文 献 精 讲

Gravitational Radiation in the Limit of High Frequency. I. The Linear Approximation and Geometrical Optics

DOI: 10.1103/PhysRev.166.1263

Reporter：郭志伟

March 2，2021

Contents

1、Expansion of The Riemann Tensor

2、Gauge Transformations And Invariance

3、Linear Approximation

4、WKB Analysis of The Wave Equation

5、Summary

Abstract

A formalism is developed for obtaining approximate gravitational wave solutions to the vacuum Einstein equations of general relativity in situations where the gravitational fields of interest are quite strong. To accomplish this we assume the wave to be of high frequency and expand the vacuum field equations in powers of the correspondingly small wavelength, getting an approximation scheme valid for all orders of $1/r$, for arbitrary velocities up to that of light, and for all intensities of the gravitational field. To lowest order in the wavelength, we obtain a gauge-invariant linearized equation for gravitational waves which is just a covariant generalization of that for massless spin-2 fields in a flat background space. This wave equation is solved in the WKB approximation to show that gravitational waves travel on null geodesics of the curved background geometry with their amplitude, frequency, and polarization modified by the curvature of space-time in exact analogy to light waves.

弱场线性近似下的爱因斯坦场方程失去了引力场与自身作用的可能性，并且不适用强引力场问题。对于强引力场问题，本文假设引力波是高频的（波前可以看成是平面波），对场方程进行求解，利用WKB近似，得出引力波是沿着零测地线前行，这与光波类似。

1、Expansion of The Riemann Tensor

考虑一个四维流形$(M, g_{\mu\nu})$, $g_{\mu\nu}$ 是其时空度规

$$g_{\mu\nu}(x) = \gamma_{\mu\nu}(x) + \varepsilon h_{\mu\nu}(x, \varepsilon) \tag{1.1}$$
$$\varepsilon \ll 1,\ \gamma_{\mu\nu} = O(1),\ h_{\mu\nu} = O(1)$$

$\gamma_{\mu\nu}$ 背景度规，是时空缓坐标慢变化的函数， $h_{\mu\nu}$ 引力波项， ε 极小的参数

$$\partial\gamma \sim \gamma / L \qquad \partial h \sim h / \lambda \qquad\qquad \varepsilon = \lambda / L$$

L 是背景度规的特征长度，λ 是引力波的波长，L 远远大于λ

阶数

$$\gamma_{\mu\nu,\alpha} = O(1), \qquad h_{\mu\nu,\alpha} = O(\varepsilon^{-1})$$
$$\gamma_{\mu\nu,\alpha\beta} = O(1), \quad h_{\mu\nu,\alpha\beta} = O(\varepsilon^{-2})$$

对黎曼张量展开

$$R^{\lambda}_{\mu\sigma\nu}(\gamma_{\mu\nu}+\varepsilon h_{\mu\nu})=\overset{(0)}{R}{}^{\lambda}_{\mu\sigma\nu}+\varepsilon\overset{(1)}{R}{}^{\lambda}_{\mu\sigma\nu}+\varepsilon^2\overset{(2)}{R}{}^{\lambda}_{\mu\sigma\nu}+\varepsilon^3\overset{(3+)}{R}{}^{\lambda}_{\mu\sigma\nu} \tag{1.2}$$

各阶的阶数如下

项	符号形式	阶数
$\overset{(0)}{R}_{\alpha\beta}$	$\gamma^{-1}\partial^2\gamma$	$O(1)$
$\varepsilon\overset{(1)}{R}_{\alpha\beta}$	$\gamma^{-1}\partial^2(\varepsilon h)$	$O(\varepsilon^{-1})$
$\varepsilon^2\overset{(2)}{R}_{\alpha\beta}$	$\varepsilon h\gamma^{-2}\partial^2(\varepsilon h)$	$O(1)$
$\varepsilon^3\overset{(3+)}{R}_{\alpha\beta}$	$\varepsilon^2h^2\gamma^{-3}\partial^2(\varepsilon h)$	$O(\varepsilon)$

ε 为极小值 ε^{-1} 为较大值 所以 $\overset{(1)}{R}_{\alpha\beta}$ 占主要部分

联络

$$\Gamma^{c}_{ab}=\frac{1}{2}(\gamma^{cd}-\varepsilon h^{cd})[\nabla_b(\gamma_{ad}+\varepsilon h_{ad})+\nabla_a(\gamma_{bd}+\varepsilon h_{bd})-\nabla_d(\gamma_{ab}+\varepsilon h_{ab})] \tag{1.2 a}$$

$$\overset{(0)}{\Gamma}{}^{c}_{ab}=\frac{1}{2}\gamma^{cd}(\nabla_b\gamma_{ad}+\nabla_a\gamma_{bd}-\nabla_d\gamma_{ab}) \tag{1.2 b}$$

$$\overset{(1)}{\Gamma}{}^{c}_{ab}=[\frac{1}{2}\varepsilon\gamma^{cd}(\nabla_b h_{ad}+\nabla_a h_{bd}-\nabla_d h_{ab})]+[-\frac{1}{2}\varepsilon h^{cd}(\nabla_b\gamma_{ad}+\nabla_a\gamma_{bd}-\nabla_d\gamma_{ab})] \tag{1.2 c}$$

$$\overset{(2)}{\Gamma}{}^{c}_{ab}=-\frac{1}{2}\varepsilon^2h^{cd}(\nabla_b h_{ad}+\nabla_a h_{bd}-\nabla_d h_{ab}) \tag{1.2 d}$$

黎曼张量

$$R^{\lambda}_{\mu\sigma\nu}=\nabla_{\sigma}(\overset{(0)}{\Gamma}{}^{\lambda}_{\mu\nu}+\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu}+\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu})-\nabla_{\nu}(\overset{(0)}{\Gamma}{}^{\lambda}_{\mu\sigma}+\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\sigma}+\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\sigma})+(\overset{(0)}{\Gamma}{}^{\lambda}_{\rho\sigma}+\overset{(1)}{\Gamma}{}^{\lambda}_{\rho\sigma}+\overset{(2)}{\Gamma}{}^{\lambda}_{\rho\sigma})(\overset{(0)}{\Gamma}{}^{\rho}_{\mu\nu}+\overset{(1)}{\Gamma}{}^{\rho}_{\mu\nu}+\overset{(2)}{\Gamma}{}^{\rho}_{\mu\nu})-(\overset{(0)}{\Gamma}{}^{\lambda}_{\rho\nu}+\overset{(1)}{\Gamma}{}^{\lambda}_{\rho\nu}+\overset{(2)}{\Gamma}{}^{\lambda}_{\rho\nu})(\overset{(0)}{\Gamma}{}^{\rho}_{\mu\sigma}+\overset{(1)}{\Gamma}{}^{\rho}_{\mu\sigma}+\overset{(2)}{\Gamma}{}^{\rho}_{\mu\sigma})$$

$$\overset{(0)}{R}{}^{\lambda}_{\mu\sigma\nu}=R^{\lambda}_{\mu\sigma\nu}(\gamma_{\mu\nu})$$

$$\overset{(1)}{R}{}^{\lambda}_{\mu\sigma\nu}=\nabla_{\sigma}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu}-\nabla_{\nu}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\sigma}=\frac{1}{2}(h^{\lambda}_{\nu;\mu;\sigma}+h_{\mu\sigma}{}^{;\lambda}{}_{;\nu}-h_{\mu\nu}{}^{;\lambda}{}_{;\sigma}-h^{\lambda}_{\sigma;\mu;\nu}+\overset{(0)}{R}{}^{\lambda}_{\tau\sigma\nu}\,h^{\tau}_{\mu}-\overset{(0)}{R}{}^{\mu}_{\tau\sigma\nu}\,h^{\tau}_{\lambda}) \qquad (1.3)$$

2、Gauge Transformations And Invariance

考虑一个无穷小坐标变换 $x^{\mu}\to\bar{x}^{\mu}=x^{\mu}+\varepsilon\xi^{\mu}$

$$\overset{(1)}{\bar{R}}{}^{\lambda}_{\mu\sigma\nu}=\overset{(1)}{R}{}^{\lambda}_{\mu\sigma\nu}-\mathcal{L}_{\xi}\overset{(0)}{R}{}^{\lambda}_{\mu\sigma\nu} \qquad \mathcal{L}_{\xi}\overset{(0)}{R}{}^{\lambda}_{\mu\sigma\nu}=\overset{(0)}{R}{}^{\lambda}_{\mu\sigma\nu}{}_{;\tau}\,\xi^{\tau}+\overset{(0)}{R}{}^{\tau}_{\mu\sigma\nu}\,\xi_{\tau}{}^{;\lambda}+\overset{(0)}{R}{}^{\lambda}_{\tau\sigma\nu}\,\xi^{\tau}{}_{;\mu}+\overset{(0)}{R}{}^{\lambda}_{\mu\tau\nu}\,\xi^{\tau}{}_{;\sigma}+\overset{(0)}{R}{}^{\lambda}_{\mu\sigma\tau}\,\xi^{\tau}{}_{;\nu}$$

$$\overset{(1)}{R}{}^{\lambda}_{\mu\sigma\nu}-\overset{(1)}{\bar{R}}{}^{\lambda}_{\mu\sigma\nu}\doteq\varepsilon^{2} \qquad (2.1)$$

在高频极限下 $\lambda/L=\varepsilon\to 0$

3、Linear Approximation

$$R^{(1)\lambda}_{\mu\sigma\nu} = \frac{1}{2}(h^{\lambda}_{\nu;\mu;\sigma} + h_{\mu\sigma}{}^{;\lambda}{}_{;\nu} - h_{\mu\nu}{}^{;\lambda}{}_{;\sigma} - h^{\lambda}_{\sigma;\mu;\nu} + R^{(0)\lambda}_{\tau\sigma\nu} h^{\tau}_{\mu} - R^{(0)\mu}_{\tau\sigma\nu} h^{\tau}_{\lambda}) \tag{1.3}$$

做如下定义 $\varphi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}h\,, \qquad \varphi = \gamma^{\mu\nu}\varphi_{\mu\nu}$

爱因斯坦场方程，远源情况下 $R_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R = \kappa T_{\mu\nu}(=0)$

$$\Box\varphi_{\mu\sigma} - \frac{1}{2}\gamma_{\mu\sigma}\Box\varphi - \nabla_{\sigma}\nabla^{\nu}\varphi_{\mu\nu} - \nabla_{\nu}\nabla_{\mu}\varphi^{\nu}_{\sigma} + 2R^{(0)\lambda}_{\mu\sigma\nu}\varphi^{\nu}_{\lambda} + R^{(0)}_{\tau\sigma}\varphi^{\tau}_{\mu} + R^{(0)}_{\tau\sigma}\varphi^{\tau}_{\lambda} = 0 \tag{3.1}$$

洛伦兹规范条件 $\varphi^{\mu\nu}{}_{;\nu} = 0$ 横向无迹规范 $\varphi = 0$

$$\Box\varphi_{\mu\sigma} + 2R^{(0)\lambda}_{\mu\sigma\nu}\varphi^{\nu}_{\lambda} + R^{(0)}_{\tau\sigma}\varphi^{\tau}_{\mu} + R^{(0)}_{\tau\sigma}\varphi^{\tau}_{\lambda} = 0 \tag{3.2}$$

4、WKB Analysis of The Wave Equation

在平直时空下，平面波解的形式是 $\varphi_{\mu\nu} = A_{\mu\nu} e^{ik_\alpha x_\alpha}$ 振幅和波矢都是常数

在本文中，振幅和波矢是时空坐标缓慢变化的量 $k_a = \phi_{,\alpha}$

$$\varphi_{\mu\nu} = A_{\mu\nu} e^{i\phi} \tag{4.1}$$

各项阶数 $R^{(0)}_{\mu\sigma\nu}{}^{\lambda} = O(1)$, $A^{\mu\nu}{}_{;\tau} = O(1)$, $k_\mu = O(\varepsilon^{-1})$, $k_{\mu,\nu} = O(\varepsilon^{-1})$

二阶对称的爱因斯坦张量有10个分量，为减少其自由度将波解带入到洛伦兹规范条件和TT规范条件中，得

$$\gamma^{\mu\nu} A_{\mu\nu} = 0 \tag{4.2}$$

$$k_\beta A^{\alpha\beta} = 0 \longrightarrow \text{引力波是横波} \tag{4.3}$$

此情况下，引力场张量只有两个独立的分量，即两种偏振态

将4.1式带入到3.2式中可得

$$(-k^{\beta}k_{\beta}A_{\mu\nu})+i(2k_{\beta}A_{\mu\nu}{}^{;\beta}+A_{\mu\nu}k^{\beta}{}_{;\beta})+(A_{\mu\nu}{}^{;\beta}{}_{;\beta}+2\overset{(0)}{R}{}^{\lambda}_{\mu\sigma\nu}A^{\nu}_{\lambda}+\overset{(0)}{R}_{\tau\sigma}A^{\tau}_{\mu}+\overset{(0)}{R}_{\tau\sigma}A^{\tau}_{\lambda})=0 \qquad (4.4)$$

三个括号的阶数分别是 $\varepsilon^{-2},\varepsilon^{-1},1$　　取最低阶 $k^{\beta}k_{\beta}=0$ ⟶ 类光测地线，光速传播

第二项 $k_{\beta}A_{\mu\nu}{}^{;\beta}+\frac{1}{2}A_{\mu\nu}k^{\beta}{}_{;\beta}=0$　　定义极化张量 $e_{\mu\nu}=(A^{\mu\nu}A_{\mu\nu})^{-1/2}A_{\mu\nu}$　　$\mathcal{A}=(A^{\mu\nu}A_{\mu\nu})^{1/2}$

$$(\mathcal{A}_{,\beta}k^{\beta}+\frac{1}{2}\mathcal{A}\ k^{\beta}{}_{;\beta})e_{\mu\nu}+\mathcal{A}\ e_{\mu\nu;\beta}k^{\beta}=0 \qquad (4.5)$$

$$e_{\mu\nu;\beta}k^{\beta}=\frac{\delta e_{\mu\nu}}{\delta l}=0 \qquad (4.6)$$

4.6式表明极化张量沿着类光测地线传播。 在测地线上一点，可以假设初始条件是

$$e_{\mu\nu}k^{\nu}=0 \qquad e_{\mu\nu}\gamma^{\mu\nu}=0 \qquad (4.7)$$

引力波是横波 ⟵

$$k_{\beta}A^{\alpha\beta}=0 \qquad \gamma^{\mu\nu}A_{\mu\nu}=0$$

将 $\varphi_{\mu\nu} = A_{\mu\nu} e^{i\phi}$ 直接带入到一阶黎曼张量中

$$R^{(1)\lambda}_{\mu\sigma\nu} = \frac{1}{2}(h^{\lambda}_{\nu;\mu;\sigma} + h_{\mu\sigma}{}^{;\lambda}{}_{;\nu} - h_{\mu\nu}{}^{;\lambda}{}_{;\sigma} - h^{\lambda}_{\sigma;\mu;\nu} + R^{(0)\lambda}_{\tau\sigma\nu} h^{\tau}_{\mu} - R^{(0)\mu}_{\tau\sigma\nu} h^{\tau}_{\lambda})$$

$$\Downarrow$$

$$R^{(1)}_{\beta\mu\nu\rho} - 2k_{[\beta} h_{\mu][\nu} k_{\rho]} = 0 \qquad (4.11)$$

Wely张量是四倍主空矢量，在高频极限下属于Petrov-type N，而这与横向引力辐射有关

间接证明了利用WKB近似对高频微扰分析的正确性

5、Summary

1、对强引力场下的引力波采取高频WKB近似的方法得出引力波是横波，并且以光速传播的结论

2、高频WKB近似也可以应用到弯曲时空