(u,\mu_b) - \frac{2\,Q_\ell}{u}\, C_7^{\text{eff}}(u,\mu_b)$$

$$\begin{aligned} H_{9/\bar{9}}^{(1)} &= C_9^{\text{eff}}(u,\mu_b)\, H_9^{(e+f)}(u,\mu_b) + C_7^{\text{eff}}(u,\mu_b)\, H_7^{(e+f)}(u,\mu_b) + C_{10}\, H_{10}^{(a+b+c+d)}(u,\mu_b) \\ &= (C_9^{\text{eff}} - \frac{2\,Q_\ell}{u}\, C_7^{\text{eff}})\, \frac{\alpha_{\text{em}}}{4\pi}\, Q_b Q_s \left[-\ln^2\frac{\tilde{r}}{\bar{u}} - 2\ln\frac{\tilde{r}}{\bar{u}} + \frac{1}{2}\ln^2\tilde{r} + 2\,\text{Li}_2(-\frac{u}{\bar{u}}) - 4 - \frac{\pi^2}{12}\right] \\ &+ (C_9^{\text{eff}} - \frac{2\,Q_\ell}{u}\, C_7^{\text{eff}})\, \frac{\alpha_{\text{em}}}{4\pi}\, Q_\ell^2 \left[-\ln^2\frac{-u-i0}{\tilde{r}} + 3\ln\frac{-u-i0}{\tilde{r}} - 8 + \frac{\pi^2}{6}\right] \\ &+ C_9^{\text{eff}}\, \frac{\alpha_{\text{em}}}{4\pi}\, Q_b Q_s \left[\ln\tilde{r} - \frac{\bar{u}}{u}\ln\bar{u}\right] \\ &+ C_{10}\, \frac{\alpha_{\text{em}}}{4\pi}\, Q_\ell Q_s \left[-\ln^2\frac{u}{r} - \ln^2\frac{-\bar{u}-i0}{r} + \frac{2\ln u}{\bar{u}} + \ln^2 r + 3\ln r + 2\,\text{Li}_2\left(-\frac{\bar{u}}{u}\right) + 10 + \frac{\pi^2}{6}\right] \end{aligned}$$

where $Q_\ell = -1$, $Q_q = -1/3$ and $\tilde{r} = \frac{\mu^2}{m_b^2}\, e^{\gamma_E}$

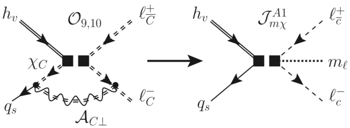
2. Integrate out intermediate hard-coll. scale $\sqrt{m_b \Lambda_{\rm QCD}}$ and match onto HQET$\otimes$SCET

$$\mathcal{O}_9(u) \to \int d\omega\, J_m(u;\omega)\, \mathcal{J}^{A1}_{m\chi}(\omega)$$

operator relevant for the **power-enhanced** contribution in the position space (ω is soft momentum),

$$\widetilde{\mathcal{J}}^{A1}_{m\chi}(v) = \left[\bar{q}_s(v\bar{n})\, Y(v\bar{n},0)\, \frac{\not{\bar{n}}}{2} P_L\, h_v(0)\right] \left[Y_+^\dagger Y_-\right] \left[\bar{\ell}_C(0)\,(4 m_\ell P_R)\,\ell_{\overline{C}}(0)\right]$$

[M.Beneke, T.Feldmann'03]



- To convert hard-collinear quark into a soft quark, we need power suppressed interaction

$$\mathcal{L}^{(1)}_{\xi q}(x) = \bar{q}_s(x_-)\left[W_{\xi C} W_C\right]^\dagger (x)\, i \not{D}_{C\perp} \xi_C(x) + \text{h.c.}$$

Small component of hard-collinear the same as soft momentum $\to$ Soft fields become delocalised along the light- cone: $\bar{u}_s(\ell_q)\ \not{\epsilon}_\perp(k) \frac{\not{n}}{2} \frac{1}{\bar{n}\cdot \ell_q}$, it is power enhaced factor.

- $\mathcal{A}_C^\perp$ should be connected with the current $\mathcal{L}^{(0)}_m(y)$ to produce the r.h.s of $\widetilde{\mathcal{J}}^{A1}_{m\chi}(v)$.

$$\mathcal{L}^{(0)}_m(y) = m_\ell\, \bar{\ell}_C \left[i \not{D}_{C\perp}, \frac{1}{i n_+ D_C}\right] \frac{\not{n}_+}{2} \ell_C \qquad \text{with } m_\tau^2 \sim \mu_{hc}$$

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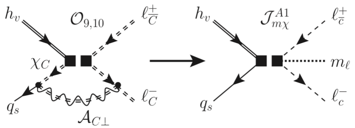

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- calculate the matrix element of l.h.s of matching Fig,

$$\left\langle \ell(p_\ell)\,\bar{\ell}(p_{\bar{\ell}}) \left| \int d^4x \int d^4y\, T\left\{\mathcal{O}_9^{\mathrm{I}}(u),\, \mathcal{L}_{\xi q}^{(1)}(x),\, \mathcal{L}_m^{(0)}(y)\right\} \right| b\,(p_b)\, q\,(\ell_q) \right\rangle$$

to extract the jet function

$$J_{O_9}^{(0)}(u;\omega) = \frac{\alpha_{em}}{4\pi}\, Q_\ell Q_s\, \frac{\bar{u}}{\omega}\, \ln\left(1 + \frac{u}{\bar{u}}\, \frac{\omega\, n\cdot p_\ell}{m_\ell^2}\right) \theta(u)\theta(\bar{u})$$

$\rightarrow$ $\omega = \bar{n}\cdot\ell_q \sim \Lambda_{QCD} \sim \lambda^2$. it is power enhanced $\frac{1}{\lambda^2}$.

3. Factorization of the amplitude in HQET $\otimes$ SCET

$$\widetilde{\mathcal{J}}_{m\chi}^{A1}(v) \quad = \left[\bar{q}_s\,(v\bar{n})\; Y\,(v\bar{n},0)\; \frac{\not{\bar{n}}}{2}\, P_L\, h_v(0)\right] \left[Y_+^\dagger Y_-\right] \left[\bar{\ell}_C(0)\,(4m_\ell P_R)\,\ell_{\overline{C}}(0)\right]$$

$$\mathcal{A} \sim \int \frac{d\omega}{\omega} \left\langle \ell_C \bar{\ell}_{\bar{C}} \left| \mathcal{J}_{m\chi}^{A1}(\omega) \right| \bar{B}_q(p) \right\rangle$$

$$= \langle \ell_C|\,[\text{hard coll. }]|0\rangle\, \langle \bar{\ell}_{\bar{C}}|\,[\text{ hard anticoll. }]|0\rangle \int \frac{d\omega}{\omega}\, \langle 0|[\text{soft}](\omega)|\bar{B}_q(p)\rangle$$

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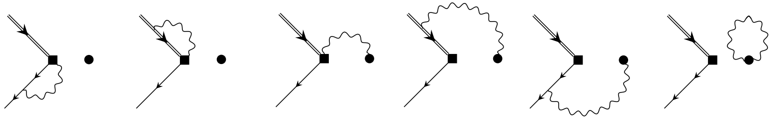
Soft function/B-meson LCDA

Soft matrix element

$$\frac{\left\langle 0 \left| \bar{q}_s\,(vn_-)\; Y\,(vn_-,0)\;\; \not{n}_- \gamma_5 h_v(0) \left[Y_+^\dagger\, Y_- \right](0) \right| \bar{B}_q(p) \right\rangle}{\left\langle 0 \left| \left[Y_+^\dagger\, Y_- \right](0) \right| 0 \right\rangle}$$

$$\equiv -i\mathscr{F}_{B_q} m_{B_q} \int_0^\infty d\omega e^{-i\omega v} \Phi_+(\omega) \quad [Beneke, Bobeth, Szafron'19]$$

- $\Phi_+(\omega)$ is the B-meson LCDA generalized to QED for $B_s \to \ell^+\ell^-$ and process dependent decay constant $\mathscr{F}_{B_q}$ can be defined similarly
- $Y\,(vn_-,0)$ is soft gauge QED-QCD invariant building blocks
- Wilson lines $\left[Y_+^\dagger\, Y_- \right](0)$ are process dependent, consequence of soft photon decoupling



$\rightarrow$ soft function becomes process dependent!

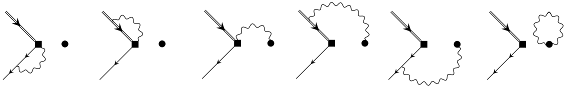
Soft function/B-meson LCDA

QED expansion at soft scale

$$\mathcal{F}_{B_q}(\mu_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_{\rm em}(\mu_s)}{4\pi}\right)^n F_{B_q}^{(n)}(\mu_s)$$

$$\mathcal{F}_{B_q}(\mu_s)\,\Phi_+(\omega;\mu_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_{\rm em}(\mu_s)}{4\pi}\right)^n F_{B_q}^{(n)}(\mu_s)\,\phi_+^{(n)}(\omega;\mu_s)$$

- The leading term is the standard B meson decay constant and LCDA.
- Higher-order terms define non-universal, non-local QCD (more precisely, HQET) matrix elements that have to be evaluated nonperturbatively. For example at one loop,



- At **LL or NLL** accuracy, only the universal objects $F_{B_q}^{(0)}(\mu_s)$ and $\phi_+^{(0)}(\omega;\mu_s)$ are needed. However, they evolve differently to $\mu \gg \mu_s$ due to QED effects.

$$F_{B_q}^{(0)}(\mu)\phi_+^{(0)}(\omega;\mu) = U_s^{\rm QCD}(\mu,\mu_s;\omega,\omega')\,U_s^{\rm QED}(\mu,\mu_s;\omega,\omega') \otimes_{\omega'} \left[F_{B_q}^{(0)}(\mu_s)\,\phi_+^{(0)}(\omega;\mu_s)\right]$$

$$= U_s^{\rm QED}(\mu,\mu_s;\omega,\omega') \otimes_{\omega'} \left[F_{B_q}(\mu)\phi_+(\omega';\mu)\right]$$

Soft function/B-meson LCDA

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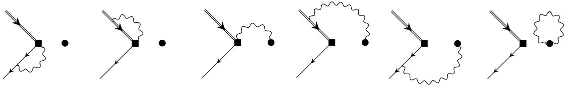

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Factorization of the amplitude

including the renormalized hard (anti)coll. on-shell matrix elements,

$$\left\langle \ell^- (p_\ell) \right| [\text{ hard coll. }] \left| 0 \right\rangle = Z_\ell \bar{u}_C (p_\ell) \, , \quad \left\langle \ell^+ (p_{\bar{\ell}}) \right| [\text{hard anticoll. }] \left| 0 \right\rangle = Z_{\bar{\ell}} v_{\bar{C}} (p_{\bar{\ell}})$$

we can now derive the factorized expression for the matrix elements of the Fourier transforms in the form,

$$\left\langle \ell^+ (p_{\bar{\ell}}) \, \ell^- (p_\ell) \left| \mathcal{J}^{A1}_{m\chi}(\omega, w) \right| \bar{B}_q(p) \right\rangle = T_+ m_{B_q} \mathcal{F}_{B_q} \Phi_+(\omega)$$

where

$$T_+(\mu) \equiv (-i) m_\ell(\mu) Z_\ell(\mu) Z_{\bar{\ell}}(\mu) \left[\bar{u}_C (p_\ell) \, P_R v_{\overline{C}} (p_{\bar{\ell}}) \right]$$

The complete expression for the power-enhanced $B \to \tau\tau$ amplitude is now obtained by adding the hard function and hard-(anti)collinear matching coefficients,

$$i\mathcal{A}_9 = T_+ \int_0^1 du \, 2 \, H_9(u) \int_0^\infty d\omega \, J_m(u; \omega) \, m_{B_q} \, \mathcal{F}_{B_q} \, \Phi_+(\omega)$$

$$\sim \; m_\ell \; \times \; \frac{m_{B_q}}{\omega} \; T'_+ \int_0^1 du \, 2 \, H_9(u) \int_0^\infty d\omega \, J'_m(u; \omega) \, m_{B_q} \, \mathcal{F}_{B_q} \, \Phi_+(\omega)$$

Helicity suppression $\times$ power enhancement factor

Factorization of the amplitude

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Helicity suppression $\times$ power enhancement factor

Resummed amplitude

- The Hard function and Soft function are evoluted to μ_{hc}
- Anomalous dimension is (almost) known
  - $\rightarrow$ SCET I: B-type current with fermion number 2 [M. Beneke, M. Garny, R. Szafron, J. Wang JHEP 1803, 001 2018]
  - $\rightarrow$ soft part [B.Lange, M. Neubert, Phys.Rev.Lett. 91, 102001, 2003]
    + additional contribution from the soft Wilson lines [Beneke, Bobeth, Szafron '19]
- The resummed result

$$i\mathcal{A}^{(1)} = -i\,A^{(1)}\,\left[\bar{\ell}_C\,(1+\gamma_5)\,\ell_{\bar{C}}\right]$$

$$\begin{aligned}
A^{(1)} = \frac{1}{2}\,\frac{\alpha_{\rm em}\,(\mu_{hc})}{4\pi}\,Q_\ell Q_q\,\mathcal{N}\,m_\ell\,m_{B_q}\,f_{B_q}(\mu_{hc},\mu_s)\,\times\int_0^1 du\,\bar{u} \\
\left\{\left[U_h(\mu_b,\mu_{hc})\,U_s^{\rm QED}\,(\mu_{hc},\mu_s;\omega) + U_{\bar{h}}(\mu_b,\mu_{hc})\,U_{\bar{s}}^{\rm QED}\,(\mu_{hc},\mu_s;\omega)\right]\left[H_9^{(0)}(\mu_b)+H_9^{(1)}(\mu_b)\right]\right\} \\
\times\int_0^\infty \frac{d\omega}{\omega}\,\phi_+\,(\omega;\mu_{hc})\,\ln\left(1+\frac{u}{\bar{u}}\,\frac{m_b\,\omega}{m_\ell^2}\right)
\end{aligned}$$

Numerical prediction: the power-enhanced correction

- The non-radiative branching fraction of $B_q \to \tau^+\tau^-$ for central values of the parameters are

$$\mathrm{Br}^{(0)}(B_d \to \tau^+\tau^-) = \left(2.07959_{(\mathrm{LO})} - 0.00094_{(\mathrm{NLO})}\right) \times 10^{-8}$$
$$\mathrm{Br}^{(0)}(B_s \to \tau^+\tau^-) = \left(6.83576_{(\mathrm{LO})} - 0.00311_{(\mathrm{NLO})}\right) \times 10^{-7}$$

$\to$ power-enhanced correction changes the branching fraction by: $\sim 0.04\%$

- compared with $B_{d,s} \to \mu^+\mu^-$, power-enhanced correction $\sim 0.4\%$, [Beneke, Bobeth, Szafron '17, '19]

$$\begin{aligned} i\mathcal{A} =& m_\ell f_{B_q} \mathcal{N} C_{10} \left[\bar{\ell}\gamma_5 \ell\right] + \frac{\alpha_{\mathrm{em}}}{4\pi} Q_\ell Q_q m_\ell m_{B_q} f_{B_q} \mathcal{N} \left[\bar{\ell}\,(1+\gamma_5)\,\ell\right] \\ & \times \left\{ \int_0^1 du (1-u) C_9^{\mathrm{eff}}\left(u m_b^2\right) \int_0^\infty \frac{d\omega}{\omega} \phi_+(\omega) \left[\ln \frac{m_b \omega}{m_\ell^2} + \ln \frac{u}{1-u}\right] \right. \\ & \left. - Q_\ell C_7^{\mathrm{eff}} \int_0^\infty \frac{d\omega}{\omega} \phi_+(\omega) \left[\ln^2 \frac{m_b\omega}{m_\ell^2} - 2\ln\frac{m_b\omega}{m_\ell^2} + \frac{2\pi^2}{3}\right] \right\} + \ldots \end{aligned}$$

Summary and outlook

- QED factorization more complicated than QCD-alone due to charged external states

- beyond the standard ultra-soft photon approximation, structure depended corrections can be calculated in **SCET** and **HQET**
  - $\rightarrow$ convolution of hard function (have included NLO), jet function and QED specific B-meson LCDAs at NLO completely for $B_q \rightarrow \tau^+\tau^-$
  - $\rightarrow$ interesting effect – power suppressed interaction $\mathcal{L}^{(1)}_{\xi q}(x)$ lead to power enhanced correction $1/\Lambda_{QCD}$

- Outlook
  - $\rightarrow$ Next-to-leading power QED corrections (non-enhanced terms)
  - $\rightarrow$ renormalization beyond one-loop level
  - $\rightarrow$ require systematic understanding of higher-order terms of the QED specific B-meson LCDAs in non-perturbative approaches: sum rules, lattice QCD beyond LL and NLL.

Thank you!

Backup-Slides

non-power enhanced operator

Field	heavy quark	light quark		leptons			photon (gluon)		
	h_v	χ_C	χ_c	q_s	ℓ_C	ℓ_c	$A_C\,(G_C)$	$A_c\,(G_c)$	$A_s\,(G$
Scaling	λ^3	λ	λ^2	λ^3	λ	λ^2	$(1,\lambda,\lambda^2)$	$(1,\lambda^2,\lambda^4)$	$\lambda^2(1,1$

SCET$_I$

$$\tilde{\mathcal{O}}_m = m_\ell\,[\bar{q}_s(0)P_R h_v(0)]\,[\bar{\ell}_C(0)\gamma_5\ell_{\bar{C}}(0)]$$

$$\widetilde{\mathcal{J}}_m^{A1} = m_\ell\bar{q}_s(0)P_R h_v(0)\left[Y_+^\dagger Y_-\right](0)\,[\bar{\ell}_c(0)\gamma_5\ell_{\bar{c}}(0)]$$

B LCDA and uncertainties

we will employ the following three-parameter model for leading twist B-meson LCDA
$\phi^+(\omega) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \frac{\omega}{\omega_0^2} e^{-\frac{\omega}{\omega_0}} U\left(\beta - \alpha, 3 - \alpha, \frac{\omega}{\omega_0}\right)$

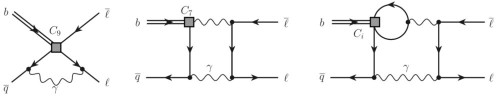
parametric origin:

B_s meson decay con- stant f_{B_s}, quark-mixing element V_{cb}, top-quark mass

Non-parametric uncertainties:

are due to the omission of higher-order corrections α_s,α_{em},$\alpha_s\alpha_{em}$ in the QCD and QED couplings α_s and α_{em}, respectively,

and also m_b^2/m_W^2 from higher- dimension operators in the weak effective Lagrangian.
$\overline{\mathrm{Br}}_{s\mu}^{(0)} = 3.677 \cdot 10^{-9} \times (1 - 0.0166 S_9 + 0.0105 S_7) = 3.660 \cdot 10^{-9}$
$\overline{\mathrm{Br}}_{d\mu}^{(0)} = 1.031 \cdot 10^{-10} \times (1 - 0.0155 S_9 + 0.0103 S_7) = 1.027 \cdot 10^{-10}$



$$i\mathcal{A} = -i\left(A^{(0)}\left[\bar{\ell}_C\,\gamma_5\,\ell_{\bar{C}}\right] + A^{(1)}\left[\bar{\ell}_C\,(1+\gamma_5)\,\ell_{\bar{C}}\right]\right) \tag{1}$$

where

$$\begin{aligned}
A^{(0)} &= m_\ell f_{B_q}\mathcal{N}\,C_{10}(\mu_b)\,U_\ell(\mu_b,\mu_{hc}) \\
A^{(1)} &= \frac{1}{2}\,\frac{\alpha_{\rm em}(\mu_{hc})}{4\pi}\,Q_\ell Q_q m_\ell\, m_{B_q} f_{B_q}\mathcal{N} \\
&\quad \int_0^1 du\,\bar{u}\,\Big\{\Big[U_h(\mu_b,\mu_{hc})\,U_s^{\rm QED}(\mu_{hc},\mu_s;\omega) - U_{\bar{h}}(\mu_b,\mu_{hc})\,U_{\bar{s}}^{\rm QED}(\mu_{hc},\mu_s;\omega)\Big]\;H_9^{(0)'}(\mu_b) \\
&\quad + \Big[U_h(\mu_b,\mu_{hc})\,U_s^{\rm QED}(\mu_{hc},\mu_s;\omega) + U_{\bar{h}}(\mu_b,\mu_{hc})\,U_{\bar{s}}^{\rm QED}(\mu_{hc},\mu_s;\omega)\Big]\;\Big[H_9^{(0)}(\mu_b) + H_9^{(1)}(\mu_b)\Big]\Big\} \\
&\quad \times \int_0^\infty \frac{d\omega}{\omega}\,\phi_+(\omega;\mu_{hc})\ln\left(1+\frac{u}{\bar{u}}\,\frac{m_b\,\omega}{m_\ell^2}\right).
\end{aligned} \tag{2}$$

with hard functions

$$H^{(0)\prime}_{9/\bar{9}} = C_{10}(\mu_b)$$

$$H^{(0)}_{9/\bar{9}} = C_9^{\text{eff}}(u,\mu_b) - \frac{2\,Q_\ell}{u}\,C_7^{\text{eff}}(u,\mu_b)$$

$$H^{(1)}_{9/\bar{9}} = C_9^{\text{eff}}(u,\mu_b)\,H_9^{(e+f)}(u,\mu_b) + C_7^{\text{eff}}(u,\mu_b)\,H_7^{(e+f)}(u,\mu_b) + C_{10}\,H_{10}^{(a+b+c+d)}(u,\mu_b)$$

$$= (C_9^{\text{eff}} - \frac{2\,Q_\ell}{u}\,C_7^{\text{eff}})\,\frac{\alpha_{\text{em}}}{4\pi}\,Q_bQ_s\left[-\ln^2\frac{\tilde{r}}{\bar{u}} - 2\ln\frac{\tilde{r}}{\bar{u}} + \frac{1}{2}\ln^2\tilde{r} + 2\,\text{Li}_2(-\frac{u}{\bar{u}}) - 4 - \frac{\pi^2}{12}\right]$$

$$+ (C_9^{\text{eff}} - \frac{2\,Q_\ell}{u}\,C_7^{\text{eff}})\,\frac{\alpha_{\text{em}}}{4\pi}\,Q_\ell^2\left[-\ln^2\frac{-u-i0}{\tilde{r}} + 3\ln\frac{-u-i0}{\tilde{r}} - 8 + \frac{\pi^2}{6}\right]$$

$$+ C_9^{\text{eff}}\,\frac{\alpha_{\text{em}}}{4\pi}\,Q_bQ_s\left[\ln\tilde{r} - \frac{\bar{u}}{u}\ln\bar{u}\right]$$

$$+ C_{10}\,\frac{\alpha_{\text{em}}}{4\pi}\,Q_\ell Q_s\left[-\ln^2\frac{u}{r} - \ln^2\frac{-\bar{u}-i0}{r} + \frac{2\ln u}{\bar{u}} + \ln^2 r + 3\ln r + 2\,\text{Li}_2\left(-\frac{\bar{u}}{u}\right) + 10 + \frac{\pi^2}{6}\right] \quad (3)$$

where $Q_\ell = -1$, $Q_q = -1/3$ and $\tilde{r} = \frac{\mu^2}{m_b^2}\,e^{\gamma_E}$ the evolution functions are

$$U_\ell(\mu_b,\mu_{hc}) = \exp\left[\int_{\mu_b}^{\mu}\frac{d\mu'}{\mu'}\Gamma_{\text{c}}\left(\mu'\right)\ln\frac{m_{B_q}}{\mu'}\right]$$

$$U_{h,\bar{h}}(\mu_b,\mu_{hc}) = \exp\left[\int_{\mu_b}^{\mu}\frac{d\mu'}{\mu'}\Gamma^{\text{I}}_{\text{cusp}}\left(\mu'\right)\ln\frac{m_{B_q}}{\mu'}\right]$$

$$U^{\text{QED}}_{s,\bar{s}}(\mu_{hc},\mu_s;\omega) = \exp\left[\frac{4\pi}{\alpha_{\text{em}}(\mu_s)}\,\frac{Q_q\,(\pm 2Q_\ell + Q_q)}{\beta^2_{0,\text{em}}}\left(g_0(\eta_{\text{em}}) + \frac{\alpha_{\text{em}}(\mu_s)}{2\pi}\beta_{0,\text{em}}\ln\eta_{\text{em}}\ln\frac{\omega}{\mu_s}\right)\right] \quad (4)$$

with

$$\Gamma^{\text{I}}_{\text{cusp}}(\alpha_s,\alpha_{\text{em}}) = \Gamma_c(\alpha_{\text{em}}) + \Gamma_s(\alpha_s,\alpha_{\text{em}})$$