

# Inverse Problem:

A novel method for calculating  
the non-perturbation quantities

**Fu-Sheng Yu**

**Lanzhou University**

In collaboration with Hsiang-nan Li, Hiroyuki Umeeda, Fanrong Xu

Ao-Sheng Xiong, Ting Wei

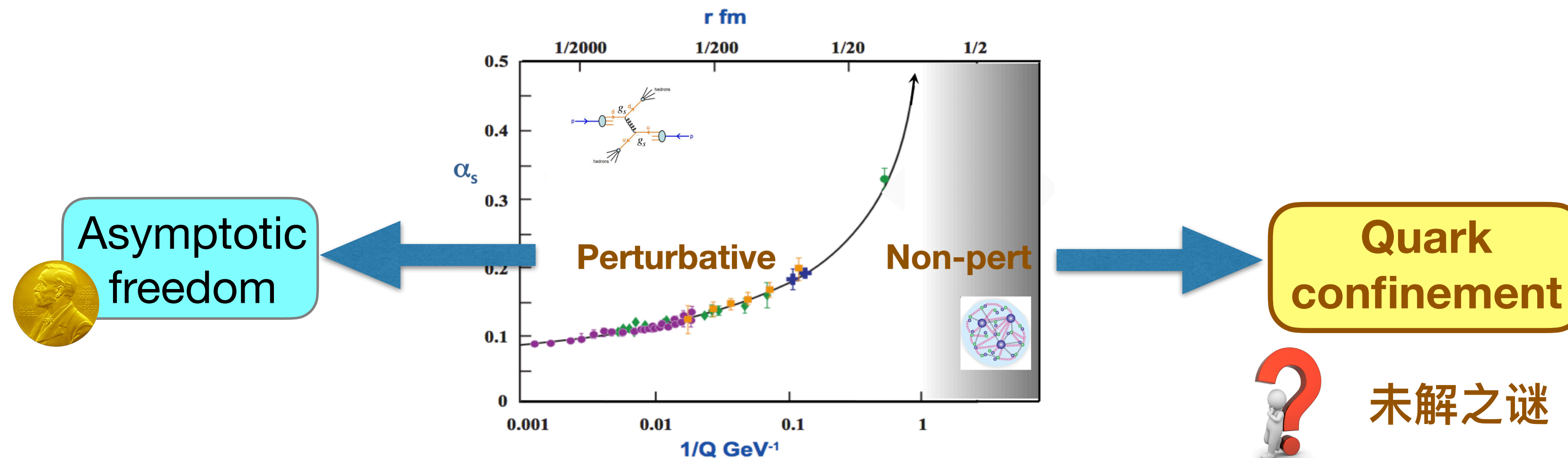
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# Outline

- What is the **Inverse Problem**. How for non-perturbation
- **Theory**: Proof of the ill-posedness
- **Proposal**:  $D^0$ - $D^0$ bar mixing
- **Applications**: muon  $g-2$ , QCD sum rules
- Outlook and Summary

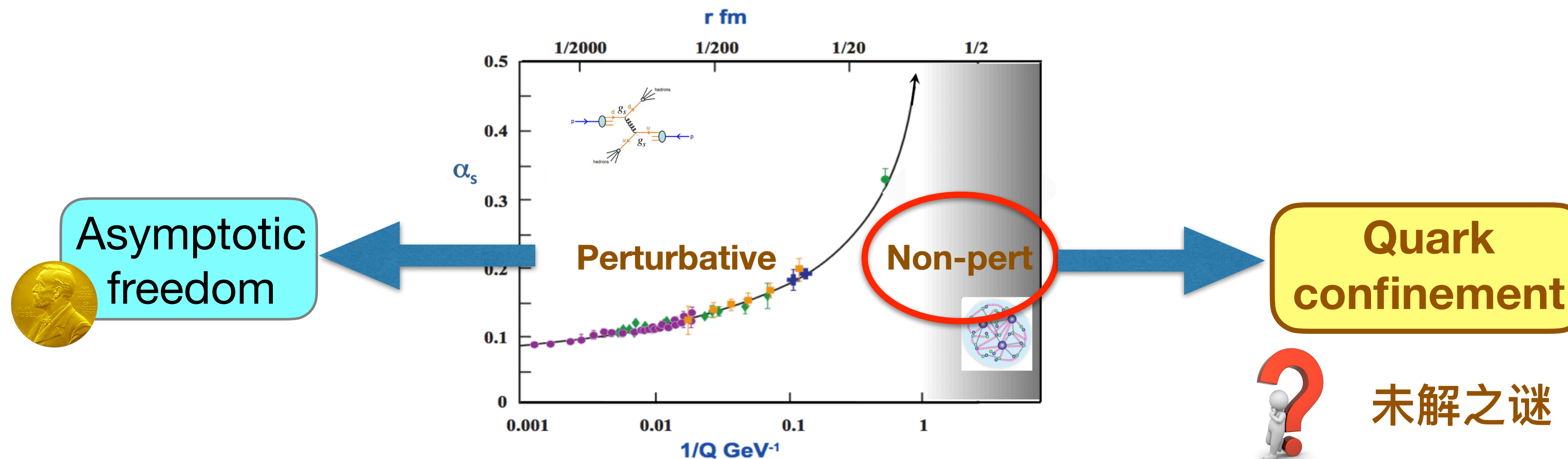
# Motivation: Problems in HEP

- Within the Standard Model: **To calculate the non-perturbative quantities**



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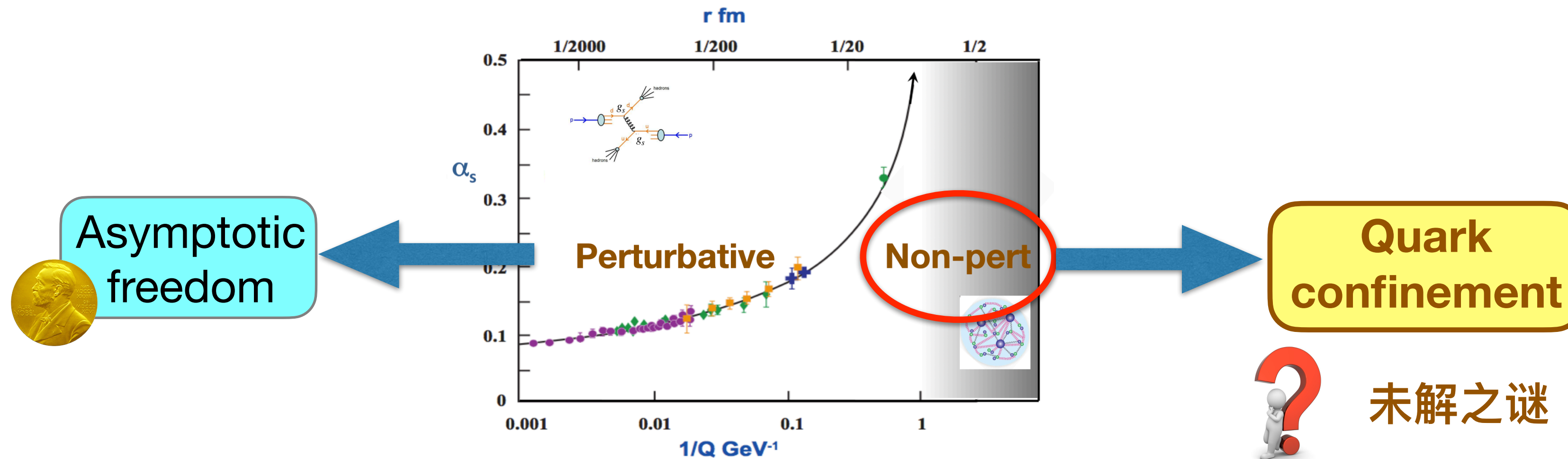
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Lattice QCD, QCD sum rules, Dyson-Schwinger Equation, ...

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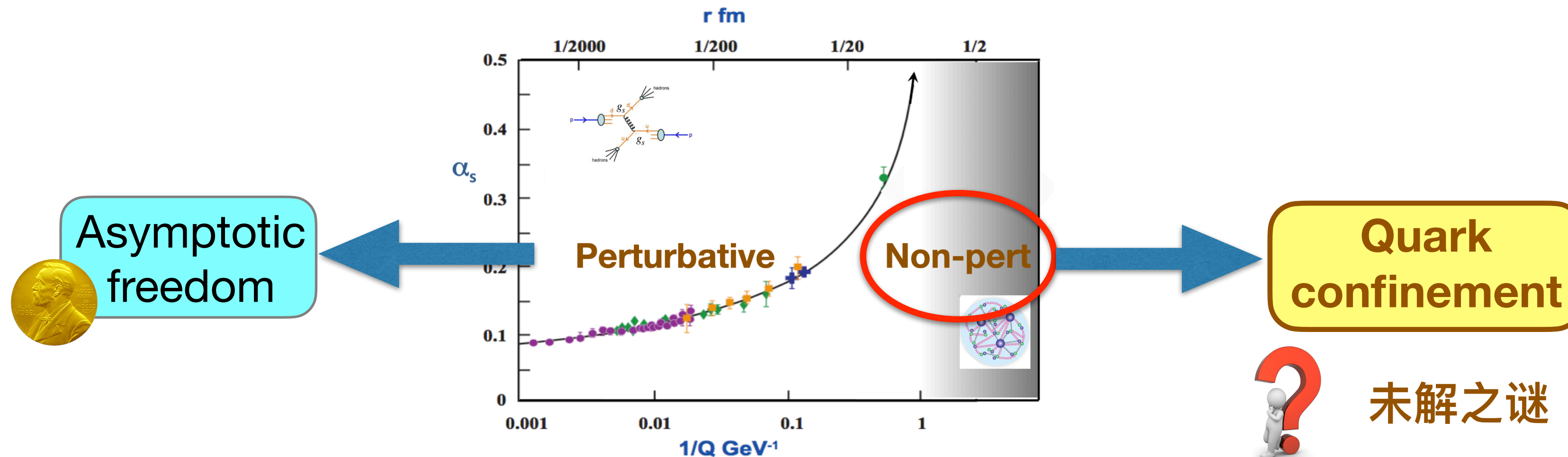


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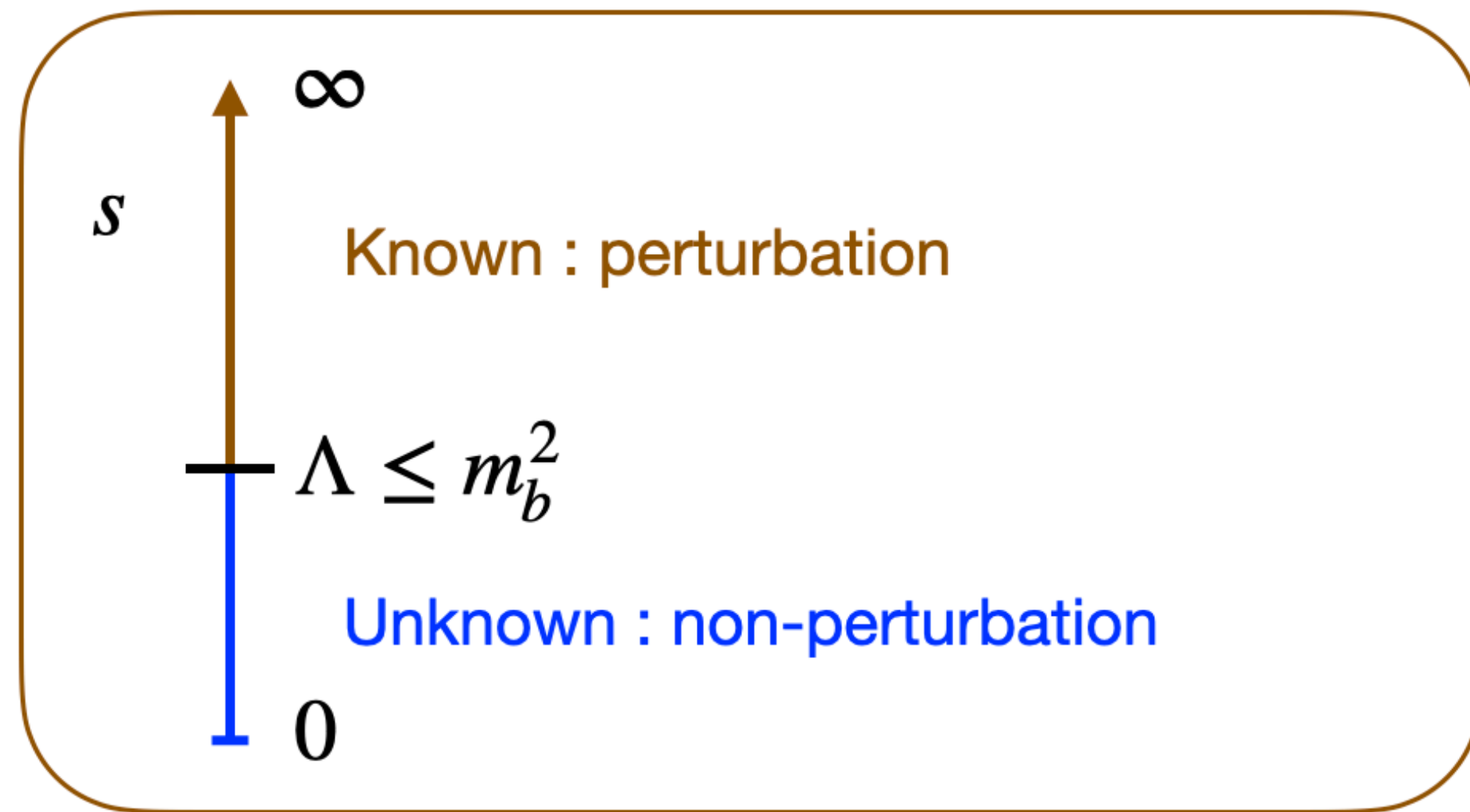
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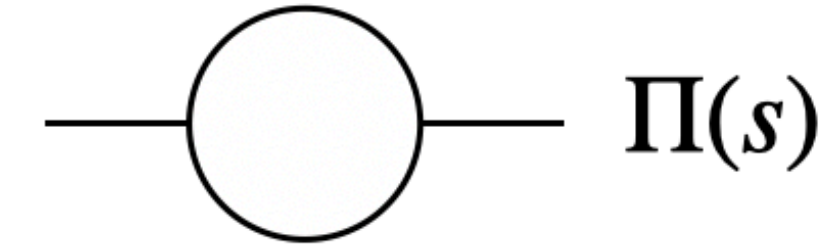
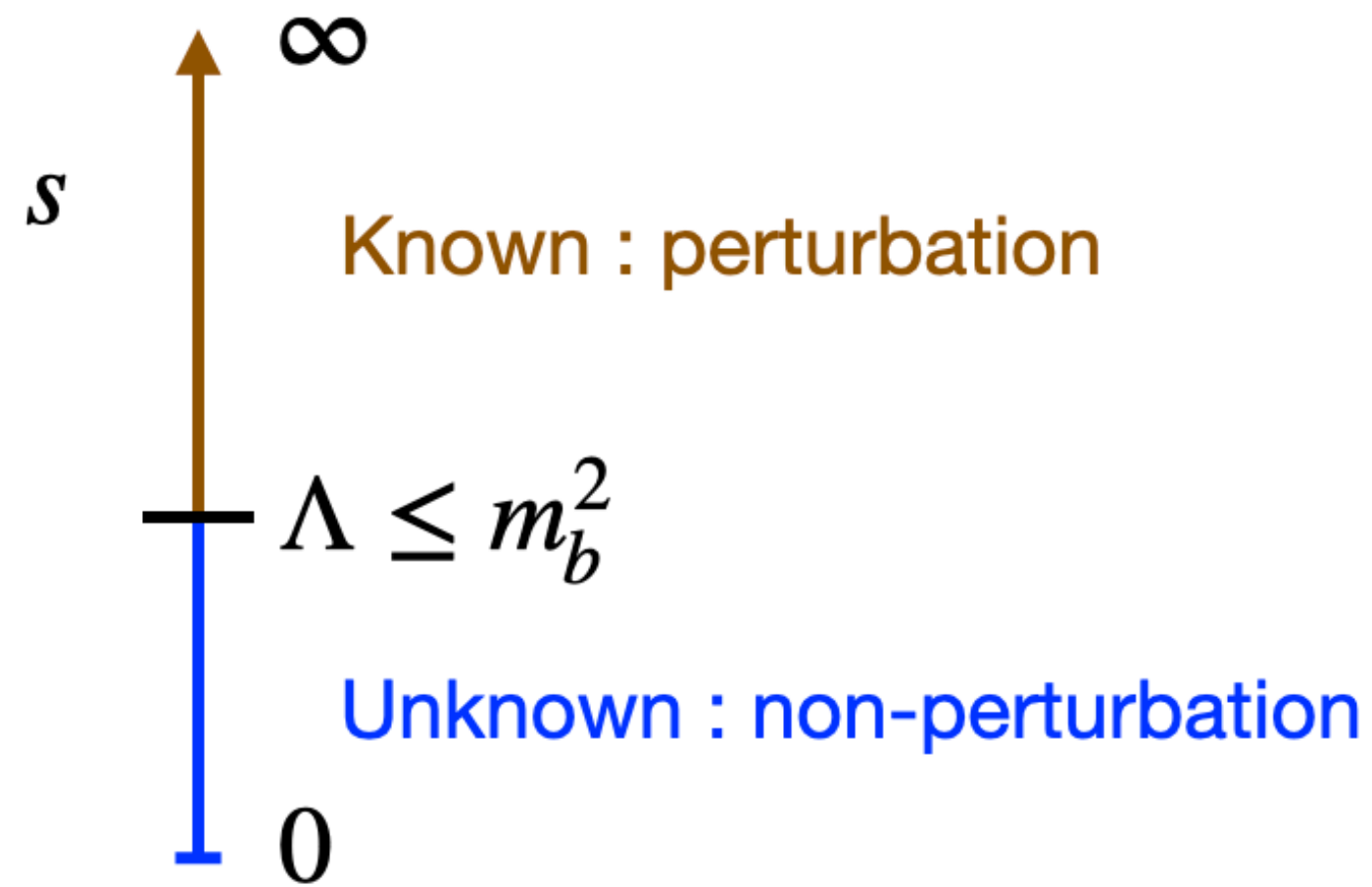
Lattice QCD, QCD sum rules, Dyson-Schwinger Equation, ...

- Beyond the Standard Model: **To search for the new physics**
  - In the present days of indirect searches, non-perturbative strong interaction play a significant role
  - High precision required for hadronic quantities: e.g. HVP of muon  $g-2$

# What is the Inverse Problem?



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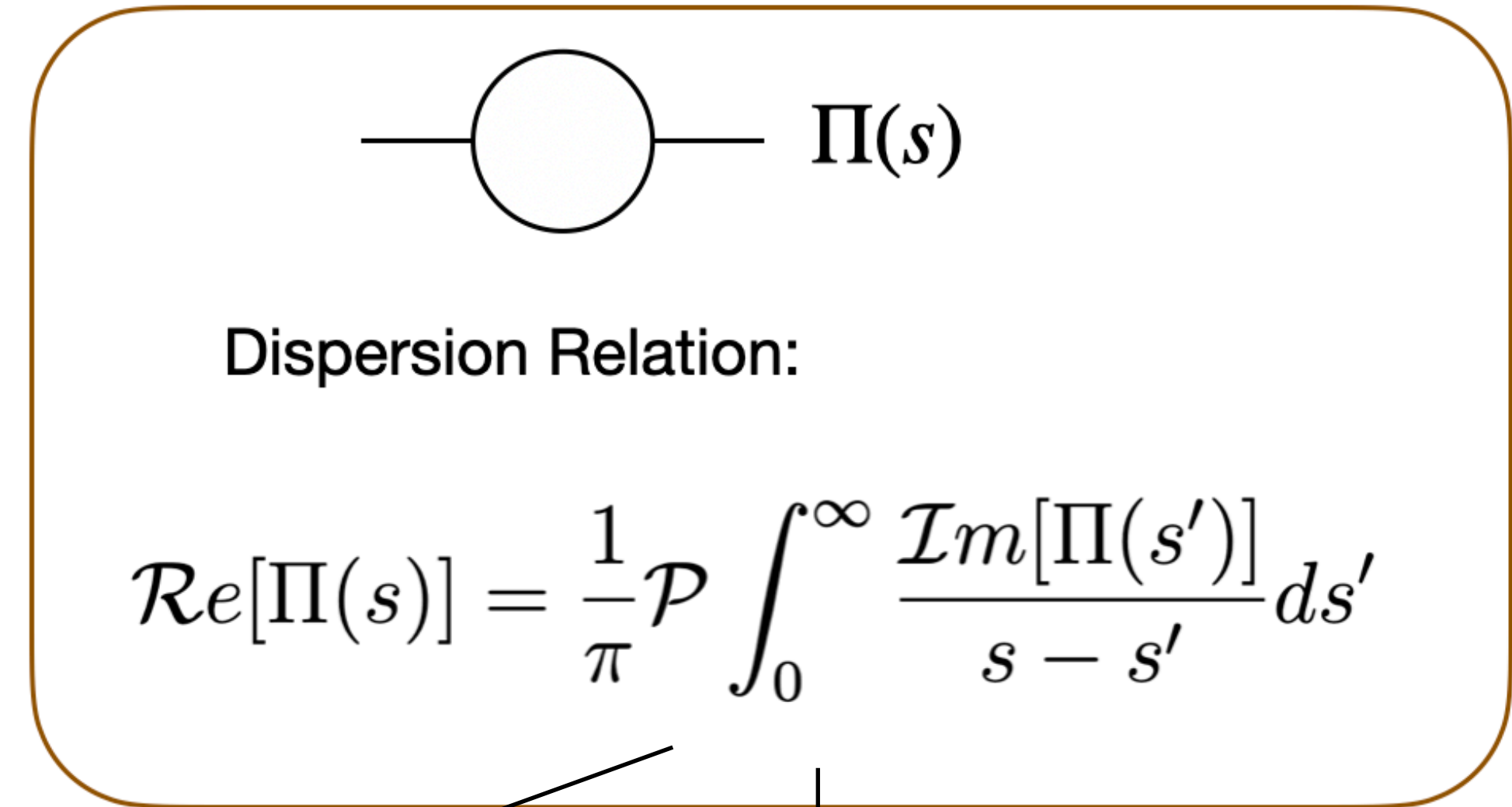
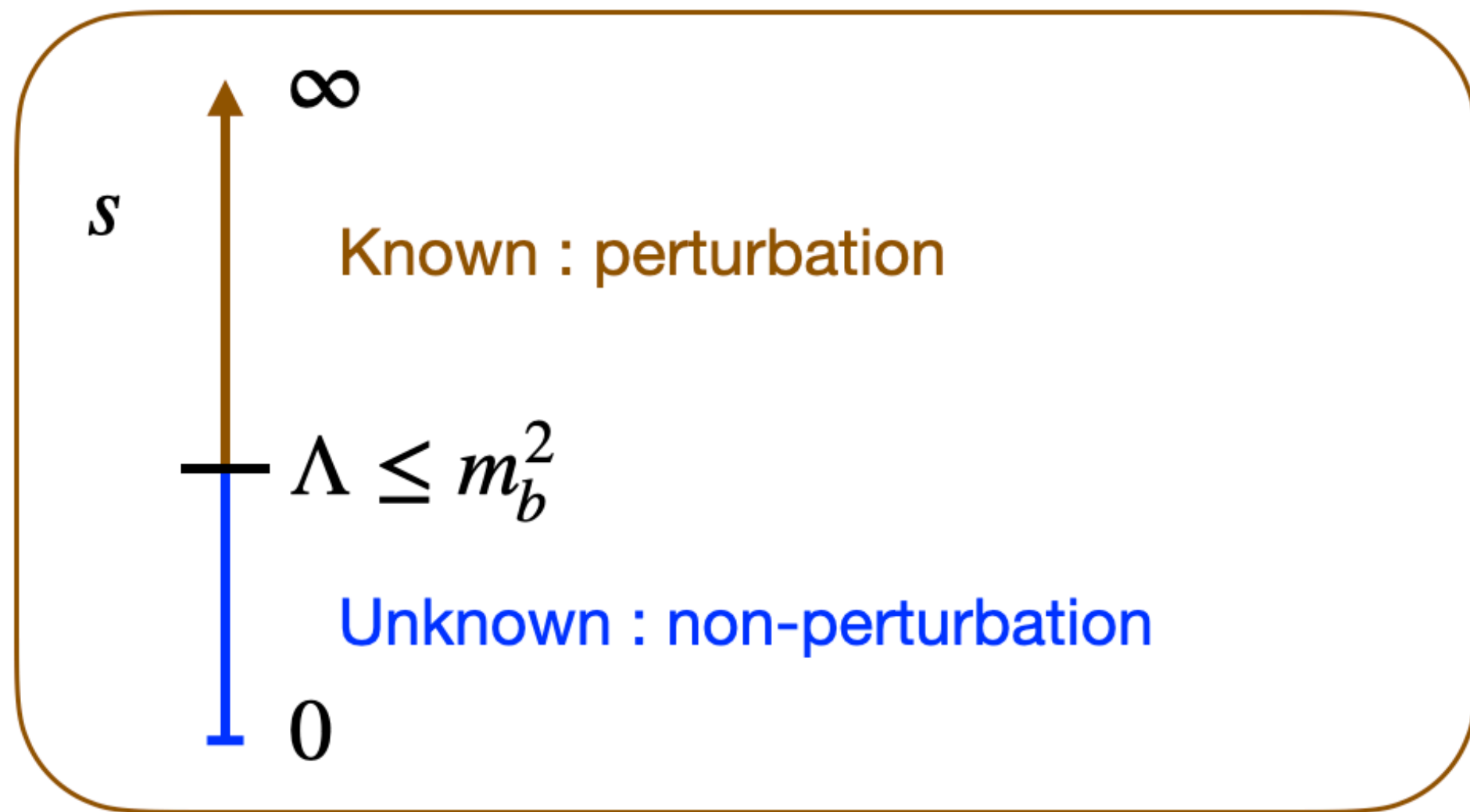


Dispersion Relation:

$$\mathcal{R}e[\Pi(s)] = \frac{1}{\pi} \mathcal{P} \int_0^{\infty} \frac{\mathcal{I}m[\Pi(s')]}{s - s'} ds'$$

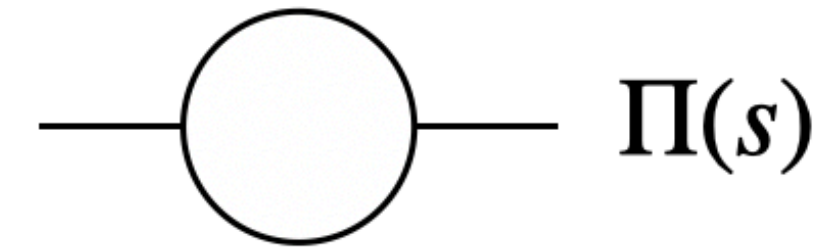
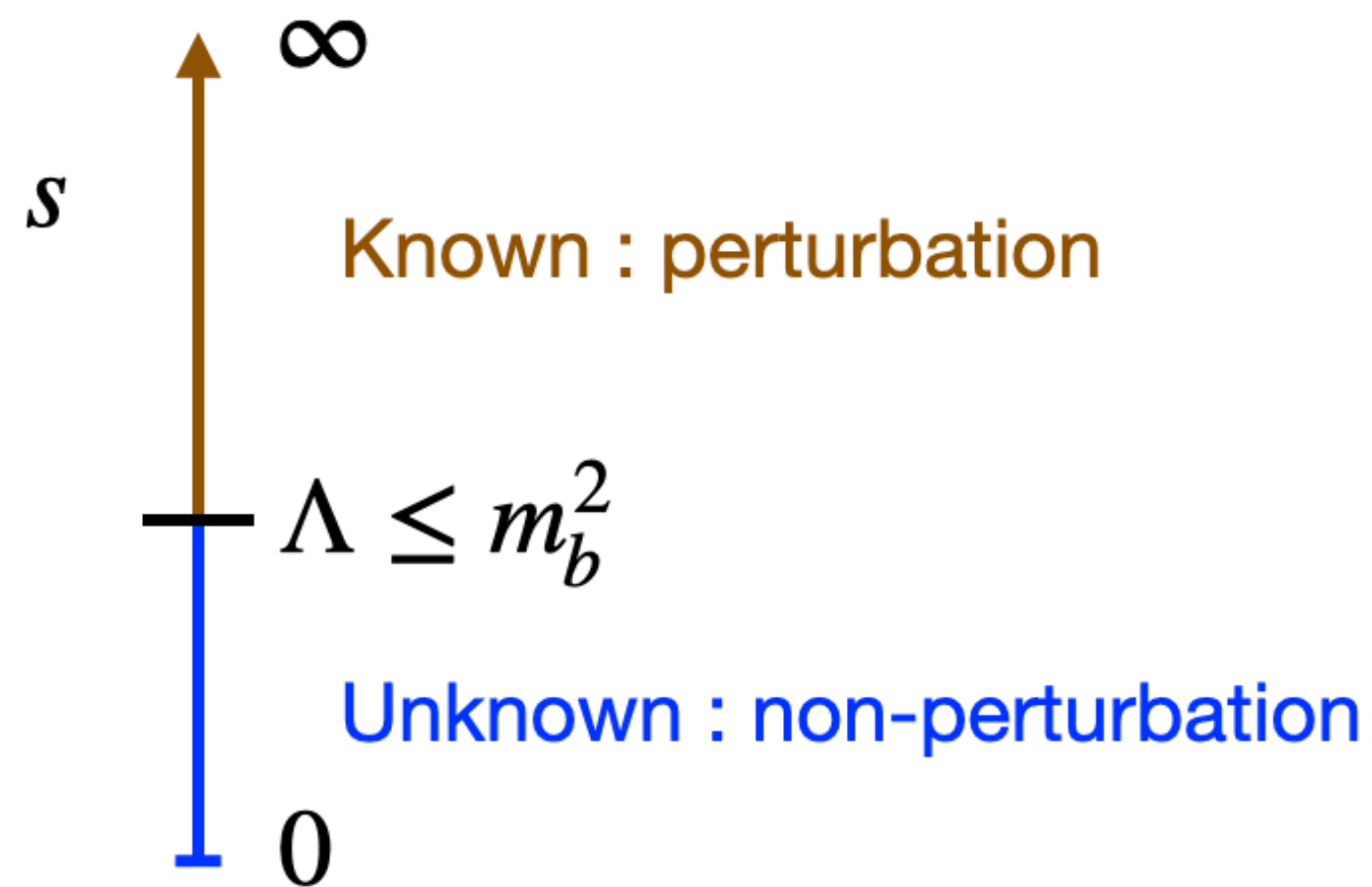


# What is the Inverse Problem?



$$\mathcal{P} \int_0^{\Lambda} \frac{\mathcal{I}m[\Pi(s')]}{s - s'} ds' = \pi \mathcal{R}e[\Pi(s)] - \mathcal{P} \int_{\Lambda}^{\infty} \frac{\mathcal{I}m[\Pi(s')]}{s - s'} ds'$$

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↑  
To be solved
←  
calculable

“charge distribution” at low  $s$



“potential” at high  $s$



$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{x}') e^{ikr}}{r} dV'$$


# What is the Inverse Problem?

- Dispersion relation: first-class Fredholm integration equation

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To be solved
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$$\int_a^b \frac{f(x)}{y - x} dx = g(y), \quad y \in [c, d], \quad c > b, \quad a > 0$$

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To be solved calculable

- 不适定问题 (ill-posed problem)

- 解的存在性 ?
- 解的唯一性 ? ?
- 解的稳定性 ? ? ?

上世纪60、70年代才开始发展的数学新分支

# 反问题的不适定性

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- 适定性(well-posedness): 上述三个条件都满足,  $K^{-1}$  在整个空间  $Y$  上存在、单值且连续, 则  $x = K^{-1}y$

## 色散关系的反问题

$$\int_a^b \frac{f(x)}{y-x} dx = g(y), \quad y \in [c, d], \quad c > b, \quad a > 0$$

1) 解的存在性：物理要求

2) 解的唯一性：证明

已知算子  $K : L^2(a, b) \rightarrow L^2(c, d)$

若  $f_1(x), f_2(x)$  都是方程  $K(x, y)f(x) = g(y)$  的解，则有  $Kf_1 = Kf_2 = g$ ，即  $K(f_1 - f_2) = 0$ 。

唯一性：证明  $f_1 = f_2$ 。只需证明  $Kf = 0$  只有零解。

$$\int_a^b \frac{f(x)}{y-x} dx = \int_a^b \frac{1}{y} \sum_{k=0}^{\infty} \left(\frac{x}{y}\right)^k f(x) dx, \quad y \in [c, d], \quad c > b, \quad a > 0,$$

控制收敛定理：
$$y \int_a^b \frac{f(x)}{y-x} dx = \sum_{k=0}^{\infty} \frac{1}{y^k} \int_a^b x^k f(x) dx = 0, \quad \forall y \in [c, d].$$

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当  $d = +\infty$ , 令  $y \rightarrow +\infty$ ,  $\int_a^b f(x) dx = 0$ 。两边同乘  $y$ ,  $\int_a^b x^k f(x) dx = 0, k = 0, 1, 2, 3, \dots$

由Weierstras定理, 存在多项式  $\int_a^b f(x) P_n(x) dx = 0$ .

$$\begin{aligned} \text{由Cauchy不等式} \quad \|f\|_{L^2(a,b)}^2 &= \int_a^b f^2(x) dx = \int_a^b [f^2(x) - f(x)P_n(x)] dx \leq \int_a^b |f(x)| |f(x) - P_n(x)| dx \\ &\leq \left[ \int_a^b f^2(x) dx \right]^{\frac{1}{2}} \left\{ \int_a^b [f(x) - P_n(x)]^2 dx \right\}^{\frac{1}{2}} \leq \|f\|_{L^2(a,b)} (\varepsilon + \sqrt{b-a}\varepsilon), \end{aligned}$$

即  $\|f\|_{L^2(a,b)} \leq \varepsilon + \sqrt{b-a}\varepsilon$ 。由于  $\varepsilon$  的任意性, 令  $\varepsilon \rightarrow 0$ , 则有  $\|f\|_{L^2(a,b)} = 0$ , 即  $f(x) = 0$ 。

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$$\|g_1(y) - g_2(y)\|_{L^2}^2 = \left\{ \int_c^a \left[ \int_a^b \frac{1}{y-x} \sin(wx) dx \right]^2 dy \right\} \rightarrow 0, w \rightarrow \infty,$$

即  $\|g_1(y) - g_2(y)\|_{L^2}^2$  在  $w \rightarrow \infty$  时可以任意小。然而  $\|f_1(x) - f_2(x)\|_{L^2}$  却不能任意小

$$\|f_1(x) - f_2(x)\|_{L^2} = \left\{ \int_a^b \sin^2(wx) dx \right\}^{\frac{1}{2}} = \sqrt{\frac{b-a}{2} - \frac{1}{2w} \sin w(b-a) \cos w(b+a)},$$

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Proposal:  $D^0 - \bar{D}^0$  mixing

Applications in and tested by: muon g-2, QCD sum rules

# $D^0 - \bar{D}^0$ Mixing

- The time evolution

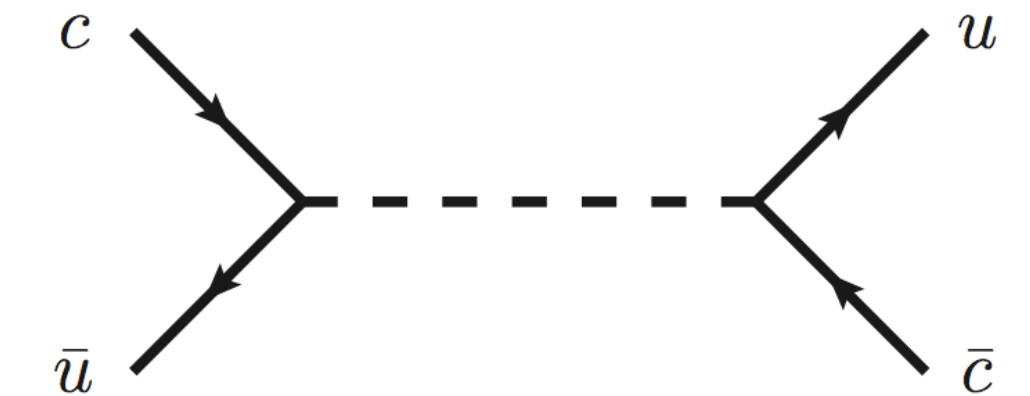
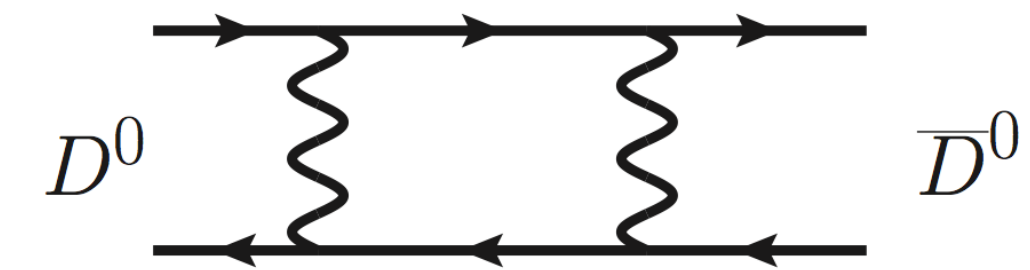
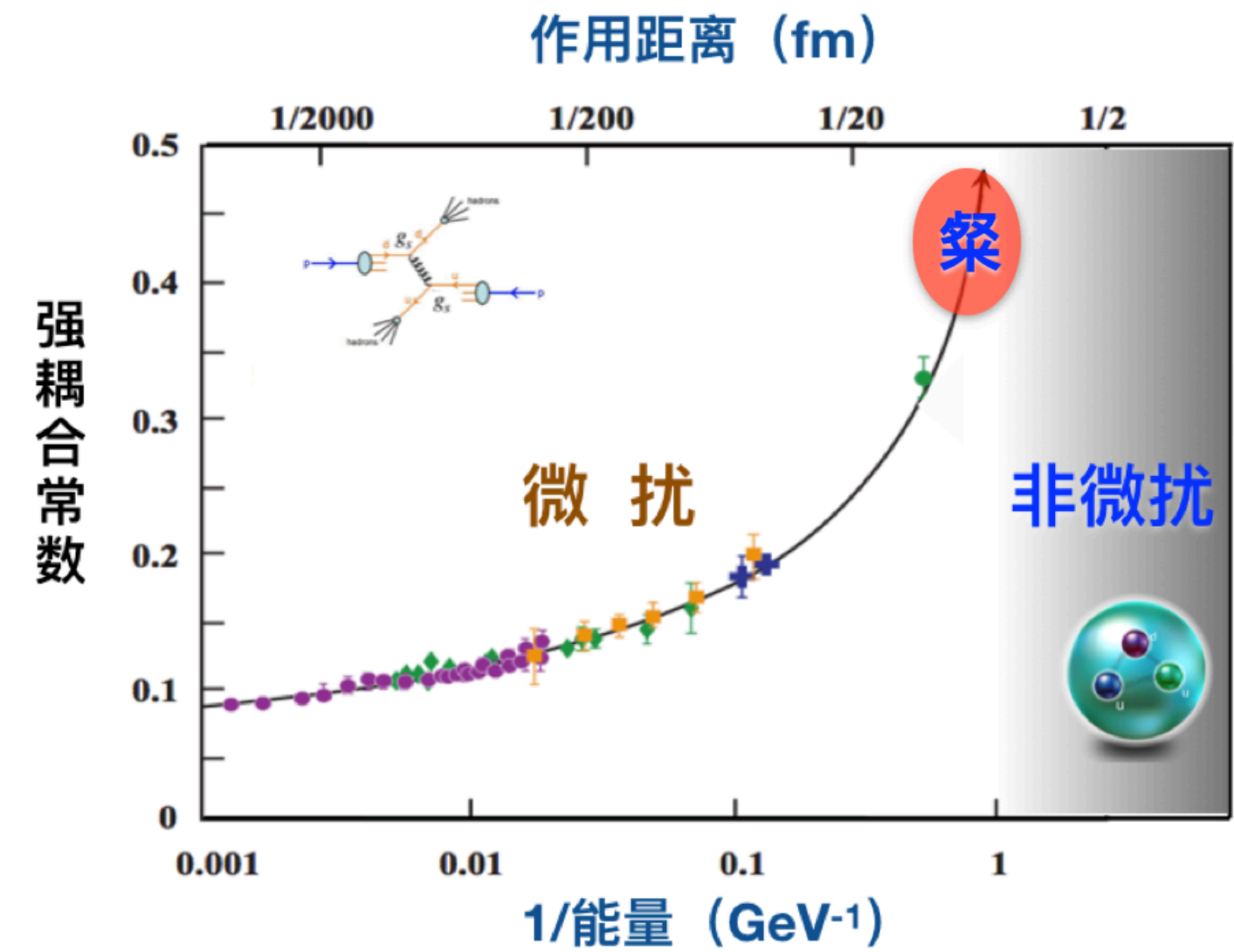
$$i \frac{\partial}{\partial t} \begin{pmatrix} D^0(t) \\ \bar{D}^0(t) \end{pmatrix} = \left( \mathbf{M} - \frac{i}{2} \mathbf{\Gamma} \right) \begin{pmatrix} D^0(t) \\ \bar{D}^0(t) \end{pmatrix}$$

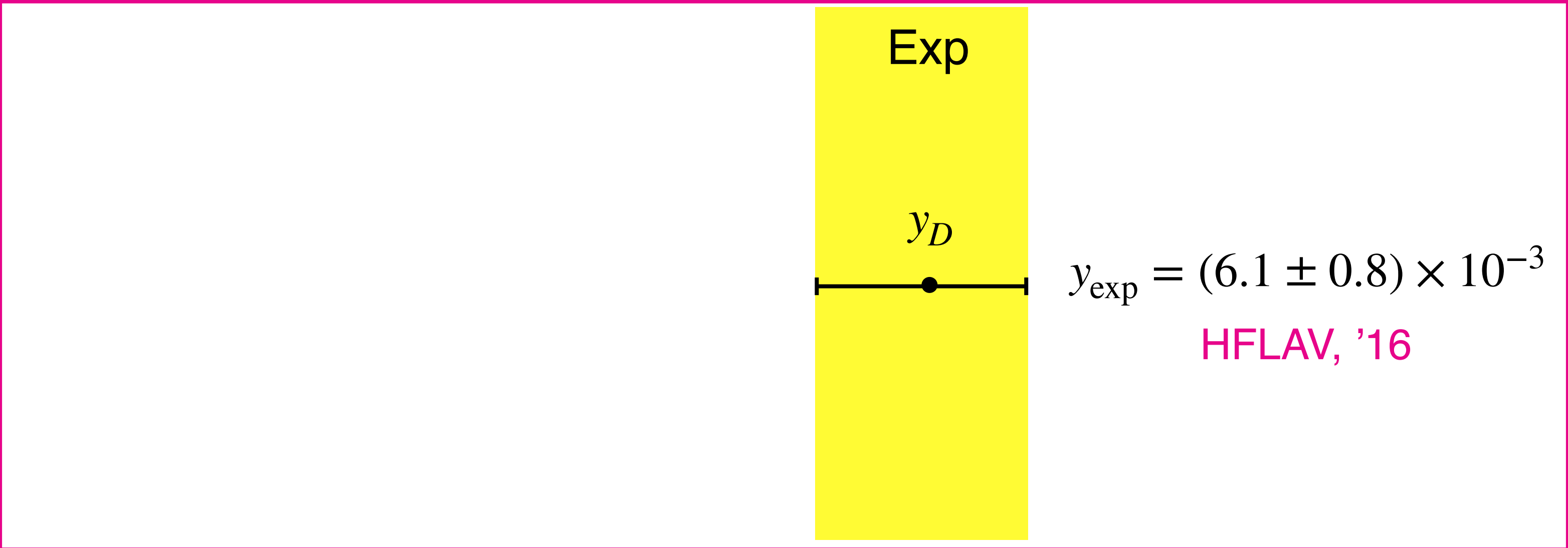
- Mixing parameters: Mass and Width differences

$$x \equiv \frac{\Delta m}{\Gamma} = \frac{m_1 - m_2}{\Gamma}$$

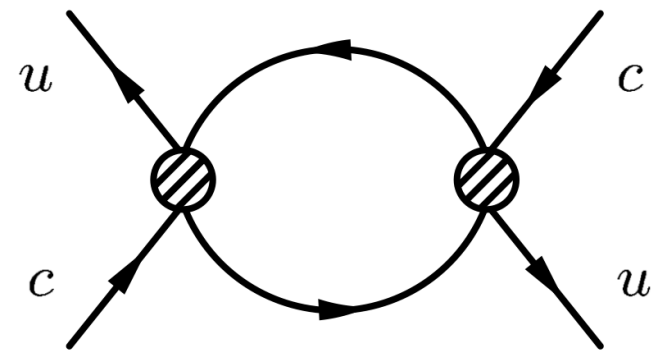
$$y \equiv \frac{\Delta \Gamma}{2\Gamma} = \frac{\Gamma_1 - \Gamma_2}{2\Gamma}$$

- Useful to search for new physics,
- but less understood in the Standard Model





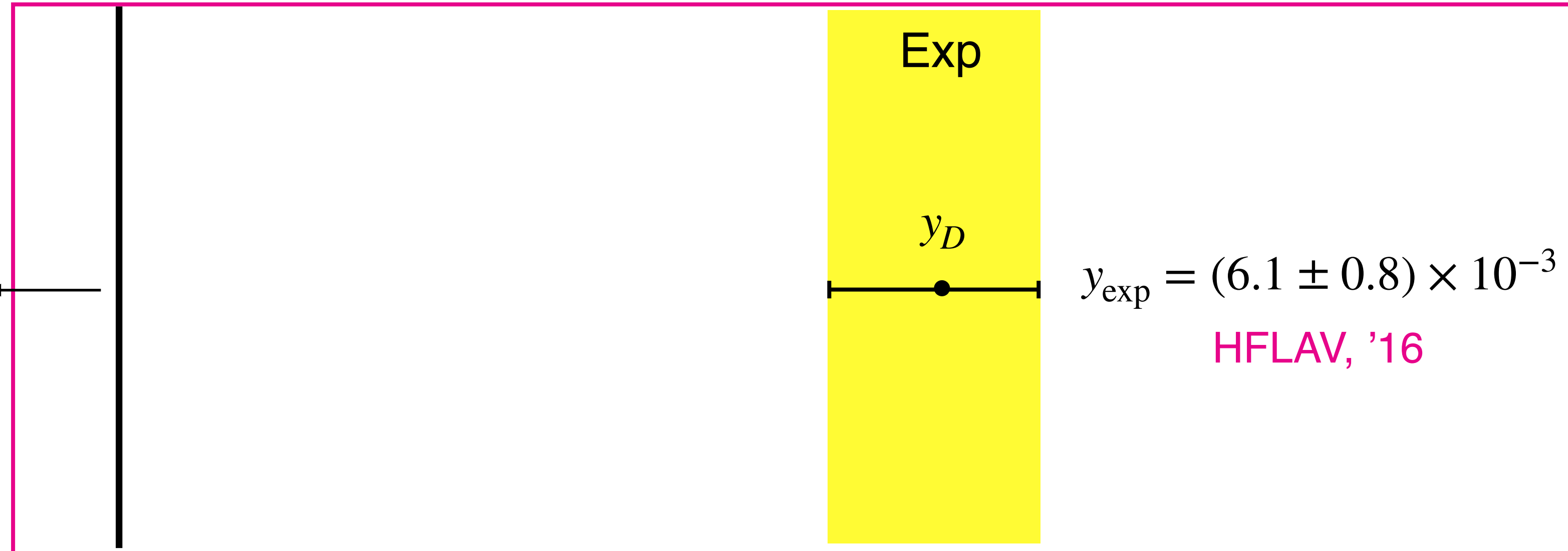




Inclusive approach  
doesn't work

$$y_{\text{incl}} \sim 10^{-7}$$

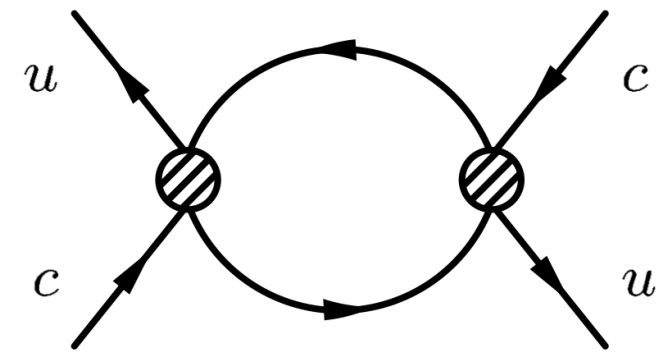
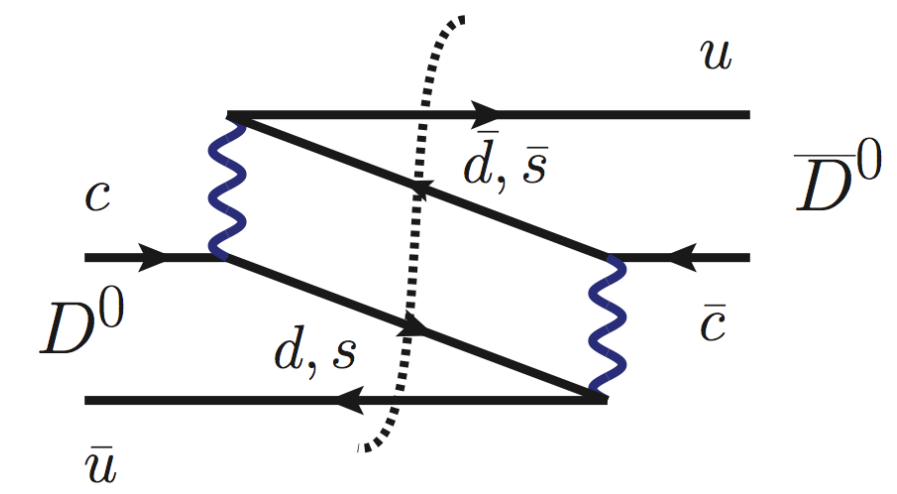
Lenz, *et al*, '12



Falk, et al, '02; Cheng, Chiang, '10

$$y_{PP+PV} = (3.6 \pm 2.6) \times 10^{-3}$$

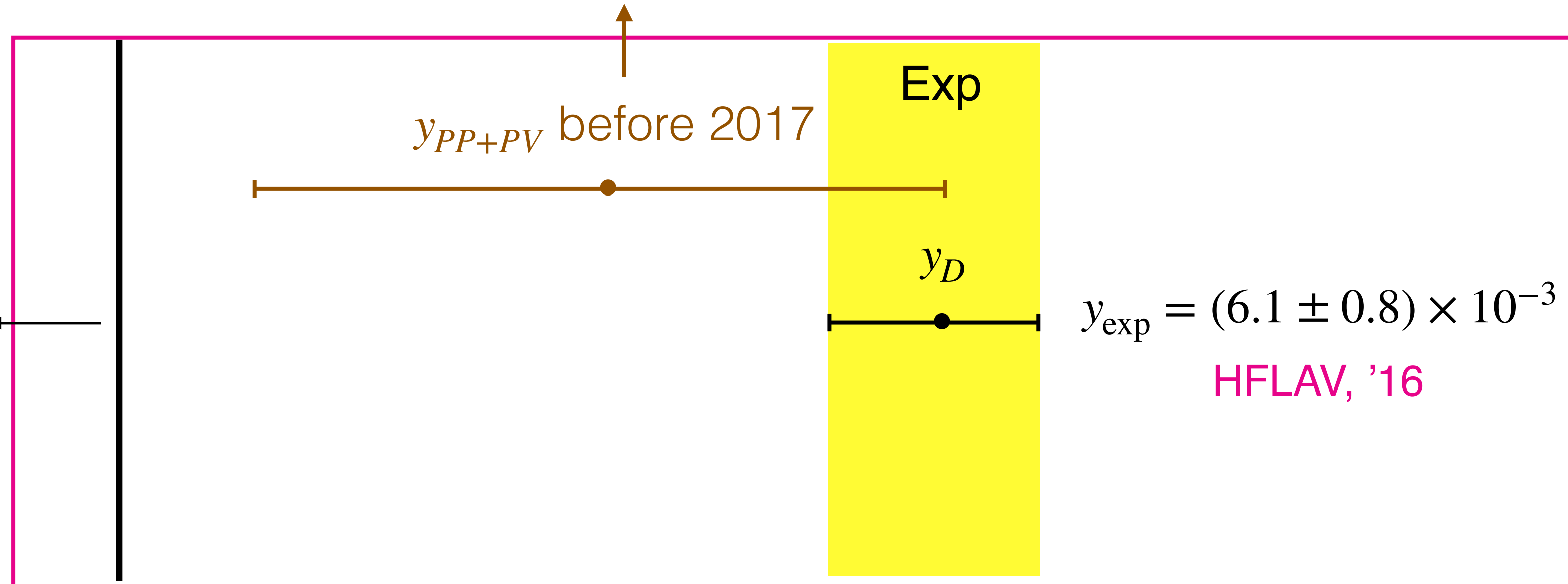
- Before 2017, exclusive approach is hopeful



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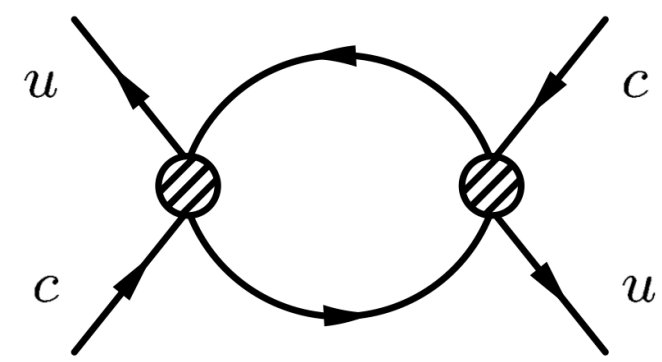
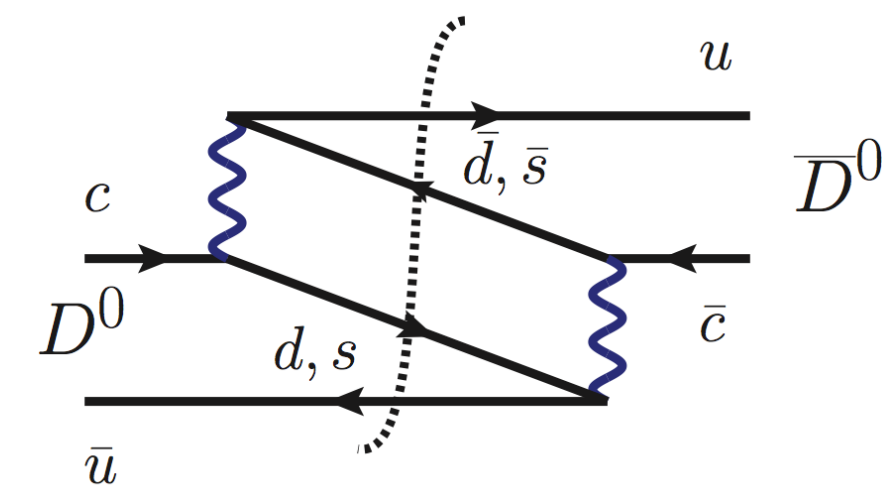
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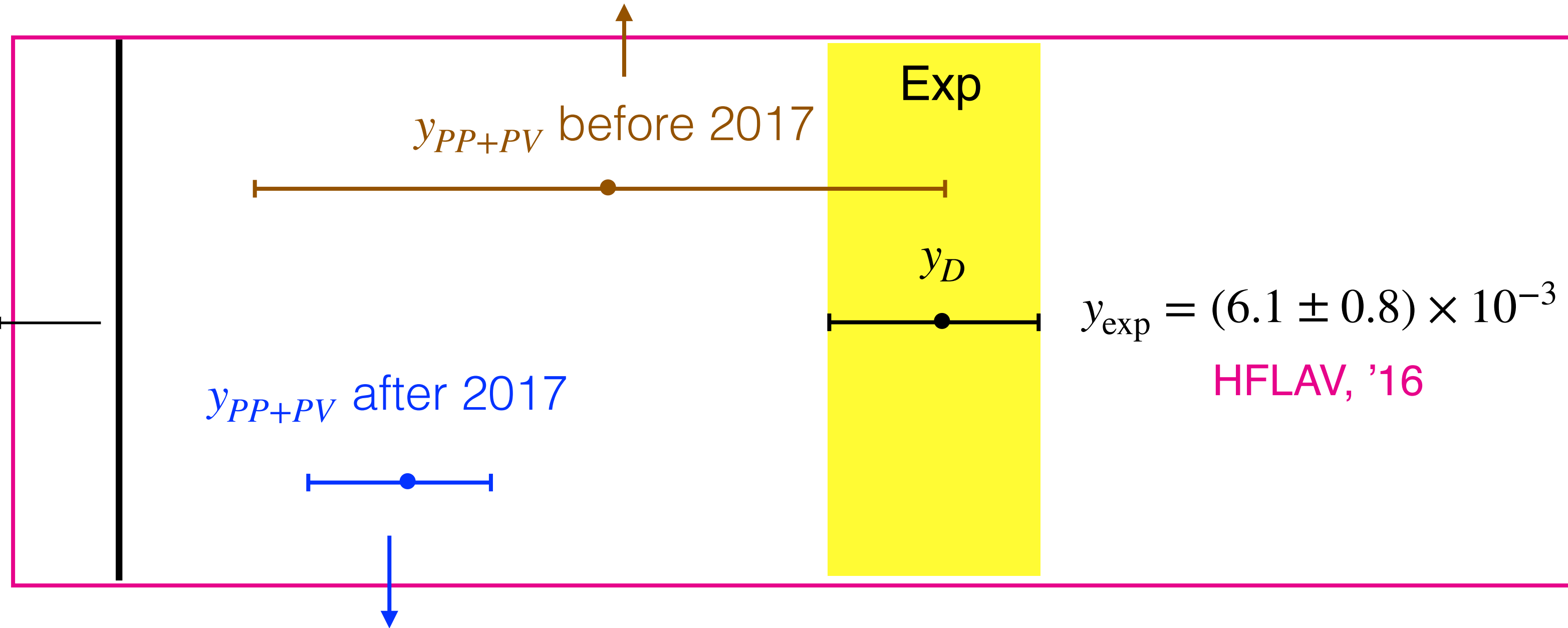
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Lenz, et al, '12



- After 2017, exclusive approach is dying

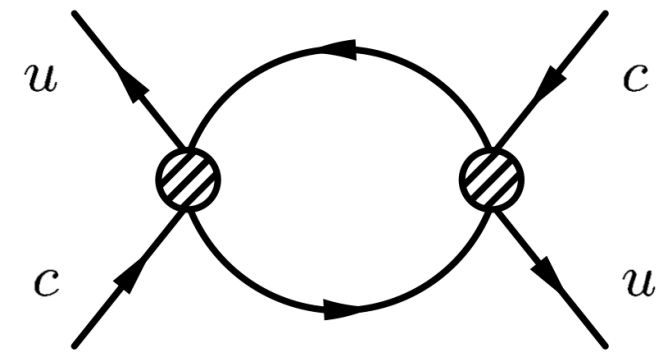
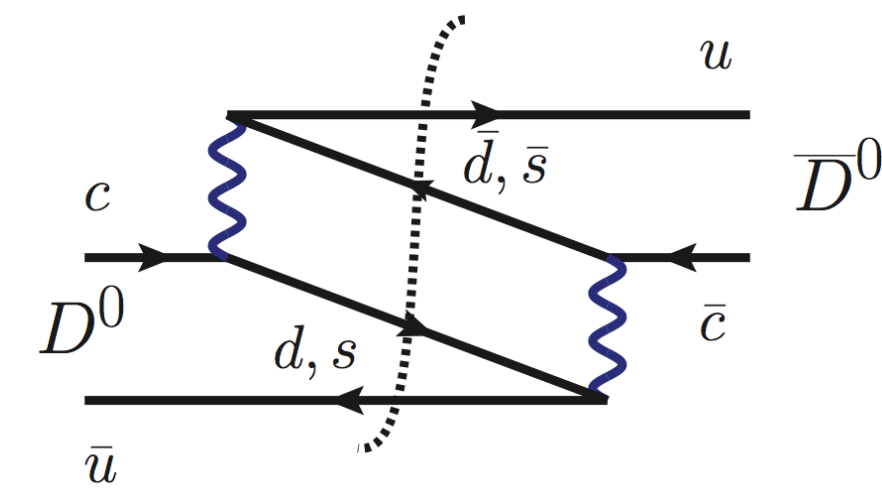
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Jiang, **FSY**, Qin, Li, Lü, '17

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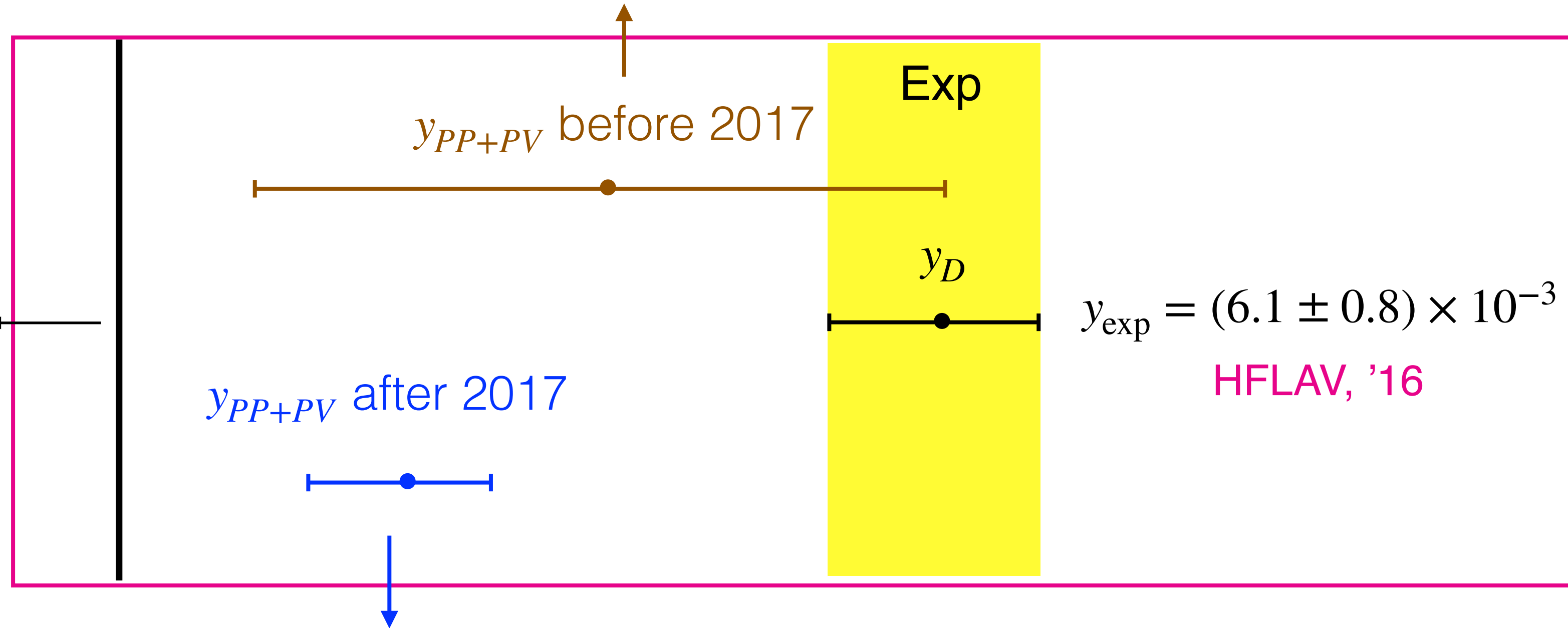
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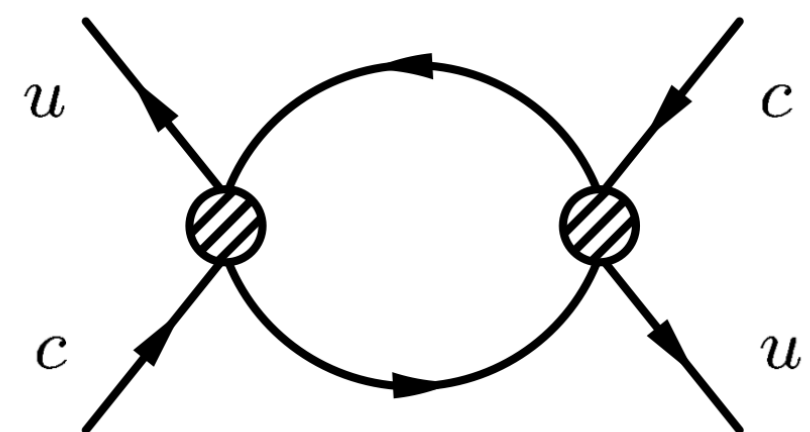
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Jiang, **FSY**, Qin, Li, Lü, '17

No theoretical methods work for D0 mixing  
No theoretical predictions for indirect CP violation

# Inclusive Approach

## Theory / Exp. comparison (for inclusive)



quark level

Short-distance

### D meson

Hagelin 1981, Cheng 1982  
 Buras, Slominski and Steger 1984  
 NLO QCD Golowich and Petrov 2005

$$\boxed{\text{SM}} \begin{cases} x \simeq 6 \times 10^{-7} \\ y \simeq 6 \times 10^{-7} \end{cases}$$

Suppressed by GIM

$$\boxed{\text{Exp.}} \begin{cases} x = (3.9^{+1.1}_{-1.2}) \times 10^{-3} \\ y = (6.51^{+0.63}_{-0.69}) \times 10^{-3} \end{cases}$$

HFLAV at Moriond2019

### $B_s$ meson

Artuso, Borissov and Lenz, 2016

$$\boxed{\text{SM}} \begin{cases} \Delta M_s = (18.3 \pm 2.7) \text{ps}^{-1} \\ \Delta \Gamma_s = (0.088 \pm 0.020) \text{ps}^{-1} \end{cases}$$

$$\boxed{\text{Exp.}} \begin{cases} \Delta M_s = (17.757 \pm 0.021) \text{ps}^{-1} \\ \Delta \Gamma_s = (0.082 \pm 0.006) \text{ps}^{-1} \end{cases}$$

HFLAV

### $B_d$ meson

Artuso, Borissov and Lenz, 2016

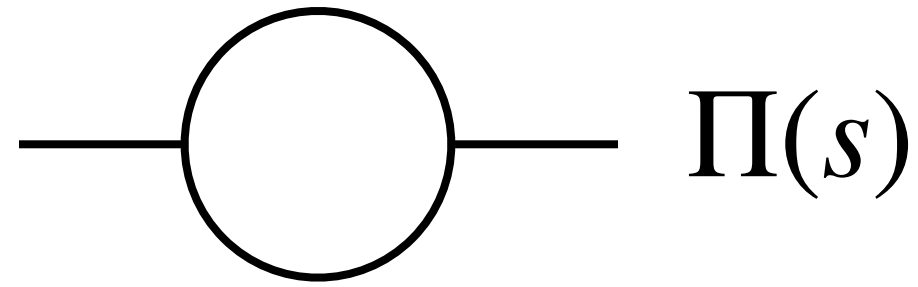
$$\boxed{\text{SM}} \begin{cases} \Delta M_d = (0.528 \pm 0.078) \text{ps}^{-1} \\ \Delta \Gamma_d = (2.61 \pm 0.59) \cdot 10^{-3} \text{ps}^{-1} \end{cases}$$

$$\boxed{\text{Exp.}} \begin{cases} \Delta M_d = (0.5055 \pm 0.0020) \text{ps}^{-1} \\ \Delta \Gamma_d = 0.66(1 \pm 10) \cdot 10^{-3} \text{ps}^{-1} \end{cases}$$

HFLAV

- For  $B_s, B_d$  mesons, the data are reproduced within  $1\sigma$ .
- For D meson, the order of magnitude is not reproduced within leading-power.

# Dispersion Relation



Dispersion Relation:

$$\text{Re}[\Pi(s)] = \frac{1}{\pi} \mathcal{P} \int_0^\infty \frac{\text{Im}[\Pi(s')]}{s - s'} ds'$$

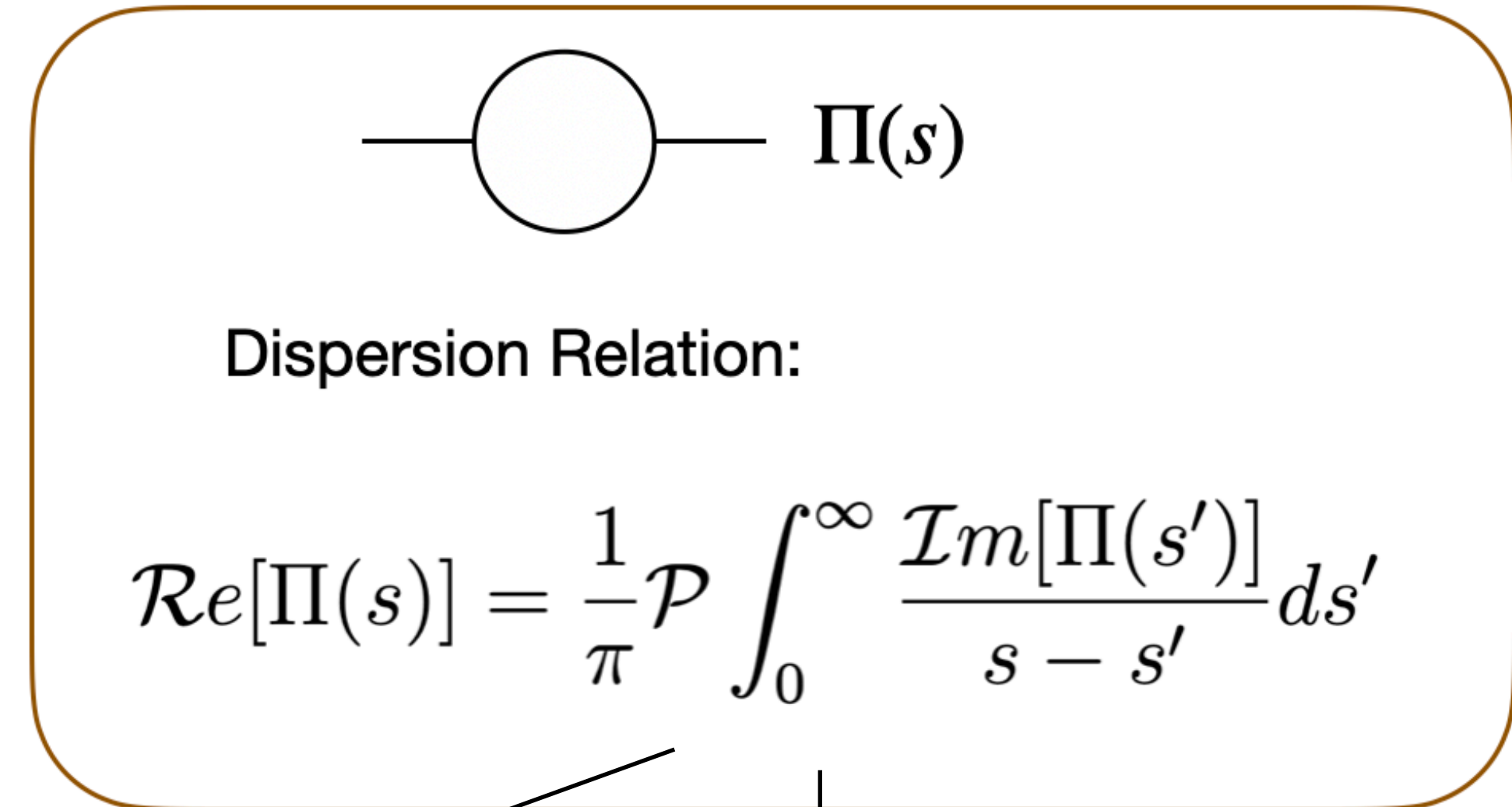
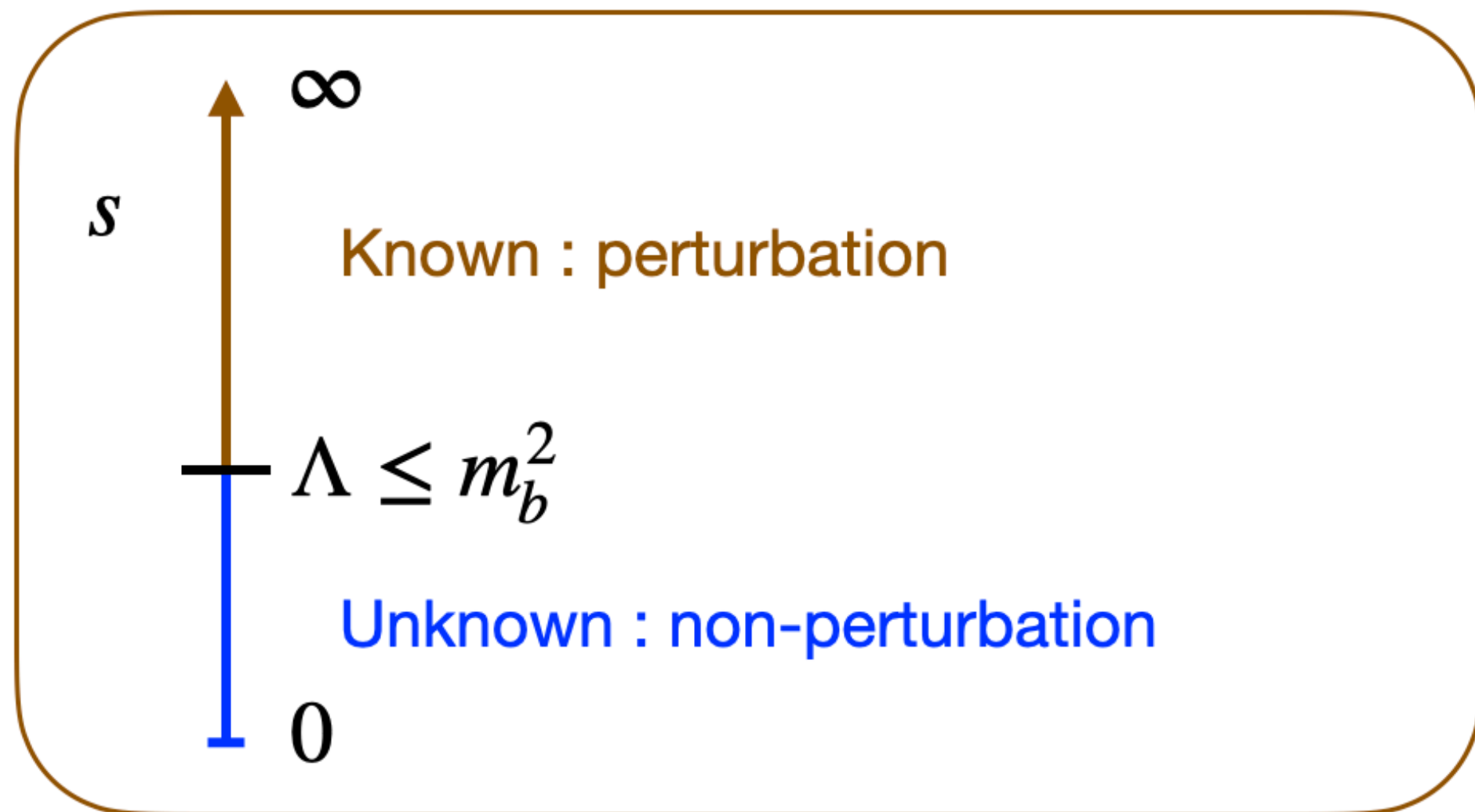
$D^0 - \bar{D}^0$  mixing:



$$\text{Re}[M_{12}(s)] = \frac{1}{\pi} \int_0^\infty \frac{\text{Im}[M_{12}(s')]}{s - s'} ds'$$

$$x(s) = \frac{1}{\pi} \int_0^\infty \frac{y(s')}{s - s'} ds'$$

# What is the Inverse Problem?



If  $s > \Lambda$ ,

$$\mathcal{P} \int_0^{\Lambda} \frac{\mathcal{I}m[\Pi(s')]}{s - s'} ds' = \pi \mathcal{R}e[\Pi(s)] - \mathcal{P} \int_{\Lambda}^{\infty} \frac{\mathcal{I}m[\Pi(s')]}{s - s'} ds'$$

To be solved
calculable

# Inverse Problem

$D^0 - \bar{D}^0$  mixing 

$$\int_0^\Lambda ds' \frac{y(s')}{s-s'} = \pi x(s) - \int_\Lambda^\infty ds' \frac{y(s')}{s-s'} \equiv \omega(s)$$

parametrization:

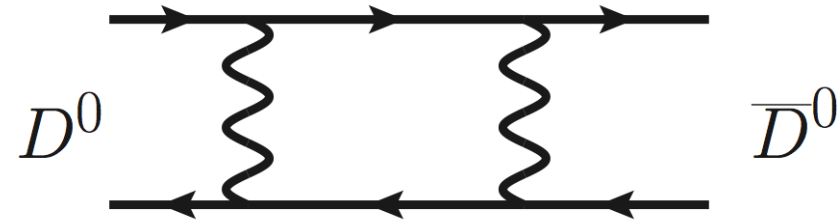
$$y(s) = \frac{Ns[b_0 + b_1(s - m^2) + b_2(s - m^2)^2]}{[(s - m^2)^2 + d^2]^2}$$

Li, Umeeda, Xu, **FSY**, PLB(2020)



# Inverse Problem

$D^0 - \bar{D}^0$  mixing

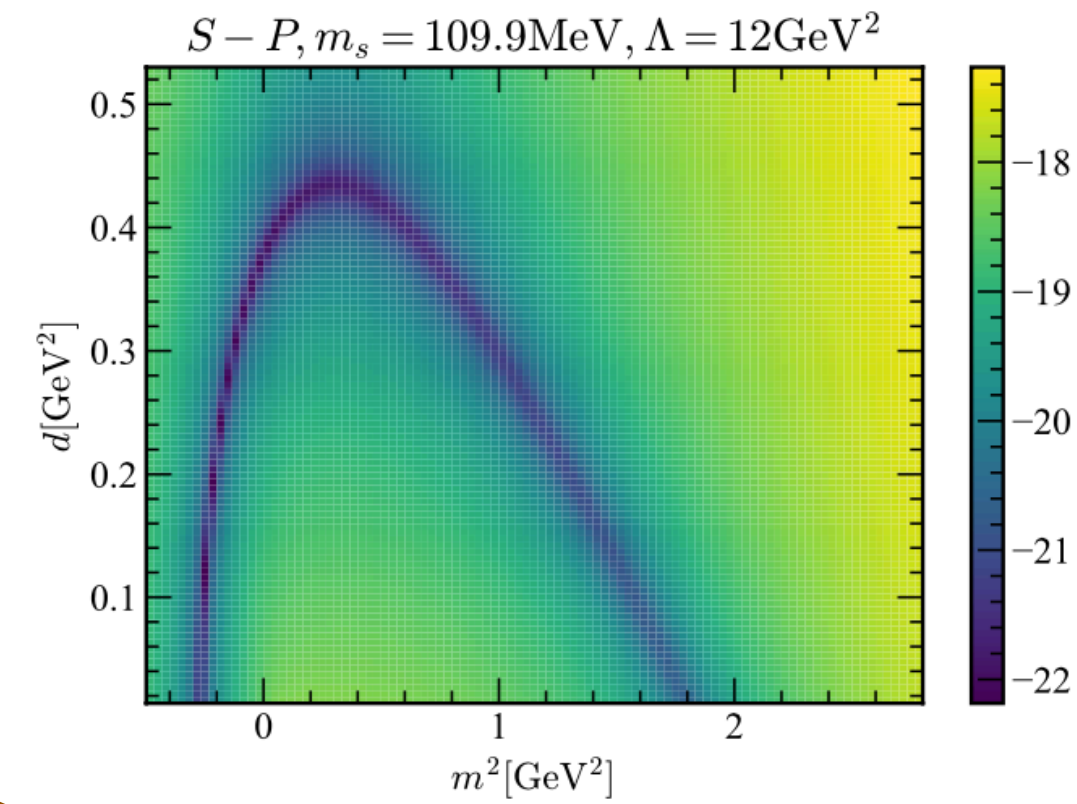


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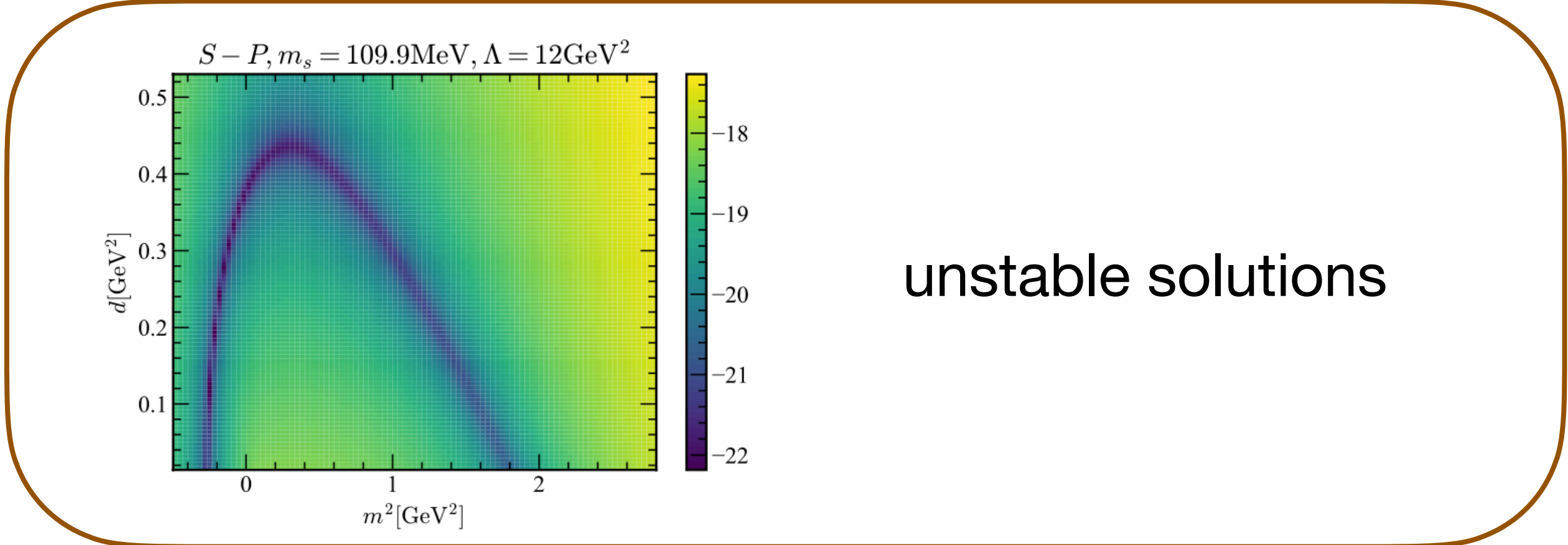
Li, Umeeda, Xu, **FSY**, PLB(2020)



unstable solutions

# Inverse Problem

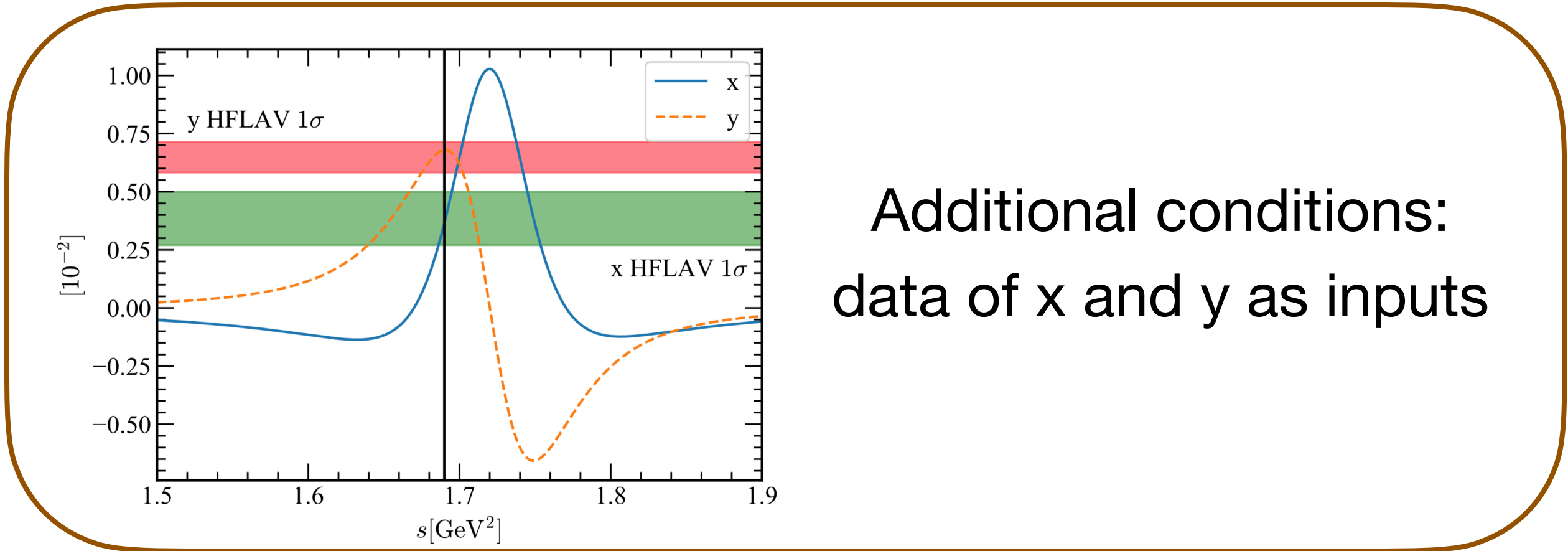
$D^0 - \bar{D}^0$  mixing 



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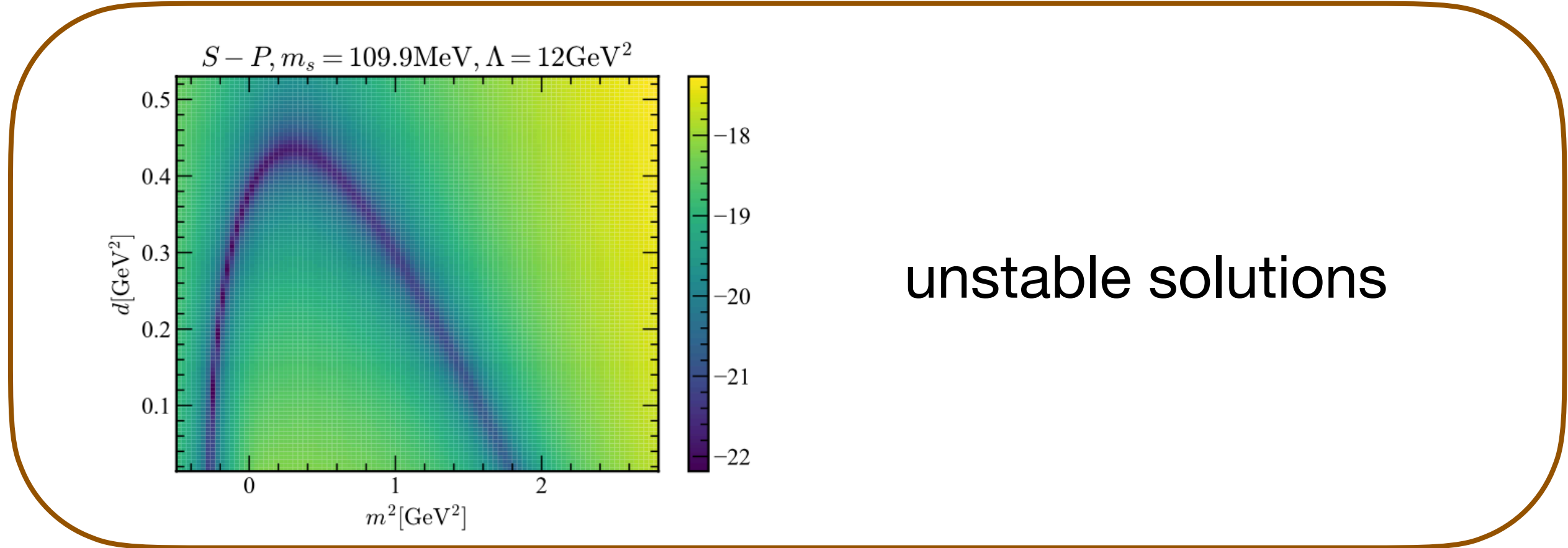
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Li, Umeeda, Xu, **FSY**, PLB(2020)

# Inverse Problem

$D^0 - \bar{D}^0$  mixing 

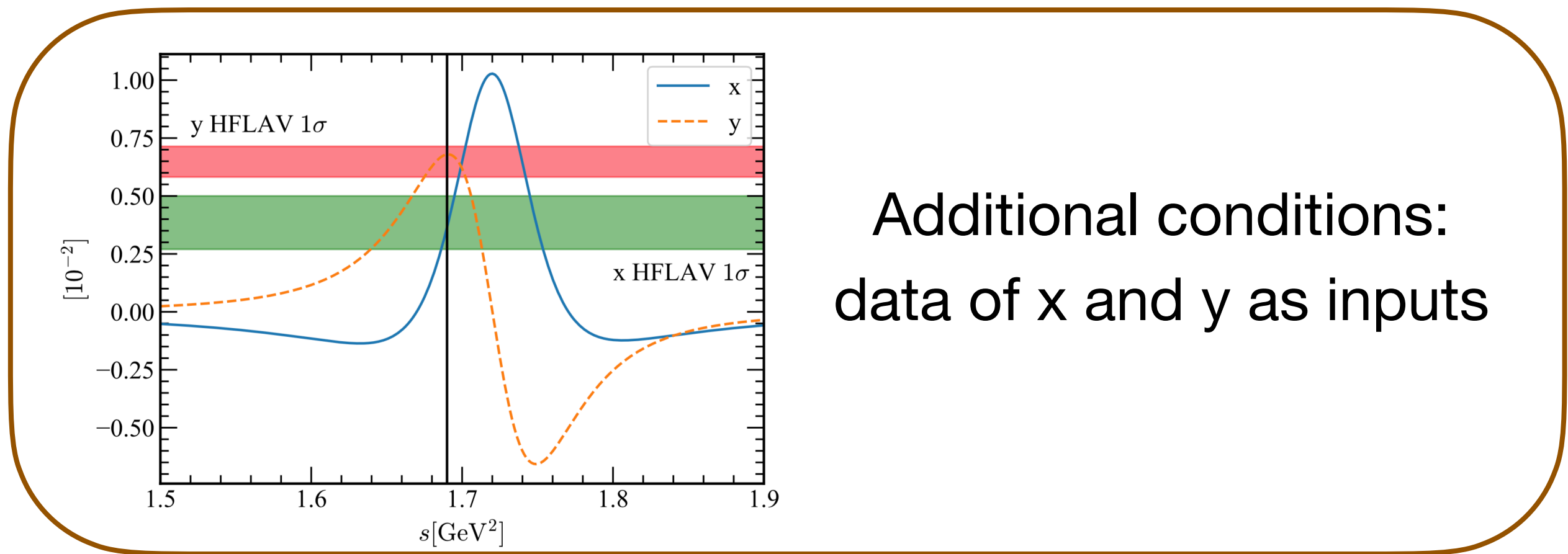


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Additional conditions:  
data of x and y as inputs

Predict indirect CPV consistent with data

$$q/p = 1.0002e^{i0.006^\circ} \quad q/p = (0.969_{-0.045}^{+0.050})e^{i(-3.9_{-4.6}^{+4.5})^\circ}$$

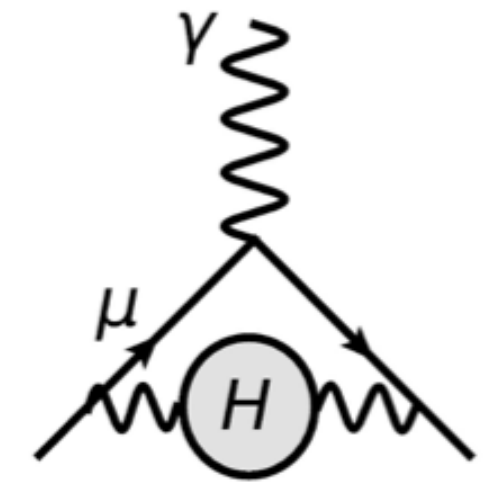
Li, Umeeda, Xu, **FSY**, PLB(2020)

# Applications of the Inverse Problem: muon $g-2$

- Muon  $g-2$ :  $4.2\sigma$  deviation from the SM
- Dominate uncertainty of the SM prediction: hadronic vacuum polarization (HVP)

Muon  $g-2$ , PRL(2021)

Aoyama, et al, Phys.Rept(2020)



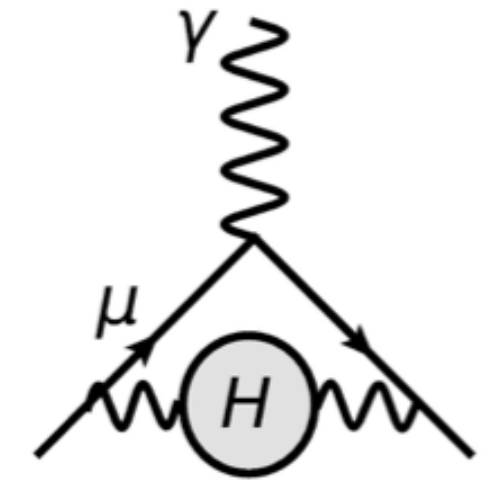
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Aoyama, et al, Phys.Rept(2020)



- Inverse Problem:

$$\int_{\lambda_r}^{\Lambda_r} ds' \frac{\text{Im}\Pi_r(s')}{s'(s'+s)} - \pi \frac{\Pi_r(0)}{s} = -\pi \frac{\Pi_r(-s)}{s} - \int_{\Lambda_r}^{\infty} ds' \frac{\text{Im}\Pi_r(s')}{s'(s'+s)}$$

$$r = \rho, \omega, \phi$$

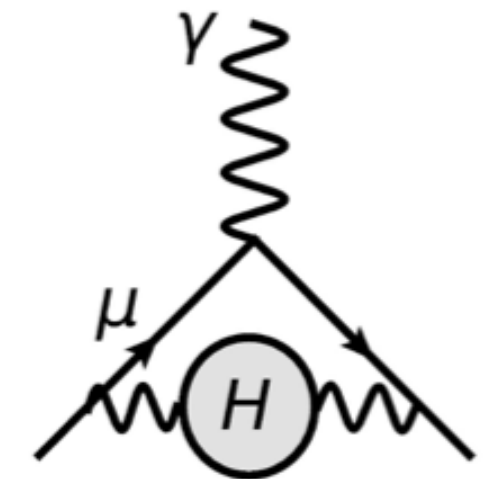
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- Result: **Inverse problem:**  $a_{\mu}^{\text{HVP}} = (641_{-63}^{+65}) \times 10^{-10}$

H.n.Li, Umeeda, '20

Data driven:  $a_{\mu}^{\text{HVP}} = (693.9 \pm 4.0) \times 10^{-10}$

Davier, Hoecker, Malaescu, Zhang, '20

Lattice QCD:  $a_{\mu}^{\text{HVP}} = (654 \pm 32_{-23}^{+21}) \times 10^{-10}$

Della Morte et al, '17

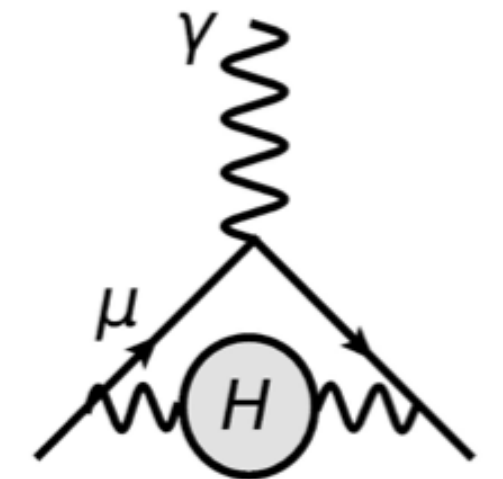
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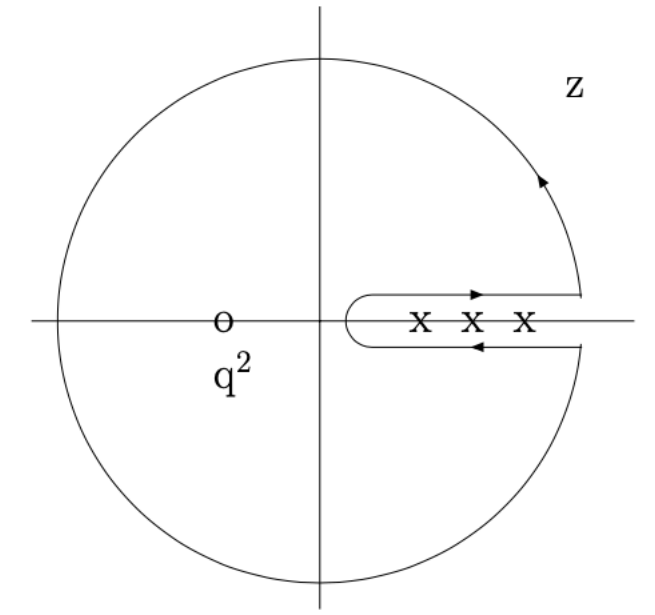
Non-perturbative properties can be revealed from asymptotic QCD by solving an inverse problem.

# Applications of the Inverse Problem: QCD sum rules

• Conventional QCD sum rules  $\Pi_{\mu\nu}(q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T [J_\mu(x) J_\nu(0)] | 0 \rangle$

Dispersion relation:  $\Pi(q^2) = \frac{1}{2\pi i} \oint ds \frac{\Pi(s)}{s - q^2} = \frac{1}{\pi} \int_{t_{min}}^{\infty} ds \frac{\text{Im } \Pi(s)}{s - q^2 - i\epsilon}$

$$\text{Im}\Pi(q^2) = \pi f_V^2 \delta(q^2 - m_V^2) + \pi \rho^h(q^2) \theta(q^2 - s_h)$$





# Applications of the Inverse Problem: QCD sum rules

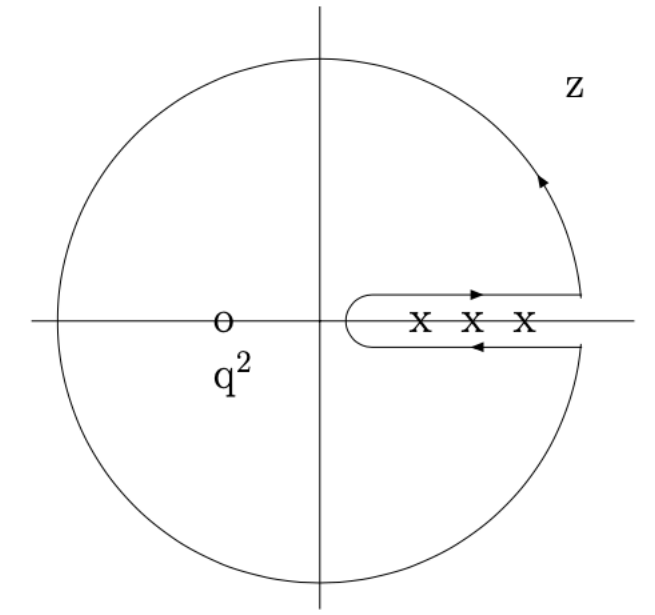
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Quark-hadron duality: 
$$\rho^h(s) = \frac{1}{\pi} \text{Im}\Pi^{\text{pert}}(s) \theta(s - s_0)$$

$$\int_{s_h}^{\infty} ds \frac{\rho^h(s)}{s - q^2} = \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\text{Im}\Pi^{\text{pert}}(s)}{s - q^2}$$



# Applications of the Inverse Problem: QCD sum rules

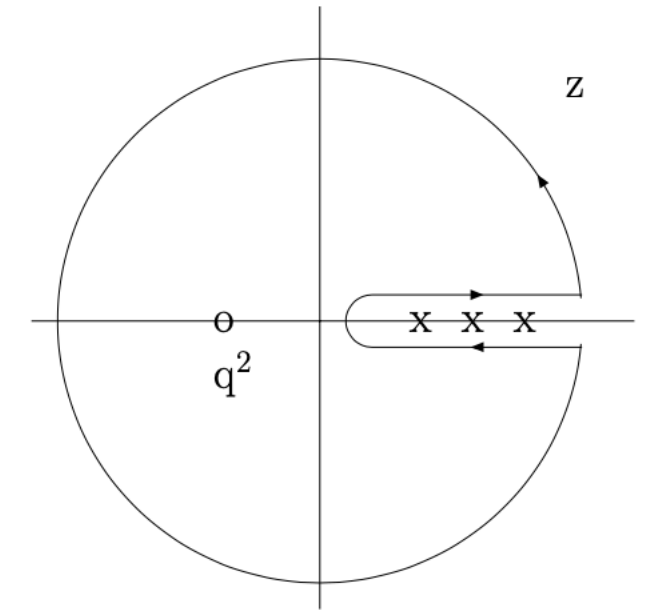
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**Uncertainty sources: quark-hadron duality and Borel transformation**

# Applications of the Inverse Problem: QCD sum rules

- Inverse-Problem QCD sum rules

$$\frac{1}{2\pi i} \oint ds \frac{\Pi(s)}{s - q^2} = \frac{1}{\pi} \int_{s_i}^{\Lambda} ds \frac{\text{Im}\Pi(s)}{s - q^2} + \frac{1}{\pi} \int_{\Lambda}^R ds \frac{\text{Im}\Pi^{\text{pert}}(s)}{s - q^2} + \frac{1}{2\pi i} \int_C ds \frac{\Pi^{\text{pert}}(s)}{s - q^2}$$

# Applications of the Inverse Problem: QCD sum rules

## • Inverse-Problem QCD sum rules

$$\frac{1}{2\pi i} \oint ds \frac{\Pi(s)}{s - q^2} = \frac{1}{\pi} \int_{s_i}^{\Lambda} ds \frac{\text{Im}\Pi(s)}{s - q^2} + \frac{1}{\pi} \int_{\Lambda}^R ds \frac{\text{Im}\Pi^{\text{pert}}(s)}{s - q^2} + \frac{1}{2\pi i} \int_C ds \frac{\Pi^{\text{pert}}(s)}{s - q^2}$$

**Involving excited states** and parameterization:

$$\begin{aligned} \text{Im}\Pi(q^2) = & \pi f_{\rho}^2 \delta(q^2 - m_{\rho}^2) + \pi f_{\rho(1450)}^2 \delta(q^2 - m_{\rho(1450)}^2) + \pi f_{\rho(1700)}^2 \delta(q^2 - m_{\rho(1700)}^2) \\ & + \pi f_V^2 \delta(q^2 - m_V^2) + \pi \rho^h(q^2), \end{aligned}$$

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$$\rho^h(y) = b_0 P_0(2y - 1) + b_1 P_1(2y - 1) + b_2 P_2(2y - 1) + b_3 P_3(2y - 1) + \dots$$

# Applications of the Inverse Problem: QCD sum rules

## • Inverse-Problem QCD sum rules

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$$\rho^h(y) = b_0 P_0(2y - 1) + b_1 P_1(2y - 1) + b_2 P_2(2y - 1) + b_3 P_3(2y - 1) + \dots$$

$$m_{\rho(770)} (m_{\rho(1450)}, m_{\rho(1700)}, m_{\rho(1900)}) \approx 0.78 (1.46, 1.70, 1.90) \text{ GeV}$$

$$f_{\rho(770)} (f_{\rho(1450)}, f_{\rho(1700)}, f_{\rho(1900)}) \approx 0.22 (0.19, 0.14, 0.14) \text{ GeV}$$

# 总结与展望

1. 反问题方法：计算非微扰物理量的新方法——用微扰QCD反解非微扰物理量
2. 在 $D^0 - \bar{D}^0$  mixing、muon g-2、QCD sum rules得到检验和应用
3. 色散关系的反问题是不适定的：解不稳定（已证明）

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  - 1) 新的数学方法将带来新的生命力
  - 2) 理论发展：误差分析—正则化方法
  - 3) 应用扩展：原则上低能非微扰都可以，首先努力在其它方法不太成功的地方，如粲物理能标、激发态和连续谱



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Thanks!

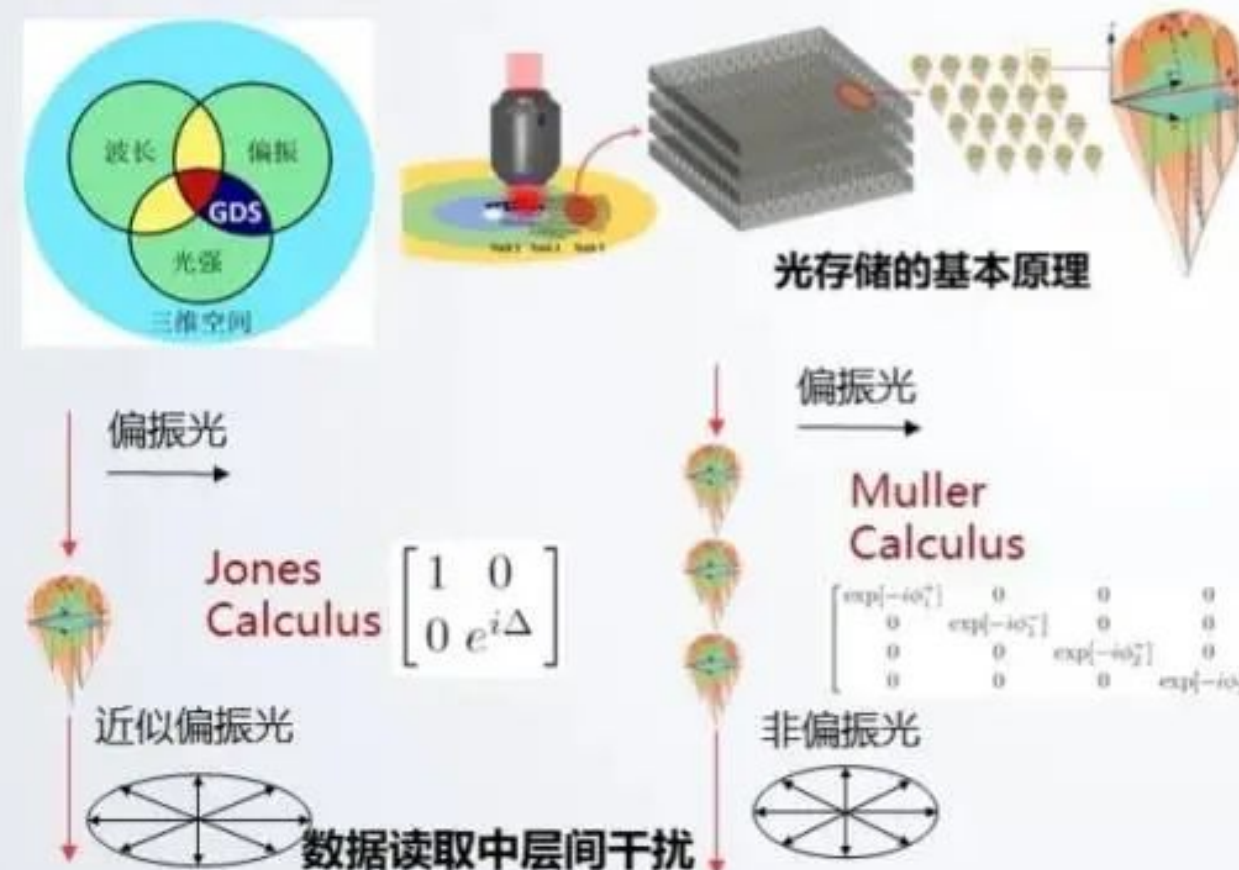
# 后香农时代，数学决定未来发展的边界

徐文伟  
华为董事、战略研究院院长



## 挑战问题7：反问题高精度快速求解

光存储在密度、存储时间、成本和存储环境要求上具备竞争力。尤其是玻璃存储能够存储超千年。  
**挑战：**高密度要求多层和多通道，不同层间或通道间的光干扰影响存储信号恢复的可靠性和精度。



### 数学模型

$$J_{out} = J_{Analyzer} \cdot J_{Polarizer} \cdot J_{sample} \cdot J_{Polarizer} \cdot J_{in}$$

$$\mathcal{L} = \sum ||j_i(\delta, \theta) - \Phi(A_i, \Lambda)||_2^2 + \mathcal{R} \leftarrow \text{正则化项}$$

### 主要挑战

- 反问题中正则化方法的选取
- 层间相互干扰的模型构建
- 数值方法的稳定性
- 基于数据的模型修正策略
- 高效求解算法构造
- 算法与硬件的适配

**问题：**探索层间相互干扰和通道间相互干扰的模型，寻找高精度、高速度、低延迟的算法，突破存储的世界纪录

## 误差分析：正则化方法

$$\int_a^b \frac{f(x)}{y-x} dx = g(y), \quad y \in [c, d], \quad c > b, \quad a > 0$$

1) 解的存在性



2) 解的唯一性



3) 解的稳定性



色散关系的反问题是不适定的

解的稳定性：**误差最关键**

正则化方法

$$(K + \lambda)x = y, \quad \lambda > 0$$

使得色散关系的反问题从不适定的变为适定的，从而得到相对稳定的解。

引入的正则化参数会带来新的误差，需要通过系统性的数学分析得到最终的理论误差。

构造正则化参数  $\lambda = \lambda(\delta_y)$  要与原始数据  $y$  的误差水平  $\delta_y$  相匹配，从而使得求解的  $x$  的误差  $\delta_x \leq f(\lambda, \delta_y)$